Brenno Gustavo Barbosa

Compactly-supported and Relative Integration in Differential Cohomology

São Carlos, SP - Brasil

June 6, 2022

Brenno Gustavo Barbosa

Compactly-supported and Relative Integration in Differential Cohomology

Monografia apresentada para obtenção do título de Doutor em Matemática pela Universidade Federal de São Carlos

Universidade Federal de São Carlos – UFSCar

Departamento de Matemática – DM

Programa de Pós-Graduação em Matemática – PPGM

Supervisor: Fábio Ferrari Ruffino

São Carlos, SP - Brasil

June 6, 2022

UNIVERSIDADE FEDERAL DE SÃO CARLOS

Centro de Ciências Exatas e de Tecnologia
Programa de Pós-Graduação em Matemática

Folha de Aprovação

Defesa de Tese de Doutorado do candidato Brenno Gustavo Barbosa, realizada em 11/04/2022.

Comissão Julgadora:

Prof. Dr. Fabio Ferrari Ruffino (UFSCar)

Prof. Dr. Daniel Vendrúscolo (UFSCar)

Prof. Dr. Pedro Luiz Queiroz Pergher (UFSCar)

Prof. Dr. Ugo Bruzzo (UFPB)

Prof. Dr. Abdelmoubine Amar Henni (UFSC)

O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

À minha tia Marilda.

Acknowledgements

Se este trabalho existe, é porque muitas pessoas participaram dele tanto de forma direta quanto indireta.

Primeiramente, gostaria de agradecer a toda a minha família e em especial aos meus pais, Tania e Domingos, e aos meus avós, Luzia e Alaor, que sempre estiveram presentes e me apoiaram. Teria sido impossível chegar aqui sem eles. Note que esse “impossível” não é, de forma alguma, uma figura hiperbólica.

Agradeço muitíssimo ao meu orientador, Fábio Ferrrari Ruffino. Seu vasto conhecimento e sua capacidade singular de transmiti-lo só encontram rival a altura na sua paciência , que é provavelmente infinita.

Sou muito grato também ao meu amigo Juan Nuñez, que é coautor deste trabalho, mas que muitas vezes atuou como um coorientador. Saliento, porém, que seu papel não se restringiu a questões profissionais: sua amizade sincera e capacidade de andar por mais de 30km em um dia me mantiveram são mentalmente e fisicamente.

Agradeço também a minha namorada Clarissa Bergo, que conseguiu transformar o meu inglês em algo minimamente inteligível e que, se isso não bastasse, ainda ouviu por horas intermináveis alguns tópicos intragáveis sobre topologia e geometria. Essa tese foi concebida durante um período muito tenso, a epidemia de Covid-19, mas não era só o período que estava tenso: eu estava provavelmente mais tenso que ele. A Cla, mesmo passando por um período de transição profissional, foi um alicerce para mim. Sem o seu apoio, compreensão, paciência e carinho, eu certamente não teria conseguido chegar ao fim deste trabalho. Ainda não sei como ela conseguiu. Eu espero um dia poder retribuir.

Tenho uma dívida de gratidão eterna com meu grande amigo Hugo Botós por mais motivos do que eu poderia escrever em um agradecimento. Para se ter uma ideia do tamanho dessa dívida, se hoje eu concluo esse doutorado é porque ele me incentivou e acreditou no meu potencial. Eu sempre fui apaixonado pela matemática, mas, por algum motivo ou outro, nunca segui a minha paixão. O senhor Hugo me mostrou que o sonho ainda podia se tornar realidade. Na verdade, essa é uma das habilidades dele: inspirar a paixão matemática nas pessoas com quem ele tem contato, o que ele só pode fazer, claro, pela incrível paixão que ele nutre pela área e pelo enorme talento matemático que possui. Obviamente, isso é apenas um dos pontos, um outro ponto é o suporte emocional que ele me deu em momentos cruciais.

Devo agradecer também ao meu amigo Filipe Fernandes pelas discussões, pelas sugestões e pelo empurrão para eu colocar um ponto final na tese. O Filipe é uma pessoa muito centrada e calma. Ele consegue transmitir essa serenidade para lidar com os problemas o que, sem dúvida, me ajudou várias vezes.

Estou também em dívida com o pessoal do DM-UFSCar. Sinto muito orgulho de ter feito parte desta comunidade na qual fui muito bem recebido e tratado. A cordialidade e a presteza da Priscila na secretaria de pós-graduação foram sem iguais. A dedicação e seriedade com que os professores conduzem as pesquisas e projetos de extensão, o prazer e atenção para com os alunos com que ministram suas aulas são realmente inspiradores. Espero um dia chegar a esse nível.

Deixo aqui registrado a minha gratidão aos meus inúmeros amigos que estiveram presentes ao longo de todos estes anos: desde a Física: Thiago, Jaque; passando pela estatística: Marina; pelo ICMC-USP: Marcelo, Nancy, Alisson; até o DM-UFSCar: Alex, Thales, Osmar, Marco e Diana.

Por fim, agradeço a CAPES pelo apoio financeiro. O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Código de Financiamento 001.

"São as águas de março fechando o verão,
é promessa de vida no teu coração"
(Águas de Março, Tom Jobim)

Resumo

Neste trabalho, foram construídas versões mais gerais da integração diferencial em teorias de cohomologia diferencial. Dentre as versões obtidas podemos citar a integração com suporte compacto, a integração com suporte verticalmente compacto e suas versões relativas. Para obter estes mapas, foi necessário construir um produto especial entre classes diferenciais paralelas e relativas e estender a teoria de cohomologia diferencial a uma categoria de sequencias de variedades.

Palavras-chave: Cohomologia diferencial. Integração diferencial. Topologia algébrica. Geometria diferencial.

Abstract

In this work, we have constructed more general versions of differential integration maps in differential cohomology theories. Among the obtained versions we can mention the integration with compact supports, the integration with vertically compact supports and their relative versions. To obtain these maps, it was necessary to construct a special product between parallel and relative differential classes and to extend differential cohomology theories to a category of sequences of manifolds.

Keywords: Differential cohomology. Differential integration. Algebraic topology. Differential geometry.

List of Figures

List of Tables

List of symbols

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$	Natural, integer, real, rational and complex numbers
$\mathbb{R}_+, \mathbb{R}_{++}$	Nonegative and positive real numbers
$\mathtt{Top}, \mathtt{Top}^2$	Category of topological spaces and its category of maps.
$\mathtt{Top}_2, \mathtt{Top}^2_\omega$	Category of pairs of topological spaces and its category of maps.
$\mathtt{Top}_\omega, \mathtt{Top}^2_\omega$	Category of finite sequences of topological spaces and its category of maps.
M_ρ, C_ρ	Mapping cylinder and mapping cone
$(h, \partial), (\widetilde{h}, S)$	Relative cohomology theory and reduced cohomology theory
$\mathfrak{h}, \mathfrak{h}_\mathbb{R}$	Coefficient group of $\widehat{h}$ and its tensorization with $\mathbb{R}$
$HG, HG^\bullet$	Eilenberg-Maclane spectrum of a graded group G and its cohomology
ch	Chern-Dold character
$\mathcal{K}(X), \mathcal{V}(f), \mathcal{VV}(f,g)$	Compact susets of X, Vertically compact sets of f, doubly-vertically compact sets of (f,g).
h_c, h_v, h_{vv}	Compact, vertically compact and doubly vertically compact cohomologies
u	Thom class
T, T_c, T_v	Classical, compactly supported and doubly-vertically-compactly supported Thom isomorphisms
$f_!, f_{c!}, f_{v!}$	Classical, compactly supported and vertically-compactly supported umkehr maps
$(F, f)_!$	Relative umkehr map
$\mathtt{Man}, \mathtt{Man}^2$	Category of Mmanifolds and its category of maps
$\mathtt{Man}_2, \mathtt{Man}^2_\omega$	Category of pairs of manifolds and its category of maps.
$\mathtt{Man}_\omega, \mathtt{Man}^2_\omega$	Category of finite sequences of manifolds and its category of maps.
$\Omega, \Omega G$	de Rham complex and de Rham complex with coefficients in G

Ω_{par}	de Rham complex of parallel forms
Ω_c, Ω_v	de Rham complexes of forms with compact and vertically compact supports
$\int_f, \int_{(F,f)}$	Fiber and fiber relative integration of differential forms
$\widehat{h}, \widehat{h}_{\text{par}}, \widehat{h}_{\text{flat}}$	Relative differential cohomology, parallel differential cohomology
R, R_{par}	Curvature and its parallel version
I, I_{par}	Forgetful map and its parallel version
a, a_{par}	Trivialization and its parallel version
$\widehat{h}_c, \widehat{h}_v, \widehat{h}_{vv}$	Compact, Vertically compact and doubly-vertically compact differential cohomologies
R_c, R_v	Compact and vertically compact curvatures
I_c, I_v	Compact and vertically compact forgetful maps
a_c, a_v	Compact and vertically compact trivializations
$\int_{S^1}, \int_{S^1}^c$	S^1 integration and compact S^1 integration (topological and differential)
$\int_{\mathbb{R}^N}^c, \int_{\mathbb{R}^N}^v$	Compact and Vertical integration maps (topological and differential)
$\widehat{u}$	Differential Thom class
$\widehat{T}, \widehat{T}_c, \widehat{T}_v$	Classical, compactly supported and doubly-vertically-compactly supported differential Thom morphisms (absolute and relative cases)
$\widehat{f}_!, \widehat{f}_{c!}, \widehat{f}_{v!}$	Classical, compact and vertical differential integration maps
$\widehat{(F,f)}_!, \widehat{f}_{!!}$	Relative differential integration map and relative differential integration with respect to boundary
S, Z, B	Singular (co)chains, (co)cycles and (co)boundaries.
E, A	Eilenberg-Maclane and Alexander-Whitney maps.
$(\chi, (\omega, \eta))$	Differential characters
$[E_1, E_2, \psi]$	Vector bundles triple
(E_n, ϵ_n)	Ω-spectrum with structure maps
$E^\bullet$	Cohomology associated to spectrum (E_n, ϵ_n)
ι_n	Fundamental cocycle

Contents

Introduction

We start this introduction by drawing attention to the two main concepts we are going to discuss in this text:

- Differential Cohomology;
- Integration maps in Differential Cohomology.

What is differential cohomology?

Differential cohomology theories are refinements of generalized cohomology theories, in the sense that they not only retain all topological information, but also carry geometric information, which is encoded in the differential structure.

Let us illustrate the idea of a differential refinement with an over-simplistic model, which still captures the central idea of differential cohomology.

Fix a manifold X and let $H\mathbb{Z}^1(X)$ denote its first singular cohomology group with integral coefficients. Let $\iota \in H\mathbb{Z}^1(S^1) \cong \mathbb{Z}$ be a generator. Every cohomology class in $H\mathbb{Z}^1(X)$ can be written as a pullback $f^*\iota$ for some continuous map $f : X \to S^1$. Moreover, the sum of two classes $f^*\iota, g^*\iota \in H\mathbb{Z}^1(X)$ is represented by $(fg)^*\iota$, where fg is the pointwise product. The abelian group $\widehat{H\mathbb{Z}}^1(X) := C^\infty(X, S^1)$ of smooth functions from X to S^1, with its pointwise product, is a differential refinement of $H^1(X)$. But what exactly does this mean?

Consider the homomorphism $I : \widehat{H\mathbb{Z}}^1(X) \to H^1(X)$, which sends a smooth function $f : X \to S^1$ to the cohomology class $f^*\iota$. This map is a **forgetful map**. When we say that $\widehat{H\mathbb{Z}}^1(X)$ refines $H^1(X)$, we mean that I is surjective. This is indeed the case, since the pullback of the generator ι by a continuous function f classifies the homotopy class of f. Moreover, in a manifold, any continuous function is homotopic to a smooth one.

So, what extra piece of information do we have?

Let $f, g \in \widehat{H\mathbb{Z}}^1(X)$ be such that $I(f) = I(g)$, *i.e.*, $f^*\iota = g^*\iota$. This means that $\frac{f}{g}$ is null-homotopic, which is equivalent to saying that there exists a function $h : X \to \mathbb{R}$ such that $\frac{f}{g} = \exp \circ h$, where $\exp : \mathbb{R} \to S^1$ is the normalized exponential map $t \mapsto e^{2\pi i t}$. We say that the function h is a global logarithm of $\frac{f}{g}$. Therefore, when two functions f and g carry the same topological information, their quotient is a global logarithm.

Given a smooth function, $h : X \to \mathbb{R}$ we define the homomorphism $a(h) = \exp \circ h$. This gives us a map $a : \Omega^0(X) \to \widehat{H\mathbb{Z}}^1(X)$, where $\Omega^0(X) = C^\infty(X)$. Observe that the kernel of this map is formed by constant $\mathbb{Z}$-valued functions. This maps gives us trivial topological classes. In fact, $H(t,x) = \exp(t \cdot h(x))$ is a homotopy between the constant function 1 and $a(h)$. Hence, the transformation a is called (topological) **trivialization**. It follows from the discussion above that $\mathrm{Im}(a) = \ker(I)$.

But there is yet another piece of geometric information, which is in some sense a mixture of the trivialization and the forgetful map. Recall that there exists a morphism from the ordinary cohomology with coefficients in $\mathbb{Z}$ to de Rham cohomology. This morphism can be roughly described as thinking of a $\mathbb{Z}$-valued class as an $\mathbb{R}$-valued class and applying the de Rham isomorphism. The existence of such a morphism suggests that we can find a closed differential 1-form $\omega \in \Omega^1_{\mathrm{cl}}(X)$ whose de Rham class $q_{\mathrm{dR}}(\omega)$ contains the same topological information as the real version of the underlying cohomology class of f, $I(f)$. Let us see how we can do that.

Let $dt \in \Omega^1(S^1)$ be the angular element[1] of S^1, *i.e.*, a normalized volume form. Consider the map $R : \widehat{H\mathbb{Z}}^1(X) \to \Omega^1_{\mathrm{cl}}(X)$ defined by $R(f) = f^*dt$, which is a homomorphism. We call R the **curvature**. There are two other ways to think about the curvature. The first is to recognized that, although the function $f : X \to S^1$ may fail to admit a global logarithm, it always has local logarithms at each point when restricted to suitable neighbourhood $U \subset X$ of it. Denoting this local logarithm by $\log f$ the form $R(f)$ is locally described by $d(\log f) = f^{-1}df$ (which is convenient in calculations). Despite this local nature of the construction, it is possible to show that these local forms can be glued to form a global form. The second way does not resort to local data. Observe that the tangent bundle of S^1 is trivial and thus can be identified with $S^1 \times \mathbb{R}$. The projection of the differential of f on $\mathbb{R}$ give us the the curvature. In particular, taking $X = S^1$ and $f = \mathrm{id}_{S^1}$, the identity map, this aproach gives a "coordinate-free" description of the angular form dt.

The de Rham isomorphism in singular cohomology tells us that the de Rham class of dt is associated to the generator $\iota \in H\mathbb{Z}^1(S^1)$. Using the de Rham isomorphism $r : H_{\mathrm{dR}} \to H\mathbb{R}$, where $H\mathbb{R}$ denotes the ordinary cohomology with real coefficients, we have the following equality

$$I(f) \otimes_{\mathbb{Z}} 1_{\mathbb{R}} = r^{-1} \circ q_{\mathrm{dR}}(R(f)) \in H\mathbb{R}^1(X).$$

Equivalent the following diagram commutes

$$\begin{tikzcd} \widehat{H\mathbb{Z}}^1(X) \arrow[r, "I"] \arrow[d, "R"] & H\mathbb{Z}^1(X) \arrow[d, "\otimes 1_{\mathbb{R}}"] \\ \Omega_{\mathrm{cl}}(X) \arrow[r, "r^{-1} \circ q_{\mathrm{dR}}"] & H\mathbb{R}^1(X) \end{tikzcd}$$

Such a diagram translates the idea that the de Rham class of de curvature and the underlying cohomology class convey the same **real** cohomological information.

[1] Needless to say that the notation is poor, since dt is not a exact form, but we stick to the usual connvention.

Completing the model, note that, since $a(h)$ is topologically trivial, we expect that $R(a(h))$ is exact, that is, $R(a(h)) = dg$ for some $g \in \Omega^0(X)$. Indeed:

$$R(a(h)) = R(\exp \circ h) = dh,$$

so that $R \circ a = d$.

So, we will define a differential refinement of H^1 as a contravariant functor $\widehat{H\mathbb{Z}}^1$ with natural transformations, R, I and a, satisfying the aforementioned properties. The reader should be warned that this simple model is a little deceiving. In fact, since $H\mathbb{Z}^1(X)$ has no torsion, the universal coefficient theorem implies that the the map $I(f) \otimes 1_{\mathbb{R}}$ is injective, hence $R(f)$ retains all cohomological information. In a general model, this is not the case, as we will see.

The classical introductory example of a differential refinement of a cohomology theory, which is quite appealing due its geometric nature, is the refinement of the second ordinary cohomology group with integral coefficients $H\mathbb{Z}^2(X)$ using hermitian line bundles with connections.

It is a well-know fact that the line bundles are classified, up to isomorphism, by $H\mathbb{Z}^2(X)$. If we equip a line bundle with a connection (compatible with a fixed hermitian metric), we add a geometric piece of information that is partially described by the corresponding curvature. Let us call $\widehat{H\mathbb{Z}}^2$ the group of (hermitian) line bundles with connection, up to isomorphism. The **curvature** map $R : \widehat{H\mathbb{Z}}^2 \to \Omega^2_{\mathrm{cl}}(X)$ can be naturally identified with the curvature of the connection, which is a closed complex 2-form in this case, up to a purely imaginary constant.

Since every line bundle admits a connection, we have a **forgetful map** I : $\widehat{H\mathbb{Z}}^2(X) \to H\mathbb{Z}^2(X)$, which is a surjective morphism. This shows that $\widehat{H\mathbb{Z}}^2(X)$ is a refinement of $H\mathbb{Z}^2$.

The datum which distinguishes two topologically equivalent line bundles with connections is precisely the difference of their connections. Recall that the difference of two connections on a vector bundle is a differential form with values on its endomorphism bundle. For a complex line bundle, the endomorphism bundle is trivial. This implies that the difference of two connections on a complex line bundle can be identified with a complex-valued form. Moreover, the compatibility of the connection with the hermitian metric implies that this form is purely imaginary. Loosely speaking, the **trivialization** map $a : \Omega^1(X) \to \widehat{H\mathbb{Z}}^2$ assigns to a differential form ω the trivial bundle with connection $d + i\omega$, where d is the exterior derivative.

Compared with the $\widehat{H\mathbb{Z}}^1$ model, this one is closer to the spirit of the subject. Let us see some cases:

- we can have a *topologically trivial line bundle* which is endowed with a *non flat* connection,

- we can have a line bundle with *flat* connection which is *topologically non trivial* (locally constant transition functions/local system),

- we can even have a *topologically trivial* line bundle whose connection is *flat*, yet its differential cohomology class is not the trivial one.

The last two scenarios were impossible in $\widehat{H\mathbb{Z}}^1$. Moreover, the last one explains why we said that the curvature describes the geometric information only partially.

Up to this point, we have discussed *absolute* cohomology. It is natural to inquire if we can refine *relative* cohomology as well. The answer to this question is affirmative. Let us see, as an example, how we can define a relative version of $\widehat{H\mathbb{Z}}^1$ described above.

Fix a manifold pair (X, A). A differential cohomology class over (X, A) can be described as a pair of function (f, k), where $f : X \to S^1$ and $k : A \to \mathbb{R}$, subject to the condition $f|_A = \exp(k)$ up to a certain equivalence. Let us denote the set of differential cohomology classes over (X, A) by $\widehat{H\mathbb{Z}}^1(X, A)$.

Given a pair $(f, k) \in \widehat{H\mathbb{Z}}^1(X, A)$, it is possible to associate a class $I(f, k) \in H\mathbb{Z}^1(X, A)$. One way to do this is to construct a function $F(f, k) : C(X, A) \to S^1$ from the cone of A over X to S^1 and define $I(f, k) := F(f, k)^*\iota$. This is the relative version of the **forgetful** map. The relative **curvature** is the relative form $R(f, k) = (f^*dt, k) \in \Omega^1(X, A)$ and the relative version of the **trivialization** is $a(h, 0) = (\exp(h), h|_A)$. In the case that $k = 0$ we say that the class $(f, 0)$ is *parallel*.

You are probably wondering: why is it called parallel? In order to answer this question, we present a relative version of $\widehat{H\mathbb{Z}}^2$.

In order to push $\widehat{H\mathbb{Z}}^1$ to the relative setting, we added new data - the function k, which acts as a global logarithm on A, making the class topologically trivial over it. This suggests that we need to add some trivializing data into the line bundles with connection over X.

This is accomplished by inputting a distinguished section s over A. It is a less well-know fact, but not a surprising one, that line bundles over X with a distinguished section over A are classified by elements of $H\mathbb{Z}^2(X, A)$.

In this case, the relative **curvature** is related to the curvature of the connection along with the local connection form associated to the section/frame s. Roughly, this means that the connection form over A in the distinguished frame s can be written as $\nabla t = B(t)s$, where B is the local connection form. In this case, we say that the class is *parallel* if $B = 0$, *i.e.*, if $\nabla s = 0$. This last condition is expressed in the literature by saying that the section s is parallel[2] and thus the choice of name.

This concludes our quick tour of simple examples. Now we discuss where the subject came from and some results which are relevant to our work.

It is not our intent to give a full account of how differential cohomology began and how it has grown, but it is our duty to at least cite some of the background material. For more information on the topic, the reader can consult the books (BäR; BECKER, 2014), (AMABEL; DEBRAY; HAINE, 2021), the note (BUNKE, 2013) and the survey (BUNKE; SCHICK, 2012).

Differential refinements seem to have originated in a seminal work of Cheeger and Simons (1985), which builds on the previous work of Chern and Simons (1974). In these papers, the authors show how to produce refinements of characteristic classes by using *differential characters*, which they then showed can be used to obtain obstructions to the existence of certain *conformal* embeddings.

The subject has grown considerably since then and now we have quite a literature on it. For example, Hopkins and Singer (2005) have shown that any generalized cohomology theory could be refined, Simons and Sullivan (2008) have provided a set of axioms characterizing the theory and Bunke and Schick (2010) have not only established conditions in which these theories are unique but also shown that the usual axioms do not guarantee the uniqueness. One of the conditions required for uniqueness was the existence of certain integration-like structures, called S^1 integrations, which pervade this work.

Already at the beginning of the subject, refinement of topological structures such as products have been in use. For example, the proof of the obstruction of conformal embeddings in (CHEEGER; SIMONS, 1985) relied on the existence of a multiplicative structure. Another example is the work of Hopkins and Singer (2005), who developed differential refinements of umkehr maps called *integration* in their work. This leads us to the next topic.

What is differential integration or differential umkehr maps?

First, the reader should be aware that this map has many names. It is called umkehr map, pushforward, Gysin map, integration, transfer, wrong way map, shriek and surprise map.

In the topological setting, the umkehr map associated to a continuous map $f : Y \to X$ between compact manifolds of dimensions n and m respectively is a "wrong way" map in cohomology $f_! : h^\bullet(Y) \to h^{\bullet-(n-m)}(X)$, which can be roughly defined as "conjugating the pushfoward map in the dual homology associated to the cohomology

[2] It is also possible to find the expression flat section, but we reserve term flat to classes with null curvature.

$h^\bullet$ by the Poincaré duality isomorphisms". When f is fiber bundle, this map provides a natural notion of integration in the cohomology theory h. In particular, in the case of the de Rham cohomology, it coincides with the cohomology class of the *fiberwise integration*[3] of a representative.

The umkehr map has encountered many applications since its conception in 1941 by Gysin (see the survey (BECKER; GOTTLIEB, 1999)). This idea was pushed forward by Grothendieck when he generalized the Riemann-Roch theorem for maps. In his treatment, the umkehr map played a prominent role. Since then, umkehr maps have flourished in the setting of index theory. For example, the topological index of a Dirac operator on a spin manifold twisted by a bundle $p : E \to X$ is neatly written as $p_!([E])$ (see (STOLZ, 2020)).

As already mentioned, Hopkins and Singer (2005) lifted these maps to the differential cohomology setting, where they are better known under the name of differential integration. Another example of the importance of these maps can be appreciated in the refinements of the index theorems that were carried in (FREED; LOTT, 2010). In this work, the authors provided a refinement of differential K-theory and a differential refinement of the index theorem.

The differential integration maps were further studied by Ruffino (2017) and Bunke (2013). These authors were able to define the maps in a general setting without resorting to models. Their constructions employ the idea of a refinement of the Thom isomorphism to a differential Thom morphism: given a smooth vector bundle $p : E \to X$ over a compact manifold and a differential cohomology class $\widehat{u}$ such that its underlying cohomology class is a Thom class, the Thom morphism becomes

$$\begin{aligned}\widehat{T} : \widehat{h}(X) &\to \widehat{h}(E)\\ \widehat{\alpha} &\mapsto \widehat{u} \cdot p^*\widehat{\alpha}.\end{aligned}$$

With the Thom morphism at hand, as long as there exists an S^1-integration, it is possible to define a differential integration.

Motivation for this work

This work is meant to fill the gap concerning the nonexistence of both the compactly supported cohomology version of the differential integration as well as the relative versions of the differential integration.

In the work of Ruffino and Barriga (2021), the setting of differential cohomology was extended from the absolute case (by which we mean differential cohomology on spaces) to the relative case (differential cohomology on maps). This was done by fixing a suitable set of axioms and constructing refinements of generalized cohomology theories for which

[3] Hence the name integration map.

those axioms hold. In that work, the authors could not perform integration in general because they did not yet have the necessary multiplicative structures. Let us give a brief idea of what this means.

It is possible to define the umkehr map without resorting to Poincaré duality. This is done using the so called *Thom-Pontryagin construction* (or collapse). This construction can be adapted to the differential setting, where Poincaré duality is not at our disposal. As already mentioned, the construction relies on the differential Thom morphism and also depends on the differential S^1-integration.

Recall that the Thom isomorphism associated to a n-dimensional vector bundle $p : E \to X$ admits a compactly supported version $T_c : h_c^\bullet(X) \to h_c^{\bullet+n}(E)$. To illustrate the idea, the Thom isomorphism on de Rham cohomology $T : H^\bullet_{dR,c}(X) \to H^{\bullet+n}_{dr,c}(E)$ is expressed at the level of differential forms as

$$\widehat{u} \wedge p^*\omega, \tag{1}$$

where $\omega \in \Omega^\bullet_{c,\mathrm{cl}}(X)$ is a closed differential form with *compact support* and $\widehat{u} \in \Omega^n_{v,\mathrm{cl}}(E)$ is a closed differential form with *vertically-compact support* whose and the de Rham class of its fiber restriction is a generator of $H^n_{dR,c}(E_x) \cong H^n_{dR,c}(\mathbb{R}^n) \cong \mathbb{R}$ at each fiber. Loosely speaking, a form has vertically-compact support if its fibers restrictions are compactly supported differential forms.

The analogue of (1) in the differential cohomology setting is a product

$$\widehat{u} \cdot p^*\widehat{\beta} \tag{2}$$

where $\widehat{u} \in \widehat{h}_v(E)$ is a differential Thom class, *i.e.*, a vertically-compactly supported differential class whose underlying topological class is a Thom class and $\widehat{\beta}$ is a compactly supported differential class.

Recall that the compactly-supported cohomology of X can be defined as the colimit over inclusion of compact sets $K \subseteq X$ of the groups[4] $h(X, K^c)$. In the differential setting this is also true except that we need to consider *parallel classes*, *i.e.*, the colimit is taken over inclusion of compact sets $K \subseteq X$ of the parallel differential cohomology groups $\widehat{h}_{\mathrm{par}}(X, K^c)$.

In the relative differential cohomology setting there exists a module structure over absolute classes. More precisely, we have a product

$$\begin{aligned} \cdot : \widehat{h}(X) \otimes \widehat{h}(X, A) &\to h(X, A) \\ \widehat{\alpha} \otimes \widehat{\beta} &\mapsto \widehat{\alpha} \cdot \widehat{\beta} \end{aligned}$$

where (X, A) is pair of manifolds. But, **there is no** product over relative classes

$$\cdot : \widehat{h}(X, A) \times \widehat{h}(X, B) \to \widehat{h}(X, A \cup B)$$

[4] K^c stands for the complement of K in X.

at least for two reasons:

P1) the set $A \cup B$ may fail to be a manifold even when A and B are submanifolds of X;

P2) we require a compatibility between the products and the curvature map, as in:

$$R(\widehat{\alpha} \cdot \widehat{\beta}) = R(\widehat{\alpha}) \wedge R(\widehat{\beta})$$

When we talked about the relative curvature in the $\widehat{H\mathbb{Z}}^1$ model, we mentioned that $R(f, k)$ was an element of $\Omega(X, A)$, but we have not defined the group $\Omega(X, A)$. In the literature, the reader can find two definitions of this group: consider the inclusion map $i : A \hookrightarrow X$, $\Omega(X, A)$ can be either defined

- as the kernel of the pullback $i^* : \Omega(X) \to \Omega(A)$, or
- as the mapping cone complex of the pullback i^* which is the cochain complex $\Omega^\bullet(i) = \Omega^\bullet(X) \oplus \Omega^{\bullet-1}(X)$ with differential given by

$$d(\omega, \theta) = (d\omega, di^*\omega - d\theta)$$

We use the second definition as our standard definition of relative forms. In our setting, *P2)* means that we should deal with products like

$$(\omega, \theta) \wedge (\omega', \theta'), \tag{3}$$

which does not seem to be naturally defined. Luckily, if the differential class $\widehat{\alpha}$ is parallel, its curvature is of the type $R(\widehat{\alpha}) = (\omega, 0)$, and thus the product (3) make sense as

$$(\omega, 0) \wedge (\omega', \theta') = (\omega \wedge \omega', \rho^*\omega \wedge \theta')$$

This last equations suggest the possibility to define a product between a parallel class and a relative class. Since the differential Thom morphism is defined at representative level as a product between a parallel class and other classes (absolute, parallel, relative), it will be available once we have these products.

The main concerns of this work is the construction of the differential integration maps. To achieve this, we construct the **parallel-relative** products. But problem *P1)* still remains and that was the starting point of our work.

What new content you will find here?

As already mentioned, you will find here the definition of a parallel-relative product, which will be used to define the differential integration maps. The problem here is that, in order to develop these products, we had to generalize differential cohomology to deal with problem *P1)*.

Without dwelling too much at this topic right now, we mention that it is possible to assign a cohomology theory to maps rather than pairs. Given a map ρ, we can associate a cohomology group $h(\rho)$, that can be identified with the reduced cohomology of its mapping cone $\widetilde{h}(C_\rho)$. We can refine the cohomology group $h(\rho)$ to a differential cohomology group $\widehat{h}(\rho)$ as well.

We were interested in a product between a parallel differential cohomology class $\widehat{\alpha} \in \widehat{h}_{\text{par}}(Y, B)$ and a relative one $\widehat{\beta} \in \widehat{h}(\rho)$ for some smooth map $\rho : A \to X$. But, it was not clear where $\widehat{\alpha} \times \widehat{\beta}$ should be defined:

$$\times : \widehat{h}_{\text{par}}(Y, B) \times \widehat{h}(\rho) \to \widehat{h}(?).$$

The solution to this problem was to consider maps of pairs. The product above would take values in $\widehat{h}(\mathrm{id}_Y \times \rho)$, where $\mathrm{id}_Y \times \rho : (Y \times A, B \times A) \to (Y \times X, B \times X)$ is a map of pairs. As presented here, this solution seems completely nonsense, but as the reader journeys through the end of Chapter 2, we hope that this felling will pass.

After treating the problem on an abstract basis, we needed to show the existence of models displaying these structures. This was done using three models:

- The Cheeger-Simmons differential character model of differential ordinary cohomology with integral coefficients;
- The Freed-Lott model of differential K-theory;
- The Hopkins-singer model for any cohomology theory represented through its spectrum.

In particular, the relative version of the second model was not available in the literature, so we developed it and it is thus a part of this work as well.

Organization of the Text

This text is divided in two parts:

1. **Foundations**, where most of the material is already well-established (with exception of the last section of Chapter 2), but the presentation is non-standard. For example, we deal with cohomology on maps and vertically-compact supported cohomology;
2. **Integration**, where the new material is presented.

It also has four appendices, where the first one is a collection of helpful results for the text. The other three are examples of relative differential cohomology theories and serve two purposes:

- to illustrate the concepts of relative differential cohomology, and
- to serve as a base material for the models in Chapter 6.

Except for the relative Freed-Lott model of K-theory, which was developed in the course of this work, the other two models can be found in the literature (BäR; BECKER, 2014) and (RUFFINO; BARRIGA, 2021).

Still, we have opted to give only a brief presentation of the relative Freed-Lott model, since its construction was already nicely done on the thesis of Nuñez (2021), with whom the present author shares this work.

The chapters are organized as follows:

Chapter 1 We presented relative cohomology on maps rather than the more common relative cohomology on pairs. We define the topological S^1-integration and the product in this setting. We recall the classical Thom isomorphism and then we introduce the compactly and vertically-compact cohomologies. We use these cohomologies to rephrase the Thom isomorphism and extend it to the the compactly supported and doubly-vertically compact supported setting. We also present the relative Thom isomorphism. After this, we discuss the Umkehr maps in a general cohomology theory.

Chapter 2 In this chapter, we review differential forms. The first section is just a very brief review of topics such as forms with compact and vertically compact supports as well as fiber integration. Next, we review the relative de Rham complex and fiber integration and we also introduce the parallel forms. After this, we discuss the Thom morphisms[5] and the differential umkehr maps, which coincide with fiber integration. Then, we briefly review de Rham cohomology and present the umkehr map in it. We close the chapter with some remarks which serve as a guide to the rest of the text.

Chapter 3 This chapter presents classical material related to differential cohomology in the relative setting as defined by Ruffino and Barriga (2021). We start by presenting the model of line bundles with connection as a refinement of the second ordinary cohomology group with integral coefficients. Using it as a motivation, we axiomatize the theory and present its structures like S^1-integration and a module structure (which we call absolute-relative product). Next, we present both the differential Thom morphism and the integration map only in the particular case of fiber bundles with compact fibers, which is already a generalization, since we have dropped the compactness hypothesis.

[5] Not to be confused with Thom isomorphism.

Chapter 4 We introduce the topological setting that will be used to define the integration maps. The solution requires us to develop a cohomology theory on maps of finite sequences of spaces. This chapter parallels the first one: we prove the existence of this extended cohomology, present a collection of useful lemmas, discuss the S^1-integration, a special case of the product, and construct the Thom and Umkehr maps in this language.

Chapter 5 With the ground prepared, we tackle the problem of existence of the integration. We define a differential cohomology on maps of finite sequences of smooth manifolds, define the parallel-relative product and use it to define the Thom morphisms which are needed to define the integration maps. After defining these maps, we come to the main result of this work: the axiomatization of the differential integration maps, which are also shown to be unique.

Chapter 6 In this last chapter, we construct models of the differential cohomology theory on maps of sequence (or of pairs). This chapter is short since it was already discussed in detail in (NUñEZ, 2021). We briefly present the Cheeger-Simmons model of ordinary cohomology theory and the Freed-Lott model of K-theory, focusing in the construction of the parallel-relative product. Next, we present in greater detail the Hopkins-Singer model, since this model accounts for the existence of a differential model for any cohomology theory.

Appendices

- Appendix A is a collection of useful facts;
- Appendix B is an account of the relative Cheeger-Simons model of ordinary differential cohomology with integral coefficients;
- Appendix C gives a short presentation of the relative Freed-Lott model of differential K-theory;
- Appendix D is a review the relative Hopkins-Singer model of differential cohomology, which can be used to refine any cohomology theory.

How should one read this text?

In general, when one writes something, she or he believes that the reader will follow a straight path, but we should not be idealists about it. Everyone has a different background and this should be taken into account, especially in mathematics. If you are comfortable with differential cohomology and cohomology on maps, as well as compactly supported cohomology, you could skip the first Part I without much trouble. Yet, I would recommend that you read Section 2.6, because it contains the main ideas for Part II. If you are at ease with the topological setting, you can just skim forward to the last section of Chapter 2. If you are using this material to get a first glimpse of differential cohomology, the three last appendices may be helpful.

List of results

We collect here the main results of the thesis for convenience:

- The parallel relative product in Definition 5.3.3 as well as the models presented in Chapter 6.
- The integration maps on section 5.4 and section 5.5.
- Theorem 5.4.13 and Theorem 5.4.8 which characterizes the integration maps.

This was a joint work with Juan Carlos Nuñez Maldonado.

Part I

Foundations

1 Topological Preliminaries

1.1 Introduction

In this chapter we introduce some definitions which appear throughout the text as well as some notation. The material is basic, but the presentation is not standard. In this text, we use a version of cohomology based on maps which is equivalent to the usual one. Besides, we present some topics which are not so standard such as the umkehr map. We also introduce a "new" framework for the Thom isomorphism based on the concept of vertically compact cohomology. This will help us develop other version of the umkehr map in the next chapters. We provide proofs of some standard results which can be a little troublesome to find or results which are not of main interest in the Section A.2 in Appendix A.

1.2 The topological categories

We start by fixing a convenient category of topological spaces `Top`. By convenient, we mean a category which has the basic objects that we are going to need. The reader may stick to the category of *compactly generated weakly Hausdorff spaces*, as it seems convenient for our need[1]. We shall refer to topological spaces and continuous maps just as spaces and maps whenever there is no risk of confusion.

Let $f_0 : X \to Y$ and $f_1 : X \to Y$ be two maps. Recall that a *homotopy* between f_0 and f_1 is a continuous map $F : I \times X \to Y$ such that $F_0 = f_0$ and and $F_1 = f_1$, where $F_t := F \circ i_t$ with

$$\begin{aligned} i_t : X &\to I \times X \\ x &\mapsto (t, x). \end{aligned}$$

In this case we say that f_0 and f_1 are *homotopic* and denote this relation by $f_0 \sim f_1$ (or $f_0 \sim_F f_1$ whenever we want to stress the homotopy F). This is an equivalence relation and we denote the class of f by $[f]$. Moreover, this relation is compatible with composition in the following sense: given $f, f' : Y \to Z$ and $g, g' : X \to Y$ such that $[f] = [f']$ and

---

1 Our main task in this text is to construct Umkher maps which usually depend on Thom-Pontryagin construction. According to (STRICKLAND, 2009, Section 6.8, p.22) the construction itself is discontinuous (not the map, but the construction). This is not relevant here, since we will not "pertubate" the construction. Nevertheless the reader should be warned that the "convenient" here should be taken lightly.

$[g] = [g']$, one has $[f \circ g] = [f' \circ g']$. This allows us to define the homotopy category[2] `HoTop` whose objects are spaces and the morphisms are equivalence classes of homotopic maps with composition $[f] \circ [g]$ given by the class $[f \circ g]$. An isomorphism in this category is called a *homotopy equivalence* and isomorphic objects are said to be of the *same homotopy type*. We write $X \simeq Y$ if X and Y have the same homotopy type.

We consider the following categories:

- $\mathtt{Top}_2$, whose objects are pairs of topological spaces (X, A), with $A \subseteq X$ being a subspace of X, and where a morphism $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$. We call (X, A) a pair and $f : (X, A) \to (Y, B)$ a map of pairs.

- $\mathtt{Top}_*$, whose objects are spaces with a distinguished point, *i.e*, (X, x) with $x \in X$ and morphisms $f : (X, x) \to (Y, y)$ with $f(x) = y$. We call a (X, x) a pointed space and $f : (X, x) \to (Y, y)$ a pointed map.

- $\mathtt{Top}^2$, the arrow category of `Top`, whose objects are the morphisms in `Top` and where a morphism $(f, g) : \rho \to \xi$ between $\rho : A \to X$ and $\xi : B \to Y$ is pair of continuous maps $f : X \to Y$ and $g : A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\rho"] & B \arrow[d, "\xi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

  The identity at $\rho : A \to X$ is given by $(\mathrm{id}_X, \mathrm{id}_A)$ and the composition is defined componentwise $(f, g) \circ (f', g') = (f \circ f', g \circ g')$.

Each of these categories comes with a notion of a homotopy category associated to it:

- A homotopy between $f_0 : (X, A) \to (Y, B)$ and $f_1 : (X, A) \to (Y, A)$ is just a morphism $F : (I \times X, I \times A) \to (Y, B)$ which is a homotopy in the usual sense.

- A homotopy between $f_0 : (X, x) \to (Y, y)$ and $f_1 : (X, x) \to (Y, y)$ is a morphism $F : (I \wedge X, *) \to (I \wedge Y, *)$ such that $F_0 = f_0$ and $F_1 = f_1$, where $F_t := F \circ i_t$ and

$$\begin{aligned} i_t : (X, x) &\to (I \wedge X, *) \\ v &\mapsto [(t, v)], \end{aligned}$$

  which can be seen as usual homotopy between f_0 and g_0 with $F(t, x) = F(0, v) = F(1, v) = y$ for every $v \in X$. Here $I \wedge X$ is the smash product $(I, 0) \wedge (X, x)$ described in Section 1.2.1.

[2] Perhaps we should call it naïve homotopy category as opposed to the homotopy category arising from a model category, but, since we never use these structures explicitly in this text, we stick to homotopy category.

- A homotopy between $(f_0, g_0) : \rho \to \xi$ and $(f_1, g_1) : \rho \to \xi$ is a morphism $(F, G) : \mathrm{id}_I \times \rho \to \xi$, such that $(F, G)_0 = (f_0, g_0)$ and $(F, G)_1 = (f_1, g_1)$, with $(F, G)_t := (F, G) \circ (i_t, j_t)$, where $i_t : X \mapsto I \times X$ is the inclusion of X in the slice $\{t\} \times X$ and $j_t : A \mapsto I \times A$ is the inclusion of A in the slice $\{t\} \times A$.

Remark 1.2.1. Even if f_0 is homotopic to f_1 and g_0 is homotopic to g_1, there is no guarantee that $(f_0, g_0) : \rho \to \xi$ and $(f_1, g_1) : \rho \to \xi$ are homotopic. See (BROWN; BROWN, 2006, Section 7.4) or (MAY, 1999, p.47,p.53).

The homotopy categories of $\mathtt{Top}_2$, $\mathtt{Top}_*$ and $\mathtt{Top}^2$ will be denoted by $\mathtt{HoTop}_2$, $\mathtt{HoTop}_*$ and $\mathtt{HoTop}^2$ respectively.

We regard $\mathtt{Top}$ as a subcategory of $\mathtt{Top}_2$ and $\mathtt{Top}_2$ as a subcategory of $\mathtt{Top}^2$ through the embeddings $I : \mathtt{Top} \to \mathtt{Top}^2$ and $I_2 : \mathtt{Top}_2 \to \mathtt{Top}^2$ given by

$$I(X \xrightarrow{f} Y) = (X, \emptyset) \xrightarrow{f} (X, \emptyset) \text{ and } I_2((X, A) \xrightarrow{f} (Y, B)) = i_A \xrightarrow{(f, f_A)} i_B$$

where $\emptyset$ is the empty set, $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow Y$ are inclusions. We employ the notation $\varnothing_A : \emptyset \to A$ to denote the unique map from $\emptyset$ to A[3]. We also have the inclusion $I_* : \mathtt{Top}_* \to \mathtt{Top}_2$ given by

$$I((X, x) \xrightarrow{f} (Y, y)) = (X, \{x\}) \xrightarrow{f} (Y, \{y\}).$$

Summing up, we have the following diagrams, all of which are embbedings:

$$\begin{tikzcd}
\mathtt{Top} \arrow[dr, "I_2"] & & \mathtt{Top}_* \arrow[dl, "I_*"] \\
& \mathtt{Top}_2 \arrow[d, "I^2"] & \\
& \mathtt{Top}^2. &
\end{tikzcd}$$

There is an embedding $I_* : \mathtt{Top} \to \mathtt{Top}_*$ given by

$$I(X \xrightarrow{f} Y) = (X_+, +) \xrightarrow{f_+} (Y_+, +),$$

where $X_+ = X \sqcup \{+\}$ (analogously to Y), $f_+|_X = f$ and $f_+(+) = +$. Here $\sqcup$ denotes the topological sum (disjoint union).

1.2.1 Topological Constructions

Lets recall some standard topological constructions. Given two maps

$$X \xleftarrow{f} A \xrightarrow{g} Y,$$

[3] Being completely honest, we need to employ the "standard trick" of disjointing the Hom sets in a category.

we recall that their *pushout* is a space $X \cup_{f,g} Y$ together with a pair of continuous maps $i_f : X \to X \cup_{f,g} Y$ and $i_g : Y \to X \cup_{f,g} Y$ satisfying the following universal property: given a pair of continuous maps $j_X : X \to Z$ and $j_Y : Y \to Z$ such that $j_X \circ f = j_Y \circ g$, there exists a unique $h : X \cup_{f,g} Y \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "i_Y"] \arrow[ddr, bend left, "j_Y"] & \\ X \arrow[r, "i_X"] \arrow[drr, bend right, "j_X"] & X \cup_{f,g} Y \arrow[dr, dashed, "\exists_! h"] & \\ & & Z. \end{tikzcd} \tag{1.1}$$

The standard model of a pushout is the space

$$X \cup_{f,g} Y = \frac{X \sqcup Y}{f(a) \sim g(a)}$$

endowed with the quotient topology. Here $X \sqcup Y$ denotes the topological sum and the relation $\sim$ identifies points which are images of the same point in A. The maps $i_X : X \to X \cup_{f,g} Y$ and $i_Y : Y \to X \cup_{f,g} Y$ are the composition of the inclusions $X \hookrightarrow X \sqcup Y$ and $X \hookrightarrow Y \sqcup Y$ with the quotient $q : X \sqcup Y \to X \cup_{f,g} Y$.

