Universidade Federal de São Carlos

Centro de Ciências Exatas e de Tecnologia

Departamento de Matemática

LOCAL COERCIVITY

FOR

SEMILINEAR ELLIPTIC PROBLEMS

Jose Miguel Mendoza Aranda

São Carlos

March 2018

LOCAL COERCIVITY

FOR
SEMILINEAR ELLIPTIC PROBLEMS

Universidade Federal de São Carlos
Centro de Ciências Exatas e de Tecnologia
Departamento de Matemática

LOCAL COERCIVITY FOR SEMILINEAR ELLIPTIC PROBLEMS

Jose Miguel Mendoza Aranda

Tese apresentada ao Programa de Pós-graduação em Matemática, como parte dos requisitos para a obtenção do título de Doutor em matemática.

Advisors: *Francisco Odair de Paiva*
David Arcoya

São Carlos
March 2018

ERRATA

Errata nos agradecimentos da tese -José Miguel Mendoza Aranda /Local Coercivity for Semilinear Elliptic Problems.

Página 01, linha 4 (Acknowledgment).

Onde se lê:
FAPESP for the financial support;

Leia-se:
Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) for the financial support related to the process number 2013/22044-0.

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Assinaturas dos membros da comissão examinadora que avaliou e aprovou a Defesa de Tese de Doutorado do candidato Jose Miguel Mendoza Aranda, realizada em 13/03/2018:

Prof. Dr. David Álvarez Arcoya
UGR

Prof. Dr. Gustavo Ferron Madeira
UFSCar

Prof. Dr. Ederson Moreira dos Santos
USP

Prof. Dr. Lucas Catão de Freitas Ferreira
UNICAMP

Prof. Dr. Sérgio Henrique Monari Soares
USP

Acknowledgment

First, I want to thank God.

My wife Rosalia and my parents.

My advisors Francisco Odair and David Arcoya.

FAPESP for the financial support.

Abstract

For a bounded domain Ω, a bounded Carathéodory function g in $\Omega \times \mathbb{R}$, $p > 1$, a nonnegative integrable function h in Ω which is strictly positive in a set of positive measure and a continuous function a which is superlinear with polynomial growth we prove that, contrarily with the case $h \equiv 0$, there exists a solution of the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u + g(x,u) - h(x)a(u) + f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

for every $\lambda \in \mathbb{R}$ and $f \in L^2(\Omega)$. And also give results of existence and multiplicity of similar problems, such that fractional laplacian problem, homogeneous problem and a concave perturbation of the above problem.

Resumo

Sejam Ω um domínio limitado, g uma função Carathéodory limitada em $\Omega \times \mathbb{R}$, $p > 1$, h uma função integrável não negativa em Ω e estritamente positiva num conjunto de medida positiva e a uma função continua e superlinear com crescimento polinomial provamos que, contrariamente no caso $h \equiv 0$, existe uma solução do problema elíptico semilinear

$$\begin{cases} -\Delta u &= \lambda u + g(x,u) - h(x)a(u) + f, \quad \text{em } \Omega \\ u &= 0, \qquad\qquad\qquad\qquad\qquad \text{sobre } \partial\Omega, \end{cases}$$

para cada $\lambda \in \mathbb{R}$ e $f \in L^2(\Omega)$. Também mostramos resultados de existência e multiplicidade de problemas similares como problema com laplaciano fracionário, problema homogêneo e uma perturbação do problema (0.1).

Contents

Introduction

Existence and multiplicity of solutions in Elliptic Problems are the main topic of this thesis. The first elliptic problem studied is the following:

$$\begin{cases} -\Delta u &= \lambda u + g(x,u) - h(x)a(u) + f, & \text{in } \Omega \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where Ω is a bounded domain, $\lambda \in \mathbb{R}$, g is a bounded Carathéodory function in $\Omega \times \mathbb{R}$, $f \in L^2(\Omega)$, $h \in L^1(\Omega)$ with $h \geq 0$ and a is a superlinear continuous function with polynomial growth. This problem is well-known when $h = 0$ a.e. in Ω (see [4]). Indeed, if we assume additionally that $g \equiv 0$, then the problem is linear and it has a solution of (0.2) for every datum $f(x)$ if and only if λ is not an eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ (Fredholm alternative). On the other hand, if $g \not\equiv 0$ the existence of solution remains valid for any λ which doesn't belong to the spectrum of $-\Delta$ in $H_0^1(\Omega)$. In the case that λ is an eigenvalue of this operator the existence of solution is not guaranteed, but assuming an additional hypothesis, for instance the Landesman-Lazer condition, the existence is established.

In this thesis we consider functions $h \geq 0$ which are different from zero. Respect to this case, the homogeneous semilinear elliptic equations (i.e., when $g = f = 0$) have been studied recently by several authors. In the particular case than $a(u) = |u|^{p-1}u$ Kazdan and Warner [13] obtained the first results in the context of curvature problem on compact manifolds, i.e., if $\lambda > 0$ and $h > 0$ then there is a positive solution $u > 0$ of the equation $-\Delta u = \lambda u - h|u|^{p-1}u$ on compact Riemannian manifold; Ouyang, in [15], considered the same equation that Kazdan and Warner on compact manifolds and bounded domains $\Omega \subset \mathbb{R}^n$ in case $h \leq 0$ and not only $h > 0$. He showed that there exists a $\widetilde{\lambda} > \lambda_1$ (λ_1 the first eigenvalue of the laplacian operator in Ω and $\widetilde{\lambda}$ the first eigenvalue of the laplacian operator in $\widetilde{\Omega} = \{x \in \Omega : h(x) = 0\}$) such that there is a unique positive solution $u_\lambda > 0$ of the problem

$$\begin{cases} -\Delta u &= \lambda u - h(x)|u|^{p-1}u, \quad \text{in } \Omega \\ u &= 0, \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{0.3}$$

if and only if $\lambda_1 < \lambda < \widetilde{\lambda}$. Ouyang also gave a result of the bifurcation curve of positive solutions, specifically $\lim_{\lambda \to \widetilde{\lambda}} \|u_\lambda\|_{L^2(\Omega)} = +\infty$; Del Pino and Felmer [10] deal with the existence, nonexistence and multiplicity of changing sign solutions of (0.3). Results with non power nonlinearities were obtained by Alama and Tarantello in [2], i.e., they gave similar results for the problem

$$\begin{cases} -\Delta u &= \lambda u - h(x)a(u), \quad \text{in } \Omega \\ u &= 0, \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{0.4}$$

with a being only a continuous function such that $\lim_{u\to 0} \frac{a(u)}{u} = 0$ and $\lim_{|u|\to\infty} \frac{a(u)}{u} = +\infty$. When the function $h(x)$ changes sign, the homogeneous elliptic problem (0.2) have been studied by Alama and Tarantello [1], Berestycki, Capuzzo-Dolcetta and Nirenberg [8], Ramos, Terracini and Troestler [19], among other authors.

To our knowledge, the only result on the nonhomogeneous problem (0.2) is obtained by Alama and Tarantello [3, Lemma A.3] for the case that $a(u) = |u|^{p-1}u$, where they showed existence of solution (corresponding to a minimum of the associated Euler functional) when

$$\lambda < \lambda_1(\widetilde{\Omega}) := \inf\left\{ \int_\Omega |\nabla u|^2\,dx \;:\; u \in H^1_D(\widetilde{\Omega}),\ \|u\|_2 = 1 \right\}$$

where $\widetilde{\Omega} = \{x \in \Omega : h(x) = 0\}$ and $H^1_D(\widetilde{\Omega}) := \{u \in H^1_0(\Omega) \;:\; u(x) = 0 \text{ a.e. } x \in \Omega\backslash\widetilde{\Omega}\}$. Notice that if meas $(\widetilde{\Omega}) = 0$ (i.e. $h > 0$ a.e. in Ω), then $H^1_D(\widetilde{\Omega}) = \{0\}$ and $\lambda_1(\widetilde{\Omega}) = +\infty$, while, in the case that it would be meas $(\Omega\backslash\widetilde{\Omega}) = 0$ (i.e. $h = 0$ a.e. in Ω) we would have that $\lambda_1(\widetilde{\Omega})$ would not be but the first eigenvalue λ_1 of the Laplacian operator $-\Delta$ with zero Dirichlet boundary conditions.

Thus, similarly to the case $h = 0$ a.e. in Ω in which the existence of solution of (0.2) depend on the interplay between λ and the spectrum of $-\Delta$ in $H^1_0(\Omega)$, one can think that, in the case that $h \neq 0$, the existence will depend on the relationship between λ and the spectrum of the unique self-adjoint operator H_∞ associated to the quadratic form $b(u) = \int_\Omega |\nabla u|^2\, dx$ with domain $H^1_D(\widetilde{\Omega})$. Nevertheless, we show that the presence of the nontrivial h possesses a regularizing effect with respect to the existence. Indeed, we prove that if $h \neq 0$, then there exists a solution of (0.2) for every $\lambda \in \mathbb{R}$, $f \in L^2(\Omega)$ and $p > 1$.

Next, we consider the problem (0.2) for the fractional laplacian operator:

$$\begin{cases} (-\Delta)^s u &= \lambda u + g(x,u) - h|u|^{p-1}u + f, \quad \text{in } \Omega \\ u &= 0, \qquad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{0.5}$$

where $n > 2s$ and for $s \in (0,1)$, $(-\Delta)^s$ is the nonlocal fractional Laplace operator defined on the space

$$H^s(\Omega) = \{u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dxdy < \infty\}.$$

by

$$(-\Delta)^s u(x) = C(n,s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \ \ x \in \mathbb{R}^n,$$

with

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2s}}\, d\xi \right)^{-1}.$$

For the classical Laplacian operator, the problem (3.1) was studied by Alama and Tarantello (see [2]) when $h \not\equiv 0$ and $f = g = 0$. Their obtained results about the existence and multiplicity of nontrivial solutions are based on the interaction of the parameter λ with the spectrum of the Laplacian operator in $\widetilde{\Omega}$. This is consistent with the case $h \equiv 0$ (i.e., $\widetilde{\Omega} = \Omega$) in which the existence of solutions for general f and g depends on the position of λ with respect to the spectrum of the Laplacian operator in Ω. However, recently Arcoya, Paiva and Mendoza in [5] (and in this thesis) showed that if $h \not\equiv 0$ the existence of solutions does not depends on the spectrum of the Laplacian operator in $\widetilde{\Omega}$. We extend this result to the fractional Laplacian operator by proving the existence of solution of problem (0.5) for every λ.

The last problem considered in this thesis is a concave perturbation of problem (0.4)

$$\begin{cases} -\Delta u &= -\mu|u|^{q-2}u + \lambda u - h(x)a(u), & \text{in } \Omega \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \tag{0.6}$$

where $\lambda_1 < \lambda < \lambda_1(\widetilde{\Omega})$, $\mu > 0$, $1 < q < 2$, a is a superlinear continuous function with polynomial growth and $0 \leq h \in L^\infty(\Omega)$ with $h \neq 0$. In the case that $\mu = 0$, $\lambda_1 < \lambda < \lambda_1(\widetilde{\Omega})$ and $p \in (1, +\infty)$, Alama and Tarantello in [2] showed that if $N(\lambda) = 1$ (see Chapter 3) and $\frac{a(u)}{|u|}$ is strictly increasing for $u \neq 0$, then problem (0.6) only have two nontrivial solutions (one positive and one negative) and if $N(\lambda) \geq 2$, then there exists a third nontrivial solution. Perera in [16] shows existence and multiplicity of nontrivial solutions of problem (0.6) when $h \equiv C \equiv$constant, specifically he shows that problem (0.6) have at least 4 nontrivial solutions (two positive and two negative) and if $\lambda_k < \lambda < \lambda_{k+1}$, $\lambda < \lambda_1(\widetilde{\Omega})$, then problem (0.6) have at least 5 nontrivial solutions. Thus we see that the perturbated problem obtain more solutions than the original problem. We obtain similar results than Perera when h is a $L^\infty(\Omega)$ function and not only a constant.

This thesis is organized as follows. Chapter 1 provides the proof of the existence of one solution of problem (0.2). In Section 2 we present a compactness condition, similar to the (P.S.) condition. In Section 3 we split the proof in 3 cases. Chapter 2 deal with the problem (0.5) and in Chapter 3 we consider two problems: In Section 3 we study the homogeneous case of problem (0.2) and show existence and multiplicity. In Section 4 we study problem (0.6).

Chapter 1

Preliminaries

1.1 The Space E

In this section, we are going to define the principal spaces used in this thesis and also give some results.

First, we have some notations:

- $L^p(\Omega) \equiv$ Space of Lebesgue-measurable functions $u : \Omega \to \mathbb{R}$ with finite $L^p(\Omega)$ norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{1/p}, \ 1 \leq p < \infty.$$

- We will denote the $L^2(\Omega)$ norm of $u \in L^2(\Omega)$ by $\|u\|_2 = \int_\Omega u^2 \, dx$.

- For some Lebesgue-measurable function $h \geq 0$, we denote the Banach space $L^p(\Omega, hdx) \equiv \{f : \Omega \to \mathbb{R} : f$ is a measurable function, with $\int_\Omega |f|^p h \, dx < \infty\}$, $1 \leq p < \infty$ and its norm

$$\|f\|_{L^p(\Omega, hdx)} = \left(\int_\Omega |f|^p h \, dx \right)^{1/p}.$$

- $C^m(\Omega) \equiv$ Space of m times continuosly differentiable functions $u : \Omega \to \mathbb{R}$.

- $C_0^m(\Omega) \equiv$ Space of $C^m(\Omega)-$functions with compact support in Ω.

Definition 1.1. Let Ω be a open subset of $\mathbb{R}^n$. We define the Hilbert space $H^1(\Omega)$ as

$$H^1(\Omega) = \{f \in L^2(\Omega) : f \text{ has a weak derivate, } \nabla f, \text{ with } |\nabla f| \in L^2(\Omega)\}$$

with scalar product

$$\langle u, v\rangle = \int_\Omega uv\,dx + \int_\Omega \nabla u \nabla v\,dx \quad \forall u, v \in H^1(\Omega).$$

and the associated norm

$$\|u\|_{H^1(\Omega)} = \int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx \quad \forall u \in H^1(\Omega).$$

We also define the Hilbert space $H_0^1(\Omega)$ as the closure of $C_0^1(\Omega)$ in $H^1(\Omega)$ equipped with the $H^1(\Omega)$ scalar product.

In this thesis we are going to work on bounded domains Ω. For such Ω we have the following result:

Theorema 1.2 (Poincaré's inequality)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set. Then there exists a constant $C = C(\Omega)$ such that*

$$\|u\|_2 \leq C\|\nabla u\|_2 \quad \forall u \in H_0^1(\Omega).$$

Thus we have that the expression $\|\nabla u\|_2$ is a norm on $H_0^1(\Omega)$ and it is equivalent to the norm $\|u\|_{H^1(\Omega)}$. In this thesis, we will use this norm on $H_0^1(\Omega)$ and will be denoted by $\|u\| = \|\nabla u\|_2$ for every $u \in H_0^1(\Omega)$.

Now, for some $p > 1$ and a measurable function $h : \Omega \to \mathbb{R}$ with $h \geq 0$, we define the Banach space E as

$$E = \{u \in H_0^1(\Omega) \,:\, \int_\Omega h|u|^{p+1} < +\infty\},$$

endowed with the norm

$$\|u\|_E = \|u\|_{H_0^1(\Omega)} + \left(\int_\Omega h|u|^{p+1}\,dx\right)^{1/(p+1)}.$$

The principal result about this space is that E is a Reflexive space. To show this, we are going to use the exercise 4.16 from [7] to show the following lemma:

Lemma 1.3. *Let* $1 < p < +\infty$, $\{f_n\} \subset L^p(\Omega, hdx)$, $h \geq 0$ *and measurable in* Ω *and*

a) $\|f_n\|_{L^p(\Omega,hdx)} \leq C$,

b) $f_n \to f$ *a.e. in* Ω.

Then $f \in L^p(\Omega, hdx)$ *and* $f_n \rightharpoonup f$ *in* $L^p(\Omega, hdx)$.

Proof. For the proof, we define $g_n = h^{1/p}.f_n \in L^p(\Omega)$. Then

$$\int_\Omega |g_n|^p \, dx = \int_\Omega h.|f_n|^p \, dx \leq C,$$

and $g_n \to h^{1/p}.f = g$ a.e. in Ω. Now we can apply the exercise 4.16 for g_n and so $g_n \rightharpoonup g$ in $L^p(\Omega)$. Finally calling p' such that $1/p + 1/p' = 1$ and for all $\varphi \in L^{p'}(\Omega, hdx)$ we have $\varphi.h^{1/p'} \in L^{p'}(\Omega)$ and thus

$$\int_\Omega f_n.\varphi.h \, dx = \int_\Omega g_n.\varphi.h^{1/p'} \, dx \longrightarrow \int_\Omega g.\varphi.h^{1/p'} \, dx = \int_\Omega f.\varphi.h \, dx,$$

concluding this lemma. □

Now, we use this lemma to show the reflexivity of the space E.

Lemma 1.4. *The Banach space* E *is reflexive.*

Proof. Let be $\{u_n\} \subset E$ a sequence such that $\|u_n\|_E \leq C$. Then $\{u_n\} \subset H_0^1(\Omega)$ is bounded in $H_0^1(\Omega)$ and, up to a subsequence, we can assume $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, $u_n \to u_0$ in $L^2(\Omega)$ and a.e. in Ω. Moreover, the sequence $\{u_n\} \subset L^{p+1}(\Omega, hdx)$ is bounded in $L^{p+1}(\Omega, hdx)$ and we can apply the Lemma 1.3 to obtain that $u_n \rightharpoonup u_0$ in $L^{p+1}(\Omega, hdx)$ and thus that $u_n \rightharpoonup u_0$ in E. □

1.2 Some Variational theorems

Let I be a Fréchet-differentiable functional on a Banach space B with normed dual B^* and let $dI : B \to B^*$ denote the Fréchet-derivate of E. We call a point $u \in B$ critical if

$dI(u) = 0$, otherwise, u is called regular. A number $\beta \in \mathbb{R}$ is a critical value of I if there exists a critical point u of I with $I(u) = \beta$, otherwise, β is called regular.

We also denote by $I'(u) = dI(u)$ and $I''(u) = d^2I(u)$.

Definition 1.5 (Palais-Smale sequence)**.** A sequence $\{u_n\}$ in B is a Palais-Smale sequence for I if $|I(u_n)| \leq C$ and $\|dI(u_n)\| \to 0$ as $n \to \infty$.

Definition 1.6 (Palais-Smale condition)**.** A Fréchet-differentiable functional $I : B \to \mathbb{R}$ satisfies the Palais-Smale condition (P.S.) if any Palais-Smale sequence has a convergent subsequence.

The first result is about critical points that minimizes the functional I when it is bounded below.

Theorema 1.7. *Suppose $I \in C^1(B)$ satisfies (P.S.). Then, if*

$$\beta = \inf_{u \in B} I(u)$$

is finite, $\beta = \min_{u \in B} I(u)$ is attained at a critical point of I.

The second result is the Montain Pass theorem.

Theorema 1.8. *Suppose $I \in C^1(B)$ satisfies (P.S.). Assume that*

1) $I(0) = 0$;

2) $\exists \rho > 0,\ \alpha > 0$ such that if $\|u\|_B = \rho$ then $I(u) \geq \alpha$;

3) $\exists u_1 \in B$ such that $\|u\|_B \geq \rho$ and $I(u_1) < \alpha$.

Define

$$\Gamma = \{\gamma \in C^0([0,1]; B) : \gamma(0) = 0,\ \gamma(1) = u_1\}.$$

Then

$$\beta = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} I(u) \geq \alpha$$

is a critical value.

The last result is the Rabinowitz Saddle Point theorem [18]

Theorema 1.9. *Suppose* $I \in C^1(B)$ *satisfies (P.S.). Let* $B = B_1 \oplus B_2$, *with* $\dim B_1 < \infty$ *and there exists* $R > 0$ *such that*

$$\max_{v \in B_1,\, \|v\|_B = R} I(v) < \inf_{w \in B_2} I(w).$$

If we denote by $B(0,R)$ *the ball in* B_1 *of radius* R *and center* 0 *and we define the set*

$$\Gamma = \{h \in C(B(0,R), B) \,:\, h(v) = v,\ \forall v \in B_1 \text{ with } \|v\|_B = R\}.$$

Then the number

$$c = \inf_{h \in \Gamma} \max_{v \in B(0,R)} I(h(v))$$

defines a critical value $c \geq \inf_{w \in B_2} I(w)$ *of* I.

1.3 Morse theory and Critical groups

We will give the principal results of Morse theory and critical groups (see [9]) used in this thesis.

Definition 1.10. (see [9, pag. 33]) Let H be a Hilbert space, $I : H \to \mathbb{R}$ a $C^2(H)$ functional and $u \in H$ a critical point of I. We define the Morse index of u, denoted by $m(u)$, as the dimension of the negative space corresponding to the spectral decomposing of $d^2 I(u)$.

Definition 1.11. (see [9, Definition 4.1], Chapter I) Let u be an isolated critical point of I, and set $c = I(u)$. We define the q^{th} critical group of I at u as

$$C_q(I,u) = H_q(I_c \cap U, (I_c \setminus \{u\}) \cap U),$$

$q = 0, 1, 2, ...$, where U is a neighborhood of u such that $\{v \in U \cap I_c : dI(v) = 0\} = \{u\}$, $I_c = \{v \in H : I(v) \leq c\}$ and $H_*(A,B)$ stands for the singular relative homology groups with abelian coefficient group $\mathbb{Z}$.

The following result (see [9, Corollary 5.1], Chapter I) is used to compare diferents critical points:

Theorema 1.12. *Suppose that $Ker(d^2I(u))$ is finite dimensional with dimension k and let $m = m(u)$ be the Morse index of I at u, then either*

(1)

$$C_q(I,u) = \delta_{q,m}\mathbb{Z}, \ \ or$$

(2)

$$C_q(I,u) = \delta_{q,m+k}\mathbb{Z}, \qquad or$$

(3)

$$C_q(I,u) = 0 \ for \ q \leq m, \ \ and \ q \geq m+k.$$

Next, we give two abstracts results that will be used in Chapter 4.

Theorema 1.13. *(See [17, Theorem 1.3]) Suppose that there is a direct sum decomposition $H = V \oplus W$, with V finite dimensional, such that*

$$a = \inf_W I > -\infty, \ \ b = \sup_V I < +\infty,$$

and assume that I satisfies (P.S.) condition in $[a-\epsilon, b+\epsilon]$, for some $\epsilon > 0$. Then I has a critical point u such that

$$a \leq I(u) \leq b, \ \ C_j(I,u) \neq 0$$

where $j = \dim V$.

Theorema 1.14. *(see [16, Theorem 3.1]) Let* $H = V \oplus W$ *de a Banach space with* $0 < k = \dim V < \infty$. *Suppose that* $I \in C^1(H, \mathbb{R})$ *satisfies*

I_1) *there exists* $\rho > 0$ *such that*

$$\sup_{S^1_\rho} < 0,$$

where $S^1_\rho = \{v \in V : \|v\| = \rho\}$,

I_2) $I \geq 0$ *on* W, *and*

I_3) *there exists a nonzero vector* $v_1 \in V$ *such that* I *is bounded below on the half-space* $\{sv_1 + w : s \geq 0,\, w \in W\}$.

In addition, assume that I *satisfies P.S. and has only isolated critical values with each critical value corresponding to a finite number of critical points. Then* I *has two (different) critical points* u_1, u_2 *with* $I(u_1) < 0 \leq I(u_2)$ *and* $C_{k-1}(I, u_1) \neq 0$, $C_k(I, u_2) \neq 0$.

Chapter 2

Existence of solutions for a nonhomogeneous semilinear elliptic equation

2.1 Introduction

We consider the following problem:

$$\begin{cases} -\Delta u = \lambda u + g(x,u) - h(x)a(u) + f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain, $\lambda \in \mathbb{R}$, g is a bounded Carathéodory function in $\Omega \times \mathbb{R}$, $f \in L^2(\Omega)$, $h \in L^1(\Omega)$ with $h \geq 0$ and is different from zero in a set of positive measure. Specifically, if we denote by

$$\widetilde{\Omega} = \{x \in \Omega \, : \, h(x) = 0\},$$

we assume that

$$\text{meas}\,(\Omega \backslash \widetilde{\Omega}) = \text{meas}\,\{x \in \Omega \, : \, h(x) > 0\} > 0. \tag{2.2}$$

We also assume that a is a $C(\mathbb{R})$ function such that, denoting by $A(u) = \int_0^u a(t)dt$,

$$(p+1)A(u) \leq a(u)u \quad \text{for} \quad |u| \geq R, \quad \text{for some} \quad 1 < p \quad \text{and} \quad R \quad \text{large}; \tag{2.3}$$

$$|a(u)| \leq c|u|^p + c, \quad \text{where} \quad c \quad \text{is a constant}; \tag{2.4}$$

$$\frac{a(u)}{u} > 0 \quad \forall u \neq 0, \text{ which implies that } a(0) = 0 \text{ and } A(u) > 0 \text{ for } u \neq 0; \tag{2.5}$$

$$(a(u) - a(v))(u - v) \geq C|u - v|^{p+1}, \text{ for some } C > 0 \text{ and for all } u, v \in \mathbb{R}. \tag{2.6}$$

We can observe that conditions (2.3), (2.4) and (2.5) on a implies that

$$C_1|u|^{p+1} - C_2 \leq A(u) \leq C_3|u|^{p+1} + C_4 \tag{2.7}$$

for some constans $C_i > 0$, $i = 1, 2, 3, 4$ and this inequality implies that

$$\lim_{|u| \to \infty} \frac{a(u)}{u} = \infty.$$

We obtain an inequality similar to (2.7) for the function $a(u)u$.

The function $a(u) = |u|^{p-1}u$ satisfaz all these conditions, and in this thesis we also give weak hypothesis and better results for this particular case on a.

In this chapter we prove that if condition (2.2) holds true, then there exists a solution of (2.1) for every $\lambda \in \mathbb{R}$, $f \in L^2(\Omega)$ and $p > 1$. Indeed, we prove the following result

Theorema 2.1. *If g is a bounded Carathéodory function, $p > 1$, $0 \leq h \in L^1(\Omega)$ satisfying (2.2) and a satisfaz (2.3), (2.4), (2.5) and (2.6), then the problem (2.1) has at least one solution for each $\lambda \in \mathbb{R}$ and $f \in L^2(\Omega)$.*

The above result is proved by variational tools. As usual, we need to prove that the Euler functional I_λ associated to the problem (2.1) satisfies the Palais-Smale compactness condition, as well as suitable geometrical properties. We devote Section 2 to introduce the functional I_λ and to study a general compactness condition for the family of the functionals I_λ, $\lambda \in \mathbb{R}$. The geometrical properties of the functional I_λ are studied in Section 3 which concludes the proof of Theorem 2.1.

Notation. We will denote by $\|u\| = \|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}$ (respectively, $\|u\|_2 = \left(\int_\Omega u^2\,dx\right)^{1/2}$) the norm of a function u in the space $H_0^1(\Omega)$ (respectively, $L^2(\Omega)$). In the following the letter C will denote a positive constant which can change from a line to another and even within the same formula.

2.2 A compactness condition

In order to prove the Theorem 2.1 we follow a variational approach. Specifically, we consider the reflexive space

$$E = \{u \in H_0^1(\Omega)\ :\ \int_\Omega h|u|^{p+1} < +\infty\},$$

endowed with the norm

$$\|u\|_E = \|u\|_{H_0^1(\Omega)} + \left(\int_\Omega h|u|^{p+1}\,dx\right)^{1/(p+1)}.$$

For $G(x,t) = \int_0^t g(x,s)\,ds$ and $A(t) = \int_0^t a(s)\,ds$ ($x \in \Omega$, $t \in \mathbb{R}$), we consider the C^1-functional $I_\lambda : E \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega hA(u)dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega G(x,u)dx - \int_\Omega fudx,$$

for every $u \in E$. This functional is well defined in view of (2.7) and that $h \in L^1(\Omega)$. However, for the particular case $a(u) = |u|^{p-1}u$ we can define the functional if $h \in L^1_{loc}(\Omega)$.

We say that a solution u of (2.1) is just a critical point $u \in E$ of the functional I_λ; i.e., a function $u \in E$ such that

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega h(x)a(u)\varphi dx - \lambda\int_\Omega u\varphi dx - \int_\Omega g(x,u)\varphi dx - \int_\Omega f\varphi dx = 0,\ \forall \varphi \in E.$$

In the particular case $a(u) = |u|^{p-1}u$: Since $h \in L^1_{\rm loc}(\Omega)$, we deduce that the space $C_0^\infty(\Omega)$ of C^∞-functions with compact support in Ω is a subset of E and thus any $\varphi \in C_0^\infty(\Omega)$ can be chosen as test function in the previous identity. Therefore, the notion of solution given for (2.1) is just the standard one for a Dirichlet problem, namely a solution u of the equation $-\Delta u = \lambda u + g(x,u) - ha(u) + f$ in Ω in the sense of distributions (test functions in $C_0^\infty(\Omega)$) which in addition belongs to $H_0^1(\Omega)$ (boundary condition) and satisfies that $h|u|^{p+1} \in L^1(\Omega)$.