Given some space X, we define:

- the *cylinder* of X as the space $M(X) := I \times X$.[4] We denote by $i_t : X \to M(X)$ the inclusion of X in the slice $\{t\} \times I$, i.e, $i_t(x) = (t, x)$;
- the *cone* of X as the pushout $C(X)$ of the following diagram

$$M(X) \xleftarrow{i_1} X \longrightarrow *,$$

  where $*$ is a one-point space. We generally identify $C(X)$ with $M(X) \sqcup_{i_0} *$. Observe that $C(\emptyset)$ is a one point space;
- the *suspension* of X as the pushout ΣX of the following diagram:

$$C(X) \xleftarrow{j_0} X \xrightarrow{j_0} C(X),$$

  where $j_0 : X \to C(X)$ is the inclusion of X in the base;
- the *mapping cylinder* of the map $f : X \to Y$ as the pushout M_f of the following diagram

$$M(X) \xleftarrow{i_0} X \xrightarrow{f} Y,$$

  which can be identified with the space $M(X) \sqcup_{i_0, f} Y$;

[4] We use both $M(X)$ and $I \times X$ depending on the what is the focus.

- the *double mapping cylinder* of the maps $f : A \to X$ and $g : A \to Y$ as the pushout of the following diagram:

$$M_f \xleftarrow[i_1']{} X \xrightarrow{i_1} M_g.$$

Rather than using the standard space $M_f \sqcup_{i_1, i_1} M_g$ we will use the space

$$M_{f,g} = \frac{X \sqcup A \times I \sqcup Y}{(a, 0) \sim f(a), (a, 1) \sim g(a)}$$

and the maps $i_{M_f} : M_f \to M_{f,g}$ is given by $i_{M_f}([a, t]) = [a, t]$ and $i_{M_f}[x] = x$ and analogously to i_{M_g};

- the *mapping cone* of the map $f : X \to Y$ as the pushout C_f of the following diagram

$$C(X) \xleftarrow{i_0} X \xrightarrow{f} Y$$

which can be identified with the space $C(X) \sqcup_{i_0, f} Y$.

We also have the reduced version of these constructions[5]. Given some pointed space (X, x), we define:

- the *wedge sum*[6] $(X, x) \vee (Y, y) := X \times \{y\} \cup \{x\} \times Y$, which is the categorical sum (coproduct) in $\mathtt{Top}_2$;

- the *smash product* $(X, x) \wedge (Y, y)$ of (X, x) and (Y, y) as the space

$$\left(\frac{X \times Y}{X \vee Y}, * \right),$$

where $*$ is the image of (x, y) in the quotient;

- the *reduced cone* $\widetilde{C}(X, x)$ of (X, x) as $(X, x) \wedge (I, 0)$;

- the *reduced suspension* $(\widetilde{M_f}, *)$ of (X, x) as $(S^1, 1) \wedge (X, x)$;

- the *reduced mapping cylinder* of the map $f : (X, x) \to (Y, y)$ as the pushout of the following diagram:

$$\widetilde{C}(X, x) \xleftarrow{i_0} (X, x) \xrightarrow{f} (Y, y).$$

It can be identified with the space $\widetilde{C}(X, x) \sqcup_{i_0, f} (Y, y)$, where $*$ is the equivalence class of y.

We use these constructions to define the following functors, which will help us establish the bridge between the cohomology theories in the next section:

[5] We will generally replace the point by a $*$ in order to avoid notation flooding.

[6] We will also call it cross sometimes.

- $M : \mathtt{Top}^2 \to \mathtt{Top}_2$ given by

$$M(\rho \xrightarrow{(f,g)} \xi) = (M_\rho, A) \xrightarrow{M(f,g)} (M_\xi, B), \tag{1.2}$$

where $\rho : A \to X$, $\xi : B \to Y$, the "A" in (M_ρ, A) is the top of the cylinder, and

$$M(f,g)(u) = \begin{cases} [(g(a),t)], & \text{if } u = [(a,t)] \text{ where } (a,t) \in M(A); \\ [f(x)] & \text{if }, u = [x] \text{ where } x \in X. \end{cases}$$

- The cone $C : \mathtt{Top}^2 \to \mathtt{Top}_*$ given by

$$C(\rho \xrightarrow{(f,g)} \xi) = (C_\rho, *) \xrightarrow{C(f,g)} (C_\xi, *) \tag{1.3}$$

where $\rho : A \to X$, $\xi : B \to Y$, $*$ is the vertex of the cone, and

$$C(f,g)(u) = \begin{cases} [(g(a),t)], & \text{if } u = [(a,t)] \text{ where } [a,t] \in C(A); \\ [f(x)] & \text{if }, u = [x] \text{ where } x \in X. \end{cases}$$

- The reduced suspension morphism $\Sigma : \mathtt{Top}_* \to \mathtt{Top}_*$ given by

$$\widetilde{\Sigma}((X,x) \xrightarrow{\rho} (Y,y)) = (\Sigma(X), *) \xrightarrow{\widetilde{\Sigma}(\rho)} (\Sigma(Y), *)$$

where $\Sigma(\rho)(\langle s, x\rangle) = \langle s, \rho(x)\rangle$.

1.3 Cohomology Theories

A topological cohomology theory usually comes in two flavours:

- or a cohomology theory over $\mathtt{Top}_*$, called a *Reduced Cohomology Theory*,
- or a cohomology theory over $\mathtt{Top}_2$, called a *Relative Cohomology Theory*.

Here we mainly use a less common version which is akin to the relative cohomology theory:

- a cohomology theory over $\mathtt{Top}^2$, which we also called *Relative Cohomology Theory*.

In order to differentiate from the usual relative cohomology we call this the *relative cohomology over maps* and the other the *relative cohomology over pairs*. In Section A.2.3 of Appendix A, the reader will find the other two versions, namely the reduced cohomology theory and the relative cohomology on pairs. Here we concentrate on the relative theory on maps.

The relative cohomology theory on maps can be characterized by axioms analogous to the the Eilenberg-Steenrod axioms of relative cohomology on pairs. Let $\Pi^2 : \mathtt{Top}^2 \to \mathtt{Top}_2$ denote the functor given by

$$\Pi^2(\rho \xrightarrow{(f,g)} \xi) = \varnothing_A \xrightarrow{(g,\varnothing)} \varnothing_B$$

where $\varnothing_X$ stands for the unique map $\emptyset \to X$ and we write $\varnothing$ in place of $\varnothing_\emptyset$.

Definition 1.3.1 (Relative Cohomology Theory on Maps)**.** A *relative cohomology* (h, ∂) theory over $\mathtt{Top}^2$ is a (contravariant) functor $h : \mathtt{HoTop}^{2,\mathrm{op}} \to \mathtt{GrAb}$ along with a natural transformation $\partial : h^\bullet \circ \Pi^2 \to h^{\bullet+1}$, satisfying the following axioms:

Long exact sequence Associated to the cohomology theory (h, ∂) there is a a functor from the homotopy category $\mathtt{HoTop}_2$ to the category of long exact sequences, defined in the following way: for each (homotopy class of a) map $\rho : A \to X$, we have the following long exact sequence:

$$\cdots \longrightarrow h^\bullet(\rho) \xrightarrow{(id_X, \varnothing_A)^*} h^\bullet(X) \xrightarrow{(\rho,\varnothing)^*} h^\bullet(A) \xrightarrow{\partial_{(X,A)}} h^{\bullet+1}(\rho) \longrightarrow \cdots,$$

where $(f, g)^*$, $h(X)$ and ρ^* are shorthand for $h(f, g)$, $h(\varnothing_X)$ and $h(\rho, \varnothing)$, respectively. For a morphism $(f, g) : \rho \to \xi$, where $\xi : B \to Y$, we have the following morphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & h^\bullet(\rho) \arrow[r, "(\mathrm{id}_X, \varnothing_A)^*"] & h^\bullet(X) \arrow[r, "i_A^*"] & h^\bullet(A) \arrow[r, "\partial"] & h^{n+1}(\rho) \arrow[r] & \cdots \\
\cdots \arrow[r] & h^\bullet(\xi) \arrow[r] \arrow[u, "(f,g)^*"] & h^\bullet(Y) \arrow[r] \arrow[u, "(f,\varnothing)^*"] & h^\bullet(B) \arrow[r, "\partial"] \arrow[u, "(g,\varnothing)^*"] & h^{n+1}(\xi) \arrow[r] \arrow[u, "(f,g)^*"] & \cdots .
\end{tikzcd}$$

Excision Consider the pushout diagram

$$\begin{tikzcd}
A \arrow[r, hook, "j"] \arrow[d, hook, "i"] & Y \arrow[d, "i_Y"] \\
X \arrow[r, "i_X"] & X \cup_{i,j} Y
\end{tikzcd} \tag{1.4}$$

where $i : A \hookrightarrow X$ and $j : A \hookrightarrow Y$ are embeddings.

If $\operatorname{int} i_X(X) \cup \operatorname{int} i_Y(Y) = X \cup_{i,j} Y$, then both morphisms

$$(i_X, j) : i \to i_Y, \text{ and } (i_Y, i) : j \to i_X$$

induce isomorphisms in cohomology.

Additive: Given a family of maps $\{(\rho_\lambda)\}_{\lambda\in\Lambda}$, let $(i_\lambda, j_\lambda) : \rho_\lambda \to \bigsqcup_{\lambda\in\Lambda} \rho_\lambda$ be the inclusions, where $\bigsqcup_{\lambda\in\Lambda} \rho_\lambda : \bigsqcup_{\lambda\in\Lambda} A_\lambda \to \bigsqcup_{\lambda\in\Lambda} X_\lambda$ denotes the map

$$\bigsqcup_{\lambda\in\Lambda} \rho_\lambda(a) = \rho_\lambda(a) \text{ if } a \in A_\lambda.$$

Then the group $(h(\bigsqcup_{\lambda\in\Lambda} \rho_\lambda), (i_\lambda, j_\lambda)^*_{\lambda\in\Lambda})$ is the direct product of the groups $\{h(\rho_\lambda)\}_{\lambda\in\Lambda}$.

The three definitions of cohomology theory are equivalent. This is a well know fact for the case of the reduced cohomology and the relative cohomology on pairs: If (h, ∂) is a relative cohomology theory on pairs, then $(h \circ I_*, S)$, where S is a suspension isomorphism, is a reduced cohomology theory. Reciprocally, if $(\widetilde{h}, s)$ is a reduced cohomology theory, then $(h \circ C \circ I_2, D)$ is a relative cohomology theory (for a proof of this result, the reader can see Switzer (2002, 7.35))[7]. Here we assume these results, since we are mainly interested in cohomology of maps.

Proposition 1.3.2. *If (h', ∂') is a relative cohomology theory on pairs as in Definition A.2.7, then $(h' \circ M, \partial')$ is a relative cohomology theory on maps. Reciprocally, if (h, ∂) is a relative cohomology theory on maps, then $(h \circ I_2, \partial)$ is a cohomology on pairs.*

Proof. We will verify that $(h' \circ M, \partial')$ satisfy the axioms. Let's call $h = h' \circ M$.

Long Exact Sequence Given a map $\rho : A \to X$, we have $h(\rho) = h'(M_\rho, A)$. By the long exact sequence for relative cohomology on pairs,

$$\cdots \longrightarrow h'^{\bullet}(M_\rho, A) \xrightarrow{id^*_{M_\rho}} h'^{\bullet}(M_\rho, \emptyset) \xrightarrow{i_1^*} h'^{\bullet}(A, \emptyset) \xrightarrow{\partial} h'^{\bullet+1}(M_\rho, A) \longrightarrow \cdots .$$

The inclusion of the base map $i_X : X \to M_\rho$ is a homotopy equivalence with homotopy inverse given by the collapse map $c_x : M_\rho \to X$ (see (A.1) in Appendix (A)) and thus we can rewrite the sequence above as

$$\begin{tikzcd}
\cdots \arrow[r] & h'^{\bullet}(M_\rho, A) \arrow[r, "id^*_{M_\rho}"] \arrow[dr, "i_X^*"'] & h'^{\bullet}(M_\rho, \emptyset) \arrow[r, "i_1^*"] \arrow[d, shift right, "i_X^*"'] & h'^{\bullet}(A, \emptyset) \arrow[r, "\partial"] & h'^{\bullet+1}(M_\rho, A) \arrow[r] & \cdots \\
& & h'^{\bullet}(X, \emptyset) \arrow[u, shift right, "c^*"'] \arrow[ur, "\rho^*"'] & & &
\end{tikzcd}$$

Observe that $M(\varnothing_X) = (X, \emptyset)$, $M(\varnothing_A) = (A, \emptyset)$, $M(\mathrm{id}_X, \varnothing_A) = i_X$, and $M(\rho, \varnothing) = \rho$. Replacing this in

$$\begin{tikzcd}
\cdots \arrow[r] & h(\rho) \arrow[r, "id^*_{M_\rho}"] \arrow[dr, "M(id_X{,}\emptyset)^*"'] & h'^{\bullet}(M_\rho, \emptyset) \arrow[r, "i_1^*"] \arrow[d, shift right, "i_X^*"'] & h'^{\bullet}(A, \emptyset) \arrow[r, "\partial"] & h'^{\bullet+1}(M_\rho, A) \arrow[r] & \cdots \\
& & h'^{\bullet}(X, \emptyset) \arrow[u, shift right, "c^*"'] \arrow[ur, "M(\rho{,}\varnothing)^*"'] & & &
\end{tikzcd}$$

we get the desired sequence:

$$\cdots \longrightarrow h^{\bullet}(\rho) \xrightarrow{(\mathrm{id}_X, \varnothing_A)} h'^{\bullet}(X) \xrightarrow{(\rho, \varnothing)^*} h'^{\bullet}(A) \xrightarrow{\partial} h'^{\bullet+1}(\rho) \longrightarrow \cdots .$$

Excision Let $i : A \hookrightarrow X$ and $j : A \hookrightarrow Y$ be embeddings such that $\operatorname{int} i_X(A) \cup \operatorname{int} i_Y(B) = X \cup_{i,j} Y$. Consider the following commutative diagram,

$$\begin{tikzcd}
A \arrow[r, "i_1'"'] \arrow[rr, bend left, "i"] \arrow[d, "i_1"'] & M_i \arrow[r, "c"'] \arrow[d, "i_{M_i}"] & Y \arrow[d, "i_1''"] \\
M_j \arrow[r, "i_{M_j}"] \arrow[rr, bend right, "M(i_X{,}i)"'] & M_{i,j} \arrow[r, "\theta"] & M_{i_Y}
\end{tikzcd}$$

[7] In this reference, the author uses the reduced cone. This slightly changes the definition we are employing.

where M_{i_Y} is the mapping cone of $i_Y : Y \to X \cup_{i,j} Y$ and the map $\theta : M_{i,j} \to M_{i_Y}$ is the composition of the collapse $\phi : M_{i,j} \to X \cup_{i,j} Y$ and the inclusion on the base $i_{X\cup_{i,j}Y} : X \cup_{i,j} Y \to M_{i_Y}$.

In the proof of Proposition 1.3.9 below, we will verify that the pair (M_i, M_j), seen as subsets of $M_i \cup M_j$ "glued" by their top, satisfies excision in the sense that $i^* : h^\bullet(M_i \cup M_j, M_i) \to h^\bullet(M_j, A)$ is an isomorphism. Since i_1 is a cofibration, Proposition A.2.5 says that ϕ is a homotopy equivalence. Observe that $i_{X\cup_{i,j}Y}$ is a homotopy equivalence as well. Now, we use the five lemma (Proposition A.4.1) in the diagram

$$\begin{tikzcd}
{} \arrow[r] & h(M_{i,j}) \arrow[r] & h(M_i) \arrow[r] & h(M_{i,j}, M_i) \arrow[r] & h(M_{i,j}) \arrow[r] & h(M_i) \arrow[r] & {} \\
{} \arrow[r] & h(M_{i_Y}) \arrow[r] \arrow[u, "\theta"] & h(Y) \arrow[r] \arrow[u, "c"] & h(i_Y) \arrow[r] \arrow[u, "{(\theta,c)}"] & h(M_{i_Y}) \arrow[r] \arrow[u, "\theta"] & h(Y) \arrow[r] \arrow[u, "c"] & {}
\end{tikzcd}$$

where all vertical arrows, except the middle one, are known to be isomorphisms. The composition gives us

$$h'^\bullet(M_{i_y}, Y) \xrightarrow[\cong]{M(i_X,i)^*} h'^\bullet(M_j, A)$$

which translates to

$$h^\bullet(i_Y) \xrightarrow[\cong]{(i_X,i)^*} h^\bullet(j)$$

Additivity This follows directly from $M_{\bigsqcup_\lambda \rho_\lambda} = \bigsqcup_\lambda M_{\rho_\lambda}$.

The reverse implication, that is, that $(h \circ I_2, \partial)$ is a cohomology theory on pairs, is clear. □

Since we can identify a pair (X, A) with the inclusion map $i : A \hookrightarrow X$, we will write $h(X, A)$ even when we want to mean $h(i)$, henceforth always treating the relative cohomology on pairs as a particular case of the relative cohomology on maps.

Remark 1.3.3. The relation between the reduced cohomology groups and the relative cohomology groups on maps is given by the functor $C : \mathtt{Top}^2 \to \mathtt{Top}_*$ described in (1.3). Given a reduced cohomology theory $(\widetilde{h}, s)$ as in Definition A.2.9, the functor $h := \widetilde{h} \circ C$ is part of the data of a relative cohomology theory on maps. It is possible to define ∂ from $(\widetilde{h}, s)$ as well, but we will not need it.

Remark 1.3.4. The reader shall recall that excision can be present in another form (see A.2.8). This other version of excision can be described in the relative cohomology of maps in the following way: given embeddings $j : A \hookrightarrow X$ and $k : U \hookrightarrow A$ such that $\overline{j \circ k(U)} \subseteq \mathrm{int}(j(A))$, the inclusion morphism $(i, i') : j' \to j$, where j' is the restriction of j to $A \setminus k(U)$, as described in the following commutative diagram

$$\begin{tikzcd}
A \setminus k(U) \arrow[r, hook, "i'"] \arrow[d, hook, "j'"] & A \arrow[d, hook, "j"] \\
X \setminus j \circ k(U) \arrow[r, hook, "i"] & X
\end{tikzcd}, \tag{1.5}$$

induces isomorphism in cohomology. This is equivalent to excision as stated in (1.4).

The uniqueness of cohomology theories (at least for spaces with the homotopy type of CW-complexes, which ensures that weak homotopy equivalence implies strong equivalence) can be established in the usual treatment of cohomology (usually using CW-approximation and spectra).

The reader may wonder why should we use relative cohomology on maps since there is apparently no gain. This will be clear when we introduce differential cohomology in Chapter 3.

1.3.1 Some basic facts

Now we collect some facts we are going to use throughout this text.

Proposition 1.3.5. *Given $\rho : A \to X$, the induced homomorphism in cohomology $\rho^* : h(X) \to h(A)$ is an isomorphism if and only if $h(\rho) = 0$.*

Proof. Consider the following piece of the long exact sequence of ρ:

$$\cdots \longrightarrow h^\bullet(X) \xrightarrow{\rho^*} h^\bullet(A) \xrightarrow{\partial_\rho} h^{\bullet+1}(\rho) \xrightarrow{(id_X, \varnothing_A)^*} h^{\bullet+1}(X) \xrightarrow{\rho^*} h^{\bullet+1}(A) \longrightarrow \cdots$$

If $\rho^* : h^{\bullet+1}(X) \to h^{\bullet+1}(A)$ is an isomorphism, then $\ker(\rho^*) = 0$ and therefore, by exactness, $\mathrm{Im}((id_X, \varnothing_A)^*) = 0$. It follows that $\ker((id_X, \varnothing_A)^*) = h^{\bullet+1}(\rho)$. On the other side of the sequence, since $\rho^* : h^\bullet(X) \to h^{\bullet+1}(A)$ is a isomorphism, we have $\mathrm{Im}(\rho^*) = h^\bullet(A)$ and by exactness $\ker(\partial_\rho) = h(A)$ and hence $\partial_\rho = 0$. Therefore $h^{\bullet+1}(\rho) = \mathrm{Im}(\partial_\rho) = 0$.

Reciprocally, consider the following piece of the sequence

$$\cdots \longrightarrow h^\bullet(\rho) \xrightarrow{(id_X, \varnothing_A)^*} h^\bullet(X) \xrightarrow{\rho^*} h(A) \xrightarrow{\partial_\rho} h^{\bullet+1}(\rho) \longrightarrow \cdots .$$

Since $h^\bullet(\rho) = 0$, it follows that $\mathrm{Im}((\mathrm{id}_X, \varnothing_A)^*)) = 0$ and therefore $\ker(\rho) = 0$, which shows that ρ is a injective. Since $h^{\bullet+1}(\rho) = 0$, we have $\ker(\partial_\rho) = h^\bullet(A)$ and by exactness $\mathrm{Im}(\rho^*) = h^\bullet(A)$, showing that ρ^* is surjective. □

Proposition 1.3.6. *Let $(f, g) : \rho \to \eta$ be a morphism between $\rho : A \to X$ and $\eta : B \to Y$. If both $f : X \to Y$ and $g : A \to B$ induce isomorphisms in cohomology, then $(f, g)^*$ is an isomorphism. In particular, this holds whenever both f and g are homotopy equivalences.*

Proof. We apply the five lemma A.4.1 to the following piece of the long exact sequence of ρ and η

$$\begin{tikzcd}
\cdots \arrow[r] & h^\bullet(X) \arrow[r, "\rho^*"] & h^\bullet(A) \arrow[r, "\partial_\rho"] & h^\bullet(\rho) \arrow[r, "(id_X, \varnothing_A)^*"] & h^{\bullet+1}(X) \arrow[r, "\rho^*"] & h^{\bullet+1}(A) \arrow[r] & \cdots \\
\cdots \arrow[r] & h^\bullet(Y) \arrow[r, "\eta^*"] \arrow[u, "f^*"] & h^\bullet(B) \arrow[r, "\partial_\eta"] \arrow[u, "g^*"] & h^\bullet(\eta) \arrow[r, "(id_X, \varnothing_A)^*"] \arrow[u, "(f,g)^*"] & h^{\bullet+1}(Y) \arrow[r, "\eta^*"] \arrow[u, "f^*"] & h^{\bullet+1}(B) \arrow[r] \arrow[u, "g^*"] & \cdots
\end{tikzcd}$$

Since the red vertical arrows are isomorphisms by assumption, it follows that $(f, g)^*$ is an isomorphism. $\square$

Given maps $\eta : B \to A$ and $\rho : A \to X$, we consider the following diagram:

$$\begin{tikzcd}
B \arrow[r, "\mathrm{id}_B"] \arrow[d, "\eta"] & B \arrow[r, "\eta"] \arrow[d, "\rho\circ\eta"] & A \arrow[d, "\rho"] \\
A \arrow[r, "\rho"] & X \arrow[r, "\mathrm{id}_X"] & X.
\end{tikzcd} \tag{1.6}$$

We have the following long exact sequence analogous to the long exact sequence of the triple in relative cohomology of pairs.

Proposition 1.3.7 (Long Exact sequence of Composition)**.** *For every pair of composable morphisms as in* (1.6)*, we have a long exact sequence*

$$\cdots \longrightarrow h^\bullet(\rho) \xrightarrow{(\mathrm{id}_X,\eta)^*} h^\bullet(\rho\circ f) \xrightarrow{(\rho,\mathrm{id}_B)^*} h^\bullet(\rho) \xrightarrow{\beta} h^{\bullet-1}(\rho) \longrightarrow .$$

Proof. We apply the braid lemma A.4.2 in appendix A in the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
& h^{\bullet-1}(B) \arrow[rr, bend left, "\partial_\eta"] \arrow[dr, "\partial_{\rho\circ\eta}"] & & h^\bullet(\eta) \arrow[rr, bend left, "\beta"] \arrow[dr, "(\mathrm{id}_A,\varnothing_B)^*"] & & h^{\bullet+1}(\rho) \arrow[rr, bend left, "(\mathrm{id}_X,\varnothing_A)^*"] \arrow[dr, "(\mathrm{id}_X,\eta)^*"] & & h^{\bullet+1}(X) \\
h^{\bullet-1}(A) \arrow[ur, "\eta^*"] \arrow[dr, "\partial_\rho"] & & h^\bullet(\rho\circ\eta) \arrow[ur, "(\rho,\mathrm{id}_B)^*"] \arrow[dr, "(\mathrm{id}_X,\varnothing_B)^*"] & & h^\bullet(A) \arrow[ur, "\partial_\rho"] \arrow[dr, "\eta^*"] & & h^{\bullet+1}(\rho\circ\eta) \arrow[ur, "(\mathrm{id}_X,\varnothing_B)^*"] & \\
& h^\bullet(\rho) \arrow[ur, "(\mathrm{id}_X,\eta)^*"] \arrow[rr, bend right, "(\mathrm{id}_X,\varnothing_A)^*"'] & & h^\bullet(X) \arrow[ur, "\rho^*"] \arrow[rr, bend right, "\rho\circ\eta^*"'] & & h^\bullet(B) \arrow[ur, "\partial_{\rho\circ\eta}"] & &
\end{tikzcd}$$

where $\beta : h^\bullet(\eta) \to h^{\bullet+1}(\rho)$ is defined as $\beta := \partial_\rho \circ (id_A, \varnothing_B)^*$. The sequences

- $h^{\bullet-1}(A) \xrightarrow{\eta^*} h^{\bullet-1}(B) \xrightarrow{\partial_\eta} h^\bullet(\eta) \xrightarrow{(\mathrm{id}_A,\varnothing_B)^*} h^\bullet(A) \xrightarrow{\eta^*} h^\bullet(B)$
- $h^{\bullet-1}(B) \xrightarrow{\partial_{\rho\circ\eta}} h^\bullet(\rho\circ\eta) \xrightarrow{(\mathrm{id}_X,\varnothing_B)^*} h^\bullet(X) \xrightarrow{\rho\circ\eta^*} h^\bullet(A) \xrightarrow{\partial_{\rho\circ\eta}} h^{\bullet+1}(\rho\circ\eta) \xrightarrow{(\mathrm{id}_X,\varnothing_B)^*} h^{\bullet+1}(X)$
- $h^{\bullet-1}(A) \xrightarrow{\partial_\rho} h^\bullet(\rho) \xrightarrow{(\mathrm{id}_X,\varnothing_A)^*} h^\bullet(X) \xrightarrow{\rho^*} h^\bullet(A) \xrightarrow{\partial_\rho} h^{\bullet+1}(\rho) \xrightarrow{(\mathrm{id}_X,\varnothing_A)^*} h^{\bullet+1}(X)$

are exact and the sequence

$$h(\rho) \xrightarrow{(\mathrm{id}_X,\eta)^*} h(\rho\circ\eta) \xrightarrow{(\rho,\mathrm{id}_B)^*} h(\eta)$$

is such that $(\rho, \mathrm{id}_B)^* \circ (\mathrm{id}_X, \eta)^* = (\rho, \eta)^*$. Since $(\rho, \eta)^*$ factors through the identity id_A as

$$\begin{tikzcd}
B \arrow[r, "\eta"] \arrow[d, "\eta"'] \arrow[rr, bend left, "\eta"] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"] & A \arrow[d, "\rho"] \\
A \arrow[r, "\mathrm{id}_A"'] \arrow[rr, bend right, "\rho"'] & A \arrow[r, "\rho"'] & X
\end{tikzcd}$$

and $h(\mathrm{id}_A) = 0$, we conclude that $(\rho, \eta)^* = 0$. Therefore, the lemma implies the exactness of the sequence

$$h^\bullet(\rho) \xrightarrow{(\mathrm{id}_X,\eta)^*} h^\bullet(\rho \circ \eta) \xrightarrow{(\rho,\mathrm{id}_B)^*} h^\bullet(\eta) \xrightarrow{\beta} h^{\bullet+1}(\rho) \xrightarrow{(\mathrm{id}_X,\eta)^*} h^{\bullet+1}(\rho \circ \eta)$$

by applying the lemma at each segment we obtain the long exact sequence

$$\cdots \longrightarrow h^\bullet(\rho) \xrightarrow{(\mathrm{id}_X,\eta)^*} h^\bullet(\rho \circ \eta) \xrightarrow{(\rho,\mathrm{id}_B)^*} h^\bullet(\eta) \xrightarrow{\beta} h^{\bullet+1}(\rho) \xrightarrow{(\mathrm{id}_X,\eta)^*} \cdots$$

as required. □

Given maps $\mu : A \to X$ and $\nu : A \to Y$, consider the following pushout diagram diagram

$$\begin{tikzcd}
A \arrow[r, "\nu"] \arrow[d, "\mu"] & Y \arrow[d, "i_Y"] \\
X \arrow[r, "i_X"] & X \cup_{\mu,\nu} Y
\end{tikzcd} \tag{1.7}$$

Definition 1.3.8 (Excisive pair)**.** We say that the pair of maps (μ, ν) as in (1.7) is *excisive* if both $(i_X, \nu) : \mu \to i_X$ and $(i_Y, \nu) : \mu \to i_Y$ induce isomorphisms in cohomology.

Proposition 1.3.9. *The pair* (i_1^μ, i_1^ν)*, where*

$$\begin{array}{c} i_1^\mu : A \to M_\mu \\ a \mapsto [(1, a)] \end{array} \quad \text{and} \quad \begin{array}{c} i_1^\nu : A \to M_\nu \\ a \mapsto [(1, a)] \end{array},$$

is excisive.

Proof. Fix some $\epsilon \in (0, 1)$ and consider the pushout diagram

$$\begin{tikzcd}
{[\epsilon, 1] \times A} \arrow[r, "i^\nu_{[\epsilon,1]}"] \arrow[d, "i^\mu_{[\epsilon,1]}"] & M_\nu \arrow[d, "i'_{M_\nu}"] \\
M_\mu \arrow[r, "i'_{M_\mu}"'] & M_\mu \cup_{(i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]})} M_\nu,
\end{tikzcd}$$

where $i^\mu_{[\epsilon,1]}(t, a) = [t, a]$ and $i^\nu_{[\epsilon,1]}(t, a) = [t, a]$.

We claim that the pair $(i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]})$ is excisive. Indeed, observe that both $i^\mu_{[\epsilon,1]}$ and $i^\nu_{[\epsilon,1]}$ are closed embeddings and

$$\operatorname{int}\left(i'_{M_\mu}(M_\mu)\right) \cup \operatorname{int}\left(i'_{M_\nu}(M_\nu)\right) = M_\mu \cup_{i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]}} M_\nu.$$

The excision axiom implies that $(i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]})$ is excisive.

Next, we show that $(i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]})$ being excisive implies that (i^μ_1, i^ν_1) is excisive. In order to see this consider the following composition

$$i^\mu_1 \xrightarrow{(\mathrm{id}_{M_\mu}, i^A_1)} i^\mu_{[\epsilon,1]} \xrightarrow{\left(i'_{M_\mu}, i^\nu_{[\epsilon,1]}\right)} i'_{M_\nu} \xrightarrow{(\mathrm{id}_{M_\nu}, \theta)} i^\nu_{M_\nu} = i^\mu_1 \xrightarrow{\left(\theta \circ i'_{M_\mu}, i_1\right)} i_{M_\nu}$$

where $\theta : M_\mu \cup_{(i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]})} M_\nu \to M_{\mu,\nu}$ is the collapse of the "shared" region $[\epsilon, 1] \times A$ of the mappings cylinder to a "shared" top $\{1\} \times A$ that sends the region $[0, \epsilon) \times A$ to $[0, 1) \times A$ by multiplying the "speed" by $\frac{1}{\epsilon}$ everything said as depicted in Figure 1.3.1.

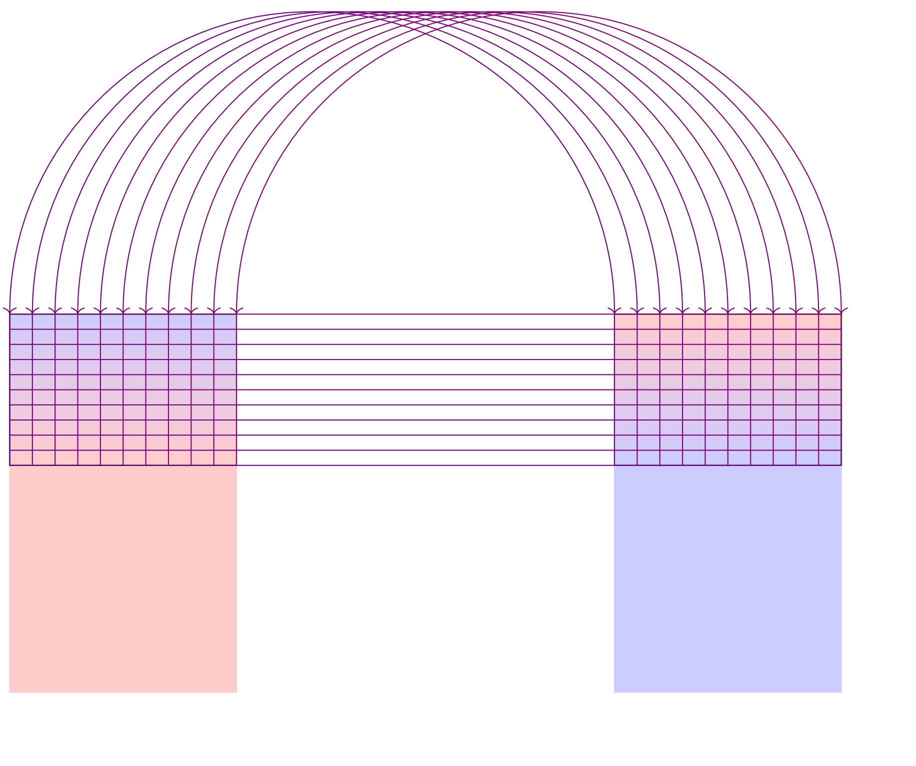

Figure 1 – Collapsing the checkered pattern region to the top.

The map θ is a homotopy equivalence. Indeed, by considering the "inclusion" $\iota : M_{\mu,\nu} \to M_\mu \cup_{i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]}} M_\nu$ we see that both $\theta \circ \iota \sim_F \mathrm{id}_{M_{\mu,\nu}}$ and $\iota \circ \theta \sim_G \mathrm{id}_{M_\mu \cup_{(i^\mu_{[\epsilon,1]}, i^\nu_{[\epsilon,1]})} M_\nu}$.

For example, the composition

$$\theta\circ\iota(p)=\begin{cases}[y] & p=[y],\ \text{where } y\in Y\\ \left[\frac{t}{\epsilon},a\right] & p=[t,a],\ \text{where } (t,a)\in[0,\epsilon]\times A\subseteq M_\nu\\ [1,a] & p=[t,a],\ \text{where } (t,a)\in[\epsilon,1]\times A\subseteq M_\nu\\ [1,a] & p=[t,a],\ \text{where } (t,a)\in[\epsilon,1]\times A\subseteq M_\rho\\ \left[\frac{t}{\epsilon},a\right] & p=[t,a],\ \text{where } (t,a)\in[0,\epsilon]\times A\subseteq M_\rho\\ [x] & p=[x],\ \text{where } x\in X\end{cases}$$

is homotopy equivalent to identity by the homotopy

$$F(s,p)=\begin{cases}[x], & p=[y] \text{ where } y\in X\\ \left[a,(1-s)\frac{t}{\epsilon}+st\right], & p=[t,a],\ \text{where } (t,a)\in[0,\epsilon)\times A\\ [a,(1-s)+st], & p=[t,a],\ \text{where } (t,a)\in[\epsilon,1]\times A\\ [a,(1-s)+st], & p=[t,a],\ \text{where } (t,a)\in[\epsilon,1]\times A\\ \left[a,(1-s)\frac{t}{\epsilon}+st\right], & p=[t,a],\ \text{where } (t,a)\in[0,\epsilon)\times A\\ [x], & p=[x],\ \text{where } x\in X.\end{cases}$$

By using proposition (A.2.6), we see that the morphism $(\mathrm{id}_{M_\mu},i_1^A):i_1^\mu\to i_{[\epsilon,1]}^\mu$ is a homotopy equivalence, because both i_1^μ and $i_{[\epsilon,1]}^\mu$ are cofibrations and id_{M_μ} and i_1^A are homotopy equivalences. The morphism $(\mathrm{id}_{M_\nu},\theta):i'_{M_\nu}\to i_{M_\nu}$ is a homotopy equivalence too by the same reasoning (notice that both i'_{M_ν} and i_{M_ν} are cofibrations by the pushout stability of the cofibrations - lemma A.2.5, since $i_{[\epsilon,1]}^\mu$ and i_1^μ are both cofibrations). The map $\theta\circ i'_{M_\mu}$ is homotopic to i_{M_μ} via the map

$$F(s,p)=\begin{cases}\left[a,(1-s)\frac{t}{\epsilon}+st\right], & (t,a)\in[0,\epsilon)\times A\\ [a,(1-s)+st], & (t,a)\in[\epsilon,1]\times A\\ [x], & p=x\in X.\end{cases}$$

Hence $(\mathrm{id}_{M_\nu},\theta\circ i'_{M_\mu})\sim(\mathrm{id}_{M_\nu},i_{M_\mu})$ again using the fact that both i_1^μ and i_{M_ν} are cofibrations and Proposition A.2.6.

It follows that the sequence

$$i_1^\mu\xrightarrow{(\mathrm{id}_{M_\mu},i_1^A)}i_{[\epsilon,1]}^\mu\xrightarrow{\left(i'_{M_\mu},i^\nu_{[\epsilon,1]}\right)}i'_{M_\nu}\xrightarrow{(\mathrm{id}_{M_\nu},\theta)}i^\nu_{M_\nu}=i_1^\mu\xrightarrow{(\theta\circ i'_{M_\mu},i_1^A)}i$$

induces in cohomology

$$\begin{aligned}h(i_{M_\nu})&\xrightarrow{(i_{M_\mu},i_1^\mu)^*}h(i_1^\mu)=h(i_{M_\nu})\xrightarrow{(\theta\circ i'_{M_\mu},i_1^\mu)^*}h(i_1)\\&=h(i_{M_\nu})\xrightarrow{(\mathrm{id}_{M_\nu},\theta)^*}h(i'_{M_\nu})\xrightarrow{\left(i'_{M_\mu},i^\nu_{[\epsilon,1]}\right)^*}h(i^\mu_{[\epsilon,1]})\xrightarrow{(\mathrm{id}_{M_\mu},i_1^A)^*}h(i_1^\mu),\end{aligned}$$

where each red morphism is a isomorphism: the first and last being homotopy equivalences and the middle one by the excision property of the pair $(i^{\mu}_{[\epsilon,1]}, i^{\mu}_{[\epsilon,1]})$. By carrying out the same proof with i^{ν}, we conclude that the pair (i^{μ}, i^{ν}) is excisive. $\square$

Consider the mapping $\phi : M_{\mu,\nu} \to X \cup_{\mu,\nu} Y$ which collapses the cylinder to the base, that is,

$$\phi(p) = \begin{cases} [x], & p = [x] \text{ where } x \in X \\ [\mu(a)], & p = [t,a] \text{ where } (t,a) \in I \times A \\ [y], & p = [y] \text{ where } y \in Y. \end{cases}$$

This map is the only one that makes the following diagram commutative:

$$\begin{tikzcd}
A \arrow[r, "i_1^{\nu}"] \arrow[d, "i_1^{\mu}"] \arrow[rr, bend left, "\nu"] \arrow[dd, bend right, "\mu"'] & M_{\nu} \arrow[r, "c_Y"] \arrow[d, "i_{M_{\nu}}"] & Y \arrow[dd, "i_Y"] \\
M_{\rho} \arrow[r, "i_{M_{\mu}}"] \arrow[d, "c_X"] & M_{\mu,\nu} \arrow[dr, dashed, "\phi"] & \\
X \arrow[rr, "i_X"] & & X \cup_{\mu,\nu} Y.
\end{tikzcd}$$

Proposition 1.3.10. *The pair (μ, ν) is excisive if and only of $h(\phi) = 0$.*

Proof. Applying the long exact sequence of the composition (Proposition 1.3.7) to $\phi \circ i_{M_{\nu}}$, we have

$$\cdots \to h(\phi) \to h(\phi \circ i_{M_{\nu}}) \to h(i_{M_{\mu}}) \to \cdots .$$

By doing the same to $\rho = c_X \circ i_1$ and using the fact that $h(c_X) = 0$, since it is a homotopy equivalence, we see that $h(\rho) \cong h(i_1)$. By the previous proposition, the pair (i_1, i_1') is excisive and we obtain $h(\rho) \cong h(i_1) \cong h(i_{M_{\rho}})$.

The long exact sequence of the composition $i_Y \circ c_Y$ gives us an isomorphism $h(i_y) \cong h(i_Y \circ c_Y)$ (as c_Y is a homotopy equivalence).

Summing up, we have

- $h(\rho) \cong h(i_{M_{\mu}})$
- $h(i_y) \cong h(i_Y \circ c_y)$
- $\phi \circ c_Y = \phi \circ i_{M_{\mu}}$ (which comes from the comutativity)

With these at hand, we are led to the following sequence (where omitted maps are compositions with isomorphisms):

$$\cdots \to h(\phi) \to h(i_Y) \to h(\rho) \to \cdots$$

Now, we conclude that $h(\phi) = 0$ if and only if $h(i_Y) \cong h(\rho)$. The map which gives us the isomorphism is precisely $(i_X, \nu) : \rho \to i_Y$. □

As an immediate consequence of the previous proposition, we have

Corollary 1.3.11. *If either μ or ν is a cofibration, then (μ, ν) is excisive. If either one of $\rho : A \to X$ and $\eta : B \to Y$ is a cofibration, then $h(\rho \wedge \eta) \cong h(\rho \cup \eta)$, where $\rho \wedge \eta : M_{\rho\times \mathrm{id}_Y, \mathrm{id}_X \times\eta} \to X \times Y$ is the map defined by*

$$(\rho \wedge \eta)(p) = \begin{cases} (\rho(a), x), & \text{if } p = [a, y] \\ (\rho(a), \rho(b)), & \text{if } p = [a, b, t] \\ (y, \eta(b)), & \text{if } p = [x, b] \end{cases}$$

and $\rho \cup \eta : A \times X \cup_{\rho\times \mathrm{id}_Y, \mathrm{id}_X \times\eta} X \times B \to X \times Y$ is the map defined by

$$(\rho \cup \eta)(p) = \begin{cases} (\rho(a), x), & \text{if } p = [a, y] \\ (\rho(a), \rho(b)), & \text{if } p = [a, b] \\ (y, \eta(b)), & \text{if } p = [x, b] \end{cases}$$

Proof. The first part follows from proposition A.2.5, since ϕ is a homotopy equivalence. The other statement follows from proposition A.2.4, property iii, which shows that $\rho \times \mathrm{id}_Y$ (or $\mathrm{id}_X \times\eta$) is a cofibration provided that ρ (or η) is a cofibration. □

1.4 Topological S^1 Integration

A topological S^1 integration can be seem as a very simple form of both the suspension and the Thom isomorphism. Consider the following functor $S : \mathtt{Top}^2 \to \mathtt{Top}^2$

$$S(\rho \xrightarrow{(f,g)} \eta) = \rho \times \mathrm{id}_{S^1} \xrightarrow{(f\times \mathrm{id}_{S^1}, g\times \mathrm{id}_{S^1})} \eta \times \mathrm{id}_{S^1} .$$

Given any functor $F : \mathtt{Top}^2 \to \mathtt{C}$, where $\mathtt{C}$ is any category, we put $SF := F \circ S$. Consider the conjugation map $t : S^1 \to S^1$ given by $t(s) = \overline{s}$ and define a morphism $t_\sharp\rho : S\rho \to S\rho$ by $t_\sharp(\rho) = (\mathrm{id}_X \times t, \mathrm{id}_A \times t)$.

Definition 1.4.1 (Topological S^1 integration)**.** A *topological S^1 integration* is a natural transformation $\textstyle\int_{S^1} : Sh^\bullet \to h^{\bullet-1}$ which satisfies the following two properties:

S1) $(\textstyle\int_{S^1})_\rho \circ (\pi_X, \pi_A)^* = 0$,

S2) $(\textstyle\int_{S^1})_\rho \circ t_\sharp\rho = -(\textstyle\int_{S^1})_\rho$.