We prove the following compactness condition:

Lemma 2.2. *Let g be a bounded Carathéodory function, $p > 1$, $f \in L^2(\Omega)$ and $0 \le h \in L^1(\Omega)$ satisfying (2.2) and a satisfaz (2.3), (2.4), (2.5) and (2.6). Assume that $\{\alpha_n\} \subset \mathbb{R}$ is a bounded sequence and $\{\epsilon_n\} \subset (0,\infty)$ is a sequence converging to zero. If $\{u_n\}$ is a sequence in E such that $I_{\alpha_n}(u_n) \ge -C$ and $|dI_{\alpha_n}(u_n)(\varphi)| \le \epsilon_n \|\varphi\|_E$ for all $\varphi \in E$, then $\{u_n\}$ is bounded in E and admits a convergent subsequence in E.*

Remark 2.3. If we take $\alpha_n = \lambda$ for every n in this lemma then the functional I_λ satisfies the Palais-Smale compactness condition for every $\lambda \in \mathbb{R}$.

Proof of Lemma 2.2. For a such sequence, it follows that

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega hA(u_n)\,dx - \frac{\alpha_n}{2}\int_\Omega u_n^2\,dx - \int_\Omega G(x,u_n)\,dx - \int_\Omega f u_n\,dx \ge -C \quad (2.8)$$

and

$$\begin{aligned}\Bigg|\int_\Omega \nabla u_n \cdot \nabla\varphi\,dx + \int_\Omega ha(u_n)\varphi\,dx - \alpha_n\int_\Omega u_n\varphi\,dx \\ - \int_\Omega g(x,u_n)\varphi\,dx - \int_\Omega f\varphi\,dx\Bigg| \le \epsilon_n\|\varphi\|_E,\end{aligned} \quad (2.9)$$

for every $\varphi \in E$.

We claim that the sequence $\{u_n\}$ is bounded in E. Otherwise, up to a subsequence, we can assume that $\|u_n\|_E \to +\infty$, $\alpha_n \to \alpha$ and if we define $v_n := u_n/\|u_n\|_E$, then $\|v_n\|_E = 1$ and, by the reflexivity of E, there is a subsequence of $\{v_n\}$ (still denoted by v_n) and a $v_0 \in E$ such that $v_n \rightharpoonup v_0$ in E, $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$, $v_n \rightharpoonup v_0$ in $L^{p+1}(\Omega, hdx)$ and $v_n \to v_0$

in $L^2(\Omega)$ and a.e. in Ω. Taking $\varphi = \dfrac{u_n}{\|u_n\|_E^2}$ in (2.9), we deduce that v_n satisfies

$$\begin{aligned}\int_\Omega |\nabla v_n|^2\,dx + \int_\Omega h\frac{a(u_n)u_n}{\|u_n\|_E^2}\,dx \leq & \frac{\epsilon_n}{\|u_n\|_E} + \alpha_n \int_\Omega v_n^2\,dx \\ & + \int_\Omega \frac{g(x,u_n)}{\|u_n\|_E} v_n\,dx + \frac{\|f\|_2\|v_n\|_2}{\|u_n\|_E}\end{aligned} \tag{2.10}$$

which implies by the boundedness of g and the hypotheses on a that

$$\|u_n\|_E^{p-1} \int_\Omega h|v_n|^{p+1}\,dx \leq C.$$

In particular, since $p > 1$ we have

$$\lim_{n\to\infty} \int_\Omega h|v_n|^{p+1}\,dx = 0.$$

Using this and that $\|v_n\|_E = \|v_n\|_{H_0^1(\Omega)} + \left(\int_\Omega h|v_n|^{p+1}\,dx\right)^{1/p+1} = 1$ we have that $\lim\limits_{n\to\infty} \int_\Omega |\nabla v_n|^2\,dx = 1$ and from (2.10), using again the boundedness of g, we obtain

$$1 \leq \alpha \int_\Omega v_0^2\,dx,$$

which implies that $v_0 \neq 0$. In addition, Fatou lemma $\left(\int_\Omega h|v_0|^{p+1}\,dx \leq \liminf\limits_{n\to\infty} \int_\Omega h|v_n|^{p+1}\,dx\right)$ and the non-negativeness of h give

$$\lim_{n\to\infty} \int_\Omega h|v_n|^{p+1}\,dx = 0 = \int_\Omega h|v_0|^{p+1}\,dx$$

and $h|v_0|^{p+1} = 0$. If $\operatorname{meas}(\widetilde{\Omega}) = 0$, then $v_0 = 0$ a.e. in Ω and we get a contradiction and it is proved that the sequence $\{u_n\}$ is bounded in E in this case.

On the other hand, if $\operatorname{meas}(\widetilde{\Omega}) > 0$, then $v_0 = 0$ a.e. in $\Omega\backslash\widetilde{\Omega}$ and thus $v_0 \in H_D^1(\widetilde{\Omega})$.

Taking $\varphi = u_n/2$ in (2.9) and subtracting (2.8), we obtain

$$\int_\Omega h\left(\frac{a(u_n)u_n}{2} - A(u_n)\right) dx + \frac{1}{2}\int_\Omega f u_n\, dx \le C + \frac{\epsilon_n\|u_n\|_E}{2} + \int_\Omega \left(\frac{1}{2}g(x,u_n)u_n - G(x,u_n)\right) dx$$

In particular, dividing by $\|u_n\|_E$ and using that $p > 1$, the boundedness of g and the hypotheses on a, we have

$$\frac{1}{\|u_n\|_E}\int_\Omega h|u_n|^{p+1}\, dx \le C.$$

By using this and the Hölder inequality, for every $\varphi \in E$ we get

$$\left|\int_\Omega h|u_n|^p\varphi\, dx\right| \le \left(\int h\,\varphi^{p+1}\, dx\right)^{\frac{1}{p+1}}\left(\int h\,|u_n|^{p+1}\, dx\right)^{\frac{p}{p+1}} \le \left(\int h\,\varphi^{p+1}\, dx\right)^{\frac{1}{p+1}} C\|u_n\|_E^{\frac{p}{p+1}}$$

and

$$\lim_{n\to\infty}\frac{1}{\|u_n\|_E}\int_\Omega h|u_n|^p\varphi\, dx = 0.$$

Using the hypotheses on a and the last equality we also have

$$\lim_{n\to\infty}\frac{1}{\|u_n\|_E}\int_\Omega h a(u_n)\varphi\, dx = 0.$$

Hence, if we divide (2.9) by $\|u_n\|_E$ and pass to the limit as $n \to \infty$ we deduce by the boundedness of g that

$$\int_\Omega \nabla v_0 \cdot \nabla\varphi\, dx = \alpha\int_\Omega v_0\varphi\, dx,$$

for every $\varphi \in E$. By density of E into $H_0^1(\Omega)$ (due to the local integrability of h), the above equality holds true for every $\varphi \in H_0^1(\Omega)$; i.e., $v_0 \ne 0$ is a solution of the problem

$$\begin{cases} -\Delta v = \alpha\, v, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

which, in addition, vanishes on the set $\Omega\backslash\widetilde{\Omega}$. However, this is impossible by (2.2) and the unique continuation property (see Proposition 3 and Remark 2 in [12]). Therefore, we conclude that the sequence $\{u_n\}$ is bounded in E also when meas$(\widetilde{\Omega}) > 0$.

Using that E is reflexive we have that there exists $u_0 \in E$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in E, $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, $u_n \rightharpoonup u_0$ in $L^{p+1}(\Omega, hdx)$, $u_n \to u_0$ in $L^2(\Omega)$ and a.e. in Ω . Since the sequence $a(u_n)$ is bounded in $L^{\frac{p+1}{p}}(\Omega, hdx)$ and converges a.e. to $a(u_0)$, we deduce that it converges weakly to $a(u_0)$ in $L^{\frac{p+1}{p}}(\Omega, hdx)$ [7, Exercise 4.16], which implies that

$$\int_\Omega ha(u_n)\varphi\, dx \longrightarrow \int_\Omega ha(u_0)\varphi\, dx,\ \forall\varphi \in L^{p+1}(\Omega, hdx), \tag{2.11}$$

Using this, if we take the limit in (2.9) as $n \to \infty$ we deduce that

$$\int_\Omega \nabla u_0 \nabla\varphi\, dx + \int_\Omega ha(u_0)\varphi\, dx - \alpha\int_\Omega u_0\varphi\, dx - \int_\Omega g(x, u_0)\varphi\, dx - \int_\Omega f\varphi\, dx = 0,$$

for every $\varphi \in E$. Substracting it from (2.9) we get

$$\left|\int_\Omega \nabla(u_n - u_0)\cdot\nabla\varphi\, dx + \int_\Omega h(a(u_n) - a(u_0))\varphi\, dx \right.$$
$$\left. - \int_\Omega (\alpha_n u_n - \alpha u_0)\varphi\, dx - \int_\Omega (g(x, u_n) - g(x, u_0))\varphi\, dx\right| \le \epsilon_n\|\varphi\|_E,$$

which by the choice $\varphi = u_n - u_0$ implies that

$$\left|\int_\Omega |\nabla(u_n - u_0)|^2 dx + \int_\Omega h(a(u_n) - a(u_0))(u_n - u_0)dx \right.$$
$$\left. - \int_\Omega (\alpha_n u_n - \alpha u_0)(u_n - u_0)dx - \int_\Omega (g(x, u_n) - g(x, u_0))(u_n - u_0)dx\right|$$
$$\le \epsilon_n\|(u_n - u_0)\|_E.$$

Noting that the third and fourth terms are going to 0 as $n \to \infty$ (by the convergence of u_n to u in $L^2(\Omega)$) and using (2.6), we have that $\|u_n - u_0\|_{H_0^1(\Omega)} \to 0$ and

$$\int_\Omega h|u_n - u_0|^{p+1}\, dx \to 0.$$

Consequently $u_n \to u_0$ in E. □

2.3 Proof of Theorem 2.1

We will see that the variational nature of the solution given by Theorem 2.1 depends on the relationship of λ with the spectrum of the operator H_∞ (associated to the quadratic form $b(u) = \int_\Omega |\nabla u|^2 \, dx$ with domain $H^1_D(\widetilde{\Omega})$). Notice that a particular example corresponds with the case in which meas $(\widetilde{\Omega}) > 0$ and meas $(\partial\widetilde{\Omega}) = 0$. In this case, the measure of the interior $\widetilde{\Omega}_\circ$ of $\widetilde{\Omega}$ has to be positive (i.e. meas $(\widetilde{\Omega}_\circ) > 0$) and we have

$$h(x) > 0 \text{ a.e. in } \Omega\backslash\widetilde{\Omega}_\circ.$$

Therefore, if we assume in addition that the interior $\widetilde{\Omega}_\circ$ of $\widetilde{\Omega}$ satisfies an exterior cone condition at every point of its boundary, then $H^1_D(\widetilde{\Omega}) = H^1_0(\widetilde{\Omega}_\circ)$ and H_∞ is nothing but the classical Laplace operator $H^1_0(\widetilde{\Omega}_\circ)$ (i.e., with zero Dirichlet condition on the boundary of $\widetilde{\Omega}_\circ$).

In the general case, when we only assume that meas $(\widetilde{\Omega}) > 0$, we denote by $\{\lambda_i(\widetilde{\Omega})\}_{i\in\mathbb{N}}$ the spectrum of H_∞ ordered by the min-max principle with eigenvalues repeated according to their multiplicity and by $\widetilde{\varphi}_i$ the associated eigenfunctions to $\lambda_i(\widetilde{\Omega})$, normalized so that $\int_{\widetilde{\Omega}} \widetilde{\varphi}_i.\widetilde{\varphi}_j \, dx = \delta_{i,j}$.

The proof of Theorem 2.1 is split in cases in the following subsections.

2.3.1 Case $\lambda < \lambda_1(\widetilde{\Omega})$.

We devote this subsection to prove Theorem 2.1 when $\lambda < \lambda_1(\widetilde{\Omega})$.

Theorema 2.4. *Let g be a bounded Carathéodory function, $p > 1$, $f \in L^2(\Omega)$, $0 \leq h \in L^1(\Omega)$ satisfying (2.2) and a satisfaz (2.3), (2.4), (2.5) and (2.6). If $\lambda < \lambda_1(\widetilde{\Omega})$, then the problem (2.1) has at least one solution.*

Remark 2.5. As it has been mentioned in the introduction, the above theorem is proved in [3] for the particular case $a(u) = |u|^{p-1}u$. Since the authors only indicate the steps for their proof, we will include here a detailed proof for completeness.

Remark 2.6. If $H^1_D(\widetilde{\Omega}) = \{0\}$, then $\lambda_1(\widetilde{\Omega})$ is infinite and we deduce from Theorem 2.4 the existence of solution for every $\lambda \in \mathbb{R}$. Hence the Theorem 2.1 is deduced in this case from the above theorem. Note that a sufficient condition to have $H^1_D(\widetilde{\Omega}) = \{0\}$ is that meas $(\widetilde{\Omega}) = 0$, i.e., that $h > 0$ a.e. in Ω. In addition, this observation also shows that the Theorem 2.4 can not be extended to the case $p = 1$ (think in the simple case that h is a positive constant).

Therefore to conclude the proof of the Theorem 2.1, in the rest of this chapter we can assume that $H^1_D(\widetilde{\Omega}) \neq \{0\}$ (which implies that all the eigenvalues $\lambda_i(\widetilde{\Omega})$ of the operator H_∞ are finite) and that $\lambda \geq \lambda_1(\widetilde{\Omega})$.

Proof. (of Theorem 2.4.) The existence of a solution of the problem (2.1) is deduced by proving that the C^1-functional I_λ has a global minimum in E.

To show this, first we show that the functional I_λ is bounded from below and we argue by contradiction assuming that there exists a sequence $\{u_n\} \subset E$ such that $0 > I_\lambda(u_n) \to -\infty$. Since

$$\begin{aligned} I_\lambda(u_n) \geq & -\frac{\lambda}{2}\int_\Omega u_n^2\,dx - \int_\Omega G(x,u_n)\,dx - \int_\Omega f u_n\,dx \\ \geq & -\frac{\lambda}{2}\|u_n\|_2^2 - (C + \|f\|_2)\|u_n\|_2, \end{aligned}$$

we deduce that $\|u_n\|_2 \to \infty$. In particular, $\|u_n\|_{H^1_0(\Omega)} \to \infty$. If we consider the normalized sequence $v_n = u_n/\|u_n\|_{H^1_0(\Omega)}$, we can also assume, up to a subsequence, that there exists $v_0 \in E$ such that $v_n \rightharpoonup v_0$ in $H^1_0(\Omega)$, $v_n \to v_0$ in $L^2(\Omega)$ and a.e in Ω. Using that $I_\lambda(u_n)$ is negative, we obtain

$$\begin{aligned} 0 > \frac{I_\lambda(u_n)}{\|u_n\|^2_{H^1_0(\Omega)}} \geq & \frac{1}{2} + C\|u_n\|^{p-1}_{H^1_0(\Omega)}\int_\Omega h|v_n|^{p+1}\,dx - \frac{C}{\|u_n\|^2_{H^1_0(\Omega)}}\int_\Omega h\,dx \\ & -\frac{\lambda}{2}\|v_n\|_2^2 - \int_\Omega \frac{G(x,u_n)}{\|u_n\|^2_{H^1_0(\Omega)}}\,dx - \frac{1}{\|u_n\|_{H^1_0(\Omega)}}\int_\Omega f v_n\,dx. \end{aligned}$$

From this inequality and the boundedness of g, we deduce the following:

1. By taking limits as $n \to +\infty$, we have

$$1 \leq \lambda\|v_0\|_2^2$$

   and $v_0 \neq 0$.