Proposition 1.4.2. *A topological S^1 integration exists for any cohomology theory.*

Proof. Fix the point $1 \in S^1$ and consider the morphism $(i_X, i_A) : \rho \to S\rho$ which includes X (resp. A) in the slice $X \times 1$ (resp. $A \times 1$) of $X \times S^1$ (resp. $A \times S^1$). As in Remark 1.3.3, we use $h(\rho) = \widetilde{h}(C_\rho)$. Consider the following exact sequence in reduced cohomology (see Proposition A.2.10):

$$\cdots \longrightarrow \widetilde{h}^\bullet(\widetilde{C}_{C(i_X,i_A)}, *) \xrightarrow{i^*} \widetilde{h}^\bullet(C_{S\rho}, *) \xrightarrow{C(i_X,i_A)^*} \widetilde{h}^\bullet(C_\rho, *) \longrightarrow \cdots,$$

which is the cofiber sequence of the map $C(i_X, i_A) : C_\rho \hookrightarrow C_{S\rho}$, which is just the inclusion. This sequence is split exact since $C(\mathrm{pr}_X, \mathrm{pr}_A)$ is a left inverse of $C(i_X, i_A)$. We write $h : \widetilde{h}^\bullet(C_{S\rho}, *) \to \widetilde{h}^\bullet(\widetilde{C}_{C_{(i_X,i_A)}}, *)$ for the split at i^*:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{h}^\bullet(\widetilde{C}_{C_{(i_X,i_A)}}, *) \arrow[r, "i"] & \widetilde{h}^\bullet(C_{S\rho}, *) \arrow[r, "C(i_X,i_A)^*"] \arrow[l, bend left, "h"] & \widetilde{h}^\bullet(C_\rho) \arrow[r] \arrow[l, bend left, "C(\mathrm{pr}_X,\mathrm{pr}_A)^*"] & 0 \end{tikzcd}.$$

As $C(i_X, i_A)$ is a cofibration, we have $\widetilde{h}(\widetilde{C}_{C(i_X,i_A)}), *) \cong \widetilde{h}\left(\dfrac{C_{S\rho}}{C_\rho}, *\right)$. It turns out that the space $\left(\frac{C_{S\rho}}{C_\rho}, *\right)$ is homeomorphic to the reduced suspension of C_ρ. Indeed,

$$\phi : \left(\frac{C_{S\rho}}{C_\rho}, *\right) \to (\widetilde{\Sigma}(C_\rho), *)$$
$$p \mapsto \begin{cases} [s,p], & \text{if } p = [s,p] \text{ with } (s,p) \in S^1 \\ *, & \text{otherwise.} \end{cases}$$

Using the suspension isomorphism, we write $s : \widetilde{h}^\bullet(\widetilde{\Sigma}(C_\rho)) \to \widetilde{h}^{\bullet-1}(C_\rho, *)$ so that

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{h}^{\bullet-1}(C_\rho, *) \arrow[r, "i^*"] & \widetilde{h}^\bullet(C_{S\rho}, *) \arrow[r, "C(i_X,i_A)^*"] \arrow[l, bend left, "s\circ\phi^*\circ h"] & \widetilde{h}^\bullet(C_\rho) \arrow[r] \arrow[l, bend left, "C(\mathrm{pr}_X,\mathrm{pr}_A)^*"] & 0. \end{tikzcd}$$

we define the Topological S^1-integration

$$\int_{S^1} : Sh^\bullet \to h^{\bullet-1}$$

as the natural transformation

$$\left(\int_{S^1}\right)_\rho = s \circ \phi^* \circ h.$$

Its naturality follows from the the naturality of the sequences once fixed the point $1 \in S^1$. We now prove that this is a satisfies both properties:

1. This is by the definition, since $u \circ (\pi_X, \pi_A)^* = 0$.

2. This follows from the fact that $s_X \circ (t^* \wedge \mathrm{id}_X) = -s_X$ where $s_X : \widetilde{h}^\bullet(\widetilde{\Sigma}X) \to \widetilde{h}^{\bullet-1}(X)$ is the suspension isomorphism.

∎

Remark 1.4.3. The topological S^1 integration is related to the cross product, to be defined in Section 1.5 ahead (Definition 1.5.4), in the following way: given $e \in h^1(S^1, 1)$ such that $s(e) = 1$, where $s : h^1(S^1, 1) \to h^0(S^0, 1)$ is the suspension isomorphism and $\alpha \in h^{\bullet-1}(\rho)$, due to the compatibility between the suspension isomorphism and the cross product, we have

$$\int_{S^1} ((\mathrm{id}_{S^1}, \varnothing_1)^* e \times \alpha) = \alpha,$$

where $(\mathrm{id}_{S^1}, \varnothing_1)^* e \times \alpha \in h(\rho \wedge (S^1, \varnothing_1)) = h(S\rho)$. The proof can be carried in a similar way as in (DIECK, 2008, Proposition 17.1.3,p. 414) (see also (BUNKE; SCHICK, 2010, section 4, p.22)).

1.5 Multiplicative Structures

In the setting of relative cohomology of pairs, it is usual to define the *external product* of two cohomology classes $\alpha \in h^p(X, A)$ and $\beta \in h^q(Y, B)$ as a class $\alpha \times \beta \in h^{p+q}(X \times Y, A \times Y \cup X \times B)$. This product makes sense at least in the case in which the pair $(X \times B, A \times Y)$ is excisive. We know that, if (X, A) or (Y, B) is a cofibration, the pair is excisive. If (X, A) is not a cofibration, we can replace it by a the cofibration $(M(X, A), A)$[8], where A denotes the top of the cylinder, and use the same definition.

We use this usual definition to motivate where the product of two classes $\alpha \in h(\rho)$ and $\beta \in h(\eta)$, with $\rho : A \to X$ and $\eta : B \to Y$, should be placed. Since $h(\rho)$ and $h(\eta)$ can be identified with $h(M_\rho, A)$ and $h(M_\eta, B)$ respectively, as in Proposition 1.6.13, we know that the product should “live” in $h(M_\rho \times M_\eta, A \times M_\eta \cup_{A\times B} M_\rho \times B)$[9]. If there was a function $\rho \wedge \eta$ such that $h(\rho \wedge \eta) = h(M_\rho \times M_\eta, A \times M_\eta \cup_{A\times B} M_\rho \times B)$, we would expect its domain to be $A \times M_\eta \cup_{A\times B} M_\rho \times B$ and its codomain to have the same homotopy type as $M_\rho \times M_\eta$, which is $X \times Y$.

Lets examine the set $M_\rho \times B \cup_{A\times B} A \times M_\eta$ more closely. We have the following homeomorphism

Proposition 1.5.1. *There exists an homeomorphism $\phi : M_\rho \times B \to M_{\rho\times \mathrm{id}_B}$.*

Proof. The homeomorphism is given by

$$\phi_\rho(u, b) = \begin{cases} [x, b], & u = [x], x \in X \\ [a, b, t], & u = [a, t], (a, t) \in A \times I \end{cases}$$

whose inverse is

$$\psi_\rho(u) = \begin{cases} ([x], b), & u = [x, b], (x, b) \in X \times B \\ ([a, t], b), & u = [a, b, t], (a, b, t) \in A \times B \times I. \end{cases}$$

[8] This is a classical result, see for instance (DIECK, 2008, Proposition 5.3.1, p. 111)
[9] We just use $\cup_{A\times B}$ to emphasize where they are glued, but is not really necessary here.

These maps are well defined and continuous (by the quotient property). $\square$

It follows that $M_\rho \times B \cup_{A\times B} A \times M_\eta$ can be identified with $M_{\rho\times \mathrm{id}_B} \cup_{A\times B} M_{\mathrm{id}_A \times\eta}$. The intersection $M_{\rho\times \mathrm{id}_B} \cap M_{\mathrm{id}_A \times\eta}$ is precisely the top of both cylinders as depicted in Figure 2. This shows that we are indeed forming a pushout of the following diagram

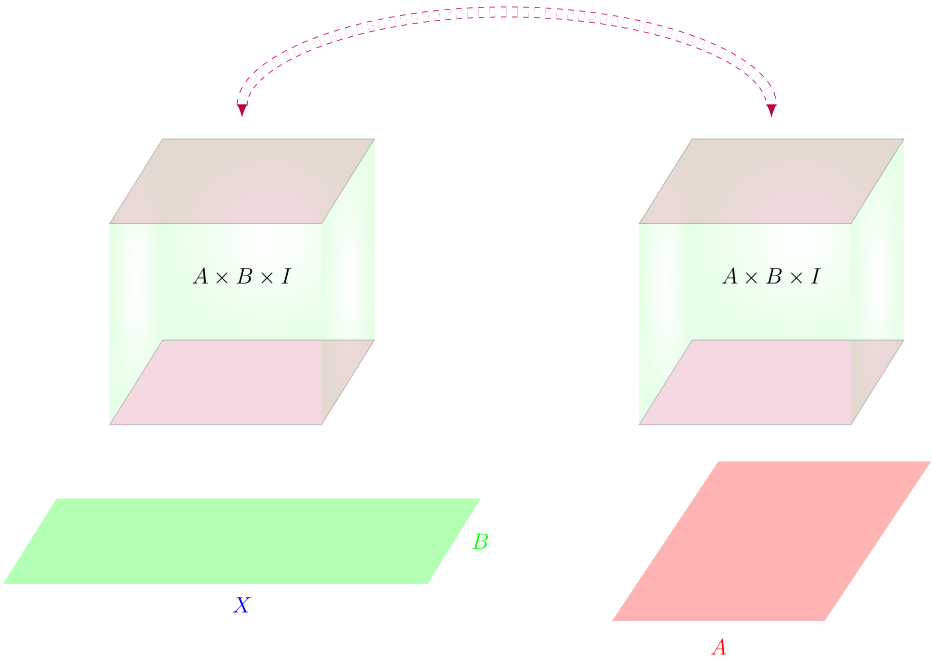

Figure 2 – Domain of $\rho \wedge \eta$, which is the double mapping cylinder $M_{\rho\times \mathrm{id}_B, \mathrm{id}_A \times\eta}$

$$M_{\rho\times \mathrm{id}_B} \xleftarrow{i_1} A \times B \xrightarrow{i_1'} M_{\mathrm{id}_A \times\rho}.$$

The set $M_{\rho\times \mathrm{id}_B} \cup_{A\times B} M_{\mathrm{id}_A \times\eta}$ is just the double mapping cylinder of the maps $\rho \times \mathrm{id}_B : A \times B \to X \times B$ and $\mathrm{id}_A \times\eta : A \times B \to A \times Y$ as defined in Section 1.2.1.

Consider the following map:

$$\rho \wedge \eta : M_{\rho\times \mathrm{id}_B, \mathrm{id}_A \times\eta} \to X \times Y$$
$$u \mapsto \begin{cases} (x, \eta(b)), & \text{if } u = [x, b] \text{ with } (x, b) \in X \times B \\ (\rho(a), \eta(b)), & \text{if } u = [a, b, t] \text{ with } (a, b, t) \in A \times B \times I \\ (\rho(a), y), & \text{if } u = [a, y] \text{ with } (a, y) \in A \times Y. \end{cases}$$

Lemma 1.5.2. *There exists a homeomorphism $\phi : M_\rho \cup B \times A \cup M_\eta \to M_{\rho\times \mathrm{id}_B, \mathrm{id}_A \times\eta}$ such that the following diagram commutes*

$$\begin{tikzcd} M_\rho \times B \cup A \times M_\eta \arrow[r, "\phi"] \arrow[d, hook] & M_{\rho\times \mathrm{id}_B, \mathrm{id}_A \times\eta} \arrow[d, "\rho\wedge\eta"] \\ M_\rho \times M_\eta \arrow[r, "c_X \times c_Y"] & X \times Y, \end{tikzcd}$$

where $c_X : M_\rho \to X$ and $c_Y : M_\eta \to Y$ are the collapse maps defined in section 1.2.1.

Proof. We define

$$\phi(p) = \begin{cases} [a, b, t/2], & \text{if } p = ([a, t], [b, 1]) \in M_\rho \times B \\ [a, b, 1 - t/2], & \text{if } p = ([a, 1], [b, t]) \in A \times M_\eta \\ [x, b], & \text{if } p = ([x], [b, 1]) \in M_\rho \times B \\ [a, y], & \text{if } p = ([a, 1], [y]) \in A \times M_\eta \end{cases}$$

which is well defined and continuous by the lifting property of quotients. Its inverse is given by

$$\phi^{-1}(q) = \begin{cases} ([a, 2t], [b, 1]), & q = [a, b, t], t \leq \frac{1}{2} \\ ([a, 1], [b, 2t - 1]), & q = [a, b, t], t \geq \frac{1}{2} \\ ([x], [b, 1]), & q = [x, b] \\ ([a, 1], y), & q = [a, y]. \end{cases}$$

The commutativity of the diagram can be verified in each case, for example,

$$c_X \times c_Y([x], [b, 1]) = (x, \eta(b)) = \rho([x, b]) = (\rho \wedge \eta)(\phi([x], [b, 1]),$$

and analogously for the others. □

Finally, we get the desired relation

Proposition 1.5.3. *The morphism* $(c_X \times c_Y, \phi)^* : h(\rho \wedge \eta) \to h(M_\rho \times M_\eta, M_\rho \times B \cup A \times M_\eta)$ *is an isomorphism.*

Proof. This follows immediately from Proposition 1.3.6, since $c_X \times c_Y$ is a homotopy equivalence (as it is the cartesian product of homotopy equivalences) and ϕ is a homeomorphism. □

Consider the following functor $M : \mathtt{Top}^2 \times \mathtt{Top}^2 \to \mathtt{Top}^2$ given by

$$M\left((\rho, \rho') \xrightarrow{((f,g),(f',g'))} (\eta, \eta')\right) = \rho \wedge \rho' \xrightarrow{(f \times f', M(f,f',g,g'))} \eta \wedge \eta'$$

where $M(f, f', g, g') : M_{\rho \times \mathrm{id}_B, \mathrm{id}_B \times \eta} \to M_{\rho \times \mathrm{id}_B, \mathrm{id}_B \times \eta}$ is defined as

$$M(f, f', g, g')(p) = \begin{cases} [f(x), g'(a')], & \text{if } p = [x, a'] \text{ with } (x, a') \in X \times A' \\ [g(a), g'(a'), t], & \text{if } p = [a, a', t] \text{ with } (a, a', t) \in A \times A' \times I \\ [g(a), f'(x')], & \text{if } p = [a, x'] \text{ with } (a, x') \in A \times X'. \end{cases}$$

in this context, there are some isomorphisms that merit special names:

- Given $\rho : A \to X$, $\rho' : A' \to X'$ and $\rho'' : A'' \to X''$, the *association isomorphism* is the map
$$(\alpha, \beta)_{\rho,\rho',\rho''} : (\rho \wedge \rho') \wedge \rho'' \to \rho \wedge (\rho' \wedge \rho''),$$
where $\alpha : (X \times X') \times X'' \to X \times (X' \times X'')$ is the homeomorphism $\alpha((x, x'), x'') = (x, (x', x''))$ and $\beta : M_{(\rho\wedge\rho')\times \mathrm{id}_{A''}, \mathrm{id}_{A\times A'} \times\rho''} \to M_{\rho\times \mathrm{id}_{A'\times A''}, \mathrm{id}_A \times(\rho'\wedge\rho)}$ is an homeomorphism [10].

- Given $\rho : A \to X$ and $\eta : B \to Y$ the *commutation isomorphism* is the map
$$(t, u)_{\rho,\eta} : \rho \wedge \eta \to \eta \wedge \rho,$$
where $t : X \times Y \to Y \times X$ is given by $t(x, y) = (y, x)$ and $u : M_{\rho\times \mathrm{id}_B, \mathrm{id}_A \times\eta} \to M_{\mathrm{id}_B \times\rho,\eta\times \mathrm{id}_B}$ has the following form
$$u(p) = \begin{cases} [a, b, t], & p = [b, a, t] \text{ with } (a, b, t) \in A \times I \times B \\ [b, x], & p = [x, b] \text{ with } (x, b) \in X \times B \\ [y, a], & p = [a, y] \text{ with } (a, y) \in A \times Y. \end{cases}$$

Thanks to these isomorphisms, we can canonically identify $\rho \wedge (\rho' \wedge \rho'')$ and $(\rho \wedge \rho') \wedge \rho''$ as well as $(\rho \wedge \eta)$ and $(\eta \wedge \rho)$. From now on, we will not display the isomorphisms.

Definition 1.5.4 (External Multiplicative Structure)**.** An *external multiplicative structure on a relative cohomology theory on maps* (h, ∂) is a natural transformation $\times : h \otimes_{\mathbb{Z}} h \to h \to h \to h \circ M$ satisfying the following axioms:

M1) (*Associativity*) Given $\alpha \in h(\rho)$, $\alpha' \in h(\rho')$ and $\alpha'' \in h(\rho'')$, it holds that
$$\alpha \times (\alpha' \times \alpha'') = (\alpha \times \alpha') \times \alpha'.$$

M2) (*Anticomutativity*) Given $\alpha \in h(\rho)$, $\beta \in h(\eta)$ it holds that
$$\alpha \times \eta = (-1)^{|\alpha||\beta|}(\beta \times \alpha).$$

M3) (*Compatibility*) If $(\rho \times \mathrm{id}_B, id_A \times \eta)$ is an excisive pair, then we have the following compatibility with the connecting homomorphism ∂: given $\alpha \in h^p(A)$ and $\beta \in h^q(\eta)$,
$$\partial(\alpha) \times \beta = \Delta(\alpha \times \beta).$$

Here, $\Delta : h^{p+q}(\mathrm{id}_A \times\eta) \to h^{p+q+1}(\rho \wedge \eta)$ is given by the composition
$$h^{p+q}(\mathrm{id}_A \times\eta) \xrightarrow[(1)]{\sim} h^{p+q}(M_{\rho\times \mathrm{id}_B, \mathrm{id}_A \times\eta}, A \times Y) \xrightarrow{\beta} h^{p+q+1}(\rho \wedge \eta),$$

[10] We do not present it here since its expression is a little involved to describe and will not be used.

where (1) is the inverse of the excision isomorphism $((i_{A\times X}, \rho \times \mathrm{id}_B)^*)^{-1}$ composed with the isomorphism $((\phi, \mathrm{id}_B)^*)^{-1}$, as in the commutative diagram

$$\begin{tikzcd}
A \times B \arrow[r, hook, "\rho\times\mathrm{id}_B"] \arrow[d, hook, "\mathrm{id}_A \times\eta"] & X \times B \arrow[r, equal, "\mathrm{id}_B"] \arrow[d, hook, "i_{X\times B}"] & X \times B \arrow[d, hook, "i'_{X\times B}"] \\
A \times Y \arrow[r, hook, "i_{A\times X}"] & A \times Y \cup_{\rho\times\mathrm{id}_B, \mathrm{id}_A \times\eta} X \times B \arrow[r, "\phi"] & M_{\rho\times\mathrm{id}_B, \mathrm{id}_A\times\eta}
\end{tikzcd},$$

and $\beta : h^{p+q}(M_{\rho\times\mathrm{id}_B,\mathrm{id}_A \times\eta}, X \times B) \to h^{p+q+1}(\rho\wedge\eta)$ is the connecting homomorphism of the long exact sequence of $\mathrm{id}_X \times\eta = (\rho \wedge \eta) \circ i'_{X\times B}$ given by Proposition 1.3.7.

A similar axiom holds, with the appropriate changes, if η is replaced by ρ.

M4) (*Unity*) There exists a class $1 \in \mathfrak{h}^0 = h^0(P)$, P being a singleton, such that

$$(i_X, j_A)^*(\alpha \times 1) = \alpha,$$

where $\alpha \in h^n(\rho)$ and $(i_X, i_A) : \rho \to \rho\times\mathrm{id}_P$ are the inclusions (which will be frequently omitted).

Remark 1.5.5. We could have opted to follow another path using reduced cohomology rather than relative cohomology of pairs to motivate our definition of the external product. This would lead us the same way: given two maps $\rho : A \to X$ and $\eta : B \to Y$, we would like to find a map $\rho \wedge \eta$ such that

$$\times : \widetilde{h}(C_\rho, *) \otimes_{\mathbb{Z}} \widetilde{h}(C_\eta, *) \to \widetilde{h}(C_\rho \wedge C_\eta, *) \simeq \widetilde{h}(C_{\rho\wedge\eta}, *)$$

In other words, we would like some map $\rho \wedge \eta$ such that $(C_\rho, *) \wedge (C_\eta, *)$ becomes homeomorphic to $(C_{\rho\wedge\eta}, *)$ or, at least, homotopy equivalent to it.

It turns out that there is indeed such an homeomorphism in $\mathtt{Top}_*$. As a matter of fact, this was our first approach to this problem (it even motivates the notation of wedge product). Nevertheless, in this text we follow a quicker, though less enlightened, path. For the sake of completeness, we present this method in the appendix (see section A.2.1).

Using this external product, we can define the usual *internal product* for excisive pairs. Given two excisive pairs (X, A) and (X, B) we define the external product reads

$$\times : h(X, A) \times h(X, B) \to h(X \times X, A \times X \cup X \times B).$$

Consider the diagonal morphism of pairs $\Delta : (X, A \cup B) \to (X \times X, A \times X \cup X \times B)$ defined by $\Delta(x) := (x, x)$. Given two classes $\alpha \in h(X, A)$ and $\beta \in h(Y, b)$, we define the *internal product*

$$\cdot : h(X, A) \times h(X, B) \to h(X, A \cup B)$$

by

$$\alpha \cdot \beta = \Delta^*(\alpha \times \beta). \tag{1.8}$$

The product $\times$ will often be referred as *external* or *cross product*. It can be can be recovered from the internal product in the case of pairs by taking

$$\alpha \times \beta = \mathrm{pr}_X^* \, \alpha \cdot \mathrm{pr}_Y^*,$$

where $\mathrm{pr}_X : (X \times Y, A \times Y) \to (X, A)$ and $\mathrm{pr}_Y : (X \times Y, X \times B) \to (X, B)$ are the projections.

1.6 Compactly-like Cohomology and Thom isomorphisms

In this section we review the concept of Thom isomorphism using a "new concept"[11]: cohomology with vertically compact supports.

1.6.1 Thom isomorphism in the classical case

Fix some multiplicative relative cohomology theory of maps $(h, \partial, \times)$ over `Top`, where $\times$ denotes the external product. Consider the long exact sequence associated to the composition $* \hookrightarrow \mathbb{R}^n \setminus \{0\} \hookrightarrow \mathbb{R}^n$, given by

$$\cdots \longrightarrow h^{n-1}(\mathbb{R}^n, *) \longrightarrow h^{n-1}(\mathbb{R}^n \setminus \{0\}, *) \xrightarrow{\beta} h^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \longrightarrow h^n(\mathbb{R}^n, *) \longrightarrow \cdots$$

Since $(\mathbb{R}^n, *)$ has the same homotopy type as id_*, we have $h^n(\mathbb{R}^n, *) = 0$ and thus $\beta : h^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to h^{n+1}(\mathbb{R}^n \setminus \{0\})$ is an isomorphism. Furthermore, the fact that there exists a homotopy equivalence $\phi : (R^n \setminus \{0\}, *) \to (S^1, *)$ leads us to an isomorphism

$$h^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \underset{\cong}{\xrightarrow{\beta^{-1}}} h^{n-1}(\mathbb{R} \setminus \{0\}, *) \underset{\cong}{\xrightarrow{\phi^*}} h^{n-1}(S^{n-1}, *) \underset{\cong}{\xrightarrow{s^{n-1}}} h^0(S^0) =: \mathfrak{h},$$

where $\mathfrak{h}$ is the cohomology's coefficient group and $s^{n-1} : h(S^{n-1}, *) \to h(S^0, *)$ is the suspension isomorphism iterated $n-1$ times.

We fix an n-dimensional[12] real vector bundle $p : E \to B$ and set $E_0 := E \setminus z(B)$, where $z : B \to E$ is the zero section. A *Thom class* is a cohomology class $u \in h^n(E, E_0)$ such that u_x, the restriction of u to the fiber E_x, is a generator of $h^n(E_x, E_{0,x}) \cong h^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathfrak{h}$. We say that a bundle is *h-orientable* if it admits a Thom class.

Proposition 1.6.1 (Thom isomorphism)**.** *Fix some h-orientable real vector bundle $p : E \to B$ and let $u \in h^n(E, E_0)$ be a Thom class. The Thom homomorphism $T : h^\bullet(B) \to h(E, E_0)$ defined as*

$$T(\alpha) := u \cdot p^* \alpha$$

is an isomorphism.

[11] New is a strong word. Although we could not find it anywhere, it is a rather straightforward concept when we compare with the analogous definition for differential forms. It is important to mention that this concept seems to appear in the literature of parametrized/fiberwise homotopy in other guises.

[12] Whenever we say n-dimensional vector bundle we are referring to the dimension of its fibers, that is, an n-plane bundle.

Here $\cdot$ is the internal product as defined in equation (1.8). For a proof of this classical result, the reader can consult (DIECK, 2008), for example. There, the author derives it from the Leray-Hirsch Theorem, which is derived in a geometric way. We will quote without proof the following result which will be used frequently

Proposition 1.6.2 (2 × 3 principle). *Let $q_E : E \to X$ and $q_F : F \to X$ be two bundles vector bundles with projections $\mathrm{pr}_E : E \oplus F \to E$ and $\mathrm{pr}_F : E \oplus F \to F$. Consider the triple (u, v, w) of Thom classes on E, F and $E \oplus F$ respectively, such that $w = \mathrm{pr}_E^* u \cdot \mathrm{pr}_F^* v$. Two elements of such a triple uniquely determine the third one.*

A proof of this can be found in (RUDYAK, 1998, Proposition 1.10 (iii), p.307).

In de Rham cohomology, it is usual to talk about vertically compact supported differential forms and to state the Thom isomorphism within this language (see section 2.2.3). Here we give a general definition of a cohomology with vertically compact supports. But first, we recall the definition of cohomology with compact supports.

1.6.2 Cohomology with Compact Supports

We denote the set of compact sets of a space X by $\mathcal{K}(X)$. Consider the partially ordered set $(\mathcal{K}(X), \subseteq)$. This is a directed set, since for any K and K' in $\mathcal{K}(X)$, we have $K \cup K' \in \mathcal{K}(X)$ and $K \subseteq K \cup K'$ and $K' \subseteq K \cup K'$ as depicted in Figure 3. For $K \subseteq L$,

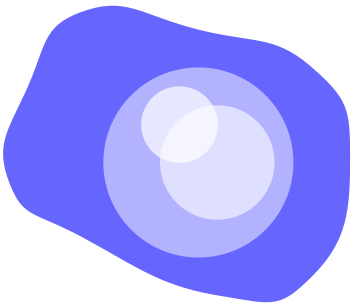

Figure 3 – The directed sets of the compacts

the inclusion $i_{KL} : L^c \to K^c$, where K^c denotes the complement in X of K, induces a map

$$i_{KL}^* : h(X, K^c) \to h(X, L^c)$$

that has the following properties:

- $i_{KK}^* = \mathrm{id}_X^*$
- $i_{KM}^* = i_{LM}^* \circ i_{KL}^*$ for $K \subseteq L \subseteq M$

From this it follows that the pair $(h(X, K^c), i_{KL})$ is a directed system.

Definition 1.6.3 (Cohomology with compact supports)**.** The *cohomology with compact supports* of a space X, denoted by $h_c(X)$, is defined as the directed limit

$$h_c(X) = \operatorname{colim}_{K \in \mathcal{K}(X)} h(X, K^c).$$

Remark 1.6.4. This is a **not** a contravariant functor $h_c : \mathtt{Top}^{\mathrm{op}} \to \mathtt{GrAb}$. The problem is that, given some map $f : X \to Y$, there is no natural way to define $h_c(f) : h_c(Y) \to h_c(X)$. Nevertheless, it is possible when $f : X \to Y$ is *proper map*, *i.e.* a continuous map such that $f^{-1}(K) \in \mathcal{K}(X)$ whenever $K \in \mathcal{K}(Y)$. Indeed, consider the morphism $f^* : h(Y, K^c) \to h(X, X \backslash f^{-1}(K)^c)$, which we compose with $i_{f^{-1}(K)} : h(X, f^{-1}(K)^c) \to h_c(X)$. The universal property of $h_c(Y)$ give us the desired map $f^* : h_c(Y) \to h_c(X)$.

In particular, this proves that h_c can be seen as functor defined on the subcategory of $\mathtt{Top}$ that has the same objects but with proper maps as morphisms. The cohomology with compact supports is invariant under proper homotopies and the excision theorem holds as well.

Remark 1.6.5. In cohomology with compact supports, given an open embedding $i : X \hookrightarrow Y$, we can define a map which goes in the "wrong" direction $i_* : h_c(X) \to h_c(Y)$. To verify this assertion, consider the isomorphism obtained through the excision triad (Y, X, K^c) where $K \in \mathcal{K}(X)$:

$$j^* : h_c(Y, K^c) \to h_c(X, K^c)$$

Observe that K is compact in Y since it is the image of a continuous map. The inverse $j^{*-1} : h(X, K^c) \to h(Y, X \cup K^c)$ composed with the morphism $i_K : h(Y, X \cup K^c) \to h_c(Y)$ induces a morphism which we denote by $i_* : h_c(X) \to h_c(Y)$.

In (MASSEY, 1978, Part 1, Chapter 7), the author gives axioms for a generalized cohomology with compact supports over locally compact Hausdorff spaces. In (MASSEY, 2019), the same author states that the above definition gives rise to such a theory in ordinary cohomology, which is indeed the case. However, we could not verify that the same holds for a generalized cohomology theory. Some progress is being made in this direction in the dissertation of Clemente (2022), which is in preparation as of the time of this writing. We will state here some of these upcoming results. The reader is invited to read the monograph when available.

Proposition 1.6.6. *Let X be a locally compact Hausdorff space and X^+ its one point compactification. Suppose that the point at infinity $+$ has a local system of good pairs (HATCHER, 2002)*[13]*. Then $h_c(X) \cong \widetilde{h}(X^+, +)$. In particular, this holds for the inclusion $j : \mathbb{R}^n \hookrightarrow S^n$, where S^n is seen as the one point compactification of $\mathbb{R}^n$.*

Proof. View Clemente's dissertation. □

[13] Local system of Absolute Neighbourhood Retracts.

In particular, this result implies that $h_c^n(\mathbb{R}^n) \cong h^n(S^n, +) \cong \mathfrak{h}$ and 0 otherwise.

Given a multiplicative cohomology theory $(h, \partial, \times)$, we can define a multiplicative structure in the associated cohomology with compact supports. In fact, more precisely, we will work with two different kinds of products.

The first product we will construct is a module structure on h_c over h. In this structure, the scalar multiplication

$$\cdot : h(X) \times h_c(X) \to h_c(X) \tag{1.9}$$

is obtained by using the universal property on the following morphism:

$$\cdot_k : h^p(X) \times h^q(X, K^c) \xrightarrow{\cdot} h^{p+q}(X, K^c) \xrightarrow{i_K} h_c^{p+1}(X),$$

where the first arrow is the internal product.

The second product is given by a ring structure on h_c

$$\cdot : h_c \times h_c \to h_c \tag{1.10}$$

and is defined by taking the double colimit[14] of the maps

$$\cdot_{K,L} : h^p(X, K^c) \times h^q(X, L^c) \xrightarrow{\cdot} h^{p+q}(X, (K \cap L)^c) \xrightarrow{i_{K\cap L}} h_c^{p+1}(X).$$

We shall see ahead that the cohomology with compact supports also has a Thom isomorphism, which admits a nice definition using a different type of cohomology, called cohomology with vertically compact supports.

1.6.3 Cohomology with Vertically Compact Support

Definition 1.6.7 (Vertically compact sets)**.** Let $f : Y \to X$ be a continuous map. We say that $V \subseteq Y$ is a *vertically compact set* if $f|_V : V \to B$ is a proper map. This means that for all $K \in \mathcal{K}(X)$ we have $V \cap f^{-1}(K) \in \mathcal{K}(Y)$.

We denote the set of vertically compact sets of $f : Y \to X$ by $\mathcal{V}(f)$.

Remark 1.6.8. One should compare this definition with the notion of fibrewise compact sets presented in (JAMES, 1989).

The idea of a vertically compact set is illustrated in Figure 4.

Compact sets can be seen as a particular case of this concept: the vertically compact sets of the constant application $c : X \to \{*\}$ are precisely the compact sets of X, that is, $\mathcal{V}(c) = \mathcal{K}(X)$. In other words, compact sets are vertically compact *to the point*.

[14] We remark that the double colimit commutes.

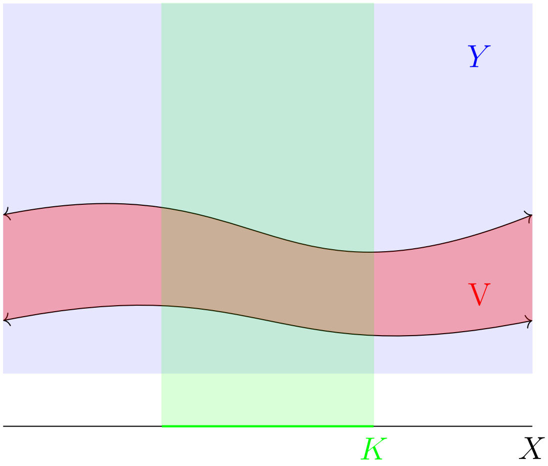

Figure 4 – The set V is an example of a vertically compact set.

Proposition 1.6.9. *The union of two vertically compact sets is vertically compact. Provided Y is Hausdorff and X is locally compact, any vertically compact set of $f : Y \to X$ is closed.*

Proof. Let V, W be vertically compact sets. For every $K \in \mathcal{K}(X)$ we have $V \cap f^{-1}(K)$ and $W \cap f^{-1}(K)$ in $\mathcal{K}(Y)$. Since the union of compact sets is compact, then $f^{-1}(K) \cap (V \cup W) = (f^{-1}(K)) \cap V) \cup (f^{-1}(K) \cap W)$ is compact.

For the second point, given $y \in V^c$, we set $x = f(y)$ and choose some compact neighbourhood K of x, which exists since X is locally compact. Since Y is Hausdorff, for each point $w \in V_K := V \cap f^{-1}(K)$, we can pick neighbourhoods W_w and W'_w of w and y, respectively, such that $W_w \cap W'_w = \emptyset$. Since $\{W_w, w \in V_K\}$ is a covering of V_K, we can extract a finite subcovering $\{W_1, \ldots, W_n\}$. The set $W'_1 \cap \ldots \cap W'_n \cap \operatorname{int}(f^{-1}(K))$ is an open neighbourhood of y which is disjoint of V. □

As we have done with cohomology with compact supports, we can also define cohomology with vertically compact supports. Note that the previous proposition makes the set $(\mathcal{V}(f), \subseteq)$ a directed set, since the union of vertically compact sets is also vertically compact.

Definition 1.6.10 (Cohomology with Vertically Compact Supports)**.** Given a map $f : Y \to X$, we define the *cohomology with vertically compact supports* as the colimit

$$h_v(X) = \operatorname{colim}_{V \in \mathcal{V}(f)} h(Y, V^c)$$

Remark 1.6.11. Let $f : Y \to X$, $g : B \to A$ be maps and $(\overline{\rho}, \rho) : g \to f$ a morphism between them. Given $\alpha \in h_v(Y)$ with respect to f, one can ask whether $\overline{\rho}^* \alpha$ belongs in $h_v(B)$. In general, the answer will be no, and a sufficient condition for this to hold is that $\rho : A \to X$ is a proper map.

Remark 1.6.12. Analogously to Remark 1.6.5, if we have $p : E \to X$ and $q : F \to X$ bundles over X such that $i : F \hookrightarrow E$ is an open embedding of vector bundles, we can define a pushfoward homomorphism

$$i_* : h_v(F) \to h_v(E)$$

using excision. This can be done provided the spaces involved are locally compact Hausdorff, since in this case V is closed, according to Proposition 1.6.9.

The following fact establishes the link between this definition and the Thom isomorphism.

Proposition 1.6.13. *Let $\pi : E \to B$ be a real vector bundle over a paracompact space. Then there exists an isomorphism between $h(E, E_0)$ and $h_v(E)$.*

The proof requires some preparation, so we will first recall some definitions:

Recall that a metric (also called bundle metric) on the vector bundle (E, p, B) is a continuous function $g : E \times_X E \to \mathbb{R}$, where $p' : E \times_X E \to X$ is the fiber product given by

$$E \times_X E = \{(v, u) \in E \times E : p(v) = p(u)\} \text{ with } p'((v, u)) := p(v) = p(u),$$

and that $g|_{p'^{-1}(x)}$ is a inner product.

It is a standard fact that every bundle over a paracompact space admits a metric (see (HUSEMöLLER, 1994, Theorem 9.5, p.38) or (RUFFINO, 2020, Proposição 7.7.2, p.179)). Moreover, given some (strictly) positive map $h : B \to \mathbb{R}_{++}$, the function $(h \cdot g)(v, u) := h(p(v)) \cdot g(v, u)$ is also a metric. We denote by $\mathcal{V}_{\text{tub}}(p)$ the set of sets of the form

$$V_g = \{v \in E : g(v, v) \leq 1\}$$

for some bundle metric g on (E, p, B). The sets V_g are the tubes around the zero section (possibly intersecting it) as displayed in Figure 5.

Definition 1.6.14 (Locally Bounded Function and cb-spaces)**.** A function $f : X \to \mathbb{R}$ is *locally bounded* if, for each point $x \in X$, there exists a neighbourhood U_x of x such that $f|_{U_x}$ is bounded, *i.e.*, $|f|_{Ux}| < c$ for some c depending on x.

A space X is said to be a continuously bounded space (*cb-space*) when for each locally bounded function $f : X \to \mathbb{R}$ there exists some continuous map $g : X \to \mathbb{R}$ such that $|f| \leq g$.

Proposition 1.6.15. *A paracompact normal space is a a cb-space.*

The proof can be found in (MACK, 1965, Corollary 2, p.469). Just notice that a paracompact space is countably paracompact. For more information on cb-spaces the reader can consult the aforementioned article.

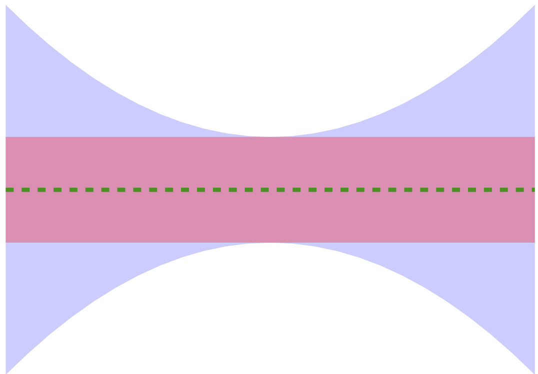

Figure 5 – Example of metric tubes. The red one is given by constant bundle metric and the blue by the bundle metric given by the function $h(x) = x^2 + 1$.

Remark 1.6.16. Notice that in a locally compact space X, every function $f : X \to \mathbb{R}$ which is *bounded on compacts*, in the sense that $|f|_K < c_K$ for every compact K, is locally bounded. This follows from the fact that for each $x \in X$ there exists a compact set K such that $|f|_{\mathrm{int}(K)} < c$.

Now, returning to the proof of Proposition 1.6.13, it will be based on two main lemmas:

- First, we prove that the tubes form a cofinal system of vertically compact sets.
- Second, we verify that the cohomology of the tubes is the same as $h(E, E_0)$ in a compatible way.

Lemma 1.6.17. *The set $\mathcal{V}_{tub}$ is a cofinal system in $\mathcal{V}(p)$.*

Proof. There are two points to be dealt with:

1. verify that the sets V_g are vertically compact;
2. verify that, for any vertically compact set V, there exists some g such that $V \subseteq V_g$.

For the first point, let $K \in \mathcal{K}(X)$. Then we have $V_g \cap E_K = \{v \in E_K : g(v, v) \leq 1\}$. Since K is compact it is possible to cover it with finitely many trivializing neighborhoods $\phi_k : E|_K \to \mathbb{R} \times U_j$ such that the following diagram commutes:

$$\begin{tikzcd}
E_{U_j \cap K} \arrow[rr, "\phi_j"] \arrow[dr, "p"'] & & (U_j \cap K) \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_{\mathbb{R}^n}"] \\
& U_j \cap K &
\end{tikzcd}$$

Defining the metric $g_j = g \circ \phi_j^{-1}$, the set $V_g \cap E_K$ is taken to the set

$$V_j = \{(x, v) \in (U_j \cap K) \times \mathbb{R}^n : g_j(x, v, v) \leq 1\}$$

V_j is closed and, since $V_j \subset (U_j \cap K) \times D$, where D is the unitary closed ball with respect to the maximum norm $\max_x(g(x, v, v))$, it is compact. From this we obtain that $V_g \cap E_{U_j \cap K}$ is compact for every j. Taking (finite) union over j, we conclude that $V_g \cap E_K$ is compact and thus V_g is vertically compact.

For the second point, we start by fixing a metric g_0 in (E, p, B), we know to exist by paracompactness and define the function

$$\begin{aligned} g' : B &\to \mathbb{R} \\ x &\mapsto \sup_{v \in E_x} g(v, v), \end{aligned}$$

which is not necessarily continuous, but is bounded on compacts and therefore locally bounded by remark 1.6.16.

Every normal paracompact space is a cb-space by proposition 1.6.15. Therefore, there exists a continuous function $h : B \to X$ such that $g' \leq h$. We define $g := h + 1$ and note that $V \subseteq V_g$. □

Lemma 1.6.18. *For each pair $(V_g, V_{g'}) \in \mathcal{V}_g(p)$ such that $V_g \subseteq V_{g'}$, one has the following commutative diagram:*

$$\begin{tikzcd} h(E, V_g^c) \arrow[d, "i^*_{V_g V_{g'}}"'] & h(E, E_0) \arrow[l, "i^*_{V_g}"'] \arrow[dl, "i^*_{V_{g'}}"] \\ h(E, V_{g'}^c) & \end{tikzcd}$$

where the inclusion $i_{V_g} : (E, V_g^c) \to (E, E_0)$ is an isomorphism.

Proof. The diagram is clearly commutative. We just verify that the inclusion induces an isomorphism. The inclusion $i_{V_g} : V_g^c \hookrightarrow E_0$ is an homotopy equivalence: an homotopy inverse of it is given by the map $\pi : E_0 \to V_g^c$ defined by $\pi(v) := \frac{v}{g(v,v)}$. Thus the map $(\mathrm{id}_E, i_{V_g}) : (E, V^g) \to (E, E_0)$ induces an isomorphism in cohomology by Proposition 1.3.6. □

Proof of Proposition 1.6.13. Since the sets $\mathcal{V}_g$ are cofinal in $\mathcal{V}(p)$ and we have the isomorphism $h(E, V_g^c) \cong h(E, E_0)$ in a way that is compatible with the inclusions, the result follows. □

1.6.4 Revisiting the Thom isomorphism

Analogously to h_c, we can also endow h_v with two different multiplicative structures:

- a scalar multiplication $\cdot : h_v(E) \times h(E) \to h_v(E)$ between a vertical class and an absolute one, that can be defined in a similar manner to the product (1.9) we had for h_c, and turns h_v into a module.
- a product $\cdot : h_v(E) \times h_v(E) \to h_v(E)$ between two vertical classes, that can be defined in a similar manner to the product (1.10) we had for h_c, and turns h_v into a ring.

It turns out that there is a third multiplicative structure which relates the vertically compact differential cohomology with the compactly supported cohomology. Given $f : X \to Y$, there is a product $\cdot f^* : h_v(Y) \times h_c(X) \to h_c(Y)$ between vertically compact classes and compact ones. The construction is roughly as follows: given a map $f : Y \to X$, $\alpha \in h_v(Y)$ and $\beta \in h_c(X)$, choose representatives $\alpha_V \in h(Y, V^c)$ and $\beta_K \in h(X, K^c)$ for α and β, respectively, where $V \in \mathcal{V}(f)$ and $K \in \mathcal{K}(X)$. The internal product of α_V and $f^*\beta_K$ is a class $\alpha_V \cdot f^*\beta_K \in h(Y, V^c \cup f^{-1}(K)^c) = h(Y, (V \cap f^{-1}(K))^c)$. Observe that $V \cap f^{-1}(K)$ is compact. Now, composing with $i_{V\cap f^{-1}(K)} : h(E, V \cap f^{-1}(K)) \to h_c(E)$ and using the universal property of the colimit in the definition of the cohomology, we get a map

$$\cdot f^* : h_v(Y) \times h_c(X) \to h_c(Y) \tag{1.11}$$

defined on the representatives as

$$\alpha \cdot f^*\beta.$$

Now, using these products, we can reinterpret the Thom class in the following way: Let $p : E \to B$ be a real vector bundle of dimension n. A cohomology class $u \in h_v^n(E)$ is a Thom class if $u|_x \in h_c^n(E_x)$ is a generator of $h_c^n(\mathbb{R}^n) = \mathfrak{h}^0$.[15] In this guise, the Thom isomorphism reads

$$\begin{aligned} T : h(B) &\to h_v(E) \\ \alpha &\mapsto u \cdot f^*\alpha, \end{aligned}$$

where $\cdot : h_v(E) \times h(E) \to h_v(E)$ is the product between vertical classes and absolute ones.

We also have the *compact Thom isomorphism*:

$$\begin{aligned} T_c : h_c(B) &\to h_c(E) \\ \alpha &\mapsto u \cdot p^*\alpha, \end{aligned}$$

where $\cdot f^* : h_v(Y) \times h_c(X)$ is the product between vertically compact classes and compact ones.

Since the proof of the compact Thom isomorphism can be hard to find (see (KAROUBI, 1978, Proposition 1.11,p.186) in K-theory), we provide a sketch here.

[15] Observe that $i_x^*\alpha$ can be regarded as a compact class.

Proof (Sketch). In the literature, it is generally the case that that there exists a long exact sequence in compactly supported cohomology[16]:

$$\cdots \longrightarrow h_c(U) \xrightarrow{i_*} h_c(X) \xrightarrow{j^*} h_c(U^c) \xrightarrow{\delta} h_c(U) \longrightarrow \cdots$$

If X is compact, the compact Thom isomorphism is just the usual Thom isomorphism. Application of the five lemma to the above sequence implies that the Thom isomorphism holds for $E|_U$, the restriction of the fiber bundle to the open set U.