2. Dividing by $\|u_n\|_{H_0^1(\Omega)}^{p-1}$ and using Fatou lemma, we get

$$0 \geq \liminf_{n\to+\infty} \int_\Omega h|v_n|^{p+1} \geq \int_\Omega h|v_0|^{p+1}\,dx.$$

and hence

$$v_0 = 0 \text{ a.e. in } \Omega\backslash\widetilde{\Omega}. \tag{2.12}$$

If it would be meas $(\widetilde{\Omega}) = 0$, then it would be concluded by (2.12) that $v_0 = 0$ a.e. in Ω, contradicting 1. Then, in this case, necessarily I_λ has to be bounded from below.

In the other case, i.e. if meas $(\widetilde{\Omega}) > 0$, then (2.12) means that $v_0 \in H_D^1(\widetilde{\Omega})$ and, by the variational characterization of $\lambda_1(\widetilde{\Omega})$ we have $\lambda_1(\widetilde{\Omega})\|v_0\|_2^2 \leq \|v_0\|_{H_0^1(\Omega)}^2$. By the weak convergence of v_n to v_0 in $H_0^1(\Omega)$ and the inequality $1 \leq \lambda\|v_0\|_2^2$, we derive that

$$\lambda_1(\widetilde{\Omega})\|v_0\|_2^2 \leq \|v_0\|_{H_0^1(\Omega)}^2 \leq \liminf_{n\to\infty} \|v_n\|_{H_0^1(\Omega)}^2 = 1 \leq \lambda\|v_0\|_2^2, \text{ with } v_0 \neq 0.$$

i.e., $\lambda_1(\widetilde{\Omega}) \leq \lambda$, contradicting our hypothesis on λ and proving, in this case, that I_λ is bounded from below.

We know that $I_\lambda \in C^1(E)$ and from Lemma 2.2 satisfies (P.S.). Thus, we can use Theorem 1.7 to show that I_λ has a critical point $u_0 \in E$ with $I(u_0) = \inf_{u\in E} I_\lambda(u)$ and then u_0 is a solution of the problem (2.1). □

2.3.2 Case $\lambda_i(\widetilde{\Omega}) < \lambda < \lambda_{i+1}(\widetilde{\Omega})$, for $i \geq 1$

In this subsection we consider the case that ($H_D^1(\widetilde{\Omega}) \neq \{0\}$ and) the parameter λ is between two consecutive eigenvalues of the operator H_∞.

Theorema 2.7. *Let g be a bounded Carathéodory function, $p > 1$, $f \in L^2(\Omega)$ and $0 \leq h \in L^1(\Omega)$ satisfying (2.2) and a satisfaz (2.3), (2.4), (2.5) and (2.6). If $H_D^1(\widetilde{\Omega}) \neq \{0\}$ and $\lambda_i(\widetilde{\Omega}) < \lambda < \lambda_{i+1}(\widetilde{\Omega})$ for $i \geq 1$, then the problem (2.1) has at least one solution u_λ.*

Proof. We are going to show that the problem (2.1) has at least one weak solution, by showing that the functional I_λ has a critical point of the form saddle point as in theorem of Rabinowitz [18, Theorem 1.2]. In order to make it, we choose $V = \langle\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_i\rangle \subset E$

and $W = \{w \in E \,/\, \int_\Omega \nabla\widetilde{\varphi}_j.\nabla w\,dx = 0 \text{ for } 1 \le j \le i\}$. Observe that W is the intersection of E with the orthogonal $V^\perp$ in $H_0^1(\Omega)$ of V and that $E = V \oplus W$. We begin by studying the geometrical properties of the functional.

First, we claim that I_λ is bounded from below on W. Otherwise, there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subset W$ such that $0 > I_\lambda(w_n) \to -\infty$. Since

$$\begin{aligned} I_\lambda(w_n) \ge& -\frac{\lambda}{2}\int_\Omega w_n^2\,dx - \int_\Omega G(x,w_n) - \int_\Omega f w_n\,dx \\ \ge& -\frac{\lambda}{2}\|w_n\|_2^2 - (C + \|f\|_2)\|w_n\|_2, \end{aligned}$$

we deduce that $\|w_n\|_2 \to \infty$. In particular, $\|w_n\|_{H_0^1(\Omega)} \to \infty$. If we consider the normalized sequence $z_n = w_n/\|w_n\|_{H_0^1(\Omega)}$, we can also assume, up to a subsequence, that there exists $z_0 \in W$ such that $z_n \rightharpoonup z_0$ in $H_0^1(\Omega)$, $z_n \to z_0$ in $L^2(\Omega)$ and a.e in Ω. Using that $I_\lambda(w_n)$ is negative, we obtain

$$\begin{aligned} 0 > \frac{I_\lambda(w_n)}{\|w_n\|^2_{H_0^1(\Omega)}} \ge& \frac{1}{2} + C\|w_n\|^{p-1}_{H_0^1(\Omega)}\int_\Omega h|z_n|^{p+1}\,dx - \frac{C}{\|w_n\|^2_{H_0^1(\Omega)}}\int_\Omega h\,dx \\ & - \frac{\lambda}{2}\|z_n\|_2^2 - \int_\Omega \frac{G(x,w_n)}{\|w_n\|^2_{H_0^1(\Omega)}} - \frac{1}{\|u_n\|_{H_0^1(\Omega)}}\int_\Omega f z_n\,dx \end{aligned}$$

From this inequality we deduce first that (by taking limits as $n \to +\infty$)

$$1 \le \lambda\|z_0\|_2^2 \text{ and } z_0 \ne 0.$$

Secondly, dividing by $\|w_n\|^{p-1}_{H_0^1(\Omega)}$ and using Fatou lemma, we also deduce that

$$0 \ge \liminf_{n\to\infty}\int_\Omega h|z_n|^{p+1} \ge \int_\Omega h|z_0|^{p+1}\,dx$$

and hence $z_0 = 0$ in $\Omega\backslash\widetilde{\Omega}$, i.e., $z_0 \in H_D^1(\widetilde{\Omega}) \cap W$. Consequently, by the weak convergence of z_n,

$$\lambda_{i+1}(\widetilde{\Omega})\|z_0\|_2^2 \le \|z_0\|^2_{H_0^1(\Omega)} \le \liminf_{n\to\infty}\|z_n\|^2_{H_0^1(\Omega)} = 1 \le \lambda\|z_0\|_2^2,$$

i.e., $\lambda_{i+1}(\widetilde{\Omega}) \leq \lambda$ contradicting our hypothesis on λ and proving that

$$\inf_{w\in W} I_\lambda(w) > -\infty.$$

On the other hand, using that the support of every function v in V is contained in $\widetilde{\Omega}$, we have $\|v\|_E = \|v\|_{H_0^1(\Omega)}$ and

$$I_\lambda(v) \leq \frac{1}{2}\|v\|^2_{H_0^1(\Omega)} + C\int_\Omega h\,dx - \frac{\lambda}{2}\|v\|_2^2 - \int_\Omega G(x,v)\,dx - \int_\Omega fv\,dx \tag{2.13}$$

$$\leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_i(\widetilde{\Omega})}\right)\|v\|^2_{H_0^1(\Omega)} + (C + \|f\|_2)\|v\|_2 + C\|h\|_{L^1(\Omega)}, \tag{2.14}$$

for all $v \in V$, and taking into account that $\lambda_i(\widetilde{\Omega}) < \lambda$, we deduce that $\lim_{v\in V,\,\|v\|_E\to+\infty} I_\lambda(v) = -\infty$. Therefore, there exists $R_\lambda > 0$ such that

$$\max_{v\in V,\,\|v\|_E = R_\lambda} I_\lambda(v) < \inf_{w\in W} I_\lambda(w).$$

Additionally, $I_\lambda \in C^1(E)$ and satisfies (P.S.) (Lemma 2.2). Using Theorem 1.9 we have that if we denote by $B_V(0, R_\lambda)$ the ball in V of radius R_λ and center 0 and

$$\Gamma_\lambda = \{h \in C(B_V(0,R_\lambda), E) \,:\, h(v) = v,\ \forall v \in V \text{ with } \|v\|_E = R_\lambda\},$$

then c_λ, defined as,

$$c_\lambda = \inf_{h\in\Gamma_\lambda}\max_{\|v\|_E\leq R_\lambda} I_\lambda(h(v)) \geq \inf_{w\in W} I_\lambda(w)$$

is a critical value of I_λ, this is, there exists $u_0 \in E$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) = c_\lambda$. Therefore u_0 is a solution of the problem (2.1). □

Remark 2.8. With the notation of the above proof, observe that if $\lambda_i(\widetilde{\Omega}) < \lambda \leq \alpha < \lambda_{i+1}(\widetilde{\Omega})$, then $I_\lambda \geq I_\alpha$ and thus $\inf_{w\in W} I_\lambda(w) \geq \inf_{w\in W} I_\alpha(w)$. Consequently, $I_\lambda(u_\lambda) = c_\lambda \geq \inf_{w\in W} I_\lambda(w) \geq \inf_{w\in W} I_\alpha(w)$.

2.3.3 Case $\lambda = \lambda_i(\widetilde{\Omega})$, for $i \geq 1$

Theorema 2.9. *Let g be a bounded Carathéodory function, $p > 1$, $f \in L^2(\Omega)$, $0 \leq h \in L^1(\Omega)$ a measurable function satisfying (2.2) and a satisfaz (2.3), (2.4), (2.5) and (2.6). If $H^1_D(\widetilde{\Omega}) \neq \{0\}$ and $\lambda = \lambda_i(\widetilde{\Omega})$ for $i \geq 1$, then the problem (2.1) has at least one solution.*

Proof. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence in the interval $(\lambda_i(\widetilde{\Omega}), \lambda_{i+1}\widetilde{\Omega}))$ which converges to $\lambda_i(\widetilde{\Omega})$. By Theorem 2.4 and Remark 3.9, for each $n \in \mathbb{N}$ there exists $u_n \in E$ such that $I'_{\alpha_n}(u_n) = 0$ and $I_{\alpha_n}(u_n) = c_{\alpha_n} \geq \inf_{w\in W} I_{\alpha_n}(w) \geq -C := \inf_{w\in W} I_{\alpha_1}(w)$. Hence, by aplying the Lemma 2.2, we deduce the existence of a subsequence u_{n_k} such that $u_{n_k} \to u_0$ in E for some $u_0 \in E$, which is a solution of the problem (2.1) for $\lambda = \lambda_i(\widetilde{\Omega})$. □

2.4 Conclution of the proof of Theorem 2.1

The proof of this theorem is now a direct consequence of the Theorems 2.4, 2.7 and 2.9. □

Chapter 3

Fractional Laplacian operator case

3.1 Introduction

For a bounded smooth domain Ω with Lipschitz boundary in $\mathbb{R}^n$, $n > 2s$, we consider the following problem:

$$\begin{cases} (-\Delta)^s u &= \lambda u + g(x,u) - h|u|^{p-1}u + f, \quad \text{in } \Omega \\ u &= 0, \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.1}$$

where for $s \in (0,1)$, $(-\Delta)^s$ is the nonlocal fractional Laplace operator defined on the space

$$H^s(\Omega) = \{u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy < \infty\}.$$

by

$$(-\Delta)^s u(x) = C(n,s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,\ \ x \in \mathbb{R}^n,$$

with

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi_1)}{|\xi|^{n+2s}}\, d\xi \right)^{-1}$$

is a constant depending on n and s (which for simplicity, we are going to take it as 1, this is, $C(n,s) = 1$ and P.V. is the principal value of the integral (which we are going to omit it in this work). (See [11] for further details on the fractional Laplace operator).

In addition, $\lambda \in \mathbb{R}$, $p > 1$, g is a bounded Carathéodory function in $\Omega \times \mathbb{R}$, $f \in L^2(\Omega)$

and $0 \le h \in L^1_{loc}(\Omega)$ is such that if we denote by

$$\widetilde{\Omega} = \{x \in \Omega : h(x) = 0\},$$

we assume that

$$\text{meas}\,(\Omega \setminus \widetilde{\Omega}) = \text{meas}\,\{x \in \Omega : h(x) > 0\} > 0. \tag{3.2}$$

We say that $u \in H^s(\mathbb{R}^n)$ is a solution for the problem (3.1) if $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dxdy =& \lambda \int_\Omega u\varphi\, dx + \int_\Omega g(x, u(x))\varphi\, dx \\ &- \int_\Omega h|u|^{p-1}u\varphi\, dx + \int_\Omega f\varphi\, dx\end{aligned}$$

for any $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

The scope of this Chapter is to extend the result in [5] to the fractional Laplacian operator by proving the existence of solution of the problem (3.1) for every λ. Specifically, we prove the following theorem.

Theorema 3.1. *If Ω is a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, $n > 2s$, $s \in (0, 1)$, $p > 1$, g is a bounded Carathéodory function in $\Omega \times \mathbb{R}$ and $0 \le h \in L^1_{loc}(\Omega)$ satisfying (3.2), then the problem (3.1) has at least one solution for each $\lambda \in \mathbb{R}$ and $f \in L^2(\Omega)$.*

3.2 Preliminary Results

We devote this section to remind (see [20] for more details) the main properties of the fractional Sobolev space

$$H^s_0(\Omega) = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathcal{C}\Omega\},$$

($\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$ is the complement of Ω) which is a Hilbert space endowed with the norm

$$\|u\|_{H^s_0(\Omega)} = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy\right)^{\frac{1}{2}},$$

where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$.

The following lemma is a sort of Poincaré-Sobolev inequality for functions in $H_0^s(\Omega)$.

Lemma 3.2 ([20], Lemma 6)**.** *There exists a constant $C > 1$, depending only on n, s and Ω, such that for any $v \in H_0^s(\Omega)$*

$$\|v\|_2 \leq C\|v\|_{H_0^s(\Omega)}.$$

The next lemma gives the compactness of $H_0^s(\Omega)$ in $L^2(\mathbb{R}^n)$.