Now, there exist a Mayer-Vietoris sequence for compact cohomology which goes the "wrong way".

$$\cdots \longrightarrow h_c^\bullet(U\cap V) \xrightarrow{i'_*\oplus j'_*} h_c^\bullet(U)\oplus h_c^\bullet(V) \xrightarrow{i_*-j_*} h_c^\bullet(X) \xrightarrow{\Delta} h_c^{\bullet+1}(U\cap V) \longrightarrow \cdots$$

Since X can be covered by open trivializing neighbourhoods, since X is Hausdorff and locally compact, then by applying the five lemma on the diagram, we have

$$\begin{tikzcd}
\cdots \arrow[r] & h_c^{\bullet+n}(E_{U\cap V}) \arrow[r] & h_c^{\bullet+n}(E_U)\oplus h_c^{\bullet+n}(E_V) \arrow[r] & h_c^{\bullet+n}(E_{U\cup V}) \arrow[r] & h_c^{\bullet+n+1}(E_{U\cap V}) \arrow[r] & \cdots \\
\cdots \arrow[r] & h_c^\bullet(U\cap V) \arrow[u, "T_c"] \arrow[r, "i'_*\oplus j'_*"] & h_c^\bullet(U)\oplus h_c^\bullet(V) \arrow[u, "T_c"] \arrow[r, "i_*-j_*"] & h_c^\bullet(U\cup V) \arrow[u, "T_c"] \arrow[r, "\Delta"] & h_c^{\bullet+1}(U\cap V) \arrow[u, "T_c"] \arrow[r] & \cdots
\end{tikzcd}$$

From this we conclude that $T : h_c^\bullet(U) \to h_c^{\bullet+n}(E_{U\cup V})$ is an isomorphism. Passing to the inverse colimit over trivializing neighbourhoods (which is possible in compactly supported cohomology), we conclude that T_c is an isomorphism. $\square$

We will also need a third version of the Thom isomorphism. Given $f : Y \to X$ and $g : Z \to Y$, the set of doubly-vertically compact sets is given by

$$\mathcal{VV}(f,g) = \mathcal{V}(g)\cap\mathcal{V}(f\circ g).$$

The doubly compacted supported cohomology of Z is defined as the colimit

$$h_{vv}(Z) = \operatorname{colim}_{V\in\mathcal{VV}(f,g)} h(Z, V^c)$$

Note that we have a map $h_{vv}(Z) \to h_v(Z)$ with respect to both g and $f\circ g$.

Consider a vector bundle $p : E \to Y$ and a map $f : Y \to X$ such that $f\circ p : E \to X$ is a vector bundle. We define the *doubly-vertically compact Thom isomorphism* $T_{vv} : h_v(Y) \to h_{vv}(E)$ as

$$T_{vv}(\alpha) = u\cdot g^*\beta,$$

where $u \in h_{vv}(u)$ is a class whose image in $h_v(X)$ is a Thom class both for p and $f\circ p$. One can verify that this is indeed an isomorphism.

[16] In particular, this holds provided $h_c(X) \cong h(X^+, +)$, but the reader should be aware that it is not always true. See Clemente's dissertation.

Remark 1.6.19. Later we will need a Thom class on trivial vector bundles. Let's show how we can endow one with a Thom class: given the product bundle $\mathrm{pr}_X : X \times \mathbb{R}^n \to X$, we have the isomorphism $i_* : h_c^n(\mathbb{R}^n) \to h^n(S^n, *)$. Pick a generator $\widetilde{u}$ of $h(S^n, *)$[17]. The projection $\mathrm{pr}_{\mathbb{R}^n}$ induces a homomorphism $\mathrm{pr}^*_{\mathbb{R}^n} : h_c(\mathbb{R}^n) \to h_v(X \times \mathbb{R}^n)$ in the following way: Given a class $\alpha \in h_c^n(\mathbb{R})$, we choose a representative $\alpha_K \in h^n(\mathbb{R}^n, K^c)$, and consider

$$\mathrm{pr}^*_{\mathbb{R}^n} \alpha_K \in h(X \times \mathbb{R}^n, \mathrm{pr}^{-1}_{\mathbb{R}^n}(K)^c). \tag{1.12}$$

Then $\mathrm{pr}^{-1}_{\mathbb{R}^n}(K)$ is vertically compact. By the universal property we get the homomorphism. We put $u = \mathrm{pr}^*_{\mathbb{R}^n} \circ i_*^{-1} \widetilde{u} \in h_c(X \times \mathbb{R}^n)$. We claim that this is a Thom class of $\mathrm{pr}_X : X \times \mathbb{R}^n \to X$. Indeed, by working backwards on (1.12), the result is clear.

1.6.5 The relative Thom isomorphism

In its usual form, the relative Thom isomorphism is stated in the following way:

Definition 1.6.20 (Relative Thom isomorphism)**.** Let (X, A) be a CW-pair and $p : E \to X$ a real vector bundle of dimension k. The map

$$\begin{aligned} T : h^\bullet(X, A) &\to h^{\bullet + k}(E, E_A \cup E_0) \\ \alpha &\mapsto u \cdot p^* \alpha \end{aligned} \tag{1.13}$$

is an isomorphism.

The proof of this fact can be found in (AGUILAR et al., 2002). In our framework, we would like to define the Thom isomorphism for a relative vector bundle over a map.

Definition 1.6.21 (Relative Vector Bundle)**.** We say that a morphism $(P, p) : \overline{\rho} \to \rho$ as in the diagram

$$\begin{array}{ccc} F & \xrightarrow{\overline{\rho}} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle P} \\ A & \xrightarrow{\rho} & X \end{array},$$

is a *relative vector bundle* over ρ, if $P : E \to X$ and $p : F \to A$ are two vector bundles and $\overline{\rho}_a : F_a \to E_{\rho(a)}$ is a linear isomorphism.

First we would like to define $h_v(\overline{\rho})$ such that an analogue of Proposition 1.6.13 holds, which would imply that $h_v(\overline{(X, A)}) = h(E, E_A \cup E_0)$ [18]. Generalizing to a map $\rho : A \to X$, it seems natural to ask for the equality $h_v(\overline{\rho}) = h(E, j(\rho^* E) \cup E_0)$, as this

[17] We remark for future use that $s^n(\widetilde{u}) = 1$, where s is the suspension isomorphism.
[18] The notation $\overline{(X, A)}$ denotes the covering of the inclusion (X, A)

particularizes well to the case $\rho = (X, A)$, where $j : \rho^*E \to E$ is the natural map depicted in the following diagram:

$$\begin{tikzcd} \rho^*E \arrow[r, "j"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ A \arrow[r, "\rho"] & X \end{tikzcd}$$

In order to have the above equality, we define[19]:

$$h_v(\overline{\rho}) = \mathrm{colim}_{V \in \mathcal{V}(p)} h(E, j(\rho^*(E)) \cup V^c).$$

With this definition, we have a well-defined product

$$\cdot : h_v(E) \times h(\overline{\rho}) \to h_v(\overline{\rho}),$$

which can be constructed in the following way:

Fix a vertical set V and a vertical class $u \in h_v(E)$ represented by $u_V \in h(E, V^c)$. Since we can write $\overline{\rho}$ as the composition

$$E \xrightarrow[\sim]{\overline{\rho}} \rho^*E \xrightarrow{j} E,$$

and $j : \rho^*E \to E$ is an embedding, we can identify $h(E, j(\rho^*E))$ with $h(\overline{\rho})$. Thus, we get

$$u \cdot \alpha \in h((E, V^c) \wedge (E, j(\rho^*))^{20},$$

provided that $((E, j(\rho^*E)), (E, V^c))$ is an excisive pair, which is always the case if either j is open or closed. This way, we have

$$u \cdot \alpha \in h(E \times E, E \times j(\rho^*E) \cup V^c \times E)$$

by pulling back along the diagonal, as in (1.8), and achieve the desired product.

We are thus led to the following Thom isomorphism analogous to (1.13), that is

$$\begin{aligned} T : h(\rho) &\to h_v(\overline{\rho}) \\ \alpha &\mapsto u \cdot (P, p)^*\alpha \end{aligned} \tag{1.14}$$

where $u \in h_v(E)$.

Unfortunately, we are not aware if there exists a compact version, since we do not know how the define $h_c(X, A)$ in general[21]. It is not yet clear how one should define it now, and we will only be able to define this in Section 4.6.

Concerning the Thom class of relative vector bundle, we have the following important fact:

[19] We will change this definition in Chapter 4, but for now it is convenient.
[20] The notation $(X, A) \wedge (Y, B)$ stands for $i_A \wedge i_B$, where $i_A : A \hookrightarrow X$ and $i_A : B \hookrightarrow Y$ are the inclusions.
[21] Alexander-Spanier cohomology seems to be a nice candidate, but we do not know how to construct an Alexander-Spanier cohomology from a generalized cohomology as we have done with compact supported cohomology.

Proposition 1.6.22. *Fix a relative vector bundle $(P, p) : \overline{\rho} \to \rho$ as in the diagram*

$$\begin{tikzcd} F \arrow[r, "\overline{\rho}"] \arrow[d, "p"] & E \arrow[d, "P"] \\ A \arrow[r, "\rho"] & X \end{tikzcd}$$

Then there is a pullback map $\overline{\rho}^* : h_v(E) \to h_v(F)$. Moreover, given a Thom class $u \in h_v(E)$, the class $\overline{\rho}^* u \in h_v(F)$ is a Thom class and one has

$$\rho^* \circ T = T \circ \rho^*$$

This is a classical result when working with the classical Thom isomorphism in the usual setting and can be translate to this setting when we use Proposition 1.6.13. We will return to it in Chapter 4.

1.7 Orientability of Maps and the Umkehr Map

Now we we wish to introduce the *umkehr map*[22] in a generalized cohomology. There are two equivalent ways to express the umkehr map: one can either use a generalized Poincaré duality or use the Thom-Pontryiagin construction. The two constructions are related through Atiyah duality, which states that the Spanier-Whitehead dual of the suspension of a manifold is the Thom spectrum of its Spivak fibration. In the differential case, this identification is done with the stable normal bundle of the smooth manifold. Since our aim is the differential case and we lack a proper definition of differential homology and thus a Poincaré duality, we are going to stick with the classical geometrical definition.

Grosso modo, the umkehr map of a map $f : Y \to X$ between an n dimensional manifold Y and an m dimensional manifold X[23] is a "wrong direction" map $f_! : h_c^\bullet(Y) \to h_c^{\bullet-(m-n)}(X)$[24]. Since we are mainly interested in smooth spaces in this text, we limit ourselves to smooth manifolds and maps. In this section, manifolds are assumed to be smooth - by which we mean C^∞ - as well as the maps. This section closely follows the article (RUFFINO, 2017).

We shall deal with manifolds with boundary and this will require us to use the concept of neat manifolds:

Definition 1.7.1 (Neat smooth map)**.** A smooth map $f : Y \to X$ is said to be *neat* if $f^{-1}(\partial X) = \partial Y$ and

$$d_y f : \frac{T_y Y}{T_y \partial Y} \to \frac{T_{f(y)} X}{T_{f(y)} \partial X}$$

[22] The umkehr map appears in the literature with many names such as Gysin map, pushfoward, transfer, shriek map and surprise map. See the review (BECKER; GOTTLIEB, 1999). We are only dealing with the standard umkehr map. For the generalized case the reader is referred to (COHEN; KLEIN, 2009).

[23] Again this is the narrow view. In principle this can be carried in Poincaré Space - also called Poincaré complexes - which are spaces in which Poincaré duality holds.

[24] Using the generalized Poincare duality, this is just $D_X \circ p_* \circ D_Y^{-1}$

is an isomorphism.

For this definition in the more general setting of manifolds with corners, see (HOPKINS; SINGER, 2005, Appendix C); and for the usual form with embeddings see (KOSINSKI, 2007). The relevant fact here is that neat embeddings admit tubular neighbourhoods.

Recall that a *tubular neighbourhood* of an embedding $\iota : Y \hookrightarrow X$ is a diffeomorphism $\phi : N(\iota(Y)) \to U$, where $N(\iota(Y))$ is the normal bundle associated to ι and $U \subseteq X$ is an open subset.

1.7.1 Umkehr Map: absolute case

In order to define the Umkehr Map, we will need the concept of cohomological orientation of a map. The elements of the definition are illustrated in Figure 6.

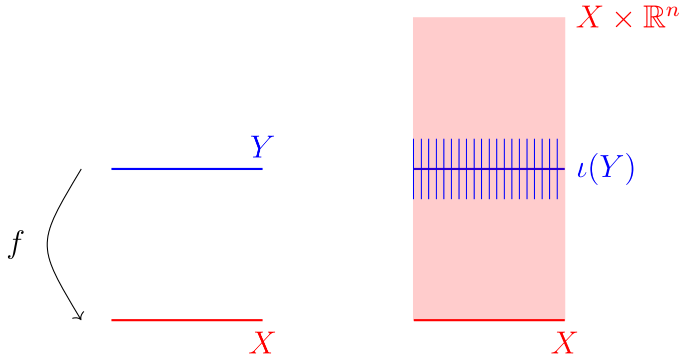

Figure 6 – The main elements of a orientation of a map f.

Definition 1.7.2 (Representative of h-orientation)**.** A *representative of an h-orientation* of a smooth neat map between compact manifolds $f : Y \to X$ is given by the following data:

1. a neat embedding $\iota : Y \to X \times \mathbb{R}^N$, for any $n \in \mathbb{N}$, such that $\mathrm{pr}_X \circ \iota = f$.

2. a Thom class u of the normal bundle $N(\iota(Y))$.

3. a tubular neighbourhood of $\iota(Y)$ in $X \times \mathbb{R}^N$ given by a diffeomorphism $\phi : N(\iota(Y)) \to U$ onto an open set U.

Remark 1.7.3. We have a few remarks about each point of the previous definition.

1. Such an embedding always exists: by the Whitney embedding theorem, there exists an embedding $j : Y \to \mathbb{R}^N$ for a large enough N. Hence, we can take $\iota(y) = (f(y), j(y))$.

2. Recall that given an embedding $j : Y \hookrightarrow X$, the normal bundle is $\frac{j^*TX}{TM}$.

3. If f is neat, the embedding is neat as well. This guarantees the existence of a tubular neighbourhood. (HIRSCH, 1976, Theorem 6.3, p.114).

We say that a representative (ι, ϕ, u) of an h-orientation is *proper* if the following diagram commutes

$$\begin{tikzcd} N(\iota(Y)) \arrow[r, "\phi"] \arrow[d, "\pi"] & U \arrow[d, "\mathrm{pr}_X"] \\ \iota(Y) \arrow[r, "\mathrm{pr}_X"] & X \end{tikzcd}$$

Now, we introduce a suitable equivalence relation among representatives of orientations. First we need a generalization of the concept of proper representative, which will only be used in the context of these equivalences. Let us consider a representative (J, U, Φ) of an h-orientation of $\mathrm{id}_I \times f : I \times Y \to I \times X$ and a neighborhood $V \subseteq I$ of $\{0, 1\}$. We say that the representative is proper on V if the following diagram commutes:

$$\begin{tikzcd} N(J(I \times Y))|_{V\times Y} \arrow[r, "\Phi|_{V\times Y}"] \arrow[d, "\pi"] & U|_{V\times Y} \arrow[d, "\mathrm{pr}_I"] \\ J(V \times Y) \arrow[r, "\mathrm{pr}_I"] & I \end{tikzcd}$$

Thanks to properness, by calling $f_0 := \mathrm{id}_I \times f$ and $f_1 := \mathrm{id}_I \times f$, we can define the restrictions $(J, U, \Phi)|_{f_0}$ and $(J, U, \Phi)|_{f_1}$.

Definition 1.7.4 (Homotopy)**.** A *homotopy between* two representatives (ι, u, ϕ) and (ι', u', ϕ') of an h-orientation of $f : Y \to X$ is a representative (J, U, Φ) of an h-orientation of $\mathrm{id}_I \times f : I \times Y \to I \times X$, such that

- (J, U, Φ) is proper over a neighborhood $V \subseteq I$ of $\{0, 1\}$;
- $(J, U, \Phi)|_{f_0} = (\iota, u, \phi)$ and $(J, U, \Phi)|_{f_0} = (\iota', u', \phi')$.

Definition 1.7.5 (Stabilization)**.** Let us consider a representative (ι, u, ϕ) with $\iota : Y \to X \times \mathbb{R}^N$. A representative (ι', u', ϕ') is said to be *equivalent* to (ι, u, ϕ) *by stabilization* if

- For any $L \in \mathbb{N}$, $\iota' : Y \to X \times \mathbb{R}^{N+L}$ is given by $\iota'(y) := (\iota(y), 0)$.
- u' on $N(\iota'(Y))$ is obtained in the following way:

  Observe that $N(\iota'(Y)) = N(\iota(Y)) \oplus (\iota(Y) \times \mathbb{R}^L)$, where $\mathrm{pr}_{\iota(Y)} : \iota(Y) \times \mathbb{R}^L \to X$ is the product bundle. We consider the canonical Thom class of Remark 1.6.19 on $\mathrm{pr}_{\iota(Y)} : \iota(Y) \times \mathbb{R}^L \to X$ and use proposition 1.6.2 to construct u'.
- For $v \in N_{\iota(Y)}$ and $w \in \mathbb{R}^L$ we have $\phi'(v, w) = (\phi(v), w) \in X \times \mathbb{R}^{N+L}$

Definition 1.7.6 (h-orientation of a map)**.** An *h-orientation* on $f : Y \to X$ is an equivalence class $[\iota, u, \phi]$ of representatives, up to the equivalence relation generated by homotopy and stabilization.

The uniqueness up to homotopy of the tubular neighbourhood tells us that this equivalence class depends only on ι and u, but not on ϕ. Because of this, we write $[\iota, u]$ for an orientation. We will use the following facts about h-orientations.

Definition 1.7.7. Let $f : Y \to X$ and $g : X \to W$ be h-oriented maps, with orientations $[\iota, u, \phi]$ and $[\kappa, v, \psi]$, where $\iota : Y \to X \times \mathbb{R}^N$ and $\kappa : X \to W \times \mathbb{R}^L$. There is a naturally induced h-orientation $[\xi, w, \chi]$ on $g \circ f : Y \to W$, which can be defined in the following way:

- the embedding ξ is given by $\xi = (\kappa, \mathrm{id}_{R^N}) \circ \iota : Y \to W \times \mathbb{R}^{N+L}$,
- the Thom class w is constructed the following way:

  On the normal bundle $N(\xi(Y)) \cong N(\iota(Y)) \oplus \iota^* N(\kappa(X) \times \mathrm{id}_{\mathbb{R}^N}) \cong N(\iota(Y)) \oplus (\mathrm{pr}_{\mathbb{R}^L})^* N(\kappa(X))$, for $\mathrm{pr}_{\mathbb{R}^L} : \mathbb{R}^{N+L} \to R^L$, we put the Thom class w induced from the ones on $N(\iota(Y))$ and $N(\kappa(X))$.
- the tubular neighbourhood $\chi : N(\xi(X)) \to U$ is arbitrary since it is unique up to homotopy.

We set $[\kappa, v][\iota, u] := [\xi, w]$.

According to (RUFFINO, 2017, Lemma 3.11, p.6), the following lemma is a consequence of of the uniqueness up to homotopy and stabilization of the embedding ι.

Proposition 1.7.8. *(2x3 principle for maps) Let $f : Y \to X$ and $g : X \to W$ be h-oriented neat maps, with orientations $[\iota, u]$ and $[\kappa, v]$, and let $[\xi, w] := [\kappa, v][\iota, u]$ be the orientation induced on $g \circ f$. Then two elements of the triple $([\iota, u], [\kappa, v], [\xi, w])$ uniquely determine the third one.*

Now consider the following maps:

$$\int_{\mathbb{R}}^{v} : h_v^{\bullet}(\mathbb{R} \times X) \to h^{\bullet - 1}(X) \quad \text{and} \quad \int_{\mathbb{R}}^{c} : h_c{\bullet}(\mathbb{R} \times X) \to h_c^{\bullet - 1}(X). \tag{1.15}$$

The first one is just

$$\int_{\mathbb{R}}^{v} := \int_{S^1} \circ (j \times \mathrm{id}_X)_*,$$

where $j : \mathbb{R} \hookrightarrow S^1$ is the open embedding in the one point compactification of the line and $(j \times \mathrm{id}_X)_* : h_v(\mathbb{R} \times X) \to h_v(S^1 \times X)$ is as defined in Remark 1.6.11 (noticing that $j \times \mathrm{id}_X : \mathbb{R} \times X \hookrightarrow S^1 \times X$ is a open inclusion of bundles over X).

The second map $\int_{\mathbb{R}}^{c}$ is defined in the following way: since the sets of the form $S^1 \times K$ are cofinal in $\mathcal{K}(S^1 \times X)$, given $\alpha \in h_c(S^1 \times X)$ we can represent it by a class $\alpha_K \in h(S^1 \times X, S^1 \times K^c)$[25] where $K \in \mathcal{K}(X)$.

Composing the maps

$$h(S^1 \times X, S^1 \times K^c) \xrightarrow{\int_{S^1}} h(X, K^c) \xrightarrow{i_K} h_c(X) \tag{1.16}$$

and using the universal property of colimits, we get a map $\int_{S^1}^{c} : h_c(S^1 \times X) \to h_c(X)$ and define $\int_{\mathbb{R}}^{c} := \int_{S^1} \circ (j \times \mathrm{id}_X)_*$, where $(j \times \mathrm{id}_X)_* : h_c(\mathbb{R} \times X) \to h_c(S^1 \times X)$ is the morphism defined in Remark 1.6.5.

We conclude by defining the integration maps over $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n}^{v} : h_v^{\bullet}(\mathbb{R}^n \times X) \to h^{\bullet - n}(X) \quad \text{and} \quad \int_{\mathbb{R}^n}^{c} : h_c^{\bullet}(\mathbb{R}^n \times X) \to h_c^{\bullet - n}(X)$$

as

$$\int_{\mathbb{R}^n}^{v} := \underbrace{\int_{\mathbb{R}}^{v} \circ \cdots \circ \int_{\mathbb{R}}^{v}}_{n} \text{ and } \int_{\mathbb{R}^n}^{c} = \underbrace{\int_{\mathbb{R}}^{c} \circ \cdots \circ \int_{\mathbb{R}}^{c}}_{n}$$

Remark 1.7.9. The compactly supported S^1-integration $\int_{S^1}^{c} : h_c^{\bullet}(S^1 \times X) \to h_c^{\bullet}(X)$, introduced in 1.16, as an intermediate step in the definition of the $\mathbb{R}$ integration map, is a natural transformation with respect to both proper maps and open embedding. In fact, for a proper map $f : X \to Y$, the following diagram is commutative:

$$\begin{tikzcd} h^{\bullet}(Y \times S^1) \arrow[r, "(f \times \mathrm{id}_{X^1})^*"] \arrow[d, "\int_{S^1}^{c}"] & h^{\bullet}(X \times S^1) \arrow[d, "\int_{S^1}^{c}"] \\ h^{\bullet - 1}(Y) \arrow[r, "f^*"] & h^{\bullet -}(X) \end{tikzcd}$$

and for an open embeddings $i : U \hookrightarrow X$, the following diagram is commutative:

$$\begin{tikzcd} h^{\bullet}(U \times S^1) \arrow[r, "(i \times \mathrm{id}_{X^1})_*"] \arrow[d, "\int_{S^1}^{c}"] & h^{\bullet}(X \times S^1) \arrow[d, "\int_{S^1}^{c}"] \\ h^{\bullet - 1}(U) \arrow[r, "i_*"] & h^{\bullet - 1}(X) \end{tikzcd}$$

Both facts can be proved by manipulating expression (1.16).

Lemma 1.7.10. *The map $\int_{\mathbb{R}}^{v}$ is an inverse to the Thom isomorphism for the trivial bundle with the canonical Thom class.*

Proof. Recall that the canonical Thom class is given by

$$u_X = \mathrm{pr}_{\mathbb{R}}^* \circ j_*^{-1}(\tilde{u}),$$

[25] Observe that $(S^1 \times K)^c = S^1 \times K^c$

where $j : \mathbb{R} \to S^1$ is the one point compactification and

$$\int_{\mathbb{R}}^{v} := \int_{S^1} (j_* \times \mathrm{id}_X).$$

It follows that

$$\begin{aligned}\int_{S^1} (j_* \times \mathrm{id}_X)(u_X \cdot \mathrm{pr}_X^* \alpha) &= \int_{S^1} (j_* \times \mathrm{id}_X)(\mathrm{pr}_{\mathbb{R}}^* \circ (j_*)^{-1} \widetilde{u} \cdot \mathrm{pr}_X^* \alpha) \\ &= \int_{S^1} (j_* \times \mathrm{id}_X^*)((j_*)^{-1} \widetilde{u} \times \alpha) \\ &= \int_{S^1} \widetilde{u} \times \alpha \\ &= \alpha,\end{aligned}$$

where this last equality follows from an observation similar to that of Remark 1.4.3. For details in the absolute case, see (BUNKE; SCHICK, 2010, Section 4, p.22). □

Definition 1.7.11 (Umkehr Map for Compact Fibers)**.** Given an h-oriented neat smooth map with compact fibers $f : Y \to X$ between manifolds of dimension n and m, respectively, and orientation representative (ι, u, ϕ), we define the *Umkehr map*

$$f_! : h^{\bullet}(Y) \to h^{\bullet-(n-m)}(X)$$

by the following composition:

$$h^{\bullet}(Y) \xrightarrow[\sim]{\iota *} h^{\bullet}(\iota(Y)) \xrightarrow{T_N} h_{\mathrm{v}}^{\bullet-(l+n-m))}(N_{\iota}(X)) \xrightarrow{(\phi^*)^{-1}}$$
$$\xrightarrow{(\phi^*)^{-1}} h_{v}^{\bullet-(l+n-m)}(U) \xrightarrow{k_*} h_{\mathrm{v}}^{\bullet-(l+n-m))}(X \times \mathbb{R}^l) \xrightarrow{\int_{\mathbb{R}^l}^{v}} h^{\bullet-(n-m)}(X)$$

in other words,

$$f_!(\alpha) = \int_{\mathbb{R}^l}^{v} k_*(\phi^{-1})^* T_N(\alpha), \tag{1.17}$$

where we have written α instead of $(\iota^{-1})^*\alpha$. Here, $k_* : h_v(U) \to h_v(X \times \mathbb{R}^l)$ is the map between the vertically compact $h_v(U)$ with respect to $\pi_N \circ \phi^{-1}$ and the vertically compact supported cohomology $h_v(X \times \mathbb{R}^l)$ with respect to the projection which is well defined since the fibres of Y over X are compact by hypothesis, hypothesis.

It is not clear that this map depends only on the orientation and not on the representative. This is indeed the case as the reader can check in (KAROUBI, 1978, Proposition 5.24,p.233). In fact, this map only depends on the homotopy class of f as an oriented map. Some of the main properties of this homomorphism are summarized in the following proposition:

Proposition 1.7.12. *The umkehr map satisfies the following properties:*

- (Projection formula) *For any oriented smooth neat map which admits a proper representative, given $\alpha \in h^p(Y)$ and $\beta \in h^q(X)$,*

$$f_!(\alpha \cdot f^*\beta) = f_!(\alpha) \cdot \beta$$

- (Composition) *If $f : Y \to X$ and $g : X \to Z$ are h-oriented maps and $f \circ g$ is endowed with the composition orientation as in 1.7.7, then*

$$(f \circ g)_! = f_! \circ g_!.$$

The proof of both results can be found in (KAROUBI, 1978, Proposition 5.24, p. 233).

As with the Thom morphism, we can define the compact case, which is done in the exact same way, by replacing the usual Thom isomorphism and the integration by their compact versions:

Definition 1.7.13 (Compact supported umkehr map)**.** Given an h-oriented neat smooth map $f : Y \to X$ between manifolds of dimension n and m respectively, and an orientation representative (ι, u, ϕ), we define the *compact umkehr map* of f as

$$\begin{aligned} f_{c!} : h_c^\bullet(Y) &\to h_c^{\bullet-(n-m)}(X) \\ \alpha &\mapsto \int_{\mathbb{R}^l}^c i_*(\phi^{-1})^* T_{N,c}(\alpha) \end{aligned}$$

where $i_* : h_c(U) \to h_c(X \times \mathbb{R}^l)$ is the same as in remark 1.6.5.

An analogue of Proposition 1.7.12 holds.

Finally, we have the *vertical umkehr map* $f_{v!} : h_v^\bullet(Y) \to h^{\bullet-(n-m)}(X)$ defined in the exact same way, just replacing the Thom isomorphism by the doubly-vertical version $T_{N,v}$.

Definition 1.7.14 (Vertically supported umkehr map)**.** Given an h-oriented neat smooth map $f : Y \to X$ between manifolds of dimension n and m respectively, and an orientation representative (ι, u, ϕ), we define the *vertical umkehr map* as

$$\begin{aligned} f_{v!} : h_v^\bullet(Y) &\to h^{\bullet-(n-m)}(X) \\ \alpha &\mapsto \int_{\mathbb{R}^l}^v i_*(\phi^{-1})^* T_{N,v}(\alpha), \end{aligned}$$

where $i_* : h_v(U) \to h_v(X \times \mathbb{R}^l)$ is the same as in remark 1.6.12

Remark 1.7.15. In order to see why the definition makes sense, we observe that since $(\phi^{-1}) * T_{N,v}(\alpha) \in h_{vv}(U)$ over $\iota(Y)$, the doubly vertically compactness implies that $(\phi^{-1}) * T_{N,v}(\alpha)$ is in $h_v(U)$ over pr_X. Formally, this means we have a composition with a map $j_* : h_{vv}(U) \to h_v(U)$, where the first is vertically compact with respect to both π_N and pr_X.

Remark 1.7.16. Consider an oriented vector bundle $p_E : E \to X$ over a smooth manifold with Thom class u_E. We would like to define an h-orientation $[\iota, u, \phi]$ of the map p_E. Now, we know that there exists a vector bundle $p_F : F \to X$ such that $p_E \oplus p_F : E \oplus F \to X$ is trivial (see (HIRSCH, 1976))[26]. We then put $[\iota, u, \phi]$ is as follows:

[26] The result is also true for the topological case. See the answers of (MITS314, 2021, James Cameron) and (MUKHERJEE, 2015, Igor Belegradek))

- $\iota : E \to X \times \mathbb{R}^N$ is defined as the composition

$$E \xrightarrow{i_E} E \oplus F \xrightarrow{\sim} X \times \mathbb{R}^N,$$

  where ι_E is the inclusion in direct sum fiberwise.

- the Thom class u is constructed as follows:

  We can identify the normal bundle of $i_E(E) = E \oplus 0$ with the bundle $\mathrm{pr}'_E : E \oplus F \to E \oplus 0$. By the 2 out of 3 principle for vector bundles (Proposition 1.6.2), we have a Thom class u_F on F which is induced by the Thom class u_E of E, and the canonical class in the trivial bundle $E \oplus F$, which is given by $u_{E\oplus F} = u_E \times u_F$. (Remark 1.6.19). We define a Thom class u in $\mathrm{pr}_F^* F$ by $\mathrm{pr}_F^*(u_F)$.

- The morphism $\phi : E \oplus F \to X \times \mathbb{R}^N$ is just the identity composed with the isomorphism $E \oplus F \xrightarrow{\sim} X \times \mathbb{R}^N$.

The following result plays a crucial role in this work.

Proposition 1.7.17. *For an oriented vector bundle $p : E \to X$, there exists an h-orientation of the map p such that the vertical umkehr map and the compact umkehr maps are inverses of the Thom isomorphism and the compact Thom isomorphism, respectively.*

Proof. We show this only for the compact case, the other one being analogous. We prove that the umkehr map is a left inverse of the Thom morphism. Note that

$$\begin{aligned}
T_N((i_E^{-1})^* \circ T_E)(\alpha)) &= \mathrm{pr}_F^*(u_F) \cdot \mathrm{pr}'^*_E \circ (i_E^{-1})^*(u_E \cdot p_E^*(\alpha)) \\
&= \mathrm{pr}_F^*(u_F) \cdot \mathrm{pr}_E^*((u_E) \cdot p_E^*(\alpha)) \\
&= \mathrm{pr}_F^*(u_F) \cdot \mathrm{pr}_E^*(u_E) \cdot \mathrm{pr}_E^* \circ p_E^*(\alpha) \\
&= (u_F \times u_F) \cdot (p_E \circ \mathrm{pr}_E)^*(\alpha) \\
&= u_{E\oplus F} \cdot p_{E\oplus F}^*(\alpha),
\end{aligned}$$

where $p_{E\oplus F} = p_E \circ \mathrm{pr}_E$. Using the isomorphism to $E \oplus F \xrightarrow{\sim} X \times \mathbb{R}^N$ and integrating in $\mathbb{R}^n$ we get

$$\int_{\mathbb{R}^n} u_{X\times\mathbb{R}^N} \cdot \mathrm{pr}_X^*(\alpha) = \alpha$$

where the equality is due to Lemma 1.7.10. □

1.7.2 Umkehr Map: relative case

It is not common to find a relative version of the umkehr morphisms in the literature. In view of this, we will give some definitions whose origins will be become more evident in the next chapters. We follow (RUFFINO; BARRIGA, 2021, Section 7.1) with some minor modifications.

Remark 1.7.18. The aforementioned work remarks that, as the absolute Umkehr map in the compact setting can be introduced with the aid of Poincaré duality, the relative case with respect to boundary can be defined using Lefschetz duality.

If $L_X : h^\bullet(X, \partial X) \to h_{n-\bullet}(X)$ is the Lefschetz duality, one can define the relative umkehr map $f_{!!} : h^\bullet(Y, \partial Y) \to h^{\bullet-(n-m)}(X, \partial X)$ as

$$f_{!!} = L_X^{-1} \circ f_* \circ L_Y,$$

where $f : (Y, \partial Y) \to (X, \partial X)$ is a smooth neat map.

Definition 1.7.19 (Relative Fiber Bundle)**.** A *relative fiber bundle* over a map $\rho : A \to X$ is a morphism $(F, f) : \overline{\rho} \to \rho$

- $F : Y \to X$ and $f : B \to A$ are fiber bundles;
- $\overline{\rho} : B \to Y$ is the covering of ρ as in the diagram

$$\begin{tikzcd} B \arrow[r, "\overline{\rho}"] \arrow[d, "f"] & Y \arrow[d, "F"] \\ A \arrow[r, "\rho"] & X \end{tikzcd}$$

  such that $\overline{\rho}$ is a fiberwise diffeomorphism[27].

The umkehr map is defined in the exact same way as the absolute case. But first we will need to define the concept of relative orientation. Given an h-orientation $[\iota, u, \phi]$ of $F : Y \to X$, we can obtain a natural orientation $[\iota', u', \phi']$ in $f : B \to A$ in the following way:

- If $\iota(y) = (F(y), j(y))$, define $\iota' : B \to A \times \mathbb{R}^n$ as $\iota'(b) := (f(b), j(\overline{\rho}(b))$. This map makes the following diagram commutative:

$$\begin{tikzcd} & A \times \mathbb{R}^n \arrow[rr, "\overline{\rho}\times \mathrm{id}_{\mathbb{R}^n}"] \arrow[ddl, "\mathrm{pr}_A"'] & & X \times \mathbb{R}^n \arrow[ddl, "\mathrm{pr}_X"] \\ B \arrow[ur, hook, "\iota'"] \arrow[rr, "\overline{\rho}"] \arrow[d, "f"] & & Y \arrow[ur, hook, "\iota"] \arrow[d, "F"] & \\ A \arrow[rr, "\rho"] & & X & \end{tikzcd}$$

- Denote by $\overline{\rho} : \iota'(B) \to \iota(Y)$ the restriction of $\rho \times \mathrm{id}_{\mathbb{R}^n}$ to ι_B. Its differential induces a map $\widetilde{\rho} : N(\iota'(A)) \to N(\iota(X))$ by the diagram

$$\begin{tikzcd} T\iota(Y) \arrow[r, "q_Y"] \arrow[d, "d\rho'"] & \frac{i^*T(X\times\mathbb{R}^n)}{i(TY)} \arrow[r, "\pi_N"] \arrow[d, "\widetilde{\rho}_\iota"] & \iota(Y) \arrow[d, "\overline{\rho}_\iota"] \\ T\iota'(B) \arrow[r, "q_B"] & \frac{i'^*T(A\times\mathbb{R}^n)}{i'(TB)} \arrow[r, "\pi'_N"] & \iota(B) \end{tikzcd} .$$

[27] Which is the same as requiring that $\overline{\rho} : B \to \rho^* Y$ is isomorphism of relative vector bundles.

We define $u' := \widetilde{\rho}^* u$.

- Calling $\phi(n) = (F \circ \pi_N(n), \varphi(n))$, we set $\phi'(n') := (g \circ \pi'_N(n'), \varphi(\widetilde{\rho}(n')))$ and we call U' the image of ϕ'. Then the pair (U', ϕ') is a tubular neighbourhood of $\iota'(B) \in A \times \mathbb{R}^n$.

Summing up, we have the following commutative diagram

$$
\begin{tikzcd}
A \times \mathbb{R}^n \arrow[rrr, bend left, "\rho\times \mathrm{id}_{\mathbb{R}^n}"] \arrow[ddddr, bend right, "\mathrm{pr}_A"'] & U' \arrow[l, hook', "\subseteq"] \arrow[r, "\rho\times \mathrm{id}_{\mathbb{R}^n}"] & U \arrow[r, hook, "\subseteq"'] & X \times \mathbb{R}^n \arrow[ddddl, bend left, "\mathrm{pr}_X"] \\
 & N(\iota'(B)) \arrow[u, "\phi'"] \arrow[r, "\widetilde{\rho}"] \arrow[d, "\pi'_N"] & N(\iota(Y)) \arrow[u, "\phi"'] \arrow[d, "\pi_N"] & \\
 & \iota'(B) \arrow[uul, hook, bend left, "\subseteq"] \arrow[r, "\rho_\iota"] & \iota(Y) \arrow[uur, hook, bend right, "\subseteq"'] & \\
 & B \arrow[u, "\iota'"] \arrow[r, "\overline{\rho}"] \arrow[d, "f"] & Y \arrow[u, "\iota"'] \arrow[d, "F'"] & \\
 & A \arrow[r, "\rho"] & X &
\end{tikzcd}
$$

Next, we define an integral $\int_{\mathbb{R}^n}^v : h_v^\bullet(\mathrm{id}_{\mathbb{R}^n} \times \rho) \to h^{\bullet - n}(\rho)$, analogous to the one we had in (1.7.1), as the composition of the map

$$
\int_{\mathbb{R}}^v := \int_{S^1} \circ (\mathrm{id}_X \times j, \mathrm{id}_A \times j)_* \tag{1.18}
$$

iterated n times. Now, we define the Umkehr map for (F, f), where F and f have compact fibers, using the same definition given in (1.17) and just replacing the Thom isomorphism by its relative version. To get further than this, we need to know how to deal with things like $h_c(X, A)$. We will come back to this problem in Section 4.6.

1.8 Conclusion

In this chapter we have presented the definitions on the topological side. We conclude by drawing attention to the nonexistence of a compact supported relative Thom isomorphism as well as the absence of a relative umkehr map in the relative compact case since we do not yet have a proper definition of relative cohomology with compact supports. Both of these problems will be solved in the second part of this work as a step to to the proof of the existence of the differential versions of these results.

It is important to keep track of the umkehr maps which we have defined so far: Table 1 summarizes it all.

Umkher \ Type	Absolute	Relative
Compact Fiber	✓	✓
Compact	✓	
Vertical	✓	

Table 1 – Umkehr maps in cohomology. The ✓ denotes the existence of the umkehr map.

2 de Rham Cohomology

2.1 Introduction

This chapter presents both a review of de Rham cohomology, in the usual and the relative setting, as well as some new framework to interpret some well stabilised results. Besides, as we shall see in the next chapter, de Rham cohomology will also work as a beacon to guide our choice of definitions as well as our path to the objectives.

The material for the first part is standard and does not differ in any radical way from of that present in (BOTT; TU, 1982; GREUB; HALPERIN; VANSTONE, 1972; NICOLAESCU, 2020; MELO, 2019), with special attention given to the first three. From the first one, we lend the notion of relative forms, on the second, we base the fibered integration and on the third, we generalize integration to a variation of the ∂-bundle concept.

2.2 Manifolds and Differential Forms

We denote the category of C^∞-manifolds (possibly with boundary) and smooth maps between them by `Man`. We assume that the empty set $\emptyset$ is a manifold, but a troublesome one, as it is highly pathological by nature[1]. This forces us to treat it separately to conform to our needs. Also, for each manifold X we write $\varnothing_X : \emptyset \to X$ for the unique *smooth* map from $\emptyset$ to X as was done in the topological spaces. Moreover, when we say X is a manifold it is assumed to be non-empty unless stated otherwise.

We denote by $\pi_{TX} : TX \to X$ the tangent bundle of X, which is a smooth vector bundle in the sense that π is a *smooth* map. Rather than writing TX_x for the fiber over $x \in X$, we have kept the standard notation T_xX, but we still use TX_A for the restriction over a subset $A \subseteq X$, somewhat paradoxically. Given a smooth map $f : X \to Y$, we denote by $df : TX \to TY$ its derivative and by $d_xf : T_xX \to T_{f(x)}Y$ its fiber restriction.

The standard constructions of smooth vector bundles (HUSEMöLLER, 1994, Ch 5, sec 6, p.67),(RUFFINO, 2020, sec 7.6, p.172) enable us to construct the cotangent bundle $\pi_{T^*X} : T^*X \to X$ as well as its n-exterior power of bundle $\pi_{\bigwedge^n T^*X} : \bigwedge^n T^*X \to X$. A smooth section $\omega \in \Gamma(\bigwedge^n T^*X)$ is called a *differential form* of degree n, or n-form for short. The set of n-forms over X will be denoted by $\Omega^n(X)$ and we will denote the degree of a form ω by $|\omega|$. In the case of the empty manifold, we refrain ourselves from giving a

[1] For example, the dimension of this manifold is not well defined since $X \times \emptyset = \emptyset$.

proper definition of the tangent bundle, but we define $\Omega^n(\emptyset) = 0$ for every n. Note that $\Omega^0(X)$ can be identified with $C^\infty(X)$, the set of real valued smooth functions over X.

The set $\Omega^n(X)$ has the structure of a $C^\infty(X)$-module with the sum defined fiberwise. The *wedge product* of $\omega \in \Omega^p(X)$ and $\omega' \in \Omega^q(X)$ is a form $\omega \wedge \omega' \in \Omega^{p+q}(X)$ given by

$$(\omega \wedge \omega')\,(X_1, \ldots, X_{p+q}) = \sum_{\sigma \in \Pi_{p,q}} \operatorname{sgn}(\sigma) \omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \omega'(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})$$

where $\Pi_{p,q}$ is the set of (p,q)-shuffles, *i.e.*, permutations, such that $\sigma(i) < \sigma(i+1)$, $i \in \{1, \ldots, p\}$ and $\sigma(j) < \sigma(j+1)$ for $j \in \{p+1, \ldots, q\}$, and sgn is the signal of the permutation. This product endows the $C^\infty(X)$-module $\Omega(X) := \bigoplus_{k=0}^n \Omega^k(X)$ with a $\mathbb{Z}$-graded commutative ring[2] structure as the product is compatible with the degrees and the graded anti-comutativity holds, *i.e* $\omega \wedge \omega' = (-1)^{|\omega||\omega'|} \omega' \wedge \omega$.

The *exterior derivative* is a 1-graded homomorphism of graded rings $d : \Omega(X) \to \Omega(X)$ defined as

$$\begin{aligned} d\omega(X_0, \ldots, X_n) &:= \sum_{i=0}^{n} (-1)^i X_i \Big(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_n)\Big) + \ldots \\ &\qquad \ldots + \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n), \end{aligned}$$

where $\widehat{X_i}$ denotes the omission of the term in the sequence.

The homomorphism d satisfies the following two properties:

i) $d \circ d = 0$,

ii) $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^{|\omega||\omega'|} \omega \wedge d\omega'$.

Therefore, $(\Omega, \wedge, d)$ is a differential graded ring. We shall refer to this differential graded algebra simply as the *de Rham complex* of X.

Let $f : X \to Y$ be a smooth map and $\omega \in \Omega^n(Y)$ an n-form. We define an n-form $f^*\omega \in \Omega^n(X)$ fiberwise [3] as

$$f^*\omega_x(v_1, \ldots, v_n) = \omega_{f(x)}(d_x f \cdot v_1, \ldots, d_p f \cdot v_n)$$

In fact, the *pullback* $f^* : \Omega(Y) \to \Omega(X)$ is a morphism of differential graded rings which is compatible with d and with products in the following sense:

$$f^* \circ d = d \circ f^*, \qquad f^*(\omega \wedge \omega') = f^*\omega \wedge f^*\omega'.$$

[2] see section A.4 in Appendix A for the appropriate definitions.

[3] To see this is smooth, one can use charts.

We finish this discussion by seeing the original de Rham cohomology through a functorial lens: we have a contravariant functor $\Omega : \mathtt{Man}^{\mathrm{op}} \to \mathtt{DGA}$ given by

$$\Omega(X \xrightarrow{f} Y) = \Omega(Y) \xrightarrow{f^*} \Omega(X),$$

where $\mathtt{DGA}$ denotes the category of differential graded algebras.

Remark 2.2.1. Latter we will need to use the de Rham complex with coefficients in a $\mathbb{Z}$-graded real algebra G. This functor is just $\Omega \otimes_{\mathbb{R}} G$, as discussed in Section A.4 of the appendix. For example, considering the $\mathbb{R}$-algebra $\mathfrak{K}_{\mathbb{R}} := (\mathbb{R}[t, t^{-1}], +, \cdot)$, where $|t| = -2$, we can identify

$$\Omega^n \mathfrak{K}_{\mathbb{R}} = \begin{cases} \Omega^{\mathrm{ev}}, & \text{if } n \text{ is even} \\ \Omega^{\mathrm{odd}}, & \text{if } n \text{ is odd} \end{cases}$$

where $\Omega^{\mathrm{ev}} = \bigoplus_{n \in \mathbb{N}} \Omega^{2n}$ and $\Omega^{\mathrm{odd}} = \bigoplus_{n \in \mathbb{N}} \Omega^{2n+1}$.