Lemma 3.3 ([20], Lemma 8)**.** *If Ω is a bounded domain with Lipschitz boundary in $\mathbb{R}^n$ and $\{v_j\}$ is a bounded sequence in $H_0^s(\Omega)$, then, there exists $v \in L^2(\mathbb{R}^n)$ such that, up to a subsequence,*

$$\{v_j\} \to v \text{ in } L^2(\mathbb{R}^n) \text{ as } j \to +\infty.$$

Now, we discuss some known results for the following eigenvalue problem

$$\begin{cases} (-\Delta)^s u &= \lambda u, \quad \text{in } \mathcal{A} \\ u &= 0, \quad \text{in } \mathbb{R}^n \setminus \mathcal{A}, \end{cases} \tag{3.3}$$

where $\mathcal{A}$ is a measurable bounded set in $\mathbb{R}^n$. Specifically, if we consider the Hilbert space

$$H_D^s(\mathcal{A}) = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathcal{C}\mathcal{A}\}.$$

(note that if $\mathcal{A}$ is an open set of $\mathbb{R}^n$, then $H_D^s(\mathcal{A}) = H_0^s(\mathcal{A})$), we say that $\lambda \in \mathbb{R}$ is an eigenvalue of $(-\Delta)^s$ in $\mathcal{A}$ if there exists a non-trivial function $u \in H_D^s(\mathcal{A})$ such that

$$\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} dxdy = \lambda \int_{\mathcal{A}} u\varphi \, dx, \ \forall \varphi \in H_D^s(\mathcal{A}),$$

and, in this case, u is called an eigenfunction of $(-\Delta)^s$ in $\mathcal{A}$ corresponding to λ .

It is standard that the existence of a first eigenvalue of $(-\Delta)^s$ in $\mathcal{A}$, denoted by $\lambda_1(\mathcal{A})$, is related to the attainability of the following infimum

$$\lambda_1(\mathcal{A}) = \inf_{u \in H_D^s(\mathcal{A}),\, \|u\|_{L^2(\mathcal{A})}=1} \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy.$$

However, it is clear that this infimum $\lambda_1(\mathcal{A}) = \infty$ provided that $H^s_D(\mathcal{A}) = \{0\}$. On the other hand; i.e., if $H^s_D(\mathcal{A}) \neq \{0\}$, this infimun is attained and thus it is the first eigenvalue of $(-\Delta)^s$ in $\mathcal{A}$.

Indeed, the following lemma gather the main properties of the eigenvalues and eigenfunctions of (3.3) in the case that $H^s_D(\mathcal{A}) \neq \{0\}$. It is proved in [21] in the case that $\mathcal{A}$ is an open bounded set in $\mathbb{R}^n$. We observe that the proof given in [21] also works for the general case in which it is only assumed that $\mathcal{A}$ is a measurable bounded set in $\mathbb{R}^n$.

Lemma 3.4 ([21], Proposition 9). *Let $s \in (0,1)$, $n > 2s$ and suppose that $H^s_D(\mathcal{A}) \neq \{0\}$. Then,*

1. *problem (3.3) admits an eigenvalue $\lambda_1(\mathcal{A})$ which is positive and that can be characterized as follows*

$$\lambda_1(\mathcal{A}) = \min_{u \in H^s_D(\mathcal{A}),\, \|u\|_{L^2(\mathcal{A})}=1} \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy = \min_{u \in H^s_D(\mathcal{A})\setminus\{0\}} \frac{\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy}{\int_{\mathcal{A}} |u(x)|^2 dx};$$

2. *there exist a non-negative function $\varphi_1^{\mathcal{A}} \in H^s_D(\mathcal{A})$, which is an eigenfunction corresponding to $\lambda_1(\mathcal{A})$, attaining the minimum in the item 1., that is,*

$$\lambda_1(\mathcal{A}) = \int_{\mathbb{R}^{2n}} \frac{|\varphi_1^{\mathcal{A}}(x) - \varphi_1^{\mathcal{A}}(y)|^2}{|x-y|^{n+2s}} dxdy, \text{ with } \|\varphi_1^{\mathcal{A}}\|_{L^2(\mathcal{A})} = 1.$$

3. *$\lambda_1(\mathcal{A})$ is simple; i.e., if $u \in H^s_0(\mathcal{A})$ is an eigenfunction corresponding to $\lambda_1(\mathcal{A})$, then $u = \alpha\varphi_1^{\mathcal{A}}$, for some $\alpha \in \mathbb{R}$;*

4. *the set of the eigenvalues of problem (3.3) consists of a sequence $\{\lambda_k(\mathcal{A})\}_{k\in\mathbb{N}}$ with*

$$0 < \lambda_1(\mathcal{A}) < \lambda_2(\mathcal{A}) \leq \dots \leq \lambda_k(\mathcal{A}) \leq \lambda_{k+1}(\mathcal{A}) \leq \dots$$

where every eigenvalue is repeated according its finite multiplicity and

$$\lambda_k(\mathcal{A}) \to +\infty \text{ as } k \to +\infty.$$

Moreover, for any $k \in \mathbb{N}$ the eigenvalues can be characterized as follows:

$$\lambda_{k+1}(\mathcal{A}) = \min_{u\in\mathbb{P}_{k+1},\,\|u\|_{L^2(\mathcal{A})}=1} \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy = \min_{u\in\mathbb{P}_{k+1}\setminus\{0\}} \frac{\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy}{\int_{\mathcal{A}} |u(x)|^2dx},$$

where

$$\mathbb{P}_{k+1} = \{u \in H^s_D(\mathcal{A}) : \langle u, \varphi^{\mathcal{A}}_j\rangle_{H^s_D(\mathcal{A})} = 0 \ \forall j = 1,\dots,k\}.$$

And for any $k \in \mathbb{N}$, $\varphi^{\mathcal{A}}_{k+1} \in \mathbb{P}_{k+1}$ is an eigenfunction corresponding to $\lambda_{k+1}(\mathcal{A})$ with $\|\varphi^{\mathcal{A}}_{k+1}\|_{L^2(\mathcal{A}} = 1$ and

$$\lambda_{k+1}(\mathcal{A}) = \int_{\mathbb{R}^{2n}} \frac{|\varphi^{\mathcal{A}}_{k+1}(x)-\varphi^{\mathcal{A}}_k(y)|^2}{|x-y|^{n+2s}}dxdy;$$

5. *the sequence $\{\varphi^{\mathcal{A}}_k\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to $\lambda_k(\mathcal{A})$ is an orthonormal basis of $L^2(\mathcal{A})$ and an orthogonal basis of $H^s_D(\mathcal{A})$.*

Remark 3.5. From the item 5. of the above lemma, we can deduce that

$$\|u\|^2_{H^s_D(\mathcal{A})} \le \lambda_k(\mathcal{A})\|u\|^2_2,\ \forall u \in \text{span}\{\varphi^{\mathcal{A}}_1,\dots,\varphi^{\mathcal{A}}_k\}.$$

Remark 3.6. For the case in which $\mathcal{A} = \widetilde{\Omega}$, we denote $\varphi^{\mathcal{A}}_j$ by $\widetilde{\varphi}_j$, for every $j \in \mathbb{N}$.

Finally, we recall the Unique Continuation Property for the eigenfunctions of the problem (3.3) when $\mathcal{A} = \Omega$.

Lemma 3.7 ([14], Theorem 1.4)**.** *Let $u \in H^s_0(\Omega)$ be an eigenfunction of $(-\Delta)^s$ in Ω. If $u = 0$ on a set $E \subset \Omega$ of positive measure, then $u = 0$ in Ω.*

3.3 Proof of the Theorem 3.1

In order to prove the Theorem 3.1 we follow a variational approach. That is, we consider the reflexive space

$$E = \{u \in H^s_0(\Omega) : \int_\Omega h|u|^{p+1} < +\infty\},$$

endowed with the norm

$$\|u\|_E = \|u\|_{H_0^s(\Omega)} + \left(\int_\Omega h|u|^{p+1}\,dx\right)^{\frac{1}{p+1}}$$

and we define the C^1-functional $I_\lambda : E \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy - \frac{\lambda}{2}\int_\Omega u^2\,dx - \int_\Omega G(x,u)dx$$

$$+\frac{1}{p+1}\int_\Omega h|u|^{p+1}dx - \int_\Omega fu dx,\ \forall u \in E,$$

where $G(x,u) = \int_0^u g(x,s)ds$. Observe that the derivative of I_λ at $u \in E$ is given by

$$\langle I'_\lambda(u),\varphi\rangle = \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy - \lambda\int_\Omega u\varphi\,dx - \int_\Omega g(x,u(x))\varphi\,dx$$

$$+\int_\Omega h|u|^{p-1}u\varphi\,dx - \int_\Omega f\varphi\,dx,\quad \forall\varphi\in E.$$

Thus, critical points of I_λ are just solutions to problem (3.1).

Following the outline of the proof in [5] we split the proof in three steps.

Step 1. Case $\lambda < \lambda_1(\widetilde{\Omega})$.

The existence of a solution of the problem (3.1) is deduced by proving that the functional I_λ has a global minimum in E. This is done by showing that I_λ is coercive, bounded below and lower semicontinuous in E.

In order to make it, we first claim that if $I_\lambda(u_n)$ is bounded from above for a sequence $\{u_n\}\subset E$, then $\|u_n\|_2$ is bounded. Indeed, if we assume by contradiction that there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n\|_2 \to +\infty$ and we divide the inequality $I_\lambda(u_n) \le C$ by $\|u_n\|_2^2$ and denote $v_n = u_n/\|u_n\|_2$ it is deduced that

$$\|v_n\|^2_{H_0^s(\Omega)} + \frac{2}{p+1}\|u_n\|_2^{p-1}\int_\Omega h|v_n|^{p+1}\,dx \le \lambda + \frac{C}{\|u_n\|_2} + \frac{2\|f\|_2}{\|u_n\|_2} + \frac{C}{\|u_n\|_2^2} \le C. \tag{3.4}$$

Hence

$$\limsup_{n\to+\infty}\|v_n\|^2_{H_0^s(\Omega)} \le \lambda \text{ and } \lim_{n\to+\infty}\int_\Omega h|v_n|^{p+1}\,dx = 0$$

and by Lemma 3.3 there is a subsequence of $\{v_n\}$, denoted by the same v_n, which is weakly convergent to some v_0 in $H_0^s(\Omega)$, $v_n \to v_0$ in $L^2(\Omega)$ and a.e. in Ω with $\|v_0\|_{L^2(\Omega)} = 1$, $\|v_0\|^2_{H_0^s(\Omega)} \leq \lambda < \lambda_1(\widetilde{\Omega})$ and $\int_\Omega h|v_0|^{p+1}\, dx = 0$, which implies that $v_0 = 0$ in $\Omega \backslash \widetilde{\Omega}$ and $H_D^s(\widetilde{\Omega})$. We show that then we get a contradiction. Indeed, if would be $H_D^s(\widetilde{\Omega}) = \{0\}$, then $v_0 = 0$ in Ω, contradicting that $\|v_0\|_{L^2(\Omega)} = 1$; while if $H_D^s(\widetilde{\Omega}) \neq \{0\}$, then we have $\lambda_1(\widetilde{\Omega}) \leq \|v_0\|^2_{H_0^s(\Omega)} \leq \lambda < \lambda_1(\widetilde{\Omega})$, obtaining a contradiction. Therefore, we conclude that $\|u_n\|_2$ is necessarily bounded.

By the above claim, if a sequence $\{u_n\} \subset E$ satisfies that $I_\lambda(u_n)$ is bounded from above, then $\|u_n\|_2$ is bounded and consequently, by (3.4), $\|u_n\|_E$ is also bounded. This means that I_λ is coercive in E. The claim also shows that I_λ is bounded from below. Otherwise, there exists a sequence $\{u_n\} \subset E$ such that $I_\lambda(u_n) \to -\infty$. In particular, $I_\lambda(u_n)$ is bounded from above and then $\|u_n\|_2$ is bounded and thus I_λ would be bounded from below, which contradicts the fact that $I_\lambda(u_n) \to -\infty$.

To prove that I_λ is w.l.s.c., let $\{u_n\} \subset E$ be a sequence weakly converging to u_0 in E. Then $u_n \rightharpoonup u_0$ in $H_0^s(\Omega)$ and $u_n \rightharpoonup u_0$ in $L^{p+1}(\Omega, hdx)$ which imply that $\|u_0\|^2_{H_0^s(\Omega)} \leq \liminf_{n\to+\infty} \|u_n\|^2_{H_0^s(\Omega)}$ and $\int_\Omega |u_0|^{p+1} h\, dx \leq \liminf_{n\to+\infty} \int_\Omega |u_n|^{p+1} h\, dx$. By the Lemma 3.3, we also deduce that $\lim_{n\to+\infty} \|u_n\|_2 = \|u_0\|_2$ and $\lim_{n\to+\infty} \int_\Omega f u_n\, dx = \int_\Omega f u_0\, dx$. Therefore, the weak lower semicontinuity of I_λ is proved and the proof of Step 1 is concluded.

Remark 3.8. If $H_D^s(\widetilde{\Omega}) = \{0\}$ (for example, if $h > 0$ a.e. in Ω; i.e., $\text{meas}(\widetilde{\Omega}) = 0$) we have $\lambda_1(\widetilde{\Omega}) = +\infty$. Therefore, in this case, the proof of this step also proves the Theorem 3.1 for all $\lambda \in \mathbb{R}$.

Step 2. Case $\lambda_i(\widetilde{\Omega}) < \lambda < \lambda_{i+1}(\widetilde{\Omega})$, for $i \geq 1$.

Here, we prove the Theorem 3.1 in the case that $H^s_D(\widetilde{\Omega}) \neq \{0\}$ and $\lambda_i(\widetilde{\Omega}) < \lambda < \lambda_{i+1}(\widetilde{\Omega})$, for $i \geq 1$. We are going to show that the problem (3.1) has at least one weak solution, by applying the saddle point theorem of Rabinowitz [18, Theorem 1.2]. In order to make it, we choose $V = \langle \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_i \rangle \subset E$ and $W = \{w \in E : \langle \widetilde{\varphi}_j, w \rangle = 0 \text{ for } 1 \leq j \leq i\}$ to obtain that $E = V \oplus W$. First, we claim that I_λ is bounded from below on W. Otherwise, there exists a sequence $\{w_n\}_{n \in \mathbb{N}} \subset W$ such that $0 > I_\lambda(w_n) \to -\infty$ and then $\|w_n\|_2 \to \infty$. In particular, $\|w_n\|_{H^s_0(\Omega)} \to \infty$. If we consider the normalized sequence $z_n = w_n / \|w_n\|_{H^s_0(\Omega)}$, we can also assume, up to a subsequence by the Lemma 3.3, that there exists $z_0 \in W$ such that $z_n \rightharpoonup z_0$ in $H^s_0(\Omega)$, $z_n \to z_0$ in $L^2(\Omega)$ and a.e in Ω. Dividing the inequality $0 > I_\lambda(w_n)$ by $\|w_n\|^{p+1}_{H^s_0(\Omega)}$ and $\|w_n\|^2_{H^s_0(\Omega)}$ we deduce, by taking $n \to +\infty$, that $0 = \int_\Omega h|z_0|^{p+1}\, dx$ and hence $z_0 = 0$ in $\Omega \backslash \widetilde{\Omega}$, i.e., $z_0 \in H^1_D(\widetilde{\Omega}) \cap W$ and that $1 \leq \lambda \|z_0\|_2^2$. Consequently

$$\lambda_{i+1}(\widetilde{\Omega}) \|z_0\|_2^2 \leq \|z_0\|^2_{H^s_0(\Omega)} \leq \liminf_{n \to \infty} \|z_n\|^2_{H^s_0(\Omega)} = 1 \leq \lambda \|z_0\|_2^2, \text{ with } z_0 \neq 0,$$

i.e., $\lambda_{i+1}(\widetilde{\Omega}) \leq \lambda$ contradicting our hypothesis on λ and proving the claim.

On the other hand, using that the support of every function v in V is contained in $\widetilde{\Omega}$ and the Remark 3.5, we have $\|v\|_E = \|v\|_{H^s_0(\Omega)}$ and

$$I_\lambda(v) \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_i(\widetilde{\Omega})}\right) \|v\|^2_{H^s_0(\Omega)} - \int_\Omega G(x, v) dx - \int_\Omega f v \, dx, \quad \forall v \in V,$$

and taking into account that $\lambda_i(\widetilde{\Omega}) < \lambda$, we deduce that $\lim_{v \in V, \|v\|_E \to +\infty} I_\lambda(v) = -\infty$. Therefore, there exists $R_\lambda > 0$ such that

$$\max_{v \in V, \|v\|_E = R_\lambda} I_\lambda(v) < \inf_{w \in W} I_\lambda(w).$$

Now we prove that the functional I_λ satisfies the Palais-Smale compactness condition.

Specifically, if $\{u_n\} \subset E$ satisfies

$$I_\lambda(u_n) = \frac{1}{2}\|u_n\|^2_{H^s_0(\Omega)} - \frac{\lambda}{2}\int_\Omega u_n^2\,dx + \frac{1}{p+1}\int_\Omega h|u_n|^{p+1}\,dx - \int_\Omega G(x,u_n)dx - \int_\Omega f u_n\,dx \le C \tag{3.5}$$

and, for a real sequence $\epsilon_n \to 0$, that $|I'_\lambda(u_n)(\varphi)| \le \epsilon_n\|\varphi\|_E$; i.e.,

$$\left|\langle u_n,\varphi\rangle_{H^s_0(\Omega)} - \lambda\int_\Omega u_n\varphi\,dx + \int_\Omega h|u_n|^{p-1}u_n\varphi\,dx - \int_\Omega g(x,u_n)\varphi dx - \int_\Omega f\varphi\,dx\right| \le \epsilon_n\|\varphi\|_E,$$

for every $\varphi \in E$; then $\{u_n\}$ admits a convergent subsequence in E. Indeed, we first claim that the sequence $\|u_n\|_2$ is bounded. Otherwise, up to a subsequence, we can assume that $\|u_n\|_2 \to +\infty$ and dividing (3.5) by $\|u_n\|_2^2$, we deduce that $v_n := u_n/\|u_n\|_2$ satisfies

$$\frac{1}{2}\|v_n\|^2_{H^s_0(\Omega)} + \frac{1}{p+1}\int_\Omega h\frac{|u_n|^{p+1}}{\|u_n\|_2^2}\,dx \le \frac{C}{\|u_n\|_2^2} + \frac{\lambda}{2} + \frac{\|f\|_2}{\|u_n\|_2} + \frac{C}{\|u_n\|_2}$$

which implies that

$$\limsup_{n\to\infty}\|v_n\|^2_{H^s_0(\Omega)} \le \lambda \text{ and } \lim_{n\to\infty}\int_\Omega h|v_n|^{p+1}\,dx = 0.$$

In particular, passing to a subsequence, we can also assume that $v_n \rightharpoonup v_0$ in $H^s_0(\Omega)$, $v_n \to v_0$ in $L^2(\Omega)$ and a.e. in Ω with $\int_\Omega h|v_0|^{p+1}\,dx = 0$ and $v_0 \in H^1_D(\widetilde{\Omega})$.

On the other hand, by (3.6) and the weak convergence of v_n to v_0, we deduce that

$$0 = \lim_{n\to\infty}\langle v_n,\varphi\rangle_{H^s_0(\Omega)} - \lambda\int_\Omega v_n\varphi\,dx = \langle v_0,\varphi\rangle_{H^s_0(\Omega)} - \lambda\int_\Omega v_0\varphi\,dx,$$

for every $\varphi \in H^1_0(\widetilde{\Omega}) \subset E$. Thus, $v_0 \in H^1_0(\widetilde{\Omega})$ is a solution of

$$\begin{cases} (-\Delta)^s v = \lambda v, & \text{in } \widetilde{\Omega} \\ v = 0 & \text{in } \mathbb{R}^n \setminus \widetilde{\Omega} \end{cases}$$

which implies that $\lambda \in \{\lambda_i(\widetilde{\Omega}) : i = 1,2,\dots\}$, contradicting that $\lambda_i(\widetilde{\Omega}) < \lambda < \lambda_{i+1}(\widetilde{\Omega})$, and proving that $\|u_n\|_2 \le C$.

From the boundedness of u_n in $L^2(\Omega)$ and (3.5) we deduce that u_n is also bounded in E and using that E is reflexive we have that, up to a subsequence, $u_n \rightharpoonup u_0$ in E. Since the sequence $|u_n|^{p-1}u_n$ is bounded in $L^{\frac{p+1}{p}}(\Omega, hdx)$ and converges a.e. to $|u_0|^{p-1}u_0$, we deduce that it converges weakly to $|u_0|^{p-1}u_0$ in $L^{\frac{p+1}{p}}(\Omega, hdx)$, which implies that

$$\int_\Omega h|u_n|^{p-1}u_n\varphi\,dx \longrightarrow \int_\Omega h|u_0|^{p-1}u_0\varphi\,dx,\ \forall\varphi\in L^{p+1}(\Omega,hdx). \tag{3.6}$$

Using this, if we take the limit as $n\to\infty$ in (3.6) we deduce that

$$\langle u_0,\varphi\rangle_{H_0^s(\Omega)} - \lambda\int_\Omega u_0\varphi\,dx + \int_\Omega h|u_0|^{p-1}u_0\varphi\,dx - \int_\Omega f\varphi\,dx - \int_\Omega g(x,u_0)\varphi dx = 0,$$

for every $\varphi\in E$. Subtracting it from (3.6), taking $\varphi = u_n - u_0$ and by taking $n\to\infty$ we get that $\|(u_n-u_0)\|_{H_0^s(\Omega)}\to 0$ and that $\int_\Omega h|u_n|^{p+1}\to\int_\Omega h|u|^{p+1}$ which, by using the Fatou lemma, implies that $\int_\Omega h|u_n-u_0|^{p+1}\to 0$ and consequently $u_n\to u_0$ in E. This complete the proof of the Palais-Smale condition of I_λ and thus of all hypotheses of the Rabinowitz saddle point theorem. Applying this theorem, there is a critical point $u_\lambda\in E$ of the functional I_λ with $I_\lambda(u_\lambda) = c_\lambda \geq \inf_{w\in W} I_\lambda(w)$.

Remark 3.9. With the notation of the above proof, observe that if $\lambda_i(\widetilde{\Omega}) < \lambda \leq \alpha < \lambda_{i+1}(\widetilde{\Omega})$, then $I_\lambda \geq I_\alpha$ and thus $I_\lambda(u_\lambda) = c_\lambda \geq \inf_{w\in W} I_\lambda(w) \geq \inf_{w\in W} I_\alpha(w)$.

Step 3. Case $\lambda = \lambda_i(\widetilde{\Omega})$, for $i\geq 1$.

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence in the interval $(\lambda_i(\widetilde{\Omega}), \lambda_{i+1}\widetilde{\Omega}))$ which converges to $\lambda_i(\widetilde{\Omega})$. By Remark 3.9, for each $n\in\mathbb{N}$ there exists $u_n\in E$ such that $I'_{\alpha_n}(u_n) = 0$ and $I_{\alpha_n}(u_n) = c_{\alpha_n} \geq \inf_{w\in W} I_{\alpha_n}(w) \geq -c := \inf_{w\in W} I_{\alpha_1}(w)$. Hence $c \geq \frac{1}{2}\langle I'_{\alpha_n}(u_n), u_n\rangle - I_{\alpha_n}(u_n)$ which implies that $\frac{1}{\|u_n\|_2}\int_\Omega h|u_n|^{p+1}\,dx \leq C$ and, by aplying the Hölder inequality we obtain that

$$\frac{1}{\|u_n\|_2}\left|\int_\Omega h|u_n|^{p-1}u_n\varphi\,dx\right| \leq \left(\int h\,\varphi^{p+1}\,dx\right)^{\frac{1}{p+1}}\left(\frac{C}{\|u_n\|_2^{\frac{1}{p}}}\right) \tag{3.7}$$

Now we claim that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Otherwise, up to a subsequence, we can assume that $\|u_n\|_2\to+\infty$. By defining $z_n = u_n/\|u_n\|_2$ and using $\langle I'_{\alpha_n}(u_n), \frac{u_n}{\|u_n\|_2^2}\rangle = 0$ we obtain

$$\|z_n\|^2_{H^s_0(\Omega)} + \frac{1}{\|u_n\|_2^2}\int_\Omega h|u_n|^{p+1} \le \alpha_n + \frac{\|f\|_2}{\|u_n\|_2} + \frac{C}{\|u_n\|_2}. \tag{3.8}$$

In particular, $\{z_n\}_{n\in\mathbb{N}}$ is bounded in $H^s_0(\Omega)$ and, passing to a subsequence, we can assume that there exists $z_0 \in H^s_0(\Omega)$ such that $\|z_0\|_2 = 1$, $z_n \rightharpoonup z_0$ in $H^s_0(\Omega)$, $z_n \to z_0$ in $L^2(\Omega)$ and a.e. in Ω. By (3.8), we also deduce that $\int_\Omega h|z_0|^{p+1}\,dx = 0$ and $z_0 = 0$ in $\Omega\backslash\widetilde{\Omega}$; i.e. $z_0 \in H^1_D(\widetilde{\Omega})$. Using (3.7), $\langle I'_{\alpha_n}(u_n), \frac{\varphi}{\|u_n\|_2}\rangle = 0$ for each $\varphi \in H^s_0(\Omega)$ and taking $n \to \infty$ we deduce that z_0 is a solution of

$$\begin{cases} (-\Delta)^s v = \lambda_i(\widetilde{\Omega})v & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}\setminus\Omega \end{cases}$$

which vanishes on the open set $\Omega\backslash\widetilde{\Omega}$. However, this is impossible in view of the unique continuation property (Lemma 3.7) and we conclude that $\{u_n\}$ is bounded in $L^2(\Omega)$. Thus u_n is also bounded in E and then, up to a subsequence, $u_n \rightharpoonup u_0$ in E for some $u_0 \in E$, which is a solution of the problem (3.1) for $\lambda = \lambda_i(\widetilde{\Omega})$.

Chapter 4

A result of multiplicity for the homogeneous case of the problem (2.1)

4.1 Introduction

In this chapter, we study the existence and multiplicity of nontrivial solutions from the subcritical homogeneous case of the problem (2.1):

$$\begin{cases} -\Delta u = \lambda u - h(x)a(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $\lambda \geq \lambda_1(\widetilde{\Omega})$, a is a $C^1(\mathbb{R})$ function satisfying for some $1 < p < 2^* - 1$ (p subcritical)

$$(p+1)A(u) \leq a(u)u \quad \text{for} \quad |u| \geq R, \quad \text{for some} \quad 1 < p \quad \text{and} \quad R \quad \text{large}; \tag{4.2}$$

$$|a(u)| \leq c|u|^p + c, \quad \text{where} \quad c \quad \text{is a constant}; \tag{4.3}$$

$$\frac{a(u)}{u} > 0 \quad \forall u \neq 0, \text{ which implies that } a(0) = 0 \text{ and } A(u) > 0 \text{ for } u \neq 0; \tag{4.4}$$

$$(a(u) - a(v))(u - v) \geq C|u - v|^{p+1}, \text{ for some } C > 0 \text{ and for all } u, v \in \mathbb{R}; \tag{4.5}$$

$$a'(0) = 0 \tag{4.6}$$

and also assume that the function $0 \leq h \in L^\infty(\Omega)$ satisfies an strongly condition than (2.2):

$$h > 0 \text{ a.e. in } \Omega \backslash \widetilde{\Omega} \text{ with } \widetilde{\Omega} = \operatorname{int}\{x \in \Omega \,/\, h(x) = 0\}. \tag{4.7}$$

Alama and Tarantello studied this problem for every $p > 1$ in [2]. They defined the number

$$N(\lambda) = \#\{j; \lambda_j < \lambda\} - \#\{j; \tilde{\lambda}_j \leq \lambda\}.$$

and showed the following result:

Theorema 4.1 (Theorem C in [2]). *Assume that $a \in C(\mathbb{R})$ satisfaz (4.2), (4.3), (4.4) for some $p \in (1, +\infty)$ and $\lim_{u \to 0} \frac{a(u)}{u} = 0$. Then (4.1) has a nontrivial solution if and only if $N(\lambda) \geq 1$.*

In Section 4.2 we apply Theorem 2.1 for $\lambda \geq \lambda_1(\widetilde{\Omega})$ to find a solution of the problem (4.1) and we show that if $N(\lambda) \geq 1$, this solution is a nontrivial critical point of the functional I_λ, given by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{\lambda}{2}\int_\Omega u^2\, dx + \int_\Omega A(u)h(x)\, dx$$

with $A(u) = \int_0^u a(t)dt$. The idea is to use the Morse theory and critical groups, but this theory only works on C^2 functionals defined in a Hilbert space (see [9] for the definitions). This is the reason to assume p subcritical, $h \in L^\infty(\Omega)$ and $a \in C^1(\mathbb{R})$, thus we have that $I_\lambda \in C^2(H_0^1(\Omega), \mathbb{R})$. We also show that if $N(\lambda) \geq 2$, we have two nontrivials solutions (the second solution is given using the same idea than in Theorem 4.1).

In section 4.3, we consider a concave perturbation of problem (4.1):

$$\begin{cases} -\Delta u &= -\mu|u|^{q-2}u + \lambda u - h(x)a(u), \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $\lambda_1 < \lambda < \lambda_1(\widetilde{\Omega})$, $\mu > 0$, $1 < q < 2$, a is a $C^1(\mathbb{R})$ function satisfying for some $1 < p < 2^* - 1$ (p subcritical), (4.2), (4.3), (4.4) and (4.6) and also assume that the function $0 \leq h \in L^\infty(\Omega)$ satisfies (4.7).