Remark 2.2.2. We can equivalently describe a structure which will be termed the *cross product* $\omega \times \omega' \in \Omega^{p+q}(X \times Y)$ and is defined between two forms $\omega \in \Omega^p(X)$ and $\omega' \in \Omega^q(Y)$ by

$$\omega \times \omega' := \mathrm{pr}_X^* \, \omega \wedge \mathrm{pr}_Y^* \, \omega'.$$

This product has the same properties as the wedge product, except that the commutative property has to be interpreted in the following way

$$\omega \times \omega' = (-1)^{|\omega||\omega'|} \tau^*(\omega' \times \omega),$$

where $\tau : X \times Y \to Y \times X$ is the transposition map given by $\tau(x, y) = (y, x)$, and the naturality has the following interpretation:

$$(f \times g)^*(\omega \times \omega') = f^*\omega \times g^*\omega',$$

where $f \times g : X \times Z \to Y \times W$ is the map $(f \times g)(x, z) := (f(x), g(z))$ with $f : X \to Y$ and $g : Z \to W$. This wedge product should be seen as analogue of the external (cross) product in cohomology.

Now, as in section A.4.2 of Appendix A, we can associate cohomology groups to this differential graded algebra. In order to fix notation, we say that ω is *exact* if $\omega \in \mathrm{Im}(d)$ and we say that ω is *closed* if $\omega \in \mathrm{Ker}(d)$. We denote by $\Omega_{\mathrm{ex}}(X)$ and $\Omega_{\mathrm{cl}}(X)$ the submodules of exact and *closed* forms respectively. The cohomology group associated to $\Omega(X)$ is the graded commutative ring $H_{dR}(X)$ defined as the quotient of $\frac{\Omega_{\mathrm{cl}}(X)}{\Omega_{\mathrm{ex}}(X)}$ with the natural grading which is called the *de Rham cohomology* group of X. We denote the cohomology class of a closed form $\omega \in \Omega_{\mathrm{cl}}(X)$ either by $[\omega]$ or by $q_{\mathrm{dR}}(\omega)$, where $q_{\mathrm{dR}} : \Omega_{\mathrm{cl}}(X) \to H_{\mathrm{dR}}(X)$ is the quotient map, whichever is convenient.

The commutative ring structure stems from the wedge product: given $[\omega] \in H^p_{\mathrm{dR}}(X)$ and $[\omega'] \in H^q_{\mathrm{dR}}(X)$, we define

$$[\omega] \wedge [\omega'] := [\omega \wedge \omega'],$$

which is well defined by the compatibility with d.

We also have the cross product $\times : H_{\mathrm{dR}}(X) \otimes H_{\mathrm{dR}}(Y) \to H_{dR}(X \times Y)$ defined as follows: given $[\omega] \in H^p_{\mathrm{dR}}(X)$ and $[\omega'] \in H^q_{\mathrm{dR}}(Y)$, we put

$$[\omega] \times [\omega'] := [\omega \times \omega'].$$

As before, these products are associative, commutative in the appropriated sense and natural.

Given a smooth map $f : X \to Y$, the morphism $f^* : \Omega^n(Y) \to \Omega(X)$ gives rise, by the compatibility with d, to a morphism $f^* : H_{dR}(Y) \to H_{dR}(X)$.

Summarizing, the de Rham cohomology groups define a contravariant functor $H_{\mathrm{dR}} : \mathtt{Man}^{\mathrm{op}} \to \mathtt{GrRing}$, where $\mathtt{GrRing}$ is the category of $\mathbb{Z}$-graded commutative rings, given by

$$H_{\mathrm{dR}}(X \xrightarrow{f} Y) = H_{\mathrm{dR}}(Y) \xrightarrow{f^*} H_{\mathrm{dR}}(X).$$

2.2.1 Compactly and vertically compact supported differential forms

The *support* of an n-form $\omega \in \Omega^n(X)$ is the set

$$\mathrm{supp}(\omega) = \overline{\{x \in X : \omega_x \neq 0\}}.$$

Definition 2.2.3 (Differential form with compact support)**.** We say that a differential form $\omega \in \Omega(X)$ has *compact support* if $\mathrm{supp}(\omega)$ is compact.

We denote the set of compactly-supported forms over X by $\Omega_c(X)$ and remark that that this is indeed a differential graded ideal of $\Omega(X)$[4].

It is not the case that pullbacks of compactly-supported smooth forms are always compactly-supported. For example, for an appropriate smooth bump function $f : \mathbb{R} \to \mathbb{R}$ supported on $(-2, 2)$, the support of its restriction to $(-1, 1)$ can be the whole interval, which is not compact. But if we consider a proper smooth map $f : X \to Y$, the pullback $f^* : \Omega_c(Y) \to \Omega_c(X)$ will be well defined.

In this text, a *proper map* is a continuous map $f : X \to Y$ such that $f^{-1}(K)$ is compact for every $K \subseteq Y$ compact. If f is proper and ω has compact support, then $f^*\omega$ has compact support. Indeed, since $f^{-1}(\mathrm{supp}(\omega))$ is closed, $\mathrm{supp}(f^*\omega) \subseteq f^{-1}(\mathrm{supp}(\omega))$ and X is compact, it follows that $\mathrm{supp}(f^*\omega)$ is compact.

It is also important to mention that if $i : X \hookrightarrow Y$ is a open smooth embedding, we can define a "wrong direction map" $i_* : \Omega_c(X) \to \Omega_c(Y)$ as $i_*\omega = \widetilde{\omega}$, where $\widetilde{\omega}$ is the ι-extension by zero.

From this we conclude that Ω_c is

[4] It is not actually a subring since the unit $1 \in \Omega^0(X)$ may not have compact support.

- a contravariant functor on the subcategory[5] of proper maps of `Man`, and
- a (covariant) functor on the subcategory of open smooth embeddings.

The *de Rham cohomology with compact supports* is defined as the cohomology of the cochain complex $(\Omega_c(X), d)$ and denoted accordingly by $H_{\mathrm{dR},c}(X)$. This cohomology $H_{\mathrm{dR},c}$ defines a functor in the same lines as the functor Ω_c: contravariant on proper maps and covariant on open embeddings.

Computing the cohomology with compact supports of $\mathbb{R}^n$ gives us

$$H^k_{dR,c}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n \\ 0, & k \neq n. \end{cases}$$

For a proof, see (BOTT; TU, 1982, Corollary 4.7.1, p.39), (NICOLAESCU, 2020, Theorem 7.2.1,p.247) or (MELO, 2019, Prop. 10.9).

A generator of $H^n_{\mathrm{dR},c}$ can be identified with any form

$$\omega_x = f(x) dx^1 \wedge \cdots \wedge dx^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a *bump function of unitary mass*, *i.e*, a smooth function with compact support which integrates to 1.

Definition 2.2.4. Let $f : Y \to X$ be any smooth map. We say that a differential form $\omega \in \Omega(Y)$ has *vertically compact support with respect to* p , if for every compact set $K \in X$ the set $\mathrm{supp}(\omega) \cap p^{-1}(K)$ is compact.

Remark 2.2.5. Vertically compact forms are sometimes called *fiberwise compact forms*, which can be misleading. This name should be taken with care, since it is possible for a form to have compact support on each fiber without being vertically compact supported, as graphically illustrated in Figure 7.

The set of vertically compact forms is a sub-differential graded ring of $(\Omega(X), d)$, which we denote by $\Omega_v(X)$. In order to see the ring structure, observe that we the product of two vertically compact forms $\omega \in \Omega^p_v(Y)$ and $\omega' \in \Omega^q_v(Y)$ is a vertically compact supported form $\omega \wedge \omega' \in \Omega^{p+q}_v(Y)$. Indeed, since $\mathrm{supp}(\omega \wedge \omega') \subseteq \mathrm{supp}(\omega') \cup \mathrm{supp}(\omega)$, intersecting both sides with $p^{-1}(K)$, where $K \in \mathcal{K}(X)$ gives us the result.

These classes also have a mixed type product. Let $f : Y \to X$ be a smooth fiber bundle and consider forms $\omega \in \Omega_c(X)$ and form $\omega' \in \Omega_v(E)$. Then $\omega'' := \omega' \wedge f^*\omega$ has compact supports, that is, $\omega'' \in \Omega_c(E)$.

Given an open embedding $i : Z \hookrightarrow Y$ of smooth fiber bundles $f : Y \to X$ and $f' : Z \to X$ over X, we can define a pushfoward map $i_* : h_v(Z) \to h_v(Y)$ by extending the form by zero in $i(Z)^c$.

[5] The identity is a proper map and the composition of proper maps is proper.

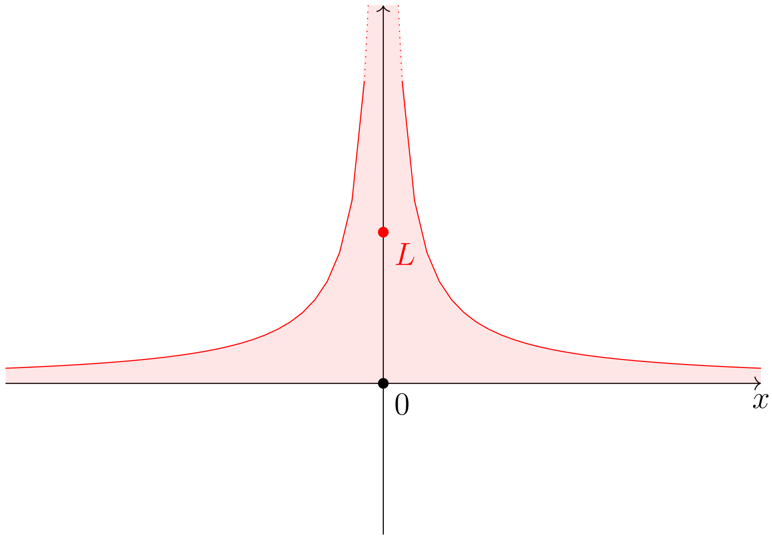

Figure 7 – The red set has compact fibers but clearly is not vertically compact.

2.2.2 Fibered Calculus

The framework in which we are going to carry integration is the one of fibered manifolds. Unfortunately, we could not find a definition of fibered manifolds with boundary in the literature[6]. We adapted the definition of ∂-bundles described in (NICOLAESCU, 2020, Definition 3.4.50, p. 136).

Definition 2.2.6 (Fibered Manifolds)**.** Let Y be an m-manifold and X be an n-manifold. We say that $f : Y \to X$ is a (m, n)-*fibered manifold* if

1. f is a surjective submersion;
2. when $\partial Y \neq \emptyset$, then $\partial p : \partial Y \to X$ is also a surjective submersion, where $\partial f = f|_{\partial Y}$.

In this case, the fibers Y_x are manifolds of dimension $m - n$. We remark that a fibered manifold is a concept slightly more general than that of a ∂-bundle. In particular, in the case without boundary, the Erehsmann's fibration theorem (DUNDAS, 2018, Section 8.5, p.182) states that, for proper maps, these concepts are the same.

We shall denote by $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}$ the semi-space in $\mathbb{R}^n$.

Definition 2.2.7 (Convenient Charts)**.** Let $f : Y \to X$ be a fibered manifold. We say that a pair of local charts $(\phi : U \to \mathbb{R}^m \times \mathbb{H}^{n-m}, \psi : V \to \mathbb{R}^m)$ around $(y, p(y))$ is convenient if

$$\psi \circ p \circ \phi^{-1} = \mathrm{pr}_m{}^{7},$$

where $\mathrm{pr}_m : \mathbb{R}^m \times \mathbb{H}^{n-m} \to \mathbb{R}^m$ is the projection in the first m-coordinates.

[6] Although we were able to find a comment in the following link <https://mathoverflow.net/questions/83013/fibred-manifolds-with-boundary>

[7] Domain restrictions implicitly understood.

A fibered manifold admits convenient charts at each point. In the case without boundary this is just the local normal form of submersions.

Two convenient charts (ϕ, ψ) and (ϕ', ψ') are said to be *orientation preserving* if the charts (ϕ, ϕ'), (ψ, ψ') preserve orientation. We say that a fibered manifold is orientable if it admits an atlas of orientation preserving convenient charts. In particular, this implies that X, B, and each fiber X_b are all orientable[8].

Definition 2.2.8 (Integration Along Fibers)**.** Let $f : Y \to X$ be a fibered manifold of dimension (m, n) and $k = m - n$ be the fiber dimension. We define

$$\int_f : \Omega_v^\bullet(M) \to \Omega^{\bullet - k}(B)$$

as

$$\left(\int_p \omega\right)_b (v_1, \dots, v_p) = \int_{Y_b} \widetilde{\omega}(v_1, \dots, v_{p-k}),$$

where

$$\widetilde{\omega}_x(v_1, \dots, v_{p-k})(u_1, \dots, u_k) = \omega_x(\widetilde{v}_1, \dots, \widetilde{v}_{p-k}, u_1, \dots, u_k)$$

is a differential form in Y_b and $\widetilde{v}$ is a vertical lift of v at x, *i.e*, $d_x f(\widetilde{v}) = v$.

We need to verify that:

- $\widetilde{\omega}(v_1, \dots, v_{p-k})$ does not depends on the lifts.
- $\int_f \omega$ is smooth.

In order to see that $\widetilde{\omega}(v_1, \dots, v_{p-k})$ is a well-defined differential form on Y_b, notice that $\widetilde{\omega}(v_1, \dots, v_{p-k})$ is a top form on Y_b, and thus it can only be non null at $x \in Y_b$ if it is applied on a basis $u_1, \dots, u_k$ of $T_x Y_b$. Therefore, we may suppose that $\{u_1, \dots, u_k\}$ is a basis of $T_x Y_b$. Now, given two vertical lifts $\widetilde{v}$ and $\widetilde{v}'$ of v, note that $\widetilde{v} - \widetilde{v}' \in \ker d_x f = T_x V_b$, which entails

$$\begin{aligned} \omega(\widetilde{v}, \widetilde{v}_2, \dots, \widetilde{v}_{n-k}, u_1, \dots, u_k) &= \omega(\widetilde{v}' + (\widetilde{v} - \widetilde{v}'), \widetilde{v}_2, \dots, \widetilde{v}_{n-k}, u_1, \dots, u_k) \\ &= \omega(\widetilde{v}', \widetilde{v}_2, \dots, \widetilde{v}_{n-k}, u_1, \dots, u_k), \end{aligned}$$

since $\widetilde{v} - \widetilde{v}'$ is a linear combination of vectors of the basis $\{u_1, \dots, u_k\}$. From the fact that we can extend $\widetilde{v}_1, \dots, \widetilde{v}_{n-k}$ to locally vertical fields it follows that $\widetilde{\omega}$ is smooth.

Next, in order to show that $\int_f \omega$ is smooth, it is enough to work in a convenient chart, where f, M and B become $f = \mathrm{pr}_{\mathbb{R}^m}$, $M = \mathbb{R}^m \times \mathbb{H}^k$ and $B = \mathbb{R}^m$. In this setting, we have a p-form $\omega \in \Omega_v^p(\mathbb{R}^m \times \mathbb{H}^k)$, which can be written as the sum of decomposable forms of the following type:

$$\omega_{(x,y)} = a(x, y) dx_I \wedge dy_J,$$

8 In (ABRAHAM; MARSDEN; RATIU, 1988, p. 472), the authors give another alternative for orientation.

where $|I| + |J| = p$ and $\mathrm{supp}(f) \in \mathbb{R}^m \times [-K, K]$.

But we only need to concentrate on the case $|I| = p - k$ and $|J| = k$ because

$$\widetilde{\omega}_{(b,y)}\left(e_{m+1}, \ldots, e_{m+k}\right) = 0$$

if $|J| < k$ (which happens due to the existence of a term $dx_i(e^j) = 0$ with $i \in I$ and $j \in J$). In this case, one has

$$\left(\int_f \omega\right)_b = \left(\int a(b, y) dy_J\right) dx_I,$$

which is smooth, since $b \mapsto \int a(b, y) dy_J$ is smooth.

Remark 2.2.9. The orientability condition is required to ensure that integration along the fiber is possible and that the obtained forms "glue" adequately.

The integration map has the following properties:

Proposition 2.2.10. *: Let $f : Y \to X$ be an (n, m)-fibered manifold, $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(N)$. The following properties hold:*

i) (Homotopy Formula)

$$\int_{\partial f} \omega = d_B \int_f \omega - \int_f d_M \omega. \tag{2.1}$$

ii) (Projection formula)

$$\int_f (\omega \wedge f^* \eta) = \int_f \omega \wedge \eta \tag{2.2}$$

iii) (Functoriality) Assume $g : Z \to Y$ is another fibered manifold without boundary. Then

$$\int_{f \circ g} = \int_f \circ \int_g \tag{2.3}$$

iv) (Stability) Let $f' : B \to A$ be another fibered manifold and $(\overline{\rho}, \rho)$ be a pair of smooth maps $\rho : A \to X$ and $\overline{\rho} : B \to Y$ such that $\rho \circ f' = p \circ \overline{\rho}$ and $\overline{\rho}$ is a fiberwise diffeomorphism, that is, $\overline{\rho}|_a : B_a \to Y_{\rho(a)}$ is a diffeomorphism. Then

$$\int_{f'} \overline{\rho}^* \omega = \rho^* \int_f \omega. \tag{2.4}$$

For a proof, see subsection A.3.2 in the Appendix A. As a direct consequence of the homotopy formula in the case of maps without boundary, the integration map descends to cohomology:

Corollary 2.2.11. *Let $f : X \to B$ be (n, m)-fibered manifold without boundary. The map $\int_f : \Omega^\bullet(X) \to \Omega^{\bullet-(n-m)}(B)$ induces a homomorphism $f_! : H^\bullet_{dR}(X) \to H^{\bullet-(n-m)}_{dR}(B)$ of degree $(m - n)$.*

Proof. For fibered manifolds without boundary, the homotopy formula (2.1) becomes

$$\int_f d_X\omega = d_B \int_f \omega,$$

which shows the required compatibility. $\square$

With the aid of the (absolute) integration along fibers[9], we construct a differential S^1-integration at the level of differential forms.

Consider the functor $S : \mathtt{Man} \to \mathtt{Man}$ given by

$$S(X \xrightarrow{f} Y) = X \times S^1 \xrightarrow{f\times \mathrm{id}_{S^1}} Y \times S^1$$

and let $S\Omega := \Omega \circ S$. We define the S^1 integration as the natural transformation $\int_{S^1} : S\Omega \to \Omega$ given by

$$\left(\int_{S^1}\right)_X = \int_{\mathrm{pr}_X}$$

at each X, where the integration is with respect to the projection $\mathrm{pr}_X : SX \to X$. Let $t : S^1 \to S^1$ be the conjugation map $t(z) = \overline{z}$.

Proposition 2.2.12. *S^1-integration has the following two properties:*

i) $(\int_{S^1})_X \circ \mathrm{pr}_X^* = 0$

ii) $\int_{S^1} \circ (id_X \times t)^* = -\int_{S^1}$

Proof. i) By the projection formula, one has

$$\int_{S^1} \mathrm{pr}_X^*\,\omega = \int_{\mathrm{pr}_X} 1 \wedge \mathrm{pr}_X^*\,\omega = \left(\int_{\mathrm{pr}_X} 1\right)\omega = 0,$$

since $\int_{\mathrm{pr}_X} 1 = 0$ for dimensional reasons.

ii) Note that

$$(\mathrm{id}_X \times t)^*(\omega)_{(s,x)}(v_1,\dots,v_n,u) = \omega_{(\overline{u},x)}(v_1,\dots,v_n,dt^*u),$$

where we use that $T(X \times S^1) = TX \oplus TS^1$, with $v_1,\dots,v_n \in TX$ and $u \in TS^1$.

In each fiber, one has

$$\widetilde{\omega}_{\overline{s}} = t^*\widetilde{\omega}_s,$$

where $\widetilde{\omega}$ is the form which appears in the definition of the integration along fibers. But the map t only inverts orientation.

$\square$

9 We have not described the absolute case of S^1 integration, but it can be seen as a particular instance of the relative one.

Remark 2.2.13. This map induces a natural transformation at the de Rham cohomology level which is closely related to the suspension isomorphism in usual topological cohomology. In fact, the following short exact sequence splits:

$$\begin{tikzcd} 0 \arrow[r] & H^{\bullet-1}_{\mathrm{dR}}(X) \arrow[r, "\times[dt]"] & H^{\bullet}_{\mathrm{dR}}(X \times S^1) \arrow[r, "i_1"] \arrow[l, bend left, "\int_{S^1}"] & H^{\bullet}_{\mathrm{dR}}(X) \arrow[r] & 0, \end{tikzcd}$$

where $i_1 : X \hookrightarrow X \times S^1$ is the inclusion at the slice $X \times \{1\}$, and the suspension isomorphism can be interpreted as the decomposition

$$H^{\bullet}(X \times S^1) \cong H^{\bullet-1}(X) \oplus H^{\bullet}(X).$$

In particular, any closed differential form $\omega \in \Omega^{\bullet}(X \times S^1)$ can be written as

$$\omega = \omega_1 \times dt + \mathrm{pr}_X^* \omega_2 + d\nu,$$

where $\omega_1 \in \Omega^{\bullet-1}(X)$ and $\omega_2 \in \Omega^{\bullet}(X)$, for some form $\nu \in \Omega^{\bullet-1}(X \times S^1)$.

2.3 Relative differential forms over smooth maps

In the last section, we have defined forms over manifolds. Now we wish to define forms over smooth maps. Denote by $\mathtt{Man}^2$ the arrow category of $\mathtt{Man}$. Its objects are smooth maps and the morphisms $(f, g) : \rho \to \eta$ between $\rho : A \to X$ and $\eta : B \to Y$ are pairs of smooth maps which make the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\rho"] & B \arrow[d, "\eta"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We also consider the category of pairs of manifolds $\mathtt{Man}_2$, where the objects are pairs of manifolds (X, A) in which A is a submanifold of X, and the morphisms $f : (X, A) \to (Y, B)$ are smooth maps $f : X \to Y$ such that $f(A) \subseteq B$. As in the topological case, we have the inclusions $I_2 : \mathtt{Man} \to \mathtt{Man}_2$ and $I^2 : \mathtt{Man}_2 \to \mathtt{Man}^2$ given by

$$I_2(X \xrightarrow{f} Y) = (X, \emptyset) \xrightarrow{f} (Y, \emptyset) \text{ and } I^2((X, A) \xrightarrow{f} (Y, B)) = i_A \xrightarrow{(f, f|_A)} i_B,$$

where $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow Y$ are the inclusions.

Given a smooth map $\rho : A \to X$, we define the *relative de Rham complex* of ρ, denoted by $\Omega(\rho)$, as the mapping cone complex of the morphism of cochains $\rho^* : \Omega(X) \to \Omega(A)$. More precisely:

$$\Omega^n(\rho) = \Omega^n(X) \oplus \Omega^{n-1}(A)$$

and $d^n : \Omega^n(\rho) \to \Omega^{n+1}(\rho)$ is given by

$$d(\omega, \theta) = (d_X\omega, \rho^*\omega - d_A\theta)$$

In this case, $\Omega(\rho)$ does not have a natural structure of a graded differential ring, but it has the structure of a $\Omega(X)$-graded differential (left) module. Indeed, one can define the following graded multiplicative structure:

$$\begin{aligned}\wedge : \Omega(X) \times \Omega(\rho) &\to \Omega(\rho)\\ (\omega', (\omega, \theta)) &\mapsto (\omega' \wedge \omega, \rho^*\omega' \wedge \theta),\end{aligned}$$

which is compatible with d in the following sense:

$$d_\rho(\omega' \wedge (\omega, \theta)) = d_X\omega' \wedge (\omega, \theta) + (-1)^{|\omega'|}\omega' \wedge \wedge d_\rho(\omega, \theta)'$$

We shall call the elements of $\Omega^n(\rho)$ *relative differential forms* of degree n over the map ρ or relative forms over ρ for short. For the sake of simplicity, we will write $\Omega(X)$ for $\Omega(\varnothing_X)$, where $\varnothing_X : \emptyset \to X$ is the empty map. Note that this notation is consistent, since $\Omega(\emptyset) = 0$.

Given a morphism $(f, g) : \rho \to \eta$ and a relative form $(\omega, \theta) \in \Omega(\eta)$, we have a pullback $(f, g)^* : \Omega(\eta) \to \Omega(\rho)$ defined by

$$(f, g)^*(\omega, \theta) = (f^*\omega, g^*\theta)$$

which is compatible with the differentials. This means that $\Omega : \mathtt{Man}^{2,\mathrm{op}} \to \mathtt{CoCh}$ is a (contravariant) functor, where $\mathtt{CoCh}_{\mathbb{R}}$ is the category of co-chain complex over $\mathbb{R}$.

Remark 2.3.1. We also have the relative cross product $\times : \Omega(Y) \times \Omega(\rho) \to \Omega(\mathrm{id}_Y \times \rho)$ defined as

$$\omega' \times (\omega, \theta) = (\omega' \times \omega, \omega' \times \theta),$$

which satisfies the compatibility

$$d(\omega' \times (\omega, \theta)) = d\omega' \times (\omega, \theta) + (-1)^{|\omega'|}\omega' \times d(\omega, \theta). \tag{2.5}$$

It is related to the usual product in a similar manner to the cross product in cohomology: through projections and diagonal maps.

Similarly to the absolute case, we define the $H_{dR}(X)$-graded module $H_{\mathrm{dR}}(\rho) := \frac{\Omega_{\mathrm{cl}}(\rho)}{\Omega_{\mathrm{ex}}(\rho)}$, where $\Omega_{\mathrm{cl}}(\rho) := \ker(d)$ and $\Omega_{\mathrm{ex}} := \mathrm{Im}(d)$, which we call the *relative de Rham cohomology* associated to ρ. We also get an analogous contravariant functor $H_{dR} : \mathtt{Man}^{2,\mathrm{op}} \to \mathtt{GrAb}$.

2.3.1 Parallel Relative Forms

We define the *de Rham complex of parallel*[10] forms $\Omega_{\mathrm{par}}(\rho)$ as

$$\Omega_{\mathrm{par}}(\rho) = \{\omega \in \Omega(X) : \rho^*\omega = 0\},$$

[10] This choice of name will be evident in the next chapter.

which is the kernel of $\rho^* : \Omega(X) \to \Omega(A)$.

The elements of $\Omega_{\mathrm{par}}(\rho)$ will be called *parallel differential forms*.

The complex $\Omega_{\mathrm{par}}(\rho)$ is a graded differential ideal with multiplicative structure given by $\omega \wedge \omega'$. Moreover, we have a natural chain morphism $i : \Omega_{\mathrm{par}} \to \Omega$ given at ρ by $i_\rho(\omega) = (\omega, 0)$. Therefore, we can view $\Omega_{\mathrm{par}}(\rho)$ as a sub-differential graded module of $\Omega(\rho)$, but it has more structure than Ω because it has a product.

The graded cohomology ring $H_{\mathrm{dR,par}}(\rho)$ associated to $\Omega_{\mathrm{par}}(\rho)$ will be termed *parallel de Rham cohomology* of ρ. The next proposition establishes the relation between these two cohomology theories in the case of closed embeddings [11].

Proposition 2.3.2. *If $\rho : A \hookrightarrow X$ is a closed embedding, then $H^n_{dR}(\rho)$ and $H^n_{dR,par}(\rho)$ are naturally isomorphic.*

Proof. Given a relative form $(\omega, \theta) \in \Omega(\rho)$, there exists a ρ-extension $\widetilde{\theta}$ of θ according to corollary A.3.3. Define the function $\phi : H_{dR}(\rho) \to H_{\mathrm{dR,par}}(\rho)$ as

$$\phi([\omega, \theta]) = [\omega - d\widetilde{\theta}],$$

which is well defined since $\rho^*(\omega - d\widetilde{\eta}) = \omega - d\eta = 0$, $\phi \circ d = 0$ and $[\omega - d\widetilde{\theta}]$ does not depend on the choice of the $\widetilde{\theta}$. Through some careful inspection, we can see that this function is a homomorphism. We now affirm that ϕ is an inverse of i (defined above) in cohomology. Indeed, on one side one has $\phi \circ i = id$ and on the other

$$\begin{aligned} i \circ \phi[\omega, \theta] &= [\omega - d\widetilde{\eta}, 0] \\ &= [(\omega, \theta) - d(\widetilde{\eta}, 0)] \\ &= [\omega, \theta] \end{aligned}$$

which shows they are inverse morphisms in cohomology. □

Remark 2.3.3. It is worth mentioning that these maps are not defined at the cochain level, but they are natural, that is, $i : \Omega_{\mathrm{par}} \to \Omega$ is a quasi-isomorphism.

Notice that the fact that these two cochain complex have the same cohomology does not mean they are necessarily isomorphic as complexes. This is one of the reasons we work with $\Omega(\rho)$ rather than $\Omega_{\mathrm{par}}(\rho)$, since the latter can be embedded in the former.

2.3.1.1 Relative Compact and Vertically Compact Forms over a map

We now give a relative version of the definition on section 2.2.1.

[11] By the uniqueness of de Rham theory in manifold pairs, they should always be the same for any embedding, but here we will only need the particular case.

Definition 2.3.4 (Fibered Smooth Map)**.** We say that a morphism $(F, f) : \overline{\rho} \to \rho$ between smooth maps $\rho : A \to X$ and $\rho : B \to Y$ is a *fibered smooth map* if both F and f are fibered manifolds, $\overline{\rho} : B \to Y$ is a fiber diffeomorphism and $\overline{\rho}^{-1}(\partial Y) \subseteq \partial B$.

$$\begin{tikzcd}
B_a \arrow[r, "\cong"] \arrow[d, hook] & Y_{\rho(a)} \arrow[d, hook] \\
B \arrow[r, "\overline{\rho}"] \arrow[d, "f"] & Y \arrow[d, "F"] \\
A \arrow[r, "\rho"] & X
\end{tikzcd}$$

Remark 2.3.5. In the case that both maps are smooth fiber bundles, we shall refer to the the pair (F, f) as a smooth relative fiber bundle over ρ, which is coherent with Definition 1.7.19.

Let $(F, f) : \overline{\rho} \to \rho$ be a fibered map. We say that a relative form $(\omega, \theta) \in \Omega(\rho)$ has *vertically compact support* if both ω and θ have (vertically) compact support. This is the same as requiring that the relative form is in the mapping cone complex of the morphism $\overline{\rho}^* : \Omega_v(X) \to \Omega_v(B)$, which is well defined since $\overline{\rho}$ is a fiber diffeomorphism (GREUB; HALPERIN; VANSTONE, 1972, Chapter VII, Section 4, p.295).

The set of vertically compact relative forms will be denoted by $\Omega_v(\overline{\rho})$ and is a sub graded differential module of $\Omega(\overline{\rho})$. We denote its de Rham cohomology, the relative vertical de Rham cohomology, by $H_{\mathrm{dR},v}(\overline{\rho})$.

Consider two relative bundles (F, f) and (F', f') over $\rho : A \to X$, and two open embeddings of vector bundles $i' : B' \hookrightarrow B$ and $i : Y' \hookrightarrow B$ as depicted in the following diagram:

$$\begin{tikzcd}
 & B \arrow[r, "\overline{\rho}"] \arrow[ddl, "f"'] & Y \arrow[ddl, "F"] \\
B' \arrow[ur, "i'"] \arrow[r, "\overline{\rho}'"] \arrow[d, "f'"'] & Y' \arrow[ur, "i"] \arrow[d, "F'"'] & \\
A \arrow[r, "\rho"] & X &
\end{tikzcd}$$

We define the pushfoward $(i, i')_* : \Omega_v(\overline{\rho}') \to \Omega_v(\overline{\rho})$ by extending the forms by zero.

The compact case is more restrictive. **For a proper map** $\rho : A \to X$, we define a compactly-supported relative form as a form $(\omega, \theta) \in \Omega(\rho)$ where ω and θ are compactly supported. We denote the complex of relative forms with compact supports by $\Omega_c(\rho)$. The differential d_ρ is the same, but it is only well defined for proper maps. Observe that this complex is just the cone of $\rho^* : \Omega_c(X) \to \Omega_c(A)$. If ρ and η are proper maps such that $(i, i') : \rho \to \eta$ is morphism with i and i' open embeddings, than we can define $(i, i')_* : \Omega_c(\rho) \to \Omega_c(\rho)$ which can be done by extending both form by zero.

2.3.2 Relative Fibered Calculus

As in the absolute case, we have notions of integration along fibers and the same construction can be carried on.

Definition 2.3.6 (Relative Fiber Integration)**.** The *relative fiber integral* associated to a fibered smooth map $(F, f) : \overline{\rho} \to \rho$ is the map

$$\begin{aligned}\int_{(F,f)} &: \Omega_v(\overline{\rho}) \to \Omega^\bullet(\rho)\\ &(\omega, \theta) \mapsto \left(\int_F \omega, \int_f \theta\right).\end{aligned}$$

A partial analogue of Proposition 2.2.10 holds

Proposition 2.3.7. *Let $p : Y \to X$ be an (n, m)-fibered manifold, $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(N)$. The following properties hold:*

i) (Homotopy Formula)

$$\int_{\partial(F,f)} \omega = d_\rho \int_{(F,f)} \omega - \int_{(F,f)} d_{\overline{\rho}}\omega, \tag{2.6}$$

where $\partial(F, f)$ is the fiber bundle obtained by restriction to the boundary.

ii) (Functoriality) If $(F, f) : \overline{\overline{\rho}} \to \overline{\rho}$ is another fibered manifold, then

$$\int_{(F,f)\circ(Q,q)} = \int_{(F,f)} \circ \int_{(Q,q)} \tag{2.7}$$

2.3.2.1 Relative S^1-Integration

Once more, we use relative fiber integration to define relative S^1-integration. Let $S : \mathtt{Man}^2 \to \mathtt{Man}^2$ be the functor

$$S(\rho \xrightarrow{(f,g)\eta}) = \mathrm{id}_{S^1} \times \rho \xrightarrow{(\mathrm{id}_{S^1} \times f, \mathrm{id}_{S^1} \times g)} \mathrm{id}_{S^1} \times \mathrm{id}_{S^1} \times \eta$$

and $S\Omega := \Omega \circ S$. As one can already guess by now, the relative differential integration $\int_{S^1} : S\Omega^\bullet \to \Omega^{\bullet-1}$ is just

$$\int_{S^1} (\omega, \theta) := \int_{(\mathrm{pr}_X, \mathrm{pr}_A)} (\omega, \theta)$$

and it has the usual two properties

- $\int_{S^1} \circ (\mathrm{pr}_X, \mathrm{pr}_A)^* = 0$, and
- $\int_{S^1} \circ (t \times \mathrm{id}_X, t \times \mathrm{id}_A)^* = -\int_{S^1}$.

2.3.3 Compactly like forms and differential Thom morphism

The parallel forms are closely related to forms with compact supports and vertically compacted forms. More precisely, one has the following:

Proposition 2.3.8. *For any manifold X and map $f : Y \to X$, we have*

$$\Omega_c(X) = \operatorname{colim}_{K \in \mathcal{K}(X)} \Omega_{par}(X, K^c)$$

and

$$\Omega_v(Y) = \operatorname{colim}_{V \in \mathcal{V}(f)} \Omega_{par}(X, V^c).$$

Proof. We prove only the first one, the second being analogous. Note that any $\omega \in \Omega_{\mathrm{par}}(X, K^c)$ is compactly supported, as $\omega|_{K^c} = 0$ implies that $\mathrm{supp}(\omega) \subseteq K$. Therefore, we have a map $i_K : \Omega_{\mathrm{par}}(X, K^c) \to \Omega_c(X)$, which is just the inclusion. Let's verify that $(\Omega_c(X), i_K)$ is a colimit of the directed system $(\Omega_{\mathrm{par}}(X, K^c), i_{KL})$.

These maps satisfies $i_K = i_L \circ i_{KL}$. Given homomorphisms $f_K : \Omega_{\mathrm{par}}(X, K^c) \to B$, such that $f_K = f_L \circ i_{KL}$, we define

$$\begin{aligned} f : \Omega_c(X) &\to B \\ \omega &\mapsto f_K(\omega), \text{ if } \mathrm{supp}(\omega) \subseteq K. \end{aligned}$$

This map is well-defined as a consequence of the compatibility between the morphisms. Moreover, one has $f_K = f \circ i_K$.

We claim that f is the only homomorphism with this property. Indeed, suppose that $g : \Omega_c(X) \to B$ is another homomorphism satisfying $f_K = g \circ i_K$. Given $\omega \in \Omega_c(X)$, one has $f(\omega) = f_{\mathrm{supp}(\omega)}(\omega) = g(i_{\mathrm{supp}(\omega)}(\omega)) = g(\omega)$ since $i_{\mathrm{supp}(\omega)}(\omega) = \omega$. □

2.3.3.1 Differential Thom Morphism and Thom Form

In analogy with section 1.6, we define a **cochain level** analogue of the Thom class.

Definition 2.3.9 (Thom form)**.** A *Thom form* of a vector bundle $p : E \to X$ is any form $\widehat{u} \in \Omega_{v,\mathrm{cl}}(X)$ whose fiber restriction $\widehat{u}_x \in \Omega_{c,\mathrm{cl}}(Y_x)$ is representative of a generator of $H^n_{\mathrm{dR},c}(E_x) \cong H^n_{\mathrm{dR},c}(\mathbb{R}^n) \cong \mathbb{R}$.

The Thom form is related to integration along fibres in the following essential way.

Proposition 2.3.10. *Let $p : E \to X$ be an oriented smooth vector bundle with Thom form $\widehat{u}$. Then the integration along fibres at chain level satisfies*

$$\int_p \widehat{u} = 1 \tag{2.8}$$

Moreover, any closed form with vertically compact supports $\widehat{u} \in \Omega_{v,cl}(E)$ satisfying (2.8) is a Thom form.

For the proof, we refer to (BOTT; TU, 1982, Proposition 6.17 and 6.18) (in the special case of bundles of finite type).

We also have a **cochain level** analogue of the two absolute Thom isomorphisms which we call (inconspicuously) differential Thom morphisms.

Definition 2.3.11 (Differential Thom morphism and Compact Supported differential Thom morphism)**.** The *differential Thom morphism* associated to a smooth oriented vector bundle $p : E \to X$ with Thom form $\widehat{u}$ is given the morphism

$$\begin{aligned} \widehat{T} : \Omega^{\bullet}_{\mathrm{cl}}(X) &\to \Omega^{\bullet+n}_{v,\mathrm{cl}}(E) \\ \omega &\mapsto \widehat{u} \wedge p^*\omega \end{aligned}$$

and the *compactly supported differential Thom morphism*

$$\begin{aligned} \widehat{T}_c : \Omega^{\bullet}_{c,\mathrm{cl}}(X) &\to \Omega^{\bullet+n}_{c,\mathrm{cl}}(E) \\ \omega &\mapsto \widehat{u} \wedge p^*\omega \end{aligned}$$

Observe that these maps may fail to be isomorphisms in general. But they are always injective, with left inverse given by the integration along fibers. Indeed, in both case we have

$$\begin{aligned} \int_p \circ \widehat{T}(\omega) &= \int_p (\widehat{u} \wedge p^*\omega) \\ &\overset{2.2}{=} \int_p \widehat{u} \wedge \omega \\ &\overset{2.8}{=} 1 \wedge \omega \\ &= \omega. \end{aligned}$$

Remark 2.3.12. To grasp why these map may fail to be surjective, see the discussion on injectivity of the integration along fibers and the proof of the Thom isomorphism in (NICOLAESCU, 2020, Proposition 7.3.32 and Theorem 7.3.34, p.285-287).

Remark 2.3.13. Lets see how we can endow the product bundle $\mathrm{pr}_X : \mathbb{R}^n \times X \to X$ with a Thom form. Consider a smooth embedding $j : \mathbb{R} \to S^1$ of $\mathbb{R}$ of its one point compactification given, say, the inverse of the stereographic projection. We define a Thom form of pr_X as $\widehat{u} := \mathrm{pr}_R^* j^* dt$, where $dt \in S^1$ is the standard volume form: $\int_{S^1} dt = 1$. Lets verify this is a Thom form. A direct computation, give us

$$\int_{\mathrm{pr}_X} \mathrm{pr}_{\mathbb{R}}^* j^* dt = \int_{S^1} dt = 1$$

this follows since $i_x^* pr_{\mathbb{R}} = j^*\omega$. By Proposition 2.3.10, this is a Thom form.

As in the topological case, we can consider the doubly-compact Thom morphism, in order to do so, we introduce the doubly compact forms. Given two maps $f : Y \to X$ and

$g : Z \to Y$ we define $\Omega_{vv}(Y)$ as the forms $\omega \in \Omega(Z)$ such that both $\mathrm{supp}(\omega) \cap (f \circ g)^{-1}(K)$ and $\mathrm{supp}(\omega) \cap f^{-1}(K')$ are compact for $K \in \mathcal{K}(X)$ and $\mathcal{K}'(Y)$.

Let $p : E \to Y$ be a vector bundle and $f : X \to Y$ be fiber bundle such that $p \circ f$ is a vector bundle. The Thom morphism $\widehat{T}_v : \Omega_v^\bullet(X) \to \Omega_{vv}^{\bullet+n}(E)$ is define in the same way, *i.e*

$$\widehat{T}_v(\omega) = \widehat{u} \wedge \pi^*\omega$$

where $\widehat{u} \in \Omega^n(E)$ is a Thom form of p.

2.3.4 Relative Vertically Compact Forms and Relative Thom Morphism

In the topological case, we have defined the vertically compact cohomology of a fiber bundle over ρ, $(F, f) : \overline{\rho} \to \rho$, as $\mathrm{colim}\, h(Y, j(\rho^*(Y)) \cup V^c)$. In view of this, we would like to define $\Omega_v(\overline{\rho})$ as $\mathrm{colim}_{V \in \mathcal{V}}\, \Omega_{\mathrm{par}}(Y, j(\rho)^*(Y) \cup V^c)$, but it is possible that $j(\rho^*(Y)) \cup K^c$ is not a manifold even if $j(\rho^*(Y))$ is a manifold. But it is natural to ask if our vertically compact forms coincide with this whenever we are in the situation in which $j(\rho^*(Y))$ is open. Turns out that this is indeed the case by an argument analogous to previous one.

Remark 2.3.14. Given $\omega' \in \Omega_v(Y)$ and $(\omega, \theta) \in \Omega(Y)$, we have $\omega' \wedge (\omega, \theta) \in \Omega_v(\overline{\rho})$. For a proper map $\rho : A \to X$, we have a product $\omega' \wedge (F, f)^*(\omega, \theta) \in \Omega_c(\overline{\rho})$ if $\omega' \in \Omega_v(Y)$ and $(\omega, \theta) \in \Omega_c(\rho)$.

Definition 2.3.15. Consider a relative vector bundle $(F, f) : \overline{\rho} \to \rho$. We define the relative Thom morphism as

$$\begin{aligned} \widehat{T} : \Omega(\rho) &\to \Omega_v(\overline{\rho}) \\ (\omega, \theta) &\mapsto \widehat{u} \wedge (F, f)^*(\omega, \theta) \end{aligned}$$

We have a analogous definition for the compact supported relative Thom morphism, provided that $\rho : A \to X$ is proper and we always have the doubly compacted Thom morphism.

2.4 Differential umkehr maps

2.4.1 Absolute umkehr maps

Now that we have a Thom morphism, we discuss the *differential umkehr maps*. The really peculiar fact is that this is just the integration along fibers. In order to define the differential umkehr map, we start by mimicking the definition of representative of h-orientation (Definition 1.7.2).

Definition 2.4.1 (Representative of a Differential Orientation of a map)**.** A de Rham *differential representative of an orientation* $(\iota, \widehat{u}, \phi)$ of a a neat map $f : Y \to X$ is given by the following data:

- a neat embedding $\iota : Y \to X \times \mathbb{R}^N$, for any $N \in \mathbb{N}$, such that $\pi_X \circ \iota = f$.
- a Thom form $\widehat{u}$ on the normal bundle $N(\iota(Y))$.
- a tubular neighbourhood of $\iota(Y)$ in $X \times \mathbb{R}^n$ given by diffeomorphism $\phi : N(\iota(Y)) \to U$ an open set.

Definition 2.4.2. A *homotopy between* two representatives $(\iota, \widehat{u}, \phi)$ and $(\iota', \widehat{u}', \phi')$ of an differential orientation of $f : Y \to X$ is a representative $(J, \widehat{U}, \Phi)$ of a differential orientation of $id_I \times f : I \times Y \to I \times X$, such that:

- $(J, \widehat{U}, \Phi)$ is proper over a neighborhood $V \subseteq I$ of $\{0, 1\}$;
- $(J, \widehat{U}, \Phi)|_{f_0} = (\iota, \widehat{u}, \phi)$ and $(J, \widehat{U}, \Phi)|_{f_0} = (\iota', \widehat{u}', \phi')$.

Definition 2.4.3 (Stabilization)**.** Let us consider a representative $(\iota, \widehat{u}, \phi)$ of a differential oriented map $f : Y \to X$ with $\iota : Y \to X \times \mathbb{R}^N$. A representative $(\iota', \widehat{u}', \phi')$ is said to be *equivalent* to (ι, u, ϕ) *by stabilization* if

- For any $L \in \mathbb{N}$, $\iota' : Y \to X \times \mathbb{R}^{N+L}$ is given by $\iota'(y) := (\iota(y), 0)$.
- Observe that $N(\iota'(Y)) = N(\iota(Y)) \oplus (\iota(Y) \times \mathbb{R}^L)$, where $\mathrm{pr}_{\iota(Y)} : \iota(Y) \times \mathbb{R}^L \to X$, is the product bundle. We put the canonical Thom form of remark on $\mathrm{pr}_{\iota(Y)} : \iota(Y) \times \mathbb{R}^L \to X$ and $\widehat{u}'$ on $N(\iota'(Y))$ is obtained using proposition 3.6.3
- For $v \in N_{\iota(Y)}$ and $w \in \mathbb{R}^L$ we have $\phi'(v, w) = (\phi(v), w) \in X \times \mathbb{R}^{N+L}$

At last we define the differential orientation

Definition 2.4.4 (de Rham differential orientation)**.** A *de Rham differential orientation* of a map $f : Y \to X$ is an equivalence class of representatives of a de Rham orientation under the equivalence relation generated by homotopy and stabilization.

Now we can define the our first differential umkehr map

Definition 2.4.5 (Differential umkehr map)**.** We define the *differential umkehr map* of a differential oriented smooth bundle with compact fibers $f : Y \to X$ between manifolds of dimension n and m, as the composition

$$
\begin{aligned}
\widehat{f}_! : \Omega^{\bullet}_{\mathrm{cl}}(Y) &\to \Omega^{\bullet-(n-m)}_{cl}(X) \\
\omega &\mapsto \int_{\mathrm{pr}_X} i_* \circ \left(\phi^{-1}\right)^* (\widehat{T}(\omega))
\end{aligned} \tag{2.9}
$$

where $(\iota, \widehat{u}, \phi)$ is representative of the differential orientation of f.

As in the topological case, this map only depends on the orientation class but now it depends on f and not only on its homotopy class.

Droping the assumptions on compact fibers, we got two other maps:

- The vertical differential umkehr map $\widehat{f}_! : \Omega_v^\bullet(Y) \to \Omega_{cl}^{\bullet-(n-m)}(X)$;
- The compact differential umkehr map $\widehat{f}_! : \Omega_{c,\mathrm{cl}}(Y) \to \Omega_{c,\mathrm{cl}}^{\bullet-(n-m)}(X)$, defined for any differential oriented fibered manifold[12].

which are defined exactly as (2.9) we just exchange the Thom morphism for the

- doubly-vertical Thom morphism in the vertical case, and
- the for the compact Thom morphism in the compact case.

Remark 2.4.6 (Relation between the integrations). The differential umkehr with compact fibers is clearly a particular case of the vertical differential umkehr map since the bundle can have compact fibers. In the setting of the compact differential umkehr, where the map $f : Y \to X$ is not necessarily a fiber bundle (it is only required to be a submersion), when X and Y are compact, f is proper, hence it is a fiber bundle by Ehresman's fibration theorem (DUNDAS, 2018, Section 8.5, p.182). Hence, in both cases we get compact differential umkehr map as a particular case, but in the latter setting the whole fiber bundle is compact, not only each fiber.

It turns out that these maps are nothing really new.

Proposition 2.4.7. *Let $f : Y \to X$ be a fibered manifold. Both the compact and vertical differential umkehr maps are equivalent to the integration along fibers.*

Proof. We prove only the vertical case, the other one being entirely analogous. Since a fibered manifold is by definition a submersion, any orientation has a proper differential representative $(\iota, \widehat{u}, \phi)$ (RUFFINO, 2017, Lemma 3.23). As long as the representative of

[12] We can drop the assumption on local triviality here, because the pushfoward always make sense.

the orientation is proper, it follows that $\mathrm{pr}_X|_U \circ \phi = \mathrm{pr}_X|_{\iota(Y)} \circ \pi_N$, one gets

$$\begin{aligned}
\int_{\mathrm{pr}_X} \iota_*(\phi^{-1})^* \left(\widehat{u} \wedge \pi_N^* \left(\left(i^{-1}\right)^* \omega\right)\right) &\overset{(1)}{=} \int_{\mathrm{pr}_X|_U} (\phi^{-1})^*(\widehat{u} \wedge \pi_N^* \left(\left(i^{-1}\right)^* \omega\right) \\
&= \int_{\mathrm{pr}_X|_{\iota(Y)} \circ \pi_N \circ \phi^{-1}} (\phi^{-1})^*(\widehat{u} \wedge \pi_N(\iota^{-1})^*(\omega)) \\
&\overset{(2.3)}{=} \int_{\mathrm{pr}_X} \circ \int_{\pi_N \circ \phi} (\phi^{-1})^* \left(\widehat{u} \wedge \pi_N^* \left(\left(i^{-1}\right)\right)^*(\omega)\right)\right) \\
&\overset{(2.4)}{=} \int_{\mathrm{pr}_X} \circ \int_{\pi_N \circ \phi} \widehat{u} \wedge \pi_N^* \left(\left(i^{-1}\right)\right)^*(\omega)\right) \\
&\overset{(2.2)}{=} \int_{\mathrm{pr}_X} \circ \int_{\pi_N} (u \cdot \pi_N^*(\omega)) \\
&\overset{(2.2)}{=} \int_{\mathrm{pr}_X} \iota^{-1}(\alpha)(\omega) \\
&\overset{(2.4)}{=} \mathrm{id}_X^* \int_f \omega = \int_f \omega
\end{aligned}$$

□

where this last equivalence was used in the following diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\iota} & \iota(Y) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{pr}_X} \\
X & \xrightarrow{\mathrm{id}_X} & X
\end{array}$$

In view of this proposition, we will just stick to the usual nomenclature in the field as in (HOPKINS; SINGER, 2005) and just call the differential umkehr maps *differential integration maps*.

2.4.2 Relative Differential Umkehr Maps

Recall Definition 1.7.19 of relative vector bundle. Given a relative smooth bundle over $\rho : A \to X$, $(F, f) : \overline{\rho} \to \rho$ where $\overline{\rho} : C \to Y$, and a differential orientation $[\iota, \widehat{u}, \phi]$ of $F : Y \to X$, we can induce a differential orientation $[\iota', \widehat{u}', \phi']$ of $f : B \to A$ in the exact same way as described in exact same way as in Section 1.7.2.

Definition 2.4.8 (Relative differential integration for compact fibers)**.** Let $\rho : A \to X$ be a proper map and $(F, f) : \overline{\rho} \to \rho$ a fiber bundle over ρ with compact fibers[13]. The relative differential integration is the map $(F, f)_! : \Omega^\bullet_{\mathrm{cl}}(\overline{\rho}) \to \Omega^{\bullet - n}_{\mathrm{cl}}(\rho)$

$$\begin{aligned}
\widehat{(F, f)}_! : \Omega_{\mathrm{cl}}(\overline{\rho}) &\to \Omega_{\mathrm{cl}}(\overline{\rho}) \\
(\omega, \theta) &\mapsto \int_{(\mathrm{pr}_X, \mathrm{pr}_A)} \kappa_*(\phi^{-1}, \phi'^{-1})^* \widehat{T}_N((\omega, \theta))
\end{aligned}$$

where we are omitting the (ι, ι'). Here $T_N(\omega, \eta) = \widehat{u} \cdot (\pi_N, \pi_{N'})^*(\omega, \theta) = (\widehat{u} \cdot \pi_N^* \omega, \widehat{u}' \cdot \pi_N'^* \theta)$.

[13] Both F and f have compact fibers.

The compact and the vertical relative integration maps are defined accordingly

Definition 2.4.9 (Relative differential integration with compact supports)**.** Let $\rho : A \to X$ be a proper map and $(F, f) : \overline{\rho} \to \rho$ a fibered map. The relative differential integration is the map $(F, f)_! : \Omega^{\bullet}_{c,\mathrm{cl}}(\overline{\rho}) \to \Omega^{\bullet - n}_{c,\mathrm{cl}}(\rho)$

$$\begin{aligned}\widehat{(F, f)}_{c!} : \Omega_{c,\mathrm{cl}}(\overline{\rho}) &\to \Omega_{c,\mathrm{cl}}(\overline{\rho}) \\ (\omega, \theta) &\mapsto \int_{(\mathrm{pr}_X, \mathrm{pr}_A)} \iota_*(\phi^{-1}, \phi'^{-1})^* \widehat{T}_{cN}((\omega, \theta))\end{aligned}$$

where $\iota_* : \Omega_c(\rho' \times \mathrm{id}_{\mathbb{R}^L}) \to \Omega_c(\rho \times \mathrm{id}_{\mathbb{R}^L})$ is the morphism discussed presented at end of Section 2.3.1.1.

Definition 2.4.10 (Relative differential integration with vertically-compact supports)**.** Let $\rho : A \to X$ be a proper map and $(F, f) : \overline{\rho} \to \rho$ a fiber bundle over ρ. The relative differential integration is the map $(F, f)_! : \Omega^{\bullet}_{c,\mathrm{cl}}(\overline{\rho}) \to \Omega^{\bullet - n}_{c,\mathrm{cl}}(\rho)$

$$\begin{aligned}\widehat{(F, f)}_{v!} : \Omega_{v,\mathrm{cl}}(\overline{\rho}) &\to \Omega_{v,\mathrm{cl}}(\overline{\rho}) \\ (\omega, \theta) &\mapsto \int_{(\mathrm{pr}_X, \mathrm{pr}_A)} \iota * (\phi^{-1}, \phi'^{-1})^* \widehat{T}_{vN}((\omega, \theta))\end{aligned}$$

where $\iota_* : \Omega_v(\rho' \times \mathrm{id}_{\mathbb{R}^L}) \to \Omega_v(\rho \times \mathrm{id}_{\mathbb{R}^L})$ is the morphism discussed at end of Section 2.3.1.1.

2.5 de Rham Cohomology

In this section we verify that the de Rham cohomology is a topological cohomology in the sense of Definition 1.3.1. More precisely, we have the following theorem

Theorem 2.5.1. *The de Rham cohomology functor $H_{dR} : \mathtt{Man}_2 \to \mathtt{GrAb}$ factors through the homotopical category of $\mathtt{Man}_2$ and satisfies*

Long Exact Sequence *For each smooth map $\rho : A \to X$, there exists a natural morphism $\partial : H^{\bullet}_{dR}(A) \to H^{\bullet+1}_{dR}(\rho)$ such that the following sequence is exact:*

$$\cdots \longrightarrow H^{\bullet}_{dR}(\rho) \xrightarrow{(\mathrm{id}_X, \varnothing_A)^*} H^{\bullet}_{dR}(X) \xrightarrow{\rho^*} H^{\bullet}_{dR}(A) \xrightarrow{\partial} H^{\bullet+1}_{dR}(\rho) \longrightarrow \cdots$$

Excision *Given some open set*[14] *$U \subseteq A$ and an smooth embedding $\rho : A \hookrightarrow X$ then $(i_{X \setminus \rho(U)}, i_{A \setminus U})^* : H_{dR}(\rho) \to H_{dR}(\rho|_{A \setminus A})$ is an isomorphism where*

$$\begin{tikzcd} A \setminus U \arrow[r, hook, "i_{A\setminus U}"] \arrow[d, hook, "\rho_{A\setminus U}"] & A \arrow[d, hook, "\rho"] \\ X \setminus \rho(U) \arrow[r, hook, "i_{X\setminus\rho(U)}"] & X \end{tikzcd}$$

[14] It is important to note that in the general case we do not U to be open, nevertheless this hypothesis was needed here.

Additivity *Given some family of smooth maps $\{\rho_\lambda\}_{\lambda\in\Lambda}$, let $(i_\lambda, j_\lambda) : \rho_\lambda \to \bigsqcup_{\lambda\in\Lambda}\rho_\lambda$ be natural inclusion. The group $(h(\bigsqcup_{\lambda\in\Lambda}\rho_\lambda), (i_\lambda, j_\lambda)^*_{\lambda\in\Lambda})$ is the directed productof the groups $h(\rho_\lambda)$.*

We prove each statement of the theorem separately.

2.5.1 Homotopy Invariance

Lemma 2.5.2. *Consider the fibrered smooth map $(\pi_X, \pi_A) : \mathrm{id}_I \times \rho \to \rho$. Give some relative form $(\omega, \theta) \in \Omega(\rho)$ one has*

$$(\iota_{0,X}, \iota_{0,A})^*(\omega,\theta) - (\iota_{0,X}, \iota_{0,A})^*(\omega,\theta) = d\int_{(\pi_X,\pi_A)}(\omega,\theta) - \int_{(\pi_X,\pi_A)} d(\omega,\theta)$$

Proof. This is just the homotopy formula (2.6) of Proposition (2.3.7) applied in this particular case bearing in mind that

$$d\int_{(\mathrm{pr}_X,\mathrm{pr}_A)}(\omega,\theta) = \int_{(\mathrm{pr}_X,\mathrm{pr}_A)} d(\omega,\theta) + \int_{\partial(\mathrm{pr}_X,\mathrm{pr}_A)}(\omega,\theta)$$

and noticing that

$$\begin{aligned}\int_{\partial(\mathrm{pr}_X,\mathrm{pr}_A)}(\omega,\theta) = \left(\int_{\partial\,\mathrm{pr}_X}\omega, \int_{\partial\,\mathrm{pr}_A}\theta\right) &= (i^*_{1,X}\omega - i^*_{0,X}\omega, i^*_{1,A}\theta - i^*_{0,A}\theta)\\ &= (i_{1,X}, i_{1,A})^*(\omega,\theta) - (i_{0,X}, i_{0,A})^*(\omega,\theta)\end{aligned}$$

□

Proposition 2.5.3 (Homotopy Invariance)**.** *If $(f_0, g_0) : \rho \to \eta$ and $(f_1, g_1) : \rho \to \eta$ are homotopic, then $(f_0, g_0)^* = (f_1, g_1)^*$*

Proof. Let $(F, G) : \mathrm{id}_I \times \rho \to \eta$ be a smooth homotopy between[15] between (f_0, g_0) e (f_1, g_1). Using the previous lemma, the homomorphism $H : \Omega^\bullet(\eta) \to \Omega^\bullet(\rho)$ defined by $H := \int_{(\pi_X,\pi_A)} \circ (F, G)^*$ is a cochain homotopy between $(f_0, g_0)^*$ e $(f_1, f_1)^*$, that is,

$$(f_1, g_1)^*(\omega,\theta) - (f_1, g_1)^*(\omega,\theta) = dH(\omega,\theta) - H(d(\omega,\theta))$$

This implies the equality of the morphisms in cohomology. Indeed, if $(\omega, \eta) \in \Omega_{\mathrm{cl}}(\rho)$, it follows that $d(\omega,\theta) = 0$, which in turns implies that $(f_1, g_1)^*(\omega,\theta) - (f_1, g_1)^*(\omega,\theta)$ is exact. □

[15] Recall that we can replace a continuous homotopy by a smooth homotopy.

2.5.2 Long Exact Sequence

In the Appendix A, we observed that to the mapping cone of a morphism of co-chain complexes is associated a long exact sequence (see Example A.4.6). Since $\Omega(\rho)$ is a mapping cone, we have the following exact sequence

$$\cdots \longrightarrow H^\bullet_{\mathrm{dR}}(\rho) \xrightarrow{(\mathrm{id}_X,\varnothing_A)^*} H^\bullet_{\mathrm{dR}}(X) \xrightarrow{\rho^*} H^\bullet_{\mathrm{dR}}(A) \xrightarrow{\partial} H^{\bullet-1}_{\mathrm{dR}}(\rho) \longrightarrow \cdots \tag{2.10}$$

The morphism ∂ is just the morphism induced by $i'_A : \Omega^{\bullet-1}(\varnothing_A) \to \Omega^\bullet(\rho)$ defined by

$$i'_A(\omega, 0) = (0, -\omega).$$

2.5.3 Excision

Proposition 2.5.4 (Excision)**.** *Let $\rho : A \hookrightarrow X$ be a smooth embedding and let $U \subseteq A$ be an open subset of A such that $\overline{U} \subseteq \mathrm{int}(\rho(A))$ and both $A \setminus U$ and $X \setminus \rho(U)$ are smooth manifolds. The morphism $(i_{X\setminus\rho(U)}, i_{A\setminus U}) : \rho_{A\setminus U} \to \rho$ given in the following diagram*

$$\begin{tikzcd} A \setminus U \arrow[r, hook, "i_{A\setminus U}"] \arrow[d, hook, "\rho_{A\setminus U}"] & A \arrow[d, hook, "\rho"] \\ X \setminus \rho(U) \arrow[r, hook, "i_{X\setminus\rho(U)}"] & X \end{tikzcd}$$

induces isomorphism in co-homology, i.e., $(i_{X\setminus\rho(U)}, i_{A\setminus U})^* : H_{dR}(\rho) \to H_{dR}(\rho_{A\setminus U})$ *is an isomorphism.*

Proof. Without loss of generality, we can assume that ρ is just an inclusion $j_A : A \hookrightarrow X$, in other words we consider the following diagram

$$\begin{tikzcd} A \setminus U \arrow[r, hook, "i_{A\setminus U}"] \arrow[d, hook, "j_{A\setminus U}"] & A \arrow[d, hook, "j_A"] \\ X \setminus U \arrow[r, hook, "i_{X\setminus U}"] & X \end{tikzcd}$$

(**Surjetivity**): Let $(\omega, \theta) \in \Omega^n_{\mathrm{dR,cl}}(j_{A\setminus U})$. First observe that θ is defined in $A \setminus U$ which is closed in A. By Corollary A.3.3, we can extend the form $\theta \in \Omega^{n-1}(A \setminus U)$ to a form $\widetilde{\theta} \in \Omega^{n-1}(A)$. Since $\overline{U} \subseteq \mathrm{int}(A)$ it follows that $\mathrm{int}(A)\setminus\overline{U}$ is a manifold and the same applies to $X \setminus \overline{U}$. By continuity of (ω, θ) it is enough to consider their restriction to $\Omega(j_{\mathrm{int}(A)\setminus\overline{U}})$ which we will still denote by (ω, θ) by a slight abuse of notation. We would like find a form $\widetilde{\omega} \in \Omega^n(X)$ such that $(\widetilde{\omega}, \widetilde{\theta}) \in \Omega^*_{cl}(j_A)$ $\widetilde{\omega}|_{X\setminus\overline{U}} = \omega$ and $\widetilde{\theta}|_{\mathrm{int}(A)\setminus\overline{U}} = \theta$.

Since $\{\mathrm{int}(A), X \setminus \overline{U}\}$ is an open cover of X, we can take a partition of unity subordinate to it

$$\varphi_{\mathrm{int}(A)} + \varphi_{X\setminus\overline{U}} = 1$$

and define

$$\widetilde{\omega}_x := (1 - \varphi_{\mathrm{int}(A)}(x)) \cdot \omega_x - \varphi_{X \setminus \overline{U}}(x) \cdot d(\widetilde{\theta}|_A)_x$$

The pair $(\widetilde{\omega}, \widetilde{\theta})$ is a closed relative form and $(i_{X \setminus \rho(U)}, i_U)^*(\widetilde{\omega}, \widetilde{\theta}) = (\omega, \theta)$ as required.

(**Injectivity**). Given $(\omega, \theta) \in \Omega^n_{\mathrm{cl}}(\rho)$ such that $\left(\omega|_{X \setminus \rho(\overline{U})}, \theta|_{\mathrm{int}(A) \setminus \overline{U}}\right) = d(\mu, \nu)$, we wish to show that there exists $(\widetilde{\mu}, \widetilde{\nu}) \in \Omega^{n-1}(\rho)$ such that $(\omega, \theta) = d(\widetilde{\mu}, \widetilde{\nu})$

$$\omega = d\widetilde{\mu} \tag{2.11}$$

$$\theta = \rho^* \widetilde{\mu} - d\widetilde{\nu} \tag{2.12}$$

First, we show (2.11) holds. By hypothesis one has

$$\omega|_{X \setminus \rho(\overline{U})} = d\mu \tag{2.13}$$

and, as (ω, θ) is closed, one gets

$$\omega|_{\rho(\mathrm{int}(A))} = \rho_* d\theta \tag{2.14}$$

where we use $\rho_* : \Omega(\mathrm{int}(A)) \to \Omega(\rho(\mathrm{int}(A)))$ the inverse of ρ^*. By (2.13) and (2.14), it follows that $\omega = d\widetilde{\mu}$ where

$$\widetilde{\mu}_x = (1 - \varphi_{\rho(\mathrm{int}(A))}(x)) \cdot \mu_x + (1 - \varphi_{X \setminus \overline{U}}(x)) \cdot (\rho_* \theta)_x$$

Now we show (2.12). By the hypothesis, we know that

$$\theta|_{\mathrm{int}(A) \setminus \overline{U}} = \rho|^*_{\mathrm{int}(A) \setminus \overline{U}} \mu - d\nu.$$

Since $(\rho|_{\mathrm{int}(U)})^* \widetilde{\mu} = \theta|_{\mathrm{int}(U}$ we get

$$\theta|_{\mathrm{int}(U} = (\rho|_{\mathrm{int}(U)})^* \widetilde{\mu}$$

we can write $\theta = \rho^* \widetilde{\mu} + d\widetilde{\nu}$ with

$$\widetilde{\nu} = \left(1 - \phi_{\mathrm{int}(U}\right) \nu$$

□

The reader maybe wondering why we have choose to prove this version of excision rather than the other one which we used as definition in the previous chapter. Perhaps not surprisingly, the answer is that we only were able to prove excision in this context and with the additional hypothesis that U is open A.

In the next chapter, we will also need a stronger version of excision which applies to the functor Ω_{par}. More precisely, we have

Proposition 2.5.5 (Excision for parallel forms)**.** *Under the same hypothesis on the maps of Proposition 2.5.4, the map*

$$(j, j')^* : \Omega_{par,cl}(X, A) \to \Omega_{par,cl}(X \setminus U, A \setminus U)$$

is a isomorphism.

Proof. First, observe that the map is injective, since $\omega|_{X \times U} = 0$ and $\omega|_A = 0$, then $\omega_U = 0$ since $U \subseteq A$. It follows that $\omega = 0$.

Next, we see the that we can define a extesion $\widetilde{\omega}$ of $\omega \in \Omega(X \setminus U, A \setminus U)$ by putting $\widetilde{\omega}_x = \omega_x$ if $x \in X \setminus U$ and 0 otherwise. □

2.5.4 Additivity

The axiom of additivity holds for countable families, since in this aces $\Omega(\bigsqcup_{n\in\mathbb{N}} X_n) = \prod_{i\in\mathbb{N}} \Omega(X_n)$. For the uncountable case, the observe that manifolds are supposed to be second countable, which precludes their disjoint union to be a manifold which makes the axiom true.

2.5.5 Multiplicative structure

De Rham's cohomology can be endowed with a multiplicative structure through the differential graded module structure of the de Rham complex in the way remarked in section A.4.3 of the Appendix A. More precisely, given a map $\rho : A \to X$, a relative class $[\omega, \theta] \in H^q_{dR}(\rho)$ and a absolute class $[\omega'] \in H^p_{dR}(X)$, we can define a product $\cdot : H^p_{dR}(X) \times H^q_{dR}(\rho) \to H^{p+q}_{dR}(\rho)$ by

$$[\omega'] \cdot [\omega, \theta] := [\omega' \wedge \omega, \rho^*\omega' \wedge \theta]$$

This module structure is natural in the following sense: given $(f, g) : \eta \to \rho$ and $\alpha \in H_{dR}(\rho)$ and $\beta \in H_{dR}(X)$,

$$(f, g)^*(\beta \cdot \alpha) = (f, g)^*\beta \cdot g^*\alpha$$

Through the use of the cross product we can also define a product structure $\times : H_{dR}(Y) \times H_{dR}(\rho) \to H_{dR}(\mathrm{id}_Y \times \rho)$ given by

$$[\omega'] \times [\omega, \theta] := [\omega' \times \omega, \omega' \times \theta]$$

This multiplicative structure is natural in following sense: if $(f, g) : \eta \to \rho$ and $h : W \to Z$

$$(h \times f, h \times g)^*(\beta \times \alpha) = h^*\beta \times (f, g)^*\alpha$$

Remark 2.5.6. Unfortunately, this is just a partial product, not a full product defined as in 1.5.4. The reason we cannot define a full product is that the function $\rho \wedge \eta$ is not a smooth function for its domain is not a manifold in general.

2.5.6 S^1-Integration

We define the functor $S : \mathtt{Man}^2 \to \mathtt{Man}^2$ as

$$S\left(\rho \xrightarrow{(F,f)} \eta\right) = \rho \times \mathrm{id}_{S^1} \xrightarrow{\left(F\times \mathrm{id}_{S^1}, f\times \mathrm{id}_{S^1}\right)} \rho \times \mathrm{id}_{S^1} .$$

Given some map $t : S^1 \to S^1$ we define a natural transformation $t_\sharp : S \to S$ that for each $\rho : A \to X$ associates the morphism $t_{\sharp,\rho} := (\mathrm{id}_X \times t, \mathrm{id}_A \times t)$. We also introduce the notation $SF := F \circ S$ for any functor $F : \mathtt{Top} \to \mathtt{C}$ where $\mathtt{C}$ is a category.

We define the natural transformation $\int_{S^1} : SH_{\mathrm{dR}} \to H_{\mathrm{dR}}$, which will be called S^1-integration, as the fiber integral of the fibered smooth map $(\pi_X, \pi_A) : S\rho \to \rho$ as in the diagram

$$\begin{array}{ccc} A \times S^1 & \xrightarrow{\mathrm{pr}_A} & A \\ \Big\downarrow{\scriptstyle \mathrm{id}_{S^1} \times \rho} & & \Big\downarrow{\scriptstyle \rho} \\ X \times S^1 & \xrightarrow{\mathrm{pr}_X} & X \end{array}$$

More precisely

$$\int_{S^1} [\omega, \theta] := \left[\int_{(\mathrm{pr}_X, \mathrm{pr}_A)} (\omega, \eta)\right]$$

This integration has the following properties

1. $\int_{S^1} \circ \pi_X^* = 0$
2. $\int_{S^1} \circ t_\sharp = -\int_{S^1}$

where $t : S^1 \to S^1$ is given by $t(z) = \overline{z}$ which are a consequence of the same properties for the integration at cochain level.

2.5.7 Compactly like Cohomology and Thom isomorphism

As we have already mentioned, the compactly supported $H_{\mathrm{dR},c}(X)$ and vertically compacted supported de Rham cohomology $H_{\mathrm{dR},v}(X)$ are defined as the cohomology of the complex Ω_c and Ω_v in both the absolute as well as the relative case[16].

In section 2.3.3.1 we introduced the differential Thom morphism, since the integration is compatible with the exterior derivative, each one of the Thom morphims induces morphisms in cohomology, more precisell we get the

- Thom isomorphism $T : H_{\mathrm{dR}}(X) \to H_{\mathrm{dR},v}(E)$
- Compact Thom isomorphism, $T_c : H_{\mathrm{dR},c}(X) \to H_{\mathrm{dR},c}(E)$
- Doubly vertical Thom isomorphism: $T_v : H_{\mathrm{dR},v}(X) \to H_{\mathrm{dR},vv}(E)$

[16] Recall that, the relative compact version only makes sense for proper maps.

- Relative Thom isomorphism, $T : H_{\mathrm{dR}}(\rho) \to H_{\mathrm{dR},v}(\overline{\rho})$
- Compact Thom isomorphism for proper maps, $T : H_{\mathrm{dR},c}(\rho) \to H_{\mathrm{dR},c}(\overline{\rho})$

which are all isomorphism in de Rham cohomology. A proof of these results in the usual absolute case can be found in any of the references (BOTT; TU, 1982),(NICOLAESCU, 2020) or (MELO, 2019). The compact case, can be proved in a analogous way. The other two can be proved using a argument similar to the one in (AGUILAR et al., 2002).

2.5.8 Umkehr map in de Rham cohomology

2.5.8.1 Absolute case

The integration maps were defined only for closed forms, but these maps can be defined for any differential form. We will call these maps, the *curvature maps*[17]

$$R_{[\iota,\widehat{u},\phi]}(\omega) = \int_{\mathrm{pr}_X} i_*\phi^{-1}\widehat{T}(\omega) \tag{2.15}$$

where T can be either the Thom morphims, the compact Thom morphism or the doubly vertical Thom morphism.

One clearly has $d \circ R_{[\iota,\widehat{u},\phi]} = \widehat{f_!} \circ d$ in either case, which means that $\widehat{f_!}$ induces the umkehr map $f_! : H_{dR}(Y) \to H_{dR}(X)$ in the Rham cohomology. The situation is better described in the following diagram

$$\begin{tikzcd}
\frac{\Omega^{\bullet-1}(Y)}{\mathrm{Im}(d)} \arrow[r, "d"] \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] & \Omega^{\bullet}_{\mathrm{cl}}(Y) \arrow[r, "q_{\mathrm{dR}}"] \arrow[d, "\widehat{f_!}"] & H^{\bullet}_{\mathrm{dR}}(Y) \arrow[d, "f_!"] \\
\frac{\Omega^{\bullet-(n-m)-1}(X)}{\mathrm{Im}(d)} \arrow[r, "d"] & \Omega^{\bullet-(n-m)}_{\mathrm{cl}}(X) \arrow[r, "q_{\mathrm{dR}}"] & H^{\bullet-(n-m)}_{\mathrm{dR}}(X)
\end{tikzcd} \tag{2.16}$$

Observe that we have quotient out the $\mathrm{Im}(d^{\bullet-2})$, which does not affect the the map d. The choice of colors will be evident in the next chapter.

2.5.9 The relative case

The same observations apply in the relative case. We define the curvature map

$$R_{[\iota,\widehat{u},\phi]}(\omega,\theta) = \int_{\mathrm{pr}_X} (i,i')_*(\phi^{-1},\phi'^{-1})^*\widehat{T}(\omega,\theta)$$

Again we have compatibility with d (using 2.6) and have the analogous diagram

$$\begin{tikzcd}
\frac{\Omega^{\bullet-1}(\overline{\rho})}{\mathrm{Im}(d)} \arrow[r, "d"] \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] & \Omega^{\bullet}_{\mathrm{cl}}(\overline{\rho}) \arrow[r, "q_{\mathrm{dR}}"] \arrow[d, "\widehat{(F,f)_!}"] & H^{\bullet}_{\mathrm{dR}}(\overline{\rho}) \arrow[d, "(F,f)_!"] \\
\frac{\Omega^{\bullet-(n-m)-1}(X)}{\mathrm{Im}(d)} \arrow[r, "d"] & \Omega^{\bullet-(n-m)}_{\mathrm{cl}}(X) \arrow[r, "q_{\mathrm{dR}}"] & H^{\bullet-(n-m)}_{\mathrm{dR}}(\rho)
\end{tikzcd}$$

[17] It seems odd to not use the same notation, but this is done in order to maintain consistence with next chapter.

2.6 Another view on the multiplicative structure

We have introduced the module product structure over relative forms in section 2.3. But the situation is unsettling: there is no natural product between $(\omega, \theta) \in \Omega(\rho)$ and $(\omega', \theta') \in \Omega(\eta)$ which lives on $\Omega(\rho \wedge \eta)$. There are two problems here

- In general, $\rho \wedge \eta$ is not a smooth function, since its domain is a mapping cylinder;
- Even if we could define some notion of smoothness in the cylinder, how should we define $(\omega, \eta) \times (\omega', \eta')$?

This second problem is in some sense inherent to the choice of the model Ω for the de Rham cohomology though. If we work with the parallel model of de Rham cohomology, we can multiply $\omega \in \Omega_{\text{par}}(\rho)$ and $\omega' \in \Omega_{\text{par}}(\eta)$ just as $\omega \times \omega' \in \Omega(X \times Y)$ such that $(\rho^* \times \mathrm{id}_Y)(\omega \times \omega') = 0$ and $(\mathrm{id}_Y \times \eta^*)(\omega \times \omega') = 0$. This works fine as long as both ρ and η are open embeddings, since in this case $\omega \times \omega' \in \Omega_{\text{par}}(\rho \times \mathrm{id}_B \cup \mathrm{id}_A \times \eta) = \Omega_{\text{par}}(X \times Y, X \times B \cup A \times Y)$, where this last equality is to be understood through identification of A with $\rho(A)$ and similarly to η. With this product we could even improve the definition of the de Rham product presented in section 2.5.5.

But this is far from good, since we would expect the same to hold in the case in which either ρ or η is a cofibration since they are excisive. But in this case $\rho \times \mathrm{id}_B \cup \mathrm{id}_A \times \eta$ is not a smooth map, since $X \times B \cup A \times Y$ is generally not a manifold, as can be seen in the simple example in Figure 8

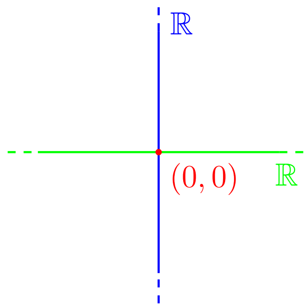

Figure 8 – The cross is the example of an union of two submanifolds of $\mathbb{R}^2$ which is not a (sub)manifold.

Nevertheless, the set $\Omega_{\text{par}}(X \times Y, X \times B, A \times Y)$, defined as

$$\Omega_{\text{par}}(X \times Y, A \times Y, X \times B) := \{\omega \in \Omega(X) : \omega|_A = 0 \text{ and } \omega|_B = 0\},$$

still make sense, regardless of $X \times B \cup A \times Y$ being a smooth manifold or not. Therefore, it would make sense to define for any pair[18] (A, B) with $A, B \subseteq X$

$$\Omega_{\text{par}}(X, A, B) := \{\omega \in \Omega(X) : \omega|_A = 0 \text{ and } \omega|_B = 0\}.$$

[18] We are mainly interested in cofibrations though.

Yet another convenient generalization is at hand: we can define a **mixed product** between $\Omega(\rho)$ and $\Omega_{\text{par}}(Y,B)$, which we call *relative-parallel product* by putting

$$\begin{aligned} \times : \Omega_{\text{par}}(Y,B) \otimes \Omega(\rho) &\to \Omega(\mathrm{id}_Y \times \rho) \\ \omega' \times (\omega, \theta) &\mapsto (\omega' \times \omega, \omega' \times \theta) \end{aligned}$$

One should observe that $\omega' \times \omega \in \Omega_{\text{par}}(Y \times X, B \times X)$ and $\omega' \times \theta \in \Omega_{\text{par}}(Y \times A, B \times A)$, which means that $(\omega' \times \omega, \omega' \times \theta)$ "lives" in the mapping cone complex of the co-chain morphism $(\mathrm{id}_Y \times \rho)^* : \Omega_{\text{par}}(Y \times X, B \times X) \to \Omega_{\text{par}}(Y \times A, B \times A)$. Put another way, what we are really considering are smooth maps between smooth pairs $\rho : (A, A') \to (X, X')$ where we are seeing $\mathrm{id}_X \times \rho : (Y \times A, B \times A) \to (Y \times X, B \times X)$ as a **map of pairs**!

Surely, for a map of pairs $\rho : (A, A') \to (X, X')$, it makes sense to talk about parallel relative forms $\Omega_{\text{par}}(\rho)$ as the forms $\omega \in \Omega_{\text{par}}(X, X')$ such that $\rho^*\omega = 0$. In this way, we can identify $\Omega_{\text{par}}(X, A, B)$ with the parallel forms $\Omega_{\text{par}}(i)$ where $i : (A, A \cap B) \to (X, B)$. Indeed, a form ω is in $\Omega_{\text{par}}(i)$ iff $\omega|_B = 0$ and $\omega|_A = 0$, therefore $\omega \in \Omega_{\text{par}}(X, A, B)$.

In order to give a clear account of these facts, lets define a new category, $\mathtt{Man}_\omega$, which is the category of finite sequences of manifolds, or more precisely, of sequences with only a finite number of non empty entries, that is, a sequence of the form $(X, X_1, \ldots, X_n, \emptyset, \ldots)$, which we write just as $(X, X_1, \ldots, X_n)$ or $(X, \vec{X})$ letting implicit the empty sets, further requiring that $X_i \subseteq X$[19] for each $i \in 1, \ldots, n$.

A morphism $\rho : (A, A_1, \ldots, A_n) \to (X, X_1, \ldots, X_m)$ is any smooth map $\rho : A \to X$ such that $\rho(A_i) \subseteq X_i$ for $i = 1, 2, \ldots$. The composition is the obvious one as well as the identity. This category comes with an associated notion of homotopy in the exact same way as in the category $\mathtt{Man}_2$. Two morphisms $f_0, f_1 : (A, A_1, \ldots, A_n) \to (X, X_1, \ldots, X_m)$ are homotopic if there exists

$$F : (I \times A, I \times A_1, \ldots, I \times A_n) \to (X, X_1, \ldots, X_m)$$

such that $F \circ i_t = f_t$, $t \in \{0, 1\}$ where i_t is the inclusion on the slice t. As in section 1.2 we define the category of maps of sequences $\mathtt{Man}^2_\omega$ as the arrow category of $\mathtt{Man}_\omega$. Its objects are maps of sequences such as $\rho : (A, \vec{A}) \to (X, \vec{X})$, which will sometimes written as $(\rho, \vec{\rho})$ in order to avoid confusion.

We define $\Omega_{\text{par}}(X, \vec{X}) := \bigcap_{i=1}^m \Omega_{\text{par}}(X, X_i)$, where $\vec{X} = (X_1, \ldots, X_m)$. In other words, a form $\omega \in \Omega_{\text{par}}(X, \vec{X})$ is a form on X such that $\omega|_{X_i} = 0$ for $i = 1, \ldots, m$. Given $\rho : (A, \vec{A}) \to (X, \vec{X})$, we define $\Omega(\rho, \vec{\rho})$ as the mapping cone complex of the morphism

$$\rho^* : \Omega_{\text{par}}(X, X_1, \ldots, X_m) \to \Omega_{\text{par}}(A, A_1, \ldots, A_n).$$

Within this setting, we can define a mixed multiplicative structure between $\Omega(\rho, \vec{\rho})$ and $\Omega_{\text{par}}(Y, \vec{B})$ which assumes values in $\Omega(\mathrm{id}_Y \times \rho)$ where

$$\mathrm{id}_Y \times \rho : (Y \times A, B \times \vec{A}, \vec{B} \times A) \to (Y \times X, B \times \vec{X}, \vec{B} \times X)$$

[19] We do note require that $X_{i+1} \subseteq X_i$.

with

$$Y \times \vec{A} := (Y \times A_1, \cdots, Y \times A_n)$$

which is just $\omega \times \omega'$.

Associated to the complex $(\Omega(\rho, \vec{\rho}), d)$ we have a de Rham cohomology which we still denote by $H_{\mathrm{dR}}(\rho, \vec{\rho})$. This cohomology satisfies the following properties

P1) *Homotopy invariance*: It is homotopy invariant with respect to homotopy in $\mathtt{Man}^2_\omega$

P2) *Long exact sequence*: For each map of tuples $\rho : (A, \vec{A}) \to (X, \vec{X})$ the sequence

$$\cdots \longrightarrow H^\bullet_{\mathrm{dR}}(\rho, \vec{\rho}) \longrightarrow H^\bullet_{\mathrm{dR}}(X, \vec{X}) \longrightarrow H^\bullet_{\mathrm{dR}}(A, \vec{A}) \xrightarrow{\partial} H^{\bullet+1}_{\mathrm{dR}}(\rho, \vec{\rho}) \longrightarrow \cdots$$

P3) *Excision*: Given a closed set $U \subseteq X$, such that $U \subseteq A_i$, for $i = 1, \ldots, n$, the inclusion map

$$i : (X \setminus U, A_1 \setminus U, \ldots, A_n \setminus U) \to (X, A_1, \ldots, A_n)$$

induces isomorphism in cohomology.

P4) *Interchangeability*: There is a natural isomorphism between $H_{\mathrm{dR}}(X, X_1, \ldots, X_n))$ and $H_{\mathrm{dR}}(X, X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for any permutation σ of $\{1, \ldots, n\}$

In fact, excision holds even in the strong sense of forms, that is,

$$(i, i')^* : \Omega_{\mathrm{par}}(X, A_1, \ldots, A_n) \to \Omega_{\mathrm{par}}(X \setminus U, A_1 \setminus U, \ldots, A_n \setminus U)$$

is a isomorphism at cochain level.

2.6.1 The relation of the extended de Rham cohomology with singular cohomology

It is natural to ask if the de Rham cohomology group $H_{\mathrm{dR}}(X, A, B)$ of the complex $(\Omega_{\mathrm{par}}(X, A, B), d)$ is isomorphic to the ordinary cohomology with real coefficients $H\mathbb{R}(X, A \cup B)$. Turns out this is indeed the case. We prove this using the smooth singular cohomology (see Appendix (B) for a review).

Proposition 2.6.1 (de Rham isomorphism). *Let A and B be submanifolds of X and $S(X, A \cup B)$ be the relative singular cochains. The map*

$$r : \Omega(X, A, B) \to S(X, A \cup B)$$
$$\omega \mapsto \left(c \mapsto \int_c \omega\right)$$

is a isomorphism.

The proof uses the following Mayer-Vietores type sequence:

Proposition 2.6.2. *Given two transverse submanifolds A and B of X. The following sequence is exact*

$$0 \longrightarrow \Omega(X,A,B) \xrightarrow{i^*} \Omega(X,A\cap B) \xrightarrow{j_A^*\oplus j_B^*} \Omega(A,A\cap B)\oplus\Omega(B,A\cap B) \longrightarrow 0$$

Proof. Exactness at $\Omega(X,A,B)$ is clear. It is also clear that $j_A^*\oplus j_B^*\circ i^*=0$. Since the kernel of $j_A^*\oplus j_B^*$ are forms in X which are null both in A and B, the exactness at $\Omega(X,A\cap B)$ is clear.

To verify that $j_A^*\oplus j_B^*$ is surjective, we start with forms $\omega_A\in\Omega(A)$ and $\omega'\in\Omega(B)$ such that $\omega_A=\omega_B=0$ in $\Omega(A\cap B)$. We can find open tubular neighbourhoods U_A and U_B of A and B which are transverse. The sets A and B are close in U_A and U_B, so the set $A\cup B$ is closed in $U_A\times U_B$. The form ω which coincides with ω_A in A and with ω_B in B can be extended to a form on $U_A\cup U_B$ and thus to the whole space X. □

Proof of Proposition. First we get rid of the hypothesis of transversality of A and B. Suppose A is not transverse to B, we can find a homotopy $F: I\times A\to X$ starting from $i: A\hookrightarrow X$ such that F_t is transverse to B and obviously homotopic to A.

So we can assume A and B to be transverse. Applying the zig-zag lemma (Proposition A.4.4) on the sequence (2.6.2). Give us the long exact sequence

$$\cdots\longrightarrow H_{\mathrm{dR}}(A,A\cap B)\oplus H(B,A\cap B)\longrightarrow H_{\mathrm{dR}}(X,A,B)\longrightarrow H_{\mathrm{dR}}(X,A\cap B)\longrightarrow\dots$$

Applying the five lemma (Proposition A.4.1) using this sequence and the morphism r, which is a isomorphism, we conclude the proof. □

2.6.2 Revisiting the Thom morphism and the integration maps

In the previous section, we have considered special models of the de Rham cohomology with compact supports and with vertically compact supports both in absolute as well as in the relative case. Now we show how to do define the Thom isomorphism without resorting to those special models: we rewrite those maps using only the definition based on colimits.

In order to define the compactly suported Thom morphism for a differentially oriented vector bundle $p: E\to X$ with Thom class $\hat{u}\in\Omega_v(E)$ we need to make sense of $\alpha\wedge p^*\beta'$ where $\alpha\in\Omega_v(E)$ and $\beta\in\Omega_c(X)$. Lets see how this can be accomplished. If $\alpha=[\omega_V]$ and $\beta=[\omega'_K]$ with $\omega_V\in\Omega_{\mathrm{par}}(E,V^c)$ and $\omega'_K\in\Omega_{\mathrm{par}}(X,K^c)$, with $V\in\mathcal{V}(p)$ and $K\in\mathcal{K}(X)$. Using the product defined above, we get

$$\omega_V\times p^*\omega_K\in\Omega(\mathrm{id}_E\times i_{p^{-1}(K^c)})$$

where $\mathrm{id}_E \times i_{p^{-1}(K^c)} : (E \times p^{-1}(K^c), V^c \times p^{-1}(K)^c) \to (E \times E, V^c \times E)$ and $i_{p^{-1}(K^c)} : p^{-1}(K^c) \hookrightarrow E$. Considering the diagonal maps $(\Delta, \Delta|_{p^{-1}(K^c)}) : i_{V^c} \to \mathrm{id}_E \times i_{p^{-1}(K^c)}$ as in the diagram

$$\begin{tikzcd}
(p^{-1}(K^c), V^c \cap p^{-1}(X)^c) \arrow[r, "\Delta|_{p^{-1}(K^c)}"] \arrow[d, hook, "i_V"] & (E \times p^{-1}(K^c), V^c \times p^{-1}(K)^c) \arrow[d, hook, "\mathrm{id}_E \times i_{p^{-1}(K^c)}"] \\
(E, V^c) \arrow[r, "\Delta"] & (E \times E, V^c \times E)
\end{tikzcd}$$

By pulling along $(\Delta, \Delta|_{p^{-1}(K^c)})$ we get $(\Delta, \Delta|_{V^c})^*(\omega_V \times p^*\omega_K) \in \Omega_{\mathrm{par}}(i_V)$ where

$$i_V : (p^{-1}(K^c), V^c \cap p^{-1}(K^c)) \hookrightarrow (E, V^c)$$

We have the following equality

$$\Omega_{\mathrm{par}}(i_V) \cong \Omega_{\mathrm{par}}(E, (V \cap p^{-1}(K))^c).$$

Note that $V \cap p^{-1}(K)$ is compact. Now we can define the compact Thom morphism by “passing" the colimit as usual.

$$\widehat{T}(\alpha) = \widehat{u} \wedge p^*\widehat{\alpha}.$$

In a complete analogous way, we obtain the doubly compact Thom morphism. And the differential umkehr maps follows immediately once we have the Thom morphisms.