We show that problem (4.8) have at least 4 nontrivial solutions (two positive and two negative) and if $\lambda_k < \lambda < \lambda_{k+1}$, $\lambda < \lambda_1(\widetilde{\Omega})$, then problem (4.8) have at least 5 nontrivial solutions.

4.2 Principal Results on the problem (4.1)

Now, we can give the main results of this Chapter and we begin with the following lemma:

Lemma 4.2. *We assume that $a \in C^1(\mathbb{R})$ satisfaz (2.4) and $\lambda_m(\widetilde{\Omega}) < \lambda$. Then every critical point u of I_λ satisfies $m(u) \geq m$, where $m(u)$ denote the Morse index of u.*

Proof. If u is a critical point of I_λ and $v \in \langle \tilde{\varphi}_1, \cdots, \tilde{\varphi}_m \rangle$, $v \neq 0$ then

$$\langle I''_\lambda(u)v, v\rangle = \int_\Omega |\nabla v|^2\, dx - \lambda \int_\Omega v^2\, dx + \int_\Omega a'(u)v^2 h\, dx = \int_\Omega |\nabla v|^2\, dx - \lambda \int_\Omega v^2\, dx < 0.$$

By the definition of $m(u)$, we can deduce that $m(u) \geq m$.

□

We give a existence result of problem (4.1).

Theorema 4.3. *Assume that $a \in C^1(\mathbb{R})$ satisfies (4.2), (4.3), (4.4) and (4.5) for some $1 < p < 2^* - 1$ and (4.6). If $N(\lambda) \geq 1$ and $\lambda \geq \lambda_1(\widetilde{\Omega})$, then problem (4.1) has a nontrivial solution.*

Proof. Assume that, for some $m \in \mathbb{N}$, $\lambda_m(\widetilde{\Omega}) \leq \lambda < \lambda_{m+1}(\widetilde{\Omega})$. The first step is to use Theorem 1.13.

To do this, we take $V = \langle \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_m \rangle$ and $W = \{w \in H_0^1(\Omega) / \int_\Omega \nabla\widetilde{\varphi}_j . \nabla w \, dx = 0 \text{ for } 1 \leq j \leq m\}$ and thus $H_0^1(\Omega) = V \oplus W$. Since $\lambda < \lambda_{m+1}(\widetilde{\Omega})$ then, as in the proof of Theorem 2.7, we have that $\inf_W I_\lambda > -\infty$. For $u \in V$ we have that $\int_\Omega |\nabla u|^2 \, dx \leq \lambda_m(\widetilde{\Omega}) \int_\Omega u^2 \, dx$ and

$$\int_\Omega A(u)h \, dx \leq C \int_\Omega |u|^{p+1} h \, dx + C \int_\Omega h \, dx = C\|h\|_{L^1(\Omega)},$$

and thus $I_\lambda(u) \leq \frac{1}{2}(\lambda_m(\widetilde{\Omega}) - \lambda)\|u\|_2 + C\|h\|_{L^1(\Omega)} \leq C\|h\|_{L^1(\Omega)}$ for every $u \in V$.

Also, by Lemma 2.2 the functional I_λ satisfies the (P.S) condition and thus we can apply the Theorem 1.13 to obtain a critical point u_1 of I_λ such that

$$C_m(I_\lambda, u_1) \neq 0. \tag{4.9}$$

In order to prove that u_1 is nontrivial, notice that $N(\lambda) \geq 1$ implies that , for some $k > m$,

$$\lambda_k < \lambda \leq \lambda_{k+1}.$$

Thus, by using $a'(0) = 0$, the Morse index of the trivial solution satisfies $m(0) = k > m$. It follows, by Theorem (1.12), that

$$C_m(I_\lambda, 0) = 0. \tag{4.10}$$

Then, comparing (4.9) and (4.10), we conclude that u is nontrivial. □

Next, we give a multiplicity result of the problem (4.1).

Theorema 4.4. *Assume that $a \in C^1(\mathbb{R})$ satisfies (4.2), (4.3), (4.4) and (4.5) for some $1 < p < 2^* - 1$ and (4.6). If $N(\lambda) \geq 2$, $\lambda \notin \{\lambda_i(\widetilde{\Omega})\}$ and $\lambda > \lambda_1(\widetilde{\Omega})$, then the problem (4.1) has at least two nontrivial solutions.*

Proof. Assume that $\lambda_m(\widetilde{\Omega}) < \lambda < \lambda_{m+1}(\widetilde{\Omega})$ and $\lambda_k < \lambda \leq \lambda_{k+1}$ with $N(\lambda) = k - m \geq 2$. By the previous theorem we have a nontrivial solution u_1 that satisfies $C_m(I_\lambda, u_1) \neq 0$.

Using Lemma (4.2) and Theorem (1.13) we obtain that

$$C_q(I_\lambda, u_1) = \delta_{q,m}\mathbb{Z}.$$

Now consider $H_0^1(\Omega) = V \oplus W$ where $V = \langle \varphi_1, \cdots, \varphi_k \rangle$. We have that $I_\lambda(w) \geq 0$ for all $w \in W$.

It follows from (4.4) and (4.6) that, given $\epsilon > 0$, there exists $C > 0$ such that

$$|A(u)| \leq \frac{\epsilon}{2}u^2 + C|u|^{p+1} \quad \forall u.$$

Taking $0 < \epsilon < \frac{\lambda - \lambda_k}{\|h\|_\infty}$ and using that $\lambda_k \|u\|_2 \geq \|u\|_{H_0^1(\Omega)}$ for $u \in V$, we have

$$\begin{aligned} I_\lambda(u) \leq & \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 + \frac{\epsilon\|h\|_\infty}{2}\|u\|_2^2 + C\|u\|^{p+1} \\ = & \frac{1}{2}\|u\|^2 - \frac{(\lambda - \epsilon\|h\|_\infty)}{2}\|u\|_2^2 + C\|u\|^{p+1} \\ \leq & \frac{1}{2}\|u\|^2 - \frac{(\lambda - \epsilon\|h\|_\infty)}{2\lambda_k}\|u\|^2 + C\|u\|^{p+1} \\ = & \frac{(\lambda_k - \lambda + \epsilon\|h\|_\infty)}{2\lambda_k}\|u\|^2 + C\|u\|^{p+1} \\ = & (\frac{(\lambda_k - \lambda + \epsilon\|h\|_\infty)}{2\lambda_k} + C\|u\|^{p-1})\|u\|^2 \end{aligned}$$

If we take $\|u\| = \rho = (\frac{\lambda - \lambda_k - \epsilon\|h\|_\infty}{4\lambda_k C})^{\frac{1}{p-1}} > 0$, we obtain that

$$I_\lambda(u) \leq \frac{(\lambda_k - \lambda + \epsilon\|h\|_\infty)}{4\lambda_k}\rho^2 < 0$$

for every $u \in V$ with $\|u\| = \rho$ and thus, for some $\delta > 0$

$$\sup_{v \in V, ||u|| = \delta} I_\lambda(v) < 0.$$

We can choose a nonzero $v_1 \in V$ such that I_λ is bounded below in $W + \langle v_1 \rangle$ (see [2, Lemma 4.4].

Now, we use the Theorem 1.14 to get a nontrivial solution u_2 such that $I_\lambda(u_2) < 0$ and

$$C_{k-1}(I_\lambda, u_2) \neq 0.$$

Since $k-1>m$, u_2 is a second nontrivial solution of the problem (4.1). $\square$

4.3 Principal results on the problem (4.8)

We define the functional associated to the problem (4.8) $I_{\mu,\lambda}: H_0^1(\Omega) \to \mathbb{R}$ by

$$I_{\mu,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\mu}{q}\int_\Omega |u|^q - \frac{\lambda}{2}\int_\Omega u^2\,dx + \int_\Omega A(u)h(x)\,dx \quad u \in H_0^1(\Omega),$$

where $\lambda_1 < \lambda < \lambda_1(\widetilde{\Omega})$, $\mu > 0$, $1 < q < 2$, a is a $C^1(\mathbb{R})$ function satisfying for some $1 < p < 2^* - 1$ (p subcritical), (4.2), (4.3), (4.4), (4.5) and (4.6) and also assume that the function $0 \le h \in L^\infty(\Omega)$ satisfies (4.7). Thus weak solutions of (4.8) correspond to critical points of the functional $I_{\mu,\lambda} \in C^1(H_0^1(\Omega),\mathbb{R})$.

We also define the functionals $I^+_{\mu,\lambda}$ and $I^-_{\mu,\lambda}$ given by

$$I^+_{\mu,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\mu}{q}\int_\Omega |u^+|^q - \frac{\lambda}{2}\int_\Omega (u^+)^2\,dx + \int_\Omega A(u^+)h(x)\,dx \quad u \in H_0^1(\Omega)$$

and

$$I^-_{\mu,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\mu}{q}\int_\Omega |u^-|^q - \frac{\lambda}{2}\int_\Omega (u^-)^2\,dx + \int_\Omega A(u^-)h(x)\,dx \quad u \in H_0^1(\Omega),$$

where $u^+ = \max\{u,0\}$ and $u^- = \min\{u,0\}$. Since $a'(0) = a(0) = 0$, by (4.4) and (4.6), we have that $I^+_{\mu,\lambda}$, $I^-_{\mu,\lambda} \in C^1(H_0^1(\Omega),\mathbb{R})$.

We begin by giving a relationship between critical points of $I_{\mu,\lambda}$, $I^+_{\mu,\lambda}$ and $I^-_{\mu,\lambda}$.

Lemma 4.5. *If u_+ and u_- are critical points of $I^+_{\mu,\lambda}$ and $I^-_{\mu,\lambda}$ respectively. Then, $u_+ \ge 0$ and $u_- \le 0$ in Ω. Moreover, u_+ and u_- are solutions of the problem (4.8) and $I_{\mu,\lambda}(u_+) = I^+_{\mu,\lambda}(u_+)$ and $I_{\mu,\lambda}(u_-) = I^-_{\mu,\lambda}(u_-)$.*

Proof. Since u_+ is a critical point of $I^+_{\mu,\lambda}$, we have that $I'^+_{\mu,\lambda}(u_+)(u_+^-) = 0$ and from this we conclude that $u_+^- = C = 0$ and thus $u_+ \ge 0$. Hence u_+ is a solution of (4.8) as well and $I_{\mu,\lambda}(u_+) = I^+_{\mu,\lambda}(u_+)$. Similarly, we obtain that $u_- \le 0$ in Ω and is a solution of the problem (4.8) with $I_{\mu,\lambda}(u_-) = I^-_{\mu,\lambda}(u_-)$. $\square$

Lemma 4.6. *The functionals $I_{\mu,\lambda}$, $I^{+}_{\mu,\lambda}$ and $I^{-}_{\mu,\lambda}$ are bounded below, coercive and satifies the (P.S.) condition.*

Proof. For every $u \in H^1_0(\Omega)$ we obtain $I_{\mu,\lambda}(u) \geq I_\lambda(u)$. From Theorem 2.4, we have that I_λ is bounded from below since $\lambda < \lambda_1(\widetilde{\Omega})$ and also I_λ is coercive (the proof is the same that bounded from below). Hence $I_{\mu,\lambda}$ is bounded from below and coercive.

Let u_n be a sequence in $H^1_0(\Omega)$ such that $I_{\mu,\lambda}(u_n)$ is bounded, i.e. $|I_{\mu,\lambda}(u_n)| \leq C$, and

$$\left| \int_\Omega \nabla u_n \cdot \nabla \varphi \, dx + \mu \int_\Omega |u_n|^{q-2} u_n \varphi \, dx - \lambda \int_\Omega u_n \varphi \, dx + \int_\Omega a(u_n) \varphi h \, dx \right| \leq \epsilon_n \|\varphi\|, \tag{4.11}$$

for some $\epsilon_n \to 0$ with $\epsilon_n > 0$ and every $\varphi \in H^1_0(\Omega)$. Since $I_{\mu,\lambda}$ is coercive, we have that $\|u_n\| \leq C$. Thus, there exists $u_0 \in H^1_0(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in $H^1_0(\Omega)$, $u_n \to u_0$ in $L^2(\Omega)$, in $L^\mu(\Omega)$, in $L^{p+1}(\Omega)$ and a.e. in Ω. Also for some function $\widetilde{g} \in L^{p+1}(\Omega)$, $|u_n| \leq \widetilde{g}$. Thus, by the dominated convergence theorem and tending $n \to \infty$ in (4.11) we deduce

$$\int_\Omega \nabla u_0 \nabla \varphi \, dx + \mu \int_\Omega |u_0|^{q-2} u_0 \varphi \, dx - \lambda \int_\Omega u_0 \varphi \, dx + \int_\Omega a(u_0) \varphi h \, dx = 0$$

for every $\varphi \in H^1_0(\Omega)$. Substracting it from (4.11) we get

$$\left| \int_\Omega \nabla (u_n - u_0) \cdot \nabla \varphi \, dx + \mu \int_\Omega (|u_n|^{q-2} u_n - |u_0|^{q-2} u_0) \varphi \, dx - \lambda \int_\Omega (u_n - u_0) \varphi \, dx + \int_\Omega (a(u_n) - a(u_0)) \varphi h \, dx \right| \leq \epsilon_n \|\varphi\|, \tag{4.12}$$

which by the choice $\varphi = u_n - u_0$ implies that

$$\left| \int_\Omega |\nabla (u_n - u_0)|^2 \, dx + \mu \int_\Omega (|u_n|^{q-2} u_n - |u_0|^{q-2} u_0)(u_n - u_0) \, dx - \lambda \int_\Omega (u_n - u_0)^2 \, dx + \int_\Omega (a(u_n) - a(u_0))(u_n - u_0) h \, dx \right| \leq \epsilon_n \|\varphi\|, \tag{4.13}$$

Using, again, the dominated convergence theorem we conclude that $u_n \to u_0$ in $H^1_0(\Omega)$

and thus $I_{\mu,\lambda}$ satisfies the P.S. condition. Similarly to this functional we show to the functionals $I^{+}_{\mu,\lambda}$ and $I^{-}_{\mu,\lambda}$. □

Lemma 4.7. *If u_{+} is a local minimizer of $I^{+}_{\mu,\lambda}$ (u_{-} is a local minimizer of $I^{-}_{\mu,\lambda}$), then it is also a local minimizer of $I_{\mu,\lambda}$ and hence the critical groups of $I_{\mu,\lambda}$ at u_{+} (u_{-}) are given by*

$$C_q(I_{\mu,\lambda}, u_{+}) = C_q(I_{\mu,\lambda}, u_{-}) = \delta_{q,0}\mathbb{Z}.$$

Proof. By a result of Brezis and Nirenberg [6], it suffices to show that u_{+} is a local minimizer of $I_{\mu,\lambda}$ in the C^1 topology. It is easily seen that u_{+} is a local minimizer of $I^{+}_{\mu,\lambda}$ in the C^1 topology also, say, $\rho > 0$ is such that $I^{+}_{\mu,\lambda}(u) \geq I^{+}_{\mu,\lambda}(u_{+})$ $\forall u \in B_{C^1}(u_{+}, \rho) = \{u \in C^1_0(\overline{\Omega}) : \|u - u_{+}\|_{C^1} < \rho\}$. Then for $u \in B_{C^1}(u_{+}, \rho)$,

$$\begin{aligned}
I_{\mu,\lambda}(u) - I_{\mu,\lambda}(u_{+}) =& I_{\mu,\lambda}(u) - I^{+}_{\mu,\lambda}(u_{+}) \\
\geq& I_{\mu,\lambda}(u) - I^{+}_{\mu,\lambda}(u) \\
=& \frac{\mu}{q}\int_{\Omega}(|u|^q - |u^{+}|^q)\,dx - \frac{\lambda}{2}\int_{\Omega}(u^2 - |u^{+}|^2)\,dx + \int_{\Omega}(A(u) - A(u^{+}))h\,dx \\
=& \frac{\mu}{q}\int_{\Omega}|u^{-}|^q\,dx - \frac{\lambda}{2}\int_{\Omega}|u^{-}|^2\,dx + \int_{\Omega}A(u^{-})h\,dx \\
\geq& \frac{\mu}{q}\int_{\Omega}|u^{-}|^q\,dx - \frac{\lambda}{2}\|u^{-}\|^{2-q}_{C^0}\int_{\Omega}|u^{-}|^q\,dx \\
=& (\frac{\mu}{q} - \frac{\lambda}{2}\|u^{-}\|^{2-q}_{C^0})\int_{\Omega}|u^{-}|^q\,dx.
\end{aligned}$$

Since $\|u - u_{+}\|_{C^1} < \rho$ and $u^{+} \geq 0$, then $\|u^{-}\|_{C^0} < \rho$. Thus taking $\widetilde{\rho} = \min\{\rho, (\frac{2\mu}{q\lambda})^{\frac{1}{2-q}}\}$, we have that u_{+} is a minimum of $I_{\mu,\lambda}$ on $B_{C^1}(u_{+}, \widetilde{\rho})$.

Since $q < 2$, the conclusion of the lemma follows (for the critical groups see Example 1 in Chapter I, Section 4 of Chang [9]). Similarly we have the same conclusion to u_{-}. □

Lemma 4.8. *$u \equiv 0$ is a local minimizer of $I_{\mu,\lambda}$, $I^{+}_{\mu,\lambda}$ and $I^{-}_{\mu,\lambda}$*

Proof. As in the proof of Lemma 4.7, we show that 0 is a local minimizer of $I_{\mu,\lambda}$ in the

C^1 topology. We have for $u \in C_0^1(\overline{\Omega})$,

$$\begin{aligned} I_{\mu,\lambda}(u) \geq& \frac{\mu}{q}\int_\Omega |u|^q\, dx - \frac{\lambda}{2}\|u\|_{C^0}^{2-q}\int_\Omega |u|^q \\ =& (\frac{\mu}{q} - \frac{\lambda}{2}\|u\|_{C^0}^{2-q})\int_\Omega |u|^q\, dx \geq 0 \end{aligned}$$

if $\|u\|_{C^0} \leq (\frac{2\mu}{q\lambda})^{\frac{1}{2-q}}$. The argument for $I^+_{\mu,\lambda}$ and $I^-_{\mu,\lambda}$ is the same. □

Lemma 4.9. *If $\lambda > \lambda_k$, then there exist μ^*, $\rho > 0$ such that*

$$\sup_{S_\rho^k} I_{\mu,\lambda} < 0$$

for $0 < \mu < \mu^*$, where $S_\rho^k = \{u \in V : \|u\| = \rho\}$ and $V = \langle \varphi_1, \ldots, \varphi_k\rangle$.

Proof. It follows from (4.4) and (4.6) that, given $\epsilon > 0$, there exists $C > 0$ such that

$$|A(u)| \leq \frac{\epsilon}{2}u^2 + C|u|^{p+1} \quad \forall u.$$

Taking $0 < \epsilon < \frac{\lambda-\lambda_k}{\|h\|_\infty}$ and using that $\lambda_k\|u\|_2 \geq \|u\|_{H_0^1(\Omega)}$ for $u \in V$, we have

$$\begin{aligned} I_{\mu,\lambda}(u) \leq& \frac{1}{2}\|u\|^2 + \frac{\mu C'}{q}\|u\|^q - \frac{\lambda}{2}\|u\|_2^2 + \frac{\epsilon\|h\|_\infty}{2}\|u\|_2^2 + C\|u\|^{p+1} \\ =& \frac{1}{2}\|u\|^2 - \frac{(\lambda - \epsilon\|h\|_\infty)}{2}\|u\|_2^2 + \frac{\mu C'}{q}\|u\|^q + C\|u\|^{p+1} \\ \leq& \frac{1}{2}\|u\|^2 - \frac{(\lambda - \epsilon\|h\|_\infty)}{2\lambda_k}\|u\|^2 + \frac{\mu C'}{q}\|u\|^q + C\|u\|^{p+1} \\ =& \frac{(\lambda_k - \lambda + \epsilon\|h\|_\infty)}{2\lambda_k}\|u\|^2 + \frac{\mu C'}{q}\|u\|^q + C\|u\|^{p+1} \\ =& (\frac{(\lambda_k - \lambda + \epsilon\|h\|_\infty)}{2\lambda_k} + C\|u\|^{p-1} + \frac{\mu C'}{q}\|u\|^{q-2})\|u\|^2 \end{aligned}$$

If we take $\|u\| = \rho = (\frac{\lambda-\lambda_k-\epsilon\|h\|_\infty}{4\lambda_k C})^{\frac{1}{p-1}}$ we obtain that

$$I_{\mu,\lambda}(u) \leq (\frac{(\lambda_k - \lambda + \epsilon\|h\|_\infty)}{4\lambda_k} + \frac{\mu C'}{q}\rho^{q-2})\rho^2$$

Finally, taking $0 < \mu < \mu^* = (\frac{q}{C'\rho^{q-2}})(\frac{\lambda-\lambda_k-\epsilon\|h\|_\infty}{4\lambda_k})$ we conclude this lemma.

□

Lemma 4.10. *If $\lambda < \lambda_{k+1}$, then $I_\lambda \geq 0$ on $W = \langle \varphi_1, \ldots, \varphi_k \rangle^\perp$.*

Proof. Using that for $u \in W$, $\lambda_{k+1}\|u\|_2 \leq \|u\|$ we have that

$$\begin{aligned} I_{\mu,\lambda}(u) \geq& \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 \\ \geq& \frac{1}{2}\|u\|^2 - \frac{\lambda}{2\lambda_k}\|u\|^2 \\ =& \frac{(\lambda_{k+1} - \lambda)}{2\lambda_{k+1}}\|u\|^2 \geq 0. \end{aligned}$$

□

Theorema 4.11. *Assume that $\lambda_1 < \lambda < \lambda_1(\widetilde{\Omega})$, $\mu > 0$, $1 < q < 2$, a is a $C^1(\mathbb{R})$ function satisfying for some $1 < p < 2^* - 1$ (p subcritical), (4.2), (4.3), (4.4), (4.5) and (4.6) and also assume that the function $0 \leq h \in L^\infty(\Omega)$ satisfies (4.7). Then there exists $\mu^* > 0$ such that problem (4.8) has at least four nontrivial solutions (two positives and two negatives) for $0 < \mu < \mu^*$.*

Proof. By Lemma 4.8, $u \equiv 0$ is a local minimizer of $I^+_{\mu,\lambda}$ and $I^-_{\mu,\lambda}$ with $I^+_{\mu,\lambda}(0) = I^-_{\mu,\lambda}(0) = 0$. By Lemma 4.9 with $k = 1$, $\inf_{H^1_0(\Omega)} I^+_{\mu,\lambda} \leq \inf_{t\geq 0} I^+_{\mu,\lambda}(t\varphi_1) < 0$ and $\inf_{H^1_0(\Omega)} I^-_{\mu,\lambda} \leq \inf_{t\geq 0} I^-_{\mu,\lambda}(-t\varphi_1) < 0$. Hence, by Theorem 1.8, $I^+_{\mu,\lambda}$ has a nontrivial critical point u_1^+ of the mountain pass type with $I^+_{\mu,\lambda}(u_1^+) > 0$. Also $I^-_{\mu,\lambda}$ has a nontrivial critical point u_1^- of the mountain pass type with $I^-_{\mu,\lambda}(u_1^-) > 0$.

Since $I^+_{\mu,\lambda}$ and $I^-_{\mu,\lambda}$ are bounded below and satisfy the (P.S.) condition, by Lemma 4.6, they also have a nontrivial global minimizer u_0^+ and u_0^- respectively, such that $I^+_{\mu,\lambda}(u_0^+) = \inf_{H^1_0(\Omega)} I^+_{\mu,\lambda} < 0$ and $I^-_{\mu,\lambda}(u_0^-) = \inf_{H^1_0(\Omega)} I^-_{\mu,\lambda} < 0$. Finally, by Lemma 4.5 we conclude this theorem. □

Theorema 4.12. *Assume that $\lambda_k < \lambda < \lambda_{k+1}$ with $k \geq 2$, $\lambda < \lambda_1(\widetilde{\Omega})$, $\mu > 0$, $1 < q < 2$, a is a $C^1(\mathbb{R})$ function satisfying for some $1 < p < 2^* - 1$ (p subcritical), (4.2), (4.3), (4.4), (4.5) and (4.6) and also assume that the function $0 \leq h \in L^\infty(\Omega)$ satisfies (4.7). Then there exists $\mu^* > 0$ such that problem (4.8) has at least five nontrivial solutions for $0 < \mu < \mu^*$.*

Proof. As in the proof of Theorem 4.11, $I^+_{\mu,\lambda}$ has a mountain pass point u_1^+ at a positive level and a global minimizer u_0^+ at a negative level and $I^-_{\mu,\lambda}$ has a mountain pass point u_1^-

at a positive level and a global minimizer u_0^- at a negative level. By Lemma 4.7, u_0^+ and u_0^- are local minimizers of $I_{\mu,\lambda}$ and the critical groups of $I_{\mu,\lambda}$ at u_0^+ and u_0^- are given by

$$C_q(I_{\mu,\lambda}, u_0^+) = C_q(I_{\mu,\lambda}, u_0^-) = \delta_{q,0}\mathbb{Z}.$$

We get one more critical point by applying Theorem 1.14 to $I_{\mu,\lambda}$ using the splitting $H_0^1(\Omega) = V \oplus W$ with $V = \langle \varphi_1, \dots, \varphi_k \rangle$. The conditions (I_1) and (I_2) have already been verified in Lemmas 4.9 and 4.10. Since $I_{\mu,\lambda}$ is bounded below, (I_3) is also satisfied. Thus $I_{\mu,\lambda}$ has two critical points u_{k-1}, u_k with $I_{\mu,\lambda}(u_{k-1}) < 0$, $I_{\mu,\lambda}(u_k) \geq 0$ and $C_{k-1}(I_{\mu,\lambda}, u_{k-1}) \neq 0$, $C_k(I_{\mu,\lambda}, u_k)$. Comparing the critical values and the critical groups of 0, u_0^+, u_0^-, u_1^+, u_1^- and u_{k-1}, and using $k \geq 2$ we see that they are all different.

$\square$

Bibliography

[1] S. Alama and G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities.* Calc. Var. Partial Differential Equations 1 (1993), 439-475.

[2] S. Alama and G. Tarantello, *On the solvability of a semilinear elliptic equation via an associated eigenvalue problem.* Math. Z. 221 (1996), 467-493.

[3] S. Alama and G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign.* J. Funct. Anal. 141 (1996), 159-215.

[4] A. Ambrosetti and D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems.* Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser 2011.

[5] D. Arcoya, F. O. de Paiva and J. M. Mendoza, *Existence of solutions for a nonhomogeneous semilinear elliptic equation*, preprint (2017).

[6] H. Brezis and L. Nirenberg, *H^1 versus C^1 local minimizers*, C. R. Acad. Sci. Paris Sér. I Math. 317, No. 5 (1993). 635-645.

[7] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer Science Business Media, 2011.

[8] H. Berestycki, I. Capuzzo-Dolcetta & L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems.* NoDEA Nonlinear Differential Equations Appl. 2 (1995), 553-572.

[9] K.C. Chang, *Infinite Dimensional Morse Theory and Multiple Solutions Problems*, Birkhäuser, Boston (1993).

[10] M. Del Pino and P.L. Felmer, *Multiple solutions for a semilinear elliptic equation.* Trans. Amer. Math. Soc. **347** (1995), 4839-4853.

[11] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. math. 136 (2012) 521-573.

[12] J.P. Gossez and D.G. de Figueiredo, *Strict monotonicity of eigenvalues and unique continuation.* Comm. P.D.E. 17, 339-346 (1992).

[13] J.L. Kazdan and F.W. Warner, *Scalar curvature and conformal deformation of Riemannian structure.* J. Differential Geometry 10 (1975), 113-134.

[14] M. Moustapha Fall and V. Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Comm. Partial Differential Equations, 39(2):354-397, 2014.

[15] T. Ouyang, *On the positive solutions of semilinear equations* $\Delta u + \lambda u - hu^p = 0$ *on the compact manifolds.* Trans. Amer. Math. Soc. **331** (1992), 503-527.

[16] K. Perera, *Multiplicity results for some elliptical problems with concave nonlinearities*, J. Diff. Equ. **140** (1997), 133-141.

[17] K. Perera and M. Schechter, *Critical groups in saddle point theorems without a finite dimensional closed loop*, Math. Nachr. 243 (2002, 156-164).

[18] P.H. Rabinowitz, *Some Minimax Theorems and Applications to Nonlinear Partial Differential Equations*, in Nonlinear Analysis: A Collection of Papers in Honor of Erich Röthe, Academic Press, New York 1978, pp. 161-177.

[19] M. Ramos, S. Terracini and C. Troestler, *Superlinear indefinite elliptic problems and Pohozaev type identities.* J. Funct. Anal. 159 (1998), 596-628.

[20] R. Servadei and E. Valdinoci, *Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators*, Rev. Mat. Iberoam. 29 (2013), no. 3, 1091-1126.

[21] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, 2010 AMS.

[22] M. Struwe, *Variational Methods, Applications to Nonlinear PDE and Hamiltonial Systems*, Springer-Verlag, Berlin, 1996.