Now, we deal with the relative Thom morphism. Consider a vector bundle map $(P, p) : \overline{\rho} \to \rho$ as in the diagram

$$\begin{tikzcd}
F \arrow[r, "\overline{\rho}"] \arrow[d, "p"] & E \arrow[d, "P"] \\
A \arrow[r, "\rho"] & X
\end{tikzcd} \tag{2.17}$$

When presented the model $\Omega_v(\rho)$ we commented that they were obtained from the cone of the map $\rho^* : \Omega_v(X) \to \Omega_v(A)$. It view of this fact it seems reasonable to give the following definition:

Definition 2.6.3 (Compactly supported and Vertically Compacted Supported relative For)**.** We define the *vertically compactly supported relative form* of the bundle over ρ in (2.17) as the colimit

$$\Omega_v(\overline{\rho}) = \mathrm{colim}_{V \in \mathcal{V}(P)} \Omega(\rho : (F, \overline{\rho}^{-1}(V^c)) \to (E, V^c))$$

and for, provided that, ρ is proper, we define the *compactly supported relative form* of ρ as the colimit

$$\Omega_c(\overline{\rho}) = \mathrm{colim}_{K \in \mathcal{K}(X)} \Omega(\rho : (F, \rho^{-1}(K^c)) \to (E, K^c)).$$

Now, we will see how we can define the relative Thom morphism. We do this by defining a product of a form $\omega' \in \Omega(\overline{\rho})$ with a class $\omega \in \Omega_v(E)$ with values in $\Omega_v(\overline{\rho})$ in the following way: given a representative $\omega_V \in \Omega_{\text{par}}(E, V^c)$ we have $\omega_V \times \omega' \in \Omega(\text{id}_E \times \overline{\rho} : (E \times F, V^c \times F) \hookrightarrow (E \times E, V^c \times E))$. Pulling along the diagonal (Δ, Δ_F) : as in the diagram

$$\begin{tikzcd} (F, \overline{\rho}^{-1}(V^c)) \arrow[r, "\Delta_F"] \arrow[d, "\overline{\rho}"] & (E \times F, V^c \times F) \arrow[d, "\text{id}_E \times \overline{\rho}"] \\ (E, V^c) \arrow[r, "\Delta"] & (E \times E, V^c \times E) \end{tikzcd}$$

where $\Delta_F(x) := (\overline{\rho}(x), x)$, give us

$$(\Delta, \Delta_F)^*(\omega_V \times \omega) \in \Omega(\overline{\rho} : (F, \rho^{-1}(V^c)) \hookrightarrow (E, V^c)).$$

By taking the colimt over $\mathcal{V}(P)$, we define a product $\cdot : \Omega_v(E) \times \Omega(\overline{\rho}) \to \Omega_v(\overline{\rho})$ which we use to define the Thom morphism

$$\widehat{T}(\omega) = \widehat{u} \cdot (P, p)^* \omega.$$

The same argument give us a compact version of the Thom morphism provided that ρ is proper. The integration maps goes exactly the same way without any changes.

2.7 Conclusion

In this chapter we have presented some flavours of the differential umkehr maps. These are listed in the Table 2 bellow This was first accomplished using particular models

Umkher \ Type	Absolute	Relative
Compact Fiber	✓	✓
Compact	✓	✓
Vertical	✓	✓

Table 2 – Integration maps in de Rham cohomology. The ✓ denotes the existence of the differential umkehr map.

of compactly supported, vertically compact supported, relative vertically compacted supported, and relative compactly supported cohomologies, namely, $\Omega_c(X)$,$\Omega_v(Y)$, $\Omega_v(\rho)$, and $\Omega_c(\rho)$ (ρ proper) respectively. At the end, we discussed how to deal with these models entirely in the framework of compact and vertically compact using parallel relative de Rham complexes rather than using special models.

In the next chapter, we will see that the closed differential forms are a special case of a differential cohomology. We will use the ideas presented here to state the main problem of this work, the existence of integration maps in a differential cohomology, as well as to devise a strategy to solve it.

3 Basic Concepts of Relative Differential Cohomology

3.1 Introduction

In this chapter, we introduce the basic concepts of *relative differential cohomology* as introduced in (RUFFINO; BARRIGA, 2021). We start by giving as a motivating example a geometric model which refines the second degree ordinary cohomology with integer coefficients in the relative scenario. After this, we introduce relative differential cohomology theory, briefly recall some standard facts and present three structures which are related to the main goal of this work:

- Differential S^1- Integration
- Multiplicative structures;
- Differential integration maps[1];

We end this chapter by stating the main goal of this work as well as presenting the plan we have conceived to achieve it.

3.2 A Motivating Example of Differential Refinement

In order to motivate the concept of relative differential cohomology, we start by refining the second degree ordinary cohomology with integer coefficients, *i.e.* $H\mathbb{Z}^2$, in a "more or less" geometric way. We do this by using the concept of a relative line bundle as in (SHAHBAZI, 2004, Def. 3.2.2, p. 24).

Consider a smooth function $\rho : A \to X$. A sectioned hermitian line bundle with connection over ρ, which we will call *relative line bundle*, is given by the following data:

- a smooth complex line bundle $p : L \to X$ (which we denote simply by L);
- an hermitian structure h over L;
- a connection ∇ compatible with the hermitian metric;
- an unitary local section $s : A \to \rho^* L$, where $\rho^* L$ is endowed with the induced metric $\rho^* h$.

[1] or differential umkehr map

We will denote a sectioned hermitian line bundle with connection by (L, h, ∇, s). For the sake of convenience, we briefly recall the meaning of these terms. The full picture can be seen, for example, in (HUYBRECHTS, 2005, Chapter 4) or (BRYLINSKI, 1993, Chapter 2) in the absolute case.

1. An hermitian structure on $p : L \to X$ is a choice of an hermitian product in each fiber $L_x = p^{-1}(x)$ of L that is "smooth" in the following sense: for each pair of local smooth sections $s, t : U \to L$, where $U \subseteq X$, the function $h(s,t) : U \to \mathbb{C}$ is smooth.

2. A connection is a $\mathbb{C}$-linear map $\nabla : \Omega^0(X; L) \to \Omega^1(X; L)$, where $\Omega^p(X; L)$ are differential forms of degree p with values in L, such that

$$\nabla fs = f\nabla s + df \otimes \nabla s$$

   The compatibility of ∇ and h is expressed in the following equation:

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t),$$

   where d is the extension of the exterior differential to the complex case defined by thinking of $\mathbb{C}$ as $\mathbb{R}^2$, that is, $d(f + ig) = df + idg$.

3. The section s is assumed to be unitary, which means that $\rho^* h(s,s) = h(s \circ \overline{\rho}, s \circ \overline{\rho}) = 1$, where $\overline{\rho} : \rho^* L \to L$ is the morphism covering ρ. Notice that this implies that the the section is nowhere vanishing.

We will say that two relative line bundles over ρ, (L, h, ∇, s) and (L', h', ∇', s'), are *isomorphic* if there exists an isomorphism of complex line bundles $\phi : L \to L'$ over X such that $h' = h \circ (\phi \times \phi)$, $\nabla' = \nabla \circ \phi$ and $s' = \phi \circ s$.

We denote the set of relative line bundles over ρ by $\widehat{H\mathbb{Z}}^2(\rho)$ and the equivalence class of (L, h, ∇, s) by $[L, h, \nabla]$. The set $\widehat{H\mathbb{Z}}^2(\rho)$ can be made an abelian group by defining the sum as

$$[L, h, \nabla, s] + [L', h', \nabla', s'] := [L \otimes_{\mathbb{C}} L, h \otimes h', \nabla \otimes \nabla', s \otimes s'],$$

where

- $(h \otimes h')(v \otimes v', u \otimes u') := h(v,u)h'(v',u')$; and
- $(\nabla \otimes \nabla')(s \otimes s') := \nabla s \otimes s' + s \otimes \nabla s'$.

The identity is the relative line bundle $[X \times \mathbb{C}, (\langle \cdot, \cdot \rangle), d, 1]$, where

- $\mathrm{pr}_X : X \times \mathbb{C} \to X$, is the trivial line bundle;

- $\langle (x,v),(x,u)\rangle = v\overline{y}$ where $\overline{y}$ denotes the complex conjugate;
- $d : \Omega^0(X;\mathbb{C}) \to \Omega^1(X;\mathbb{C})$ is the exterior derivative extended to $\mathbb{C}$ treated as $\mathbb{R}^2$;
- and 1 is the section $x \mapsto (x,1)$.

The inverse element of $[L,h,\nabla,s]$ is $[L^*,h^*,\nabla^*,s^*]$, where

- L^* is the dual line bundle;
- h^* is the dual metric, defined as $h^*(u,v) = h(v^\sharp,u^\sharp)$, where $w^\sharp(v) = h(v,w^\sharp)$ is given by the Riesz isomorphism;
- $\nabla' s' := d\circ s' - s'\circ\nabla$ is the dual connection, which is compatible with the dual metric;
- $s^*(u) = h(u,s)$.

The curvature of ∇ will be denoted by $F^\nabla \in \Omega^2(X,\mathrm{Hom}(L,L))$ and can be readily computed by

$$F^\nabla_{U,V}s = \nabla_U\nabla_V s - \nabla_V\nabla_U s - \nabla_{[U,V]}s$$

with U and V vector fields in X and s a section of L. Since $\mathrm{Hom}(L,L)$ is trivial, we can identify F with a complex valued form. Compatibility with the metric gives us

$$\langle F^\nabla s,t\rangle = -\langle s,F^\nabla t\rangle$$

and tells us that F^∇ can be identified with a purely imaginary form.

Since the distinguished section s is a global section of the bundle ρ^*L, it is a trivial line bundle. This distinguished section establishes a particular trivialization of ρ^*L. The pullback connection in this trivialization s can be expressed $d + B^{(\nabla,s)}$. The complex form $B^{\nabla,s}$ is also purely imaginary by an argument analogous to that of the curvature. We shall call this form the *covariance*. The curvature 2-form F^∇ and the covariance 1-form $B^{\nabla,s}$ are related by

$$dB^{(\nabla,s)} = \rho^*F^\nabla. \tag{3.1}$$

Given a class $[L,h,\nabla,s]$, we define its *first relative Chern form* as

$$c(L,h,\nabla,s) = \left(\frac{i}{2\pi}F^\nabla, \frac{i}{2\pi}B^{(\nabla,s)}\right),$$

which is a ρ-relative 2-form, *i.e*, an element in the relative de Rham complex $\Omega^2(\rho)$. It is a classical result of Chern-Weil theory that $\frac{i}{2\pi}F^\nabla$ is a closed form, which, together with (3.1), shows that $c(L,h,\nabla,s)$ is a closed relative form, that is, an element of $\Omega^2_{cl}(\rho)$. After some computations, one can verify that the map $c : \widehat{H}^2(\rho) \to \Omega^2_{cl}(\rho)$ is indeed a group homomorphism.

We show that the relative de Rham class $[c(L,h,\nabla,s)] \in H^2_{dR}(\rho)$ does not depend on either ∇ or s by an argument similar to those of usual Chern-Weil theory. Indeed, given pairs (∇_0, s_0) and (∇_1, s_1), we can define a connection $\widetilde{\nabla}$ on $X \times I$ and a section $\widetilde{s}$ on $A \times I$ in such a way that they coincide with (∇_0, s_0) and (∇_1, s_1) on $X \times \{0\}, A \times \{0\}$ and $X \times \{1\}, A \times \{1\}$ respectively. We thus get the following expression:

$$c(L,\nabla_1,s_1) - c(L,\nabla_0,s_0) = d\int_{(\mathrm{pr}_X,\mathrm{pr}_A)} \left(\frac{i}{2\pi}F^{\widetilde{\nabla}}, \frac{i}{2\pi}B^{\widetilde{\nabla},\widetilde{s}}\right),$$

where $\int_{(\mathrm{pr}_X,\mathrm{pr}_A)}$ denotes fiber integration of relative differential forms as in 2.3.6, where $\mathrm{pr}_X : X \times I \to X$ and $\mathrm{pr}_A : A \times I \to A$ are the projections. Also, c is a homomorphism of abelian groups, since one has

$$F^{\nabla\otimes\nabla'} = F^{\nabla} + F^{\nabla'} \text{ and } B^{(\nabla\otimes\nabla', s\otimes s')} = B^{(\nabla,s)} + B^{(\nabla',s')}.$$

We denote by $c_1^{\mathbb{R}}(L) \in H\mathbb{R}^2(\rho)$ the real ordinary cohomology class given by the image of $[c(L,\nabla,s)]$ under de Rham's relative isomorphism $r : H_{dR} \to H\mathbb{R}$. We call this class the *first real Chern class.*

The kernel of the map $c : \widehat{H\mathbb{Z}}^2(\rho) \to \Omega^2_{cl}(\rho)$ will be denoted by $\widehat{H\mathbb{Z}}^2_{\text{flat}}(\rho)$. It can be identified with the group of *flat relative line bundles* over ρ, *i.e.*, line bundles which admit a connection and a section whose curvature and covariance are both zero. Using Cech cohomology, it is possible to identify $\widehat{H\mathbb{Z}}^2_{\text{flat}}(\rho)$ with $H\frac{\mathbb{R}}{\mathbb{Z}}^1(\rho)$ (notice the shift in the degree). We also consider the subgroup $\widehat{H\mathbb{Z}}^2_{\text{par}}(\rho)$ formed by the classes $[L,h,\nabla,s] \in \widehat{H\mathbb{Z}}^2(\rho)$ such that $c([L,h,\nabla,s]) \in \Omega_{\text{cl,par}}(\rho)$, in other words, with null covariance. This occurs whenever $\rho^*\nabla s = 0$, which shows that the the section s is *parallel* and thus the name.

Another standard fact of the absolute case which carries onto the relative one is that there exists an isomorphism between the abelian group of classes of isomorphisms of hermitian line bundles with a distinguished section (with tensor product as group operation, but without connection) over ρ and $H\mathbb{Z}^2(\rho)$. The map which realizes this isomorphism is called the *first integral relative Chern class.* A proof of this fact in the relative setting is found in (SHAHBAZI, 2004, Proposition 3.0.1, p. 26).

Given some class $[L,h,\nabla,s] \in \widehat{H\mathbb{Z}}(\rho)^1$, we can just "drop" the connection, obtaining an isomorphism class of hermitian line bundles with a distinguished section. This map is surjective, since any smooth line bundle with a metric can be endowed with a compatible connection (this can be done by using a partition of unity argument). The composition of this map, which is a group homomorphism, with the first relative integral Chern class gives us a **surjective** map:

$$c_1 : \widehat{H\mathbb{Z}}^2(\rho) \to H\mathbb{Z}^2(\rho)$$

We call this the *first relative integral Chern class* of (L,s), often suppressing both the "first" and the "integral". This relative Chern class is a lift of the relative real Chern class. More precisely, its image under inclusion in the universal coefficients theorem (which amounts to the tensorization $\otimes_{\mathbb{Z}} 1$ with $1 \in \mathbb{R}$) is precisely the relative real Chern class $c_1^{\mathbb{R}}(L)$, that is, $c_1(L) \otimes_{\mathbb{Z}} 1 = r(c_1^{\mathbb{R}}(L))$ or, schematically:

$$\begin{tikzcd} \widehat{H\mathbb{Z}}^2(\rho) \arrow[r, dashed, "c_1"] \arrow[d, "c"] & H\mathbb{Z}^2(\rho) \arrow[r, dashed] \arrow[d, "\otimes_{\mathbb{Z}} 1"] & 0 \\ \Omega^2_{\mathrm{cl}}(\rho) \arrow[r, "r^{-1} \circ q"] & H\mathbb{R}^2(\rho) & \end{tikzcd} \tag{3.2}$$

where $q : \Omega^2_{\mathrm{cl}}(\rho) \to H^2_{\mathrm{dR}}(\rho)$ is the quotient map and the dashed line is exact.

Now, consider the following map:

$$a : \frac{\Omega^1(\rho)}{\mathrm{Im}(d)} \to \widehat{H\mathbb{Z}}^2(\rho)$$
$$(\omega, \theta) \mapsto [X \times \mathbb{C}, \langle \cdot, \cdot \rangle, d - 2\pi i \omega, e^{2\pi i \theta}]$$

First, observe that this map is well-defined since, if we replace (ω, θ) by $(\omega, \theta) + d(\gamma, 0) = (\omega + d\gamma, \theta + \gamma \circ \rho)$, the new bundle $(X \times \mathbb{C}, \langle \cdot, \cdot \rangle, d + 2\pi i(\omega + d\gamma), e^{-2\pi i(\theta + \gamma \circ \rho)})$ is isomorphic to $(X \times \mathbb{C}, \langle \cdot, \cdot \rangle, d - 2\pi i \omega, e^{2\pi i \theta})$ (the isomorphism is given by $(x, z) \mapsto (x, e^{2\pi i \gamma(x)} z)$). Moreover, this map is a group homomorphism as can be readily checked using the definitions of the group structure. The curvature of the relative line bundle is $F = -2\pi i d\omega$ and the covariance is $B = -2\pi i \rho^* \omega + 2\pi i d\theta$, in other words, the relative Chern form is $d(\omega, \theta)$, that is, we have the following commutative diagram:

$$\begin{tikzcd} \frac{\Omega^1(\rho)}{\mathrm{Im}(d)} \arrow[r, dashed, "a"] \arrow[dr, "d"'] & \widehat{H\mathbb{Z}}^2(\rho) \arrow[d, "c"] \\ & \Omega^2_{\mathrm{cl}}(\rho) \end{tikzcd} \tag{3.3}$$

We call the map a, the *trivialization*, since its image is topologically trivial in the sense that its $c_1 \circ a \circ a = 0$. Actually, a little more is true - the following sequence is exact:

$$\begin{tikzcd} \frac{\Omega^1(\rho)}{\mathrm{Im}(d)} \arrow[r, dashed, "a"] & \widehat{H\mathbb{Z}}^2(\rho) \arrow[r, dashed, "c_1"] & H\mathbb{Z}^2(\rho) \arrow[r, dashed] & 0 \end{tikzcd}$$

The remarkable fact here is that this maps captures what is beyond the topological information given by the cohomology of ρ, since the topological information is trivial in this case.

We have already identified the kernels of both c and c_1, now we do the same to a. Its kernel is comprised of relative forms (ω, θ) which have *integral period*, *i.e*, for every smooth 1-relative cycle (σ, δ) (for a definition of integral over relative cycles, refer to Section A.3.3),

$$\int_{(\sigma,\delta)} (\omega, \theta) = \int_\sigma \omega + \int_\delta \theta \in \mathbb{Z}.$$

Indeed, if $[X \times \mathbb{C}, \langle \cdot, \cdot \rangle, d + 2\pi i\omega, e^{2\pi i\theta}] = [X \times \mathbb{C}, \langle , \rangle, d, 1]$, there exists an isomorphism $\phi : X \times \mathbb{C} \to X \times \mathbb{C}$ of the form $\phi(x, z) = (x, e^{2\pi i\gamma(x)}z)$, where $\gamma : X \to \mathbb{R}$, such that $\omega = d\gamma$ and $\theta + \gamma \circ \rho \in \mathbb{Z}$. This is true since

$$d(e^{2\pi i\gamma}s) = e^{2\pi i\gamma}(ds + 2\pi i\omega s) \implies d\gamma = \omega$$

and

$$1 = e^{2\pi i\gamma(\rho(x))}e^{2\pi i\theta(x)} \implies \gamma(\rho(x)) - \theta(x) \in \mathbb{Z}.$$

Therefore, for any smooth relative cycle $(\sigma, \delta) \in Z_1(\rho)$ one has

$$\begin{aligned}\int_{(\sigma,\delta)}(\omega, \theta) &= \int_\sigma \omega + \int_\delta \theta = \int_\sigma d\gamma + \int_\delta \theta \\ &= \int_{\partial\sigma} \gamma + \int_\delta \theta = \int_{\rho_*\delta} \gamma + \int_\delta \theta \\ &= \int_\delta \gamma \circ \rho + \int_\delta \theta = \int_\delta (\gamma \circ \rho + \theta) \\ &= \sum_{p \in \delta} \gamma \circ \rho(p) - \theta(p) \in \mathbb{Z}.\end{aligned}$$

Also, observe that any relative form with integral period is closed (see Proposition A.3.6).

The image of a class $\alpha \in H\mathbb{Z}^1(\rho)$ under de Rham isomorphism corresponds to the de Rham class of a differential form with integral periods. From this we conclude that the following sequence is exact:

$$H\mathbb{Z}^1(\rho) \overset{r^{-1}\circ\otimes_{\mathbb{Z}}\mathbb{R}}{\dashrightarrow} \frac{\Omega^1(\rho)}{\mathrm{Im}(d)} \overset{a}{\dashrightarrow} \widehat{H\mathbb{Z}}^2(\rho) \overset{c_1}{\dashrightarrow} H\mathbb{Z}^2(\rho) \dashrightarrow 0 \qquad (3.4)$$

where $r : H^1_{dR}(\rho) \to H\mathbb{Z}^1(\rho)$ is the de Rham natural isomorphism (observe that $H_{dR}(\rho)$ is a subgroup of $\frac{\Omega^1(\rho)}{\mathrm{Im}(d)}$).

Putting it all together leads us to the following commutative diagram, in which ρ is omitted and the dashed sequence is exact:

$$\begin{tikzcd}
 & & & & & 0 \\
 & & H\frac{\mathbb{R}}{\mathbb{Z}}^1 \arrow[rr, "-\beta"] \arrow[dr, "\phi"] & & H\mathbb{Z}^2 \arrow[ur, dashed] \arrow[dr, "\otimes_{\mathbb{Z}}\mathbb{R}"] & \\
 & H\mathbb{R}^1 \arrow[ur, "q"] \arrow[dr, "r^{-1}"] & & \widehat{H\mathbb{Z}}^2 \arrow[ur, dashed, "c_1"] \arrow[dr, "c"] & & H\mathbb{R}^2 \\
H\mathbb{Z}^1 \arrow[ur, "\otimes_{\mathbb{Z}}\mathbb{R}"] \arrow[rr, dashed, "r^{-1}\circ\otimes_{\mathbb{Z}}\mathbb{R}"] & & \frac{\Omega^1}{\mathrm{Im}(d)} \arrow[ur, dashed, "a"] \arrow[rr, "d"] & & \Omega^2_{\mathrm{cl}} \arrow[ur, "r\circ q_{\mathrm{dR}}"] &
\end{tikzcd}$$

We draw attention to the red arrows, which make up one of the two axioms of relative differential cohomology. The maps q and β (with opposite signs) are the quotient homomorphism and Bockstein (connecting) homomorphism of the sequence of groups

$$0 \longrightarrow \mathbb{Z} \xhookrightarrow{i} \mathbb{R} \xrightarrow{q} \frac{\mathbb{R}}{\mathbb{Z}} \longrightarrow 0$$

and $\phi : H\frac{\mathbb{R}}{\mathbb{Z}}^1 \to \widehat{HZ}^2$ is the composition $H\frac{\mathbb{R}}{\mathbb{Z}}^1 \xrightarrow{\cong} \widehat{HZ}^2_{\text{flat}} \hookrightarrow \widehat{HZ}^2$.

Remark 3.2.1. The morphism induced by i in the sequence above coincides with the morphism $\otimes_{\mathbb{Z}}\mathbb{R}$ which appears in the universal coefficient theorem for cohomology in this case.

We will not treat the relation with the Bockstein morphism here since it will be not used and instead refer the reader to Simons and Sullivan (2008). The hexagon centered around $\widehat{H\mathbb{Z}}^2$ is characteristic of differential cohomologies, but in our language we prefer to write it as

$$\begin{tikzcd}
 & & & & & 0 \\
 & & \widehat{H\mathbb{Z}}^2_{\text{flat}} \arrow[rr, "c_1"] \arrow[dr, hook] & & H\mathbb{Z}^2 \arrow[ur, dashed] \arrow[dr, "\otimes_{\mathbb{Z}}\mathbb{R}"] & \\
 & H\mathbb{R}^1 \arrow[ur, "a \circ r^{-1}"] \arrow[dr, hook, "r^{-1}"] & & \widehat{H\mathbb{Z}}^2 \arrow[ur, dashed, "c_1"] \arrow[dr, "c"] & & H\mathbb{R}^2 \\
H\mathbb{Z}^1 \arrow[ur, "\otimes_{\mathbb{Z}}\mathbb{R}"] \arrow[rr, dashed, "r^{-1} \circ \otimes_{\mathbb{Z}}\mathbb{R}"] & & \frac{\Omega^1}{\operatorname{Im}(d)} \arrow[ur, dashed, "a"] \arrow[rr, "d"] & & \Omega^2_{\text{cl}} \arrow[ur, "r \circ q_{\text{dR}}"] &
\end{tikzcd}$$

which follows directly from the diagram

$$\begin{tikzcd}
H\mathbb{Z}^1 \arrow[r, dashed, "r^{-1} \circ \otimes_{\mathbb{Z}}\mathbb{R}"] & \frac{\Omega^1}{\operatorname{Im}(d)} \arrow[r, dashed, "a"] \arrow[dr, "d"] & \widehat{H\mathbb{Z}}^2 \arrow[r, dashed, "c_1"] \arrow[d, "c"] & H\mathbb{Z}^2 \arrow[r, dashed] \arrow[d, "\otimes_{\mathbb{Z}}\mathbb{R}"] & 0 \\
 & & \Omega^2_{\text{cl}} \arrow[r, "r \circ q_{\text{dR}}"] & H\mathbb{R}^2 &
\end{tikzcd} \tag{3.5}$$

At last, we observe that there exists a relation between the relative and absolute cases. In order to see this, we start by defining the absolute group $\widehat{H\mathbb{Z}}^2(X)$ as $\widehat{H\mathbb{Z}}(\varnothing_X)$, where $\varnothing_X : \emptyset \to X$. The differential forms on the "manifold" $\emptyset$ are regarded as 0. It is possible to verify that the following diagram is commutative:

$$\begin{tikzcd}
\widehat{H\mathbb{Z}}^2(\rho) \arrow[r, "(id_X, \varnothing_A)^*"] \arrow[d, "\text{cov}"] & \widehat{H\mathbb{Z}}^2(X) \arrow[d, "\rho^*"] \\
\frac{\Omega^1(A)}{\operatorname{Im}(d)} \arrow[r, "a"] & \widehat{H\mathbb{Z}}^2(A)
\end{tikzcd} \tag{3.6}$$

Pedantically, the map "cov" appearing in the diagram is the composition of $\mathrm{cov} : \Omega^2_{\mathrm{cl}}(\rho) \to \Omega^1(A)$ with the quotient map from $\Omega^1(A)$ to $\frac{\Omega^1(A)}{\mathrm{Im}(d)}$.

Before proceeding, we discuss some points that were not yet addressed:

- The construction $\widehat{H\mathbb{Z}}^2 : \mathtt{Man}^2 \to \mathtt{Ab}$, where $\mathtt{Ab}$ is the category of abelian groups and their homomorphisms, is functorial: given a morphism $(f, g) : \rho \to \eta$ and a class $[L, h, \nabla, s] \in \widehat{H\mathbb{Z}}^2(\eta)$, we define the group homomorphism $(f, g)^* := \widehat{H\mathbb{Z}}^2(f, g) : \widehat{H\mathbb{Z}}^2(\eta) \to \widehat{H\mathbb{Z}}^2(\rho)$ by

$$(f, g)^*[L, h, \nabla, s] = [f^*L, f^*h, f^*\nabla, g^*s] \tag{3.7}$$

- The morphisms c, c_1, a defined as above are indeed natural transformations:
  - $c : \widehat{H\mathbb{Z}}^2 \to \Omega^2_{\mathrm{cl}}$
  - $c_1 : \widehat{H\mathbb{Z}}^2 \to H\mathbb{Z}^2$
  - $a : \frac{\Omega^1}{\mathrm{Im}(d)} \to \widehat{H\mathbb{Z}}^2$

- The natural transformation $\otimes_{\mathbb{Z}}\mathbb{R} : \widehat{H\mathbb{Z}} \to \widehat{H\mathbb{Z}}$ is completely topological and corresponds to the Chern-Dold character as described in Section D.2. We opted to highlight it since this will change according to the theory.

- If we exchange $\widehat{H\mathbb{Z}}^2$ for the group of parallel classes $\widehat{H\mathbb{Z}}^2_{\mathrm{par}}(\rho)$ and Ω^1 for Ω^1_{par} in both diagrams (3.5) and (3.6), the new diagrams also commute, but there is no guarantee of exactness in the first line of (3.5). Nevertheless, for any closed embedding of manifolds, the exactness still holds, as we shall discuss in the next section.

3.3 Relative Differential Cohomology

We will work in the category $\mathtt{Man}^2$ as described in section 2.3. Fix a cohomology theory (h, ∂) over $\mathtt{Man}^2$ and denote its coefficient group by $\mathfrak{h}$. We call $\mathfrak{h}_{\mathbb{R}} := \mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{R}$ the realification of $\mathfrak{h}$ and denote its Chern-Dold[2] character by $ch : h \to H\mathfrak{h}$. We also write $\Omega\mathfrak{h}_{\mathbb{R}} := \Omega \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$, $q_{\mathrm{dR}} : \Omega_{\mathrm{cl}}\mathfrak{h}_{\mathbb{R}} \to H_{\mathrm{dR}}\mathfrak{h}_{\mathbb{R}}$ for the quotient and $r : H_{\mathrm{dR}}\mathfrak{h}_{\mathbb{R}} \to H\mathfrak{h}_{\mathbb{R}}$ for the correspondent de Rham isomorphism.

Now, using the diagrams (3.5) and (3.6) of the the previous section, we are led to the following definition:

Definition 3.3.1 (Relative Differential Cohomology)**.** A *relative differential refinement of* (h, ∂) is a (contravariant) functor $\hat{h} : \mathtt{Man}^{2,op} \to \mathtt{GrAb}$ along with the three natural transformations:

[2] For a quick review on the Chern-Dold Character, see section D.2 in Appendix D

- $R : \widehat{h} \to \Omega_{\rm cl}\mathfrak{h}_{\mathbb{R}}$, called the *curvature*;
- $I : \widehat{h} \to h$ called the *forgetful map*;
- $a : \dfrac{\Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)} \to \widehat{h}^{\bullet}$, called the *trivialization*;

satisfying the following two axioms:

A1) The following diagram is commutative and its first line is exact:

$$\begin{tikzcd} h^{\bullet-1} \arrow[r, "r^{-1}\circ ch"] & \dfrac{\Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)} \arrow[r, "a"] \arrow[dr, "d"'] & \widehat{h}^{\bullet} \arrow[r, "I"] \arrow[d, "R"] & h^{\bullet} \arrow[r] \arrow[d, "ch"] & 0 \\ & & \Omega^{n}_{\rm cl}\mathfrak{h}_{\mathbb{R}} \arrow[r, "r\circ q_{\rm dR}"] & H\mathfrak{h}^{\bullet}_{\mathbb{R}} & \end{tikzcd} \tag{3.8}$$

A2) For each $\rho : A \to X$, the following diagram is commutative:

$$\begin{tikzcd} \widehat{h}^{\bullet}(\rho) \arrow[r, "(id_X, \varnothing_A)^*"] \arrow[d, "cov"] & \widehat{h}^{\bullet}(X) \arrow[d, "\rho^*"] \\ \Omega^{\bullet-1}(A) \arrow[r, "a"] & \widehat{h}^{\bullet}(A) \end{tikzcd} \tag{3.9}$$

where $cov(\widehat{\alpha}) = \theta$ if $R(\widehat{\alpha}) = (\omega, \theta)$.

Remark 3.3.2. We will write $\widehat{h}(A)$ instead of $\widehat{h}(\varnothing_A)$ (same for $\varnothing_X$ and X) and ρ^* instead of $(\rho, \varnothing)^*$.

In the introduction of this chapter, we gave an example of a refinement of $H\mathbb{Z}^2$. The same model can be extended to higher degrees using abelian gerbes (RUFFINO, 2014). Now we give two more (extremely) simple examples which will help understand what differential cohomology theory captures.

Example 3.3.3. Let's "refine" ordinary cohomology with $\frac{\mathbb{Z}}{n\mathbb{Z}}$ coefficients. Since $\mathfrak{h} = \frac{\mathbb{Z}}{n\mathbb{Z}}$ and $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{R} = 0$, it follows that $\Omega \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}} = 0$ and thus $\widehat{H\frac{\mathbb{Z}}{n\mathbb{Z}}} = H\frac{\mathbb{Z}}{n\mathbb{Z}}$. Therefore, there is no true differential refinement! The moral of the story is simple: we cannot refine pure torsion cohomology, since there are no differential forms with torsion.

Example 3.3.4. Next, let's "refine" de Rham cohomology $H_{\rm dR}$ (or, equivalently real ordinary cohomology $H\mathbb{R}$): in this case we identify $\widehat{H}_{\rm dR}$ with $\Omega_{\rm cl}$ and take

- $R : \widehat{H}_{\rm dR} \to \Omega_{\rm cl}$ as the identity map $\mathrm{id} : \Omega_{\rm cl} \to \Omega_{\rm cl}$;
- $I : \widehat{H}_{\rm dR} \to H_{\rm dR}$ as the quotient map $q_{\rm dR} : \Omega_{\rm cl} \to H_{\rm dR}$
- $a : \frac{\Omega}{\mathrm{Im}(d)} \to \widehat{H}_{\rm dR}$ as the exterior derivative defined from the quotient $d : \frac{\Omega}{\mathrm{Im}(d)} \to \Omega^{\bullet}_{\rm cl}$

It is clear that this is indeed a differential refinement. We will refer to this model as the *de Rham toy model*.

Remark 3.3.5. The tautological de Rham refinement can be deceiving, since in this case the curvature map is an isomorphism. In a differential refinement of an arbitrary cohomology theory, it is not enough to know the curvature and the underlying topological class in order to specify a differential cohomology class. For example, in the geometric model introduced in the beginning of this section, we observed that an hermitian line bundle with connection over X (over $\varnothing_X$) of the form $[X \times \mathbb{C}, h, d + 2\pi i\omega]$ is topologically trivial, and has zero curvature provided that ω is closed. Nevertheless, if ω does not have integral periods, then the differential class is not trivial. In other words, differential classes capture *holonomy* even in the flat case.

Now, we collect some definitions and facts that will be useful ahead and also shed some light on the concept of differential cohomology. We will not prove the majority of these results, but refer to Ruffino and Barriga (2021, Section 2 and 3).

Maybe the trademark of differential cohomology is the homotopy formula. Differential cohomology is not a cohomology theory in the topological sense, since it is not homotopy invariant. In fact, we have the following:

Proposition 3.3.6 (Homotopy Formula). *Let $\rho : A \to X$ and $\eta : B \to Y$ be smooth maps and let us consider two morphisms $(f_0, g_0), (f_1, g_1) : \eta \to \rho$. If $(F, G) : \mathrm{id}_I \times \eta \to \rho$ is a homotopy between (f_0, g_0) and (f_1, g_1), then, for any $\widehat{\alpha} \in \widehat{h}(\rho)$, we have:*

$$(f_1, g_1)^*(\widehat{\alpha}) - (f_0, g_0)^*(\widehat{\alpha}) = a\left(\int_{(p_Y, p_B)} (F, G)^* R(\widehat{\alpha})\right)$$

In the de Rham toy model, this is just the homotopy formula 2.6 applied as in the proof of homotopy invariance. The proof is quite simple and can be found in (RUFFINO; BARRIGA, 2021, Corollary 2.7).

Definition 3.3.7 (Flat and Parallel Classes). Let $(\widehat{h}, R, I, a)$ be a differential cohomology theory. A class $\widehat{\alpha} \in \widehat{h}(\rho)$ is called *flat* if $R(\widehat{\alpha}) = 0$ and *parallel* if $cov(\widehat{\alpha}) = 0$.

The sets of flat and parallel classes are subgroups of $\widehat{h}(\rho)$. We denote them by $\widehat{h}_{\text{flat}}(\rho)$ and $\widehat{h}_{\text{par}}(\rho)$, respectively. They are called parallel after the model on the introduction, where parallel classes are the ones with a parallel distinguished section s, *i.e*, such that $\nabla s = 0$. In the de Rham model, the parallel classes were elements of $\Omega_{\text{cl,par}}$, hence the choice of name parallel forms.

The homotopy formula gives us a hint that the flat groups $\widehat{h}_{\text{flat}}$, which are homotopy invariant, may be a (topological) cohomology theory. This is indeed the case under a mild assumption on the existence of differential S^1-integration (to be defined in Section 3.4). We highlight this fact in the following proposition.

Proposition 3.3.8. *If $(\widehat{h}, R, I, a)$ is a differential cohomology with S^1-integration, then there exists a connecting morphism $\partial : \widehat{h}^{\bullet}_{flat}(\rho) \to \widehat{h}^{\bullet+1}_{flat}(\rho)$ which makes $(\widehat{h}_{flat}, \partial)$ a topological relative cohomology theory on maps.*

For a proof, see Ruffino and Barriga (2021, Remark 3.2).

Remark 3.3.9. In many cases $\widehat{h}^{\bullet}_{\text{flat}}$ can be identified with the cohomology theory $h^{\bullet-1}\frac{\mathbb{R}}{\mathbb{Z}}$ described by Bunke and Schick (2010, Section 5, p. 29).

Lets fix some notation: we write

- $R_{\text{par}} : \widehat{h}_{\text{par}} \to \Omega_{\text{cl,par}}\mathfrak{h}_{\mathbb{R}}$ for the transformation $R_{\text{par}}(\widehat{\alpha}) = \omega$ whenever $R(\widehat{\alpha}) = (\omega, 0)$;
- $I_{\text{par}} : \widehat{h}_{\text{par}} \to h$ for the restriction of I to $\widehat{h}_{\text{par}}$;
- and $a_{\text{par}} : \frac{\Omega^{\bullet-1}_{\text{par}}\mathfrak{h}_{\mathbb{R}}}{\text{Im}(d)} \to \widehat{h}^{\bullet}_{\text{par}}$ for the map given by $a_{\text{par}}(\omega) = a(\omega, 0)$.

In general, we cannot assert that $(\widehat{h}_{\text{par}}, R_{\text{par}}, I_{\text{par}}, a_{\text{par}})$ is a differential cohomology in the sense specified above, but we can prove so in a particular case:

Proposition 3.3.10. *Let $\texttt{Man}_{cl}$ be the subcategory of $\texttt{Man}$ with the same objects but with closed embeddings as morphisms. Restricted to $\texttt{Man}_{cl}$, $(\widehat{h}_{par}, R_{par}, I_{par}, a_{par})$ satisfies the axiom A1 (3.8) with $\Omega\mathfrak{h}_{\mathbb{R}}$ replaced by $\Omega_{par}\mathfrak{h}_{\mathbb{R}}$ and axiom A2 (3.9) written as*

$$\begin{tikzcd} \widehat{h}_{par}(\rho) \arrow[r, "(id_X, \varnothing_A)^*"] \arrow[rr, bend right, "0"'] & \widehat{h}_{par}(X) \arrow[r, "\rho^*"] & \widehat{h}_{par}(A) \end{tikzcd}$$

The proof can be found in (RUFFINO; BARRIGA, 2021, Theorem 2.12, p.11).

By convenience, we write the following definition:

Definition 3.3.11 (Parallel Differential Cohomology Theory)**.** Let $\widehat{k} : \texttt{Man}^{2,\text{op}} \to \texttt{GrAb}$ be a (contravariant) functor and consider natural transformations

- $R_{\text{par}} : \widehat{k} \to \Omega_{\text{par,cl}}\mathfrak{h}_{\mathbb{R}}$, called the parallel *curvature*;
- $I_{\text{par}} : \widehat{k} \to h$, called the parallel *forgetful map*; and
- $a_{\text{par}} : \dfrac{\Omega^{\bullet-1}_{\text{par}}\mathfrak{h}_{\mathbb{R}}}{\text{Im}(d)} \to \widehat{k}^{\bullet}$, called the parallel *trivialization*;

A quadruple $(\widehat{k}, R_{\text{par}}, I_{\text{par}}, a_{\text{par}})$ is a *parallel differential refinement of* (h, ∂) when it satisfies the following axioms:

A'1) The following diagram is commutative and its first line is exact:

$$\begin{tikzcd} h^{\bullet-1} \arrow[r, "r^{-1}\circ ch"] & \frac{\Omega^{\bullet-1}_{\mathrm{par}}\mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)} \arrow[r, "a_{\mathrm{par}}"] \arrow[dr, "d"] & \widehat{k}^{\bullet} \arrow[r, "I_{\mathrm{par}}"] \arrow[d, "R_{\mathrm{par}}"] & h^{\bullet} \arrow[r] \arrow[d, "ch"] & 0 \\ & & \Omega^{n}_{\mathrm{par,cl}}\mathfrak{h}_{\mathbb{R}} \arrow[r, "r\circ q_{\mathrm{dR}}"] & H\mathfrak{h}^{\bullet}_{\mathbb{R}} & \end{tikzcd} \tag{3.10}$$

A'2) For every class $\widehat{\alpha} \in \widehat{k}(\rho)$, one has

$$\rho^{*}\widehat{\alpha} = 0 \tag{3.11}$$

where ρ is shorthand for $(\rho, \varnothing_A) : \varnothing_A \to \rho$.

We rephrase Proposition 3.3.10 by saying that the data $(\widehat{h}_{\mathrm{par}}, R_{\mathrm{par}}, I_{\mathrm{par}}, a_{\mathrm{par}})$ given below Remark 3.3.9 is a parallel theory restricted to $\mathtt{Man}_{\mathrm{cl}}$.

Let's denote by $\Omega_{ch}\mathfrak{h}_{\mathbb{R}}(\rho) \subseteq \Omega\mathfrak{h}_{\mathbb{R}}(\rho)$ (resp. $\Omega_{\mathrm{par},ch}\mathfrak{h}_{\mathbb{R}}(\rho) \subseteq \Omega_{\mathrm{par}}\mathfrak{h}_{\mathbb{R}}(\rho)$) the subset of differential forms such that $r([\omega, \eta]) \in \mathrm{Im}(ch)$. We shall name them $\mathfrak{h}$*-forms*, in analogy with integral forms. A variant of the next lemma will be useful in the Part II.

Proposition 3.3.12. *Let* $(\widehat{h}, R, I, a)$. *For any smooth map*[3] $\rho : A \to X$, *the following sequence is exact:*

$$0 \longrightarrow \widehat{h}_{flat}(\rho) \hookrightarrow \widehat{h}_{par}(\rho) \xrightarrow{R_{par}} \Omega_{ch,par}(\rho) \longrightarrow 0$$

The proof can be found in (RUFFINO; BARRIGA, 2021, Theorem 2.13), but we will prove another version latter (Proposition 5.2.3). The following proposition plays an important role in the definition of the pushfoward morphism in compactly supported differential cohomology (Section 3.6.1).

Proposition 3.3.13 (Parallel classes satisfy excision). *If* $\widehat{h}$ *is a differential cohomology with* S^1*-integration, then the functor* $\widehat{h}_{par}$ *satisfies the excision property: if* $i : Z \hookrightarrow A$ *and* $j : A \hookrightarrow X$ *are embeddings such that the closure of* $j(i(Z))$ *is contained in the interior of* $j(A)$*, then the morphism*

$$\begin{tikzcd} A \setminus U \arrow[r, hook, "i_{A\setminus U}"] \arrow[d, hook, "\rho_{A\setminus U}"] & A \arrow[d, hook, "\rho"] \\ X \setminus \rho(U) \arrow[r, hook, "i_{X\setminus \rho U}"] & X \end{tikzcd}$$

induces an isomorphism between $\widehat{h}^{\bullet}_{par}(j)$ *and* $\widehat{h}^{\bullet}_{par}(j')$.

The proof can be found in (RUFFINO; BARRIGA, 2021, Lemma 2.15). It follows from the fact that Ω_{par} satisfies excision and is compatible with ch.

[3] Not necessarily a closed embedding

3.4 Differential S^1 integration

Analogous to the topological case and closely similar to the de Rham case, we have a the notion of differential S^1 integration. Let $S : \mathtt{Man}^2 \to \mathtt{Man}^2$ denote the same functor as in section 1.4, that is,

$$S\left(\rho \xrightarrow{(f,g)} \xi\right) = \mathrm{id}_{S^1} \times \rho \xrightarrow{\left(\mathrm{id}_{S^1} \times f, \mathrm{id}_{S^1} \times g\right)} \mathrm{id}_{S^1} \times \xi$$

and let $SF := F \circ S$ for a functor $F : \mathtt{Man}^2 \to \mathtt{C}$, with $\mathtt{C}$ any category.

Definition 3.4.1 (Differential S^1 integration)**.** *S^1-integration in a differential cohomology $\widehat{h}$ refining (h, ∂) is a natural transformation $\int_{S^1} : S\widehat{h}^\bullet \to \widehat{h}^{\bullet-1}$ satisfying the following axioms:*

S1) For any $\rho : A \to X$, we have $\int_{S^1} \circ (\mathrm{pr}_X, \mathrm{pr}_A)^* = 0$.

S2) For any $\rho : A \to X$, we have $\int_{S^1} \circ (id_X \times t, id_A \times t)^* = -\int_{S^1}$, where $t : S^1 \to S^1$ is the conjugation $t(z) = \overline{z}$;

S3) The following diagram is commutative:

$$\begin{tikzcd}
\frac{S\Omega^\bullet \mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)} \arrow[r, "a"] \arrow[d, "\int_{S^1}"] & S\widehat{h}^{\bullet+1} \arrow[r, "I"] \arrow[d, "\int_{S^1}"] \arrow[rr, bend left, "R"] & Sh^{\bullet+1} \arrow[d, "\int_{S^1}"] & S\Omega_{cl}^{\bullet+1}\mathfrak{h}_{\mathbb{R}} \arrow[d, "\int_{S^1}"] \\
\frac{\Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)} \arrow[r, "a"] & \widehat{h}^\bullet \arrow[r, "I"] \arrow[rr, bend right, "R"] & h^\bullet & \Omega_{cl}^\bullet \mathfrak{h}_{\mathbb{R}}
\end{tikzcd}$$

where both $\int_{S^1} : \frac{S\Omega^\bullet \mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)} \to \frac{\Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)}$ and $\int_{S^1} : S\Omega_{\mathrm{cl}}^{\bullet+1}\mathfrak{h}_{\mathbb{R}} \to \Omega_{\mathrm{cl}}^\bullet \mathfrak{h}_{\mathbb{R}}$ are S^1-integration of relative differential forms, and $\int_{S^1} : Sh^{\bullet+1} \to h^\bullet$ is the topological S^1 integration.

Remark 3.4.2. As remarked in (BUNKE; SCHICK, 2010, Section 4, p.23), not all differential cohomology theories admit an S^1 integration. In Remark 1.4.3, we related the topological S^1 integration to the cross product with a class $e \in h^1(S^1)$ such that $\int_{S^1} e \times \alpha = \alpha$. The problem of differential integration is related to the existence of $\widehat{e} \in \widehat{h}(S^1)$ such that

$$R(\widehat{e}) = dt, \quad I(\widehat{e}) = e, \quad t^*\widehat{e} = -\widehat{e}.$$

The following lemma will be used in the construction of a differential Thom class, as well as in the Hopkins-Singer model (Appendix D and in Section 6.4.

Lemma 3.4.3. *Let $i_1 : A \hookrightarrow S^1 \times A$ be the inclusion of A at the marked point 1. For every $\widehat{\alpha} \in \widehat{h}^\bullet(A)$ there exists a unique class $\widehat{\beta} \in \widehat{h}_{par}^{\bullet+1}(i_1)$ such that*

$$\int_{S^1} (\mathrm{id}_{S^1 \times X}, \varnothing_A)^* \widehat{\beta} = \widehat{\alpha} \ \ \textit{and} \ \ R_{par}(\widehat{\beta}) = R(\widehat{\alpha}) \times dt.$$

The proof can be found in (RUFFINO; BARRIGA, 2021, Lemma 2.16).

The simplest example of a differential S^1- integration is obviously the S^1-integration in de Rham model. We discuss the existence of more serious differential S^1-integration in a few different models in the Appendices B, C and D.

3.5 Multiplicative Structure

Unlike the topological setting and close in spirit to the case of differential forms, we cannot define a complete product $\widehat{h}(\rho) \times \widehat{h}(\eta) \to \widehat{h}(\rho \wedge \eta)$ since the $\rho \wedge \eta$ is not a smooth map. Nevertheless, as in the de Rham case (see 2.3), we can define a module structure on $\widehat{h}(\rho)$ over $\widehat{h}(X)$ by using $\rho : A \to X$. Alternatively, we could define an external multiplicative structure $\widehat{h}(\rho) \times \widehat{h}(Y) \to \widehat{h}(\rho \times \mathrm{id}_Y)$. We will construct the first one here.

Definition 3.5.1 (Relative-Absolute Multiplicative Structure in Differential Cohomology)**.** Let $(\widehat{h}, R, I, a)$ be a differential refinement of a multiplicative cohomology $(h, \partial, \cdot)$. It is called a *multiplicative differential cohomology* if, for each smooth map $\rho : A \to X$, there exists a natural map $\times : \widehat{h}^n(\rho) \otimes \widehat{h}^m(X) \to \widehat{h}^{n+m}(\rho)$ satisfying

M1) (Naturality:) For all $\widehat{\alpha} \in \widehat{h}^n(\rho)$, $\widehat{\beta} \in \widehat{h}^m(X)$, $(F, f) : \xi \to \rho$,

$$(F, f)^*(\widehat{\alpha} \times \widehat{\beta}) = (F, f)^*\widehat{\alpha} \times f^*\widehat{\beta}.$$

M2) (Associativity:) For all $\widehat{\alpha} \in \widehat{h}^n(\rho)$, $\widehat{\beta} \in \widehat{h}^m(Y)$ and $\widehat{\gamma} \in \widehat{h}^k(Z)$,

$$(\widehat{\alpha} \times \widehat{\beta}) \times \widehat{\gamma} = \widehat{\alpha} \times (\widehat{\beta} \times \widehat{\gamma}).$$

M3) (Compatibility:) For all $\widehat{\alpha} \in \widehat{h}^n(\rho)$, $\widehat{\beta} \in \widehat{Y}$, $(\omega, \eta) \in \Omega^{n-1}(\rho)$ and $\omega' \in \Omega^{n-1}(X)$,

- (*Curvature*) $R(\widehat{\alpha} \times \widehat{\beta}) = R(\widehat{\alpha}) \times R(\widehat{\beta})$
- (*Forgetful map*) $I(\widehat{\alpha} \times \widehat{\beta}) = I(\widehat{\alpha}) \times I(\widehat{\beta})$
- (*Trivialization 1*) $a((\omega, \eta) \times R(\widehat{\beta})) = a(\omega, \eta) \times \widehat{\beta}$
- (*Trivialization 2*) $a(R(\widehat{\alpha}) \times \omega') = (-1)^{|\widehat{\alpha}||\omega'|}\widehat{\alpha} \times a(\omega)$.

The compatibility conditions for R and I are intuitive. The conditions on the trivialization are inspired by de Rham toy model, where the product is given by $(\omega, \theta) \times \omega'$ as defined in 2.5. The compatibility conditions in this case are just the following:

- (*Curvature*) For $(\omega, \theta) \in \Omega_{\mathrm{cl}}(\rho)$ and $\omega' \in \Omega_{\mathrm{cl}}(Y)$,

$$\mathrm{id}((\omega, \theta) \times \omega') = \mathrm{id}((\omega, \theta)) \times \mathrm{id}(\omega');$$

- (*Forgetful map*) For $(\omega,\theta) \in \Omega_{\text{cl}}(\rho)$ and $\omega' \in \Omega_{\text{cl}}(Y)$,

$$q(\omega,\theta) \times \omega' = [(\omega,\theta) \times \omega'] = [(\omega,\theta)] \times [\omega'] = q(\omega,\theta) \times q(\omega')$$

- (*Trivialization 1*) For $(\omega,\theta) \in \Omega(\rho)$ and $\omega' \in \Omega_{\text{cl}}(Y)$,

$$d\left((\omega,\theta) \times \text{id}(\omega')\right) = d(\omega,\theta) \times \omega' + (\omega,\theta) \times d(\omega') = d(\omega,\theta) \times \omega'$$

- (*Trivialization 2*) For $(\omega,\theta) \in \Omega_{cl}(\rho)$ and $\omega' \in \Omega(Y)$,

$$\begin{aligned} d\left(\text{id}(\omega,\theta) \times (\omega')\right) &= d(\omega,\theta) \times \omega' + (-1)^{|(\omega,\theta)||\omega'|}(\omega,\theta) \times d(\omega') \\ &= (-1)^{|(\omega,\theta)||\omega'|}(\omega,\theta) \times d\omega' \end{aligned}$$

For more serious examples the reader can take a look at Appendices B, C, and D.

As it is, this situation is at most unsatisfactory, since we do not have a product between relative classes that is in disagreement with cohomology. The situation is less unsettling when we think about the de Rham toy model. In it, although we did not have a full product, there was a mixed product between a parallel class and a usual one (see Section 2.6). This was important in order to define both compact integration and vertical integration without resorting to special models.

3.6 Compactly like Cohomology and Differential Thom morphism

3.6.1 Differential Cohomology with compact and vertically compact supports

Mimicking what we have done in the topological and de Rham case, we start by defining differential cohomologies with compact and vertically compact supports. In the de Rham toy model $(\Omega_{\text{cl}}, \text{id}, q, d)$, we identified compact forms with $\text{colim}_{K\in\mathcal{K}}\, \Omega_{\text{par}}(X, K^c)$ (and analogously for the vertical compact case). This motivates the following definition:

Definition 3.6.1 (Differential Cohomology with Compact and Vertically Compact Supports)**.** The *Cohomology with compact supports of* X is defined as

$$\widehat{h}_c(X) = \text{colim}_{K\in\mathcal{K}(X)}\, \widehat{h}_{\text{par}}(E, K^c)$$

and the *Cohomology with vertically compact supports of a map* $f : Y \to X$ as

$$\widehat{h}_v(Y) = \text{colim}_{V\in\mathcal{V}(f)}\, \widehat{h}_{\text{par}}(E, K^c)$$

These functors have the exact same properties of the corresponding ones in topology and de Rham cohomology:

(i) They are contravariant functors on the subcategory of `Man` with the same objects and proper maps as morphisms;

(ii) They are covariant functors on the subcategory of `Man` with the same objects and open embeddings as morphisms (open embedding of bundle over the same base in the vertically compact case).

In the topological case, the proof of point (ii) required the excision theorem. This will not pose a problem here, since excision also holds in parallel differential cohomology, by Proposition 3.3.13.

The module-like structure of Definition 3.5.1 allows us to define products

$$\cdot : \widehat{h}_c(X) \times \widehat{h}(X) \to h_c(X) \tag{3.12}$$
$$\cdot : \widehat{h}_v(X) \times \widehat{h}(X) \to h_v(X) \tag{3.13}$$

in the exact same way as in the topological case (see 1.9), but we cannot define products

$$\cdot : \widehat{h}_c(X) \times \widehat{h}_c(X) \to \widehat{h}_c$$
$$\cdot : \widehat{h}_v(Y) \times \widehat{h}_c(X) \to \widehat{h}_v.$$

The problem here is that we cannot multiply

$$\widehat{h}_{\text{par}}(X, K^c) \times \widehat{h}_{\text{par}}(X, L^c) \to \widehat{h}_{\text{par}}(X, K^c \cup L^c)$$

One possible solution would to be to define a product directly between parallel classes. But this will not suffice in the relative case.

The maps R, I and a naturally induce maps on these cohomologies, that is, we have natural transformations

$$R_c : h_c \to \Omega_c, \quad I_c : h_c \to \Omega_c \;\text{ and }\; a_c : \Omega_c^{\bullet}\mathfrak{h}_{\mathbb{R}} \to \Omega_c \tag{3.14}$$

and analogously for the vertically compact case, by replacing c with v throughout.

3.6.2 Thom morphism

In this subsection we will define the Thom morphism and the lift of the Thom isomorphism in the differential setting, as we have done in the de Rham case (see Definition 2.3.11). The analogue of the Thom form is the differential Thom class.

Definition 3.6.2 (Differential Thom Classes). A *differential Thom class* of a smooth vector bundle $p : E \to X$ of dimension n is a class $\hat{u} \in \widehat{h}_v^n(E)$ such that $I(u) \in h_v^n(E)$ is a (topological) Thom class.

Before proceeding, lets recall some definitions in the veins of the Riemann-Roch-Grothendieck theorem. For any h-oriented vector bundle $p : E \to X$, denote the inverse of the Thom isomorphism by $\int_p$. We define the *Todd class*[4] as $Td(u) := \int_p ch(u) \in \mathfrak{h}_{\mathbb{R}}$, where u is a Thom class of E and $\int_p^{\mathbb{R}}$ is the inverse of the Thom isomorphism in ordinary cohomology with coefficients in $\mathfrak{h}_{\mathbb{R}}$. The Todd class is the "commutativity defect" of the Thom isomorphism, as discussed in (LAWSON; MICHELSOHN, 1989, p.241), that is, $\int_p^{\mathbb{R}} ch(\alpha) = Td(u) \cdot ch\left(\int_p \alpha\right)$. We define the *Todd form* as $\widehat{Td}(\widehat{u}) = \int_p R(\widehat{u}) \in \Omega^0_{\mathrm{cl}}\mathfrak{h}_{\mathbb{R}}(X)$, where $\widehat{u}$ is a differential Thom class. This definition implies the compatibility

$$Td(I(\widehat{u})) = q_R \circ (\widehat{Td}(\widehat{u})).$$

We say that two differential Thom classes $\widehat{u}_0$ and $\widehat{u}_1$ are *homotopy equivalent* if there exists a differential Thom class $\widehat{U} \in \widehat{h}_v(\mathrm{pr}_X^* E)$, where $\mathrm{pr}_X : I \times X \to X$ is the projection, such that $\widehat{u}_t = i_t^*$ for $t \in \{0, 1\}$, provided that $\widehat{Td}(\widehat{U}) = \mathrm{pr}_X^* \widehat{Td}(\widehat{u}_0)$.

There is a 2 out of 3 principle here as well:

Proposition 3.6.3 (2 × 3 principle)**.** *Given two bundles vector bundles $q_E : E \to X$ and $q_F : F \to X$ with projections $\mathrm{pr}_E : E \oplus F \to E$ and $\mathrm{pr}_F : E \oplus F \to F$, consider a triple $(\widehat{u}, \widehat{v}, \widehat{w})$ of differential Thom classes on E, F and $E \oplus F$ respectively, such that $\widehat{w} = \mathrm{pr}_E^* \widehat{u} \cdot \mathrm{pr}_F^* \widehat{v}$. Two elements of such a triple uniquely determine the third one up to a homotopy equivalence.*

A proof of this can be found in (BUNKE, 2013, Problem 4.187, p.126).

Remark 3.6.4. As in Remark 1.6.19, we can endow the trivial vector bundle with a canonical differential Thom class. In a differential cohomology theory with S^1-integration, the product line bundle $\mathrm{pr}_X : X \times \mathbb{R} \to X$ has a natural orientation defined as follows:

- Thinking of $S^1 \subset \mathbb{C}$, we fix 1 as a marked point. There exists a unique class $\widehat{v} \in \widehat{h}^1(S^1)$ such that $\int_{S^1} \widehat{\nu} = 1$ and $R(\widehat{v}) = dt$.
- Fixing an open interval $U := \exp(-\varepsilon, \varepsilon)$ around 1, where $\exp(t) := e^{2\pi i t}$, and a smooth increasing function $\varphi : I \to I$ such that $\varphi[0, \varepsilon) = 0$ and $\varphi(1 - \varepsilon, 1] = 1$, we get the smooth map of pairs $\varphi : (S^1, U) \to (S^1, 1)$, $\exp(t) \mapsto \exp(\varphi(t))$.
- Setting $D_{\mathbb{R}} := [-1, 1]$ and $D' := \mathbb{R} \setminus D_{\mathbb{R}}$, we fix a diffeomorphism

  $$\psi : (\mathbb{R}, D') \to (S^1 \setminus \{1\}, U \setminus \{1\}),$$

  that preserves the orientation in the usual sense, and consider the embedding $\iota : (S^1 \setminus \{1\}, U \setminus \{1\})Z \hookrightarrow (S^1, U)$ that induces an excision isomorphism. The class $\widehat{u}_0 := \psi^* \iota^* \phi^* \widehat{v} \in \widehat{h}^1_{\mathrm{par}}(\mathbb{R}, D')$ represents a Thom class $\widehat{u}$ of $X \times \mathbb{R}$ that depends on the choice of φ and ψ only up to homotopy.

[4] More precisely, this the "realification" of the rational Todd class.

Definition 3.6.5 (Differential Thom morphism)**.** We define the *differential Thom morphism* as the the homomorphism

$$\begin{aligned}\widehat{T} : \widehat{h}(X) &\to \widehat{h}_v(E)\\ \widehat{\alpha} &\mapsto \widehat{u} \cdot p^* \widehat{\alpha}\end{aligned}$$

where the product $\cdot$ is the one on equation (3.13).

As in the case of de Rham Thom morphism (Definition 2.3.11), this map is injective but not necessarily surjective. For a proof that $\widehat{T}$ is injective, see (BUNKE, 2013, Problem 4.184, p.124).

Note that we cannot define the compact Thom morphism now, since we are unable to multiply two parallel classes and thus cannot define a product $\cdot p^* : \widehat{h}_c(X) \to \widehat{h}_c(E)$. The same occurs in the relative case if we use the definition of vertically compact Thom as in the topological case, that is,

$$\widehat{h}_v(\overline{\rho}) = \mathrm{colim}_{V \in \mathcal{V}(p)}\, \widehat{h}_{\mathrm{par}}(E, j(\rho^*(E)) \cup V^c).$$

In order to define the Thom morphism in the usual way, one needs to define

$$\widehat{h}_{\mathrm{par}}(E, V^c) \times \widehat{h}(\overline{\rho}) \to \widehat{h}_v(\overline{\rho}),$$

which is not possible at the moment.

3.7 $\widehat{h}$-Orientation and Differential Integration

In Section 2.4.1 we constructed the differential umkehr map at cochain level in de Rham cohomology, which was identified with the fiber integration for manifolds in Proposition 2.4.7. Since de Rham cohomology is our guide in the structures of differential cohomology, we hope to be able to construct a general differential integration. The integration map in de Rham cohomology was defined for vertically compact forms, which suggests we define integration for vertically-compact supported differential cohomology.

In (RUFFINO, 2017), the author tackles the problem of integration over compact manifolds. In that particular context, the fibers were compact. Now, endowed with the definition of vertically compact classes, we can extend that to differential integration for neat smooth bundles with compact fibers.

3.7.1 Differential Orientation

Virtually almost the same definitions of an h-orientation can be carried on to the differential case: the representative of differential orientation is defined in the exact same way:

Definition 3.7.1 (Representatives of $\widehat{h}$-orientations)**.** A *representative of an $\widehat{h}$-orientation* of a smooth neat fiber bundle with compact fibers $f : Y \to X$ is given by a triple $(\iota, \widehat{u}, \phi)$ consisting of the following data:

O1) a neat embedding $\iota : Y \to X \times \mathbb{R}^N$, for any $N \in \mathbb{N}$, such that $\pi_X \circ \iota = f$.

O2) a differential Thom class $\widehat{u}$ of the normal bundle $N(\iota(Y))$.

O3) a tubular neighbourhood of $\iota(Y)$ in $X \times \mathbb{R}^n$ given by a diffeomorphism $\phi : N(\iota(Y)) \to U$, where $U \subseteq X \times \mathbb{R}^n$ is an open set.

The equivalence class of these representatives is slightly different. Consider the following map:

$$R_{[i,\widehat{u},\phi]} : \frac{\Omega^{\bullet}\mathfrak{h}_{\mathbb{R}}(Y)}{\mathrm{Im}(d)} \to \Omega^{\bullet-n}\mathfrak{h}_{\mathbb{R}}(X)$$
$$\omega \mapsto \int_{\pi} R(\widehat{u}) \wedge \omega,$$

where $\pi_N : N(\iota(Y)) \to \iota(Y)$, which we call the *curvature map*. The notion of homotopy is now adapted to:

Definition 3.7.2. A *homotopy between* two representatives $(\iota, \widehat{u}, \phi)$ and $(\iota', \widehat{u}', \phi')$ of an $\widehat{h}$-orientation of $f : Y \to X$ is a representative $(J, \widehat{U}, \Phi)$ of an h-orientation of $id_I \times f : I \times Y \to I \times X$ such that

- $(J, \widehat{U}, \Phi)$ is proper over a neighborhood $V \subseteq I$ of $\{0, 1\}$;
- $(J, \widehat{U}, \Phi)|_{f_0} = (\iota, \widehat{u}, \phi)$ and $(J, \widehat{U}, \Phi)|_{f_0} = (\iota', \widehat{u}', \phi')$.
- $\mathrm{pr}_X^* \circ R_{(\iota,\widehat{u},\phi)} = R(J, \widehat{U}, \Phi) \circ \mathrm{pr}_y^*$.

The stabilization remains the same

Definition 3.7.3 (Stabilization)**.** Consider a representative $(\iota, \widehat{u}, \phi)$ of an $\widehat{h}$-oriented map $f : Y \to X$, with $\iota : Y \to X \times \mathbb{R}^N$. A representative $(\iota', \widehat{u}', \phi')$ is said to be *equivalent* to (ι, u, ϕ) *by stabilization* if

- $\iota' : Y \to X \times \mathbb{R}^{N+L}$ is given by $\iota'(y) := (\iota(y), 0)$ for any $L \in \mathbb{N}$
- $\widehat{u}'$ is given as follows: observe that $N(\iota'(Y)) = N(\iota(Y)) \oplus (\iota(Y) \times \mathbb{R}^L)$, where $\mathrm{pr}_{\iota(Y)} : \iota(Y) \times \mathbb{R}^L \to X$, is the product bundle. We put the canonical orientation of remark 3.6.4 on $\mathrm{pr}_{\iota(Y)} : \iota(Y) \times \mathbb{R}^L \to X$ and define $\widehat{u}'$ on $N(\iota'(Y))$ using proposition 3.6.3.
- $\phi'(v, w) = (\phi(v), w) \in X \times \mathbb{R}^{N+L}$ for $v \in N_{\iota(Y)}$ and $w \in \mathbb{R}^L$

At last, we define a differential orientation:

Definition 3.7.4 ($\widehat{h}$-orientation)**.** An *$\widehat{h}$-orientation of a map $f : Y \to X$* is an equivalence class of representatives of $\widehat{h}$-orientations under the equivalence relation generated by homotopy and stabilization.

As in the topological case, we define a compatible orientation on the composition of maps. The definition is given in the exact same way as (1.7.7). We state it here for convenience.

Definition 3.7.5. Let $f : Y \to X$ and $g : X \to W$ be $\widehat{h}$-oriented maps, with orientations $[\iota, \widehat{u}, \phi]$ and $[\kappa, \widehat{v}, \psi]$, where $\iota : Y \to X \times \mathbb{R}^N$ and $\kappa : X \to W \times \mathbb{R}^L$. There is a naturally induced $\widehat{h}$-orientation $[\chi, \widehat{w}, \xi]$ on $g \circ f : Y \to W$, defined in the following way:

- ξ is the embedding given by $(\kappa, \mathrm{id}_{R^N}) \circ \iota : Y \to W \times \mathbb{R}^{N+L}$;
- $\widehat{w}$ is the differential Thom class induced from the ones on $N\iota(Y)$ and $N\kappa(X)$ on the normal bundle $N\xi(Y)(W \times \mathbb{R}^{N+L}) \cong N(\iota(Y)) \oplus \iota^* N(\kappa(X) \times \mathrm{id}_{\mathbb{R}^N}) \cong N(\iota(Y)) \oplus (\mathrm{pr}_{\mathbb{R}^L})^* N(\kappa(X))$, where $\mathrm{pr}_{\mathbb{R}^L} : \mathbb{R}^{N+L} \to \mathbb{R}^L$ is the projection.
- $\xi : N(\xi(X)) \to U$ is an arbitrary tubular neighbourhood, since it is unique up to homotopy.

These orientations satisfy a 2 out of 3 principle too:

Proposition 3.7.6 (2×3 principle for maps)**.** *Let $f : Y \to X$ and $g : X \to W$ be $\widehat{h}$-oriented neat submersions with orientations $[\iota, \widehat{u}]$ and $[\kappa, \widehat{v}]$, respectively, and let $[\xi, \widehat{w}]$ be the orientation induced on $g \circ f$ as in Definition 3.7.5 above. Two elements of the triple $([\iota, \widehat{u}], [\kappa, \widehat{v}], [\xi, \widehat{w}])$ uniquely determine the third one.*

See (RUFFINO, 2017, Lemma 3.28) for a discussion.

3.7.2 Differential Integration

We can define integration maps over $\mathbb{R}^n$

$$\int_{R^n} \widehat{h}_v^\bullet(X \times \mathbb{R}^n) \to \widehat{h}^{\bullet-n}(X) \quad \text{and} \quad \int_{R^n} \widehat{h}_c^\bullet(X \times \mathbb{R}^n) \to \widehat{h}_c^{\bullet-n}(X)$$

as in the topological case (1.7.1), by replacing topological S^1-integration by its differential counterpart.

Remark 3.7.7. For future usage, we remark that an analogue of Lemma 1.7.10 is true for the differential case, that is: the $\mathbb{R}$-integration map is a left inverse of the differential Thom morphism of the trivial bundle with its trivial differential Thom class.

Remark 3.7.8. There is a compactly supported version of the differential S^1-integration defined in (1.16), which satisfies Remark 1.7.9 and is compatible with the natural transformations R_c, I_c and a_c as well.

Now we define differential integration as in the topological case with compact fibers (Definition 1.7.11):

Definition 3.7.9 (Differential Integration with compact fibers)**.** Let $f : Y \to X$ be an $\widehat{h}$-oriented neat smooth bundle with **compact fibers**. Given a representative (ι, u, ϕ) of the orientation, we define the *differential integration*

$$\widehat{f}_!(\widehat{\alpha}) = \int_{\mathbb{R}^n} \kappa_* \circ \phi^{-1} T(\widehat{\alpha}),$$

where $\kappa_* : \widehat{h}_v{}^\bullet(U) \to \widehat{h}_v{}^\bullet(X \times \mathbb{R}^n)$ is constructed as in Definition 1.7.11 by using the fact that the fibers are compact.

It is possible to verify that this map depends only on the orientation class, despite being defined through a representative.

The standard example of integration in differential cohomology is the integration on fibers on closed forms in the de Rham toy model. Recall the commutative diagram (2.16). It suggests some compatibilities with the data R, I and a. In fact, one has the following:

Proposition 3.7.10 (Compatibility)**.** *Given an $\widehat{h}$-oriented map $f : Y \to X$ with compact fibers, its differential integration makes the following diagram commutative:*

$$\begin{tikzcd}
\dfrac{\Omega^{\bullet-1}(Y;\mathfrak{h}_{\mathbb{R}})}{\mathrm{Im}(d)} \arrow[r, "a"] \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] & \widehat{h}^\bullet(Y) \arrow[r, "I"] \arrow[d, "\widehat{f}_!"] \arrow[rr, bend left, "R"] & h^\bullet(Y) \arrow[d, "f_!"] & \Omega^\bullet_{cl}(Y;\mathfrak{h}_{\mathbb{R}}) \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] \\
\dfrac{\Omega^{\bullet-1-k}(X;\mathfrak{h}_{\mathbb{R}})}{\mathrm{Im}(d)} \arrow[r, "a"] & \widehat{h}^{\bullet-k}(X) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & h^{\bullet-k}(X) & \Omega^{\bullet-k}_{cl}(X;\mathfrak{h}_{\mathbb{R}})
\end{tikzcd}$$

For a discussion of these properties see (RUFFINO, 2017, equation (40)), (RUFFINO; BARRIGA, 2021, Section 6.8).

Proposition 3.7.11. *Let $f : Y \to X$ be an $\widehat{h}$-oriented map with compact fibers.*

- *(*Projection Formula*) The differential integration map is a morphism of $\widehat{h}(X)$-modules,* i.e., *given $\widehat{\alpha} \in \widehat{h}(Y)$ and $\widehat{\beta} \in \widehat{h}(X)$,*

$$f_!(\widehat{\alpha} \cdot f^*(\widehat{\beta})) = f_!(\widehat{\alpha}) \cdot \widehat{\beta}.$$

- *Given another neat $\widehat{h}$-oriented map $g : Z \to Y$ and endowing $f \circ g$ of the naturally induced orientation (def. 6.7), we have $\widehat{(f \circ g)_!} = \widehat{f}_! \circ \widehat{g}_!$.*

The proof of both these results can be found in (RUFFINO, 2017, lemma 3.24, lemma 3.27).

Remark 3.7.12. We note that an differential analogue of Remark 1.7.16 holds: given an $\widehat{h}$-oriented vector bundle $p_E : E \to X$ over a smooth manifold with differential Thom class $\widehat{u}_E$, we can define a natural $\widehat{h}$-orientation $[\iota, \widehat{u}, \phi]$ of p_E as a map. The definition is the same, except we use the differential 2 out of 3 principle for bundles (Proposition 3.6.3).

Since we do not have a definition of the compactly supported Thom isomorphism nor of any of its relative versions, we cannot yet come up with a sound definition of the differential integration map in these cases.

3.8 Conclusion

We have finished the preliminaries of this work. With respect to integration in differential cohomology, the situation is summarized in the following table:

Umkehr \ Type	Absolute	Relative
Compact Fibers	✓	
Compact		
Vertically Compact		

Table 3 – List of constructed differential integration maps in relative differential cohomology.

Our objective is clear by now: to complete this table. This seems attainable since the de Rham toy model has each one of these maps. Moreover, the constructions we have presented in the last section of the previous chapter (Section 2.6) indicate a possible way to accomplish this task.

Part II

Integration

4 Cohomology on maps of tuples

4.1 Introduction

In the previous chapter, we presented the problems that one faces when trying to define the following versions of differential integration maps for a smooth fibered map (or a smooth fiber bundle) $f : Y \to X$:

- the compactly supported $\widehat{f}_{c!} : h_c(Y) \to h_c(X)$;
- the vertically compact supported $\widehat{f}_{v!} : h_v(Y) \to h(X)$.

Those same problems are also a hindrance to the definition of relative integration maps even in the case of compact fibers. All in all, we can say that the problem rests on the nonexistence of products between a vertically compact differential class and

- a compactly supported differential class,
- a doubly vertically compact differential class, or
- relative classes in general,

which can all be regarded as special cases of a general product between a parallel class (the vertically compact one) and other relative classes.

Nevertheless, all these products were available in the de Rham toy model of differential cohomology. At first, this could be done because we had special models of $\widehat{h}_v(Y)$, $\widehat{h}_c(X)$, $\widehat{h}_v(\overline{\rho})$, and $\widehat{h}_c(\rho)$, with this last one only for proper maps. At the end of Chapter 2 (Section 2.6), we explored how we could define integration directly with $\widehat{h}_{\mathrm{par}}(\rho)$ without resorting to those special models. This was accomplished by defining a new chain complex $\Omega(\rho, \vec{\rho})$, where $(\rho, \vec{\rho}) : (A, \vec{A}) \to (X, \vec{X})$ is a map between finite sequences of manifolds.

Our strategy in this part of the text is to do the same for a general differential cohomology theory. In order to do so, it is convenient to extend a (topological) cohomology to the space of finite sequences of topological spaces and see how the usual product behaves in this setting[1]. This is precisely what we do in this chapter.

[1] We remark that there is no gain in the topological setting, just a a convenient presentation.

4.2 Some new categories

In section 2.6, we have considered a new category, $\mathtt{Man}^2_\omega$, and discussed how to define de Rham cohomology on it. We wish to generalize this construction to any cohomology theory. In order to do so, we introduce a new category, $\mathtt{Top}_\omega$, which we call the *category of finite sequences of topological spaces*, whose objects are sequences of topological spaces of the form

$$(X, X_1, \ldots, X_n, \emptyset, \cdots),$$

where $X_1, \ldots, X_n \subseteq X$[2]. For the sake of convenience, we shall write simply $(X, X_1, \ldots, X_n)$ or $(X, \vec{X})$, omitting the empty sets. A morphism between two sequences $f : (X, X_1, \ldots, X_n) \to (Y, Y_1, \ldots, Y_m)$ is simply a continuous map $f : X \to Y$ such that $f(X_n) \subseteq Y_n$. The morphisms will be called *maps of sequences* and will be denoted either by f or $(f, \vec{f})$[3]. A homotopy between two maps of sequences $f_0, f_1 : (X, X_1, \ldots, X_n) \to (Y, Y_1, \ldots, Y_m)$ is just a homotopy $F : I \times X \to Y$ between f_0 and f_1 such that $F_t(X_k) \subseteq Y_k$ for $k = 1, \ldots, m$. The homotopy category will be denoted by $\mathtt{HoTop}_\omega$.

Observe that we can view the category of pairs $\mathtt{Top}_2$ as a subcategory of $\mathtt{Top}_\omega$ through the inclusion functor $J_{2,\omega} : \mathtt{Top}_2 \to \mathtt{Top}_\omega$, which sends a topological pair (X, X') to the sequence $(X, X', \emptyset, \ldots)$ and a morphism to its underlying function. We also have a reverse functor $J_{\omega,2} : \mathtt{Top}_\omega \to \mathtt{Top}_2$, which acts on objects as

$$J_{\omega,2}(X, X_1, \ldots, X_m) = (X, \bigcup_{n \in \mathbb{N}} X_n).$$

and in morphisms as the identity, provided that the union of spaces is an object in $\mathtt{Top}$.

As in Chapter 1, we can form the arrow category of $\mathtt{Top}_\omega$, which we denote by $\mathtt{Top}^2_\omega$. Its objects are maps of sequences $\rho : (A, \vec{A}) \to (X, \vec{X})$. A morphism between two maps of sequences $\rho : (A, \vec{A}) \to (X, \vec{X})$ and $\eta : (B, \vec{B}) \to (Y, \vec{Y})$ is a pair of maps of sequences (f, g) which makes the following diagram commutative:

$$\begin{array}{ccc}
(A, \vec{A}) & \xrightarrow{g} & (B, \vec{B}) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \eta} \\
(X, \vec{X}) & \xrightarrow{f} & (Y, \vec{Y})
\end{array}$$

The notion of homotopy associated to this category is similar to that of $\mathtt{Top}^2_2$. A homotopy between $(f_0, g_0) : (\rho, \vec{\rho}) \to (\eta, \vec{\eta})$ and $(f_1, g_1) : (\rho, \vec{\rho}) \to (\eta, \vec{\eta})$ is a morphism (F, G) : $\mathrm{id}_I \times (\rho, \vec{\rho}) \to \eta$, where $\mathrm{id}_I \times (\rho, \vec{\rho})$ is shorthand for

$$\mathrm{id}_I \times \rho : (I \times A, I \times A_1, \ldots, I \times A_n) \to (I \times X, I \times X_1, \ldots, I \times X_n),$$

[2] Observe that we do not require $X_{n+1} \subseteq X_n$.

[3] This notation is very redundant, yet it can help differentiate between usual maps whenever there is danger of confusion.

such that F is a homotopy between f_0 and f_1 and G is a homotopy between g_0 and g_1. We denote the homotopy category by $\mathtt{HoTop}^2_\omega$.

Consider the category of maps of pairs $\mathtt{Top}^2_2$, which is just the arrow category of $\mathtt{Top}_2$. It can be viewed as a subcategory of $\mathtt{Top}^2_\omega$ through the inclusion functor $J^2_{2,\omega} : \mathtt{Top}^2_2 \to \mathtt{Top}^2_\omega$, which is defined in objects by

$$J^2_{2,\omega}((\rho, \rho') : (A, A') \to (X, X')) = (\rho, \rho') : (A, A', \emptyset, \dots) \to (X, X', \emptyset, \dots)$$

and in morphisms in the natural way. There is also a reverse functor $J^2_{\omega,2} : \mathtt{Top}^2_\omega \to \mathtt{Top}^2_2$, defined at objects by

$$J^\omega_{\omega,2}((A, \vec{A}) \xrightarrow{(\rho,\vec{\rho})} (X, \vec{X})) = (A, A') \xrightarrow{(\rho,\rho')} (X, X'), \tag{4.1}$$

where $A' = \bigcup_n A_n$, $B' = \bigcup_n B_n$ and $\rho' : A' \to B'$ is the restriction of ρ to A'.

We also need to extend the concept of cofibration to finite sequences.

Definition 4.2.1 (Cofibration sequences)**.** We say that a sequence $(X, X_1, \dots, X_n)$ is a cofibration if each pair (X, X_i), $i = 1, \dots, n$ is a cofibration.

4.3 Cohomology over $\mathtt{Top}^2_\omega$

Rather than give a set of axioms for a cohomology over $\mathtt{Top}^2_\omega$, we will do so in the category $\mathtt{Top}^2_2$ and use the functor $J^2_{\omega,2}$, defined in (4.1), to "push" it to a cohomology over $\mathtt{Top}^2_\omega$.

As in the topological case, we begin by defining the functor $\Pi^2_2 : \mathtt{Top}^2_2 \to \mathtt{Top}^2$, in objects as

$$\Pi^2_2\left((A, A') \xrightarrow{\rho} (X, X')\right) = (\emptyset, \emptyset) \xrightarrow{\varnothing_A} (A, A')$$

and in morphisms as $\Pi^2_2(f, g) = g$.

Definition 4.3.1 (Cohomology on maps of pairs)**.** A *cohomology theory over* $\mathtt{Top}^2_2$ is a (contravariant) functor $h : \mathtt{HoTop}^{2,\mathrm{op}}_2 \to \mathtt{GrAb}$ together with a natural transformation $\partial : h^\bullet \circ \Pi_\omega \to h^{\bullet+1}$ satisfying the following axioms:

Long exact sequence For each $\rho : (A, A') \to (X, X')$, we have the following long exact sequence:

$$\cdots \longrightarrow h^\bullet(\rho, \rho') \xrightarrow{(id_X, \varnothing_A)^*} h(X, X') \xrightarrow{(\rho,\varnothing)^*} h^\bullet(A, A') \xrightarrow{\partial_\rho} h^{\bullet+1}(\rho, \rho') \longrightarrow \cdots$$

where $h(X, X')$ is shorthand for $h(\varnothing_X, \varnothing_{X'})$, which is functorial with respect to morphisms.

Excision: Let $\rho : (A, A') \hookrightarrow (X, X')$ be an embedding[4]. If $Z \subseteq A'$ and $W \subseteq X'$ are subspaces such that $\overline{Z} \subseteq \operatorname{int} A'$, $\overline{W} \subseteq \operatorname{int} X'$ and $\rho(Z) \subseteq W$, then the inclusion morphism $(i, i') : \rho' \to \rho$, where ρ' denotes the restriction of ρ, induces an isomorphism in cohomology.

$$\begin{tikzcd}
(A \setminus Z, A' \setminus Z) \arrow[r, hook, "i'"] \arrow[d, hook, "\rho'"] & (X, A) \arrow[d, hook, "\rho"] \\
(X \setminus W, X' \setminus W) \arrow[r, hook, "i"] & (X, X')
\end{tikzcd}$$

Additivity Given a family of maps of sequences $\{\rho_\lambda\}_{\lambda\in\Lambda}$, let $(i_\lambda, j_\lambda) : \rho_\lambda \to \bigsqcup_{\lambda\in\Lambda} \rho_\lambda$ be the inclusion maps. The group $(h(\bigsqcup_{\lambda\in\Lambda} \rho_\lambda), (i_\lambda, j_\lambda)^*_{\lambda\in\Lambda})$ is the direct product of the groups $h(\rho_\lambda)$.

Consider the functor $C_2 : \mathtt{Top}^2_2 \to \mathtt{Top}^2$ defined on objects by

$$C_2((A, A') \xrightarrow{\rho} (X, X')) = C(A, A') \xrightarrow{C(\rho)} C(X, X'),$$

where $C(\rho)([a]) = [\rho(a)]$ and $C(\rho)([t, a']) = [t, \rho(a)]$, and similarly on morphisms.

Proposition 4.3.2. *Given a cohomology theory on maps (h, ∂), there exists a ∂' such that the pair $(h \circ C_2, \partial')$ is a cohomology on maps of pairs.*

Proof. Consider the long exact sequence of the composition 1.3.7 applied to $* \hookrightarrow C(A, A') \xrightarrow{C(\rho)} C(X, X')$, where the inclusion is the point in the vertex of the cone.

$$\cdots \to h^\bullet(C(\rho)) \to h^\bullet(* \to C(X, X')) \to h^\bullet(* \to C(A, A')) \xrightarrow{\beta} h^{\bullet+1}(C(\rho)) \to$$

This sequence translates into the sequence

$$\cdots \to h^\bullet(\rho) \to h^\bullet(X, X') \to h^\bullet(A, A') \to h^{\bullet+1}(\rho) \to \cdots$$

Taking $\partial' = \beta$, we get the long exact sequence.

To verify excision, we can assume without loss of generality that ρ is an inclusion. Consider the following diagram, where every map is a inclusion:

$$\begin{tikzcd}
h(C(A, A')) \arrow[r] \arrow[d] & h(C(A \setminus U, A' \setminus U)) \arrow[d] \\
h(C(X, X')) \arrow[r] & h(C(X \setminus U, X' \setminus U))
\end{tikzcd}$$

According to (DIECK, 2008, Propsoition 7.2.5), the horizontal maps are isomorphisms, which proves the result.

Additivity follows directly from the definition. $\square$

[4] By which we mean that the map $\rho : A \hookrightarrow X$ is an embedding

Reciprocally, given a cohomology on maps of pairs (h, ∂), we can define a cohomology on maps as $(h \circ I, \partial)$, where $I_\omega : \mathtt{Top}_2 \to \mathtt{Top}^2_2$ is given by

$$I(A \xrightarrow{\rho} X) = (A, \emptyset) \xrightarrow{\rho} (X, \emptyset).$$

which acts on morphisms as identity.

4.3.1 Some preliminary results

Before dealing with the cohomology on maps, we collect some results which will be useful ahead.

Lemma 4.3.3. *If (A, A') and (X, X') are cofibrations, then the cohomology group $h(\rho)$ of the map of pairs $\rho : (A, A') \to (X, X')$ is canonically isomorphic to $h(\overline{\rho})$, where $\overline{\rho} : \frac{A}{A'} \to \frac{X}{X'}$.*

Proof. Consider the quotient maps $q_A : (A, A') \to \left(\frac{A}{A'}, *\right)$ and $q_X : (X, X') \to \left(\frac{X}{X'}, *\right)$. Since both (A, A') and (X, X') are cofibrations, these maps induce isomorphisms in cohomology. Using the five lemma (Proposition A.4.1) in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & h^{\bullet-1}\left(\frac{X}{X'}, *\right) \arrow[r, "\overline{\rho}^*"] \arrow[d, "q_X^*"] & h^{\bullet-1}\left(\frac{A}{A'}, *\right) \arrow[r, "\partial_{\overline{\rho}}"] \arrow[d, "q_A^*"] & h^\bullet(\overline{\rho}) \arrow[r, "\left(id_{\frac{X}{X'}}, \varnothing_{\frac{A}{A'}}\right)^*"] \arrow[d, "(q_X, q_A)^*"] & h^\bullet\left(\frac{X}{X'}, *\right) \arrow[d, "q_X^*"] & \cdots \\
\cdots \arrow[r] & h^{\bullet-1}(X, X') \arrow[r, "\rho^*"] & h^{\bullet-1}(A, A') \arrow[r, "\partial_\rho"] & h^\bullet(\rho) \arrow[r, "(id_X, \varnothing_A)^*"] & h^\bullet(X, X') & \cdots
\end{tikzcd}$$

where q_X^* and q_A^* are isomorphisms, we get an isomorphism $(q_X, q_A)^* : h(\overline{\rho}) \to h(\rho)$, where $\overline{\rho} : \left(\frac{A}{A'}, *\right) \to \left(\frac{X}{X'}, *\right)$. Performing a diagram chase in the following diagram

$$\begin{tikzcd}
& 0 & 0 & & 0 & \\
\cdots \arrow[r] & h^\bullet(*, \emptyset) \arrow[r, "\mathrm{id}"] \arrow[u] & h^\bullet(*, \emptyset) \arrow[r, "0"] \arrow[u] & h^{\bullet+1}(*, *) \arrow[r, "0"] & h^{\bullet+1}(*, \emptyset) \arrow[r] \arrow[u] & {} \\
\cdots \arrow[r] & h^\bullet\left(\frac{X}{X'}, \emptyset\right) \arrow[r, "\rho^*"] \arrow[u] & h^\bullet\left(\frac{A}{A'}, \emptyset\right) \arrow[r, "\partial_\rho"] \arrow[u] & h^{\bullet+1}(\rho) \arrow[r] \arrow[u] & h^{\bullet+1}\left(\frac{X}{X'}, \emptyset\right) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & h^\bullet\left(\frac{X}{X'}, *\right) \arrow[r, "\overline{\rho}^*"] \arrow[u] & h^\bullet\left(\frac{A}{A'}, *\right) \arrow[r, "\partial_{\overline{\rho}}"] \arrow[u] & h^{\bullet+1}(\overline{\rho}) \arrow[r] \arrow[u] & h^{\bullet+1}\left(\frac{X}{X'}, *\right) \arrow[r] \arrow[u] & \cdots \\
& 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

and using that $h(*, *) = 0$, we conclude that the identity map $\mathrm{id} : h(\overline{\rho}) \to h(\rho)$ is an isomorphism. $\square$

Lemma 4.3.4. *Consider a map of pairs $(\rho, \rho') : (A, A') \to (X, X')$. The corresponding mapping cones of ρ and ρ' form the pair $(C_\rho, C_{\rho'})$. There is a canonical isomorphism $h^\bullet(\rho, \rho') \cong \widetilde{h}^\bullet(C_\rho, C_{\rho'})$.*

Proof. This follows directly from the relation $h^\bullet(\rho,\rho') \cong h^\bullet(C(\rho)) \cong \widetilde{h}^\bullet(C_{C(\rho)})$, the fact that there exist a canonical homeomorphism between $C(C_\rho, C_{\rho'})$ and $C_{C(\rho)}$, and the relation $\widetilde{h}(C(C_\rho, C_{\rho'})) \cong h(C_\rho, C_{\rho'})$. $\square$

Corollary 4.3.5. *If (A,A') and (X,X') are cofibrations and $\rho' : A' \to X'$ is a homeomorphism, then the cohomology groups of $(\rho,\rho') : (A,A') \to (X,X')$ are canonically isomorphic to the ones of $\rho : A \to X$.*

Proof. It follows from lemma 4.3.4 that $h^\bullet(\rho,\rho') \cong h^\bullet(C\rho, C\rho')$. Since ρ' is a homeomorphism, $C\rho' \cong CA'$ and so it is contractible. Since (A,A') and (X,X') are cofibrations, the embedding $C\rho' \hookrightarrow C\rho$ is a cofibration too, hence

$$h^\bullet(C_\rho, C_{\rho'}) \cong \widetilde{h}^\bullet\left(\frac{C_\rho}{C_{\rho'}}\right) \overset{(1)}{\cong} \widetilde{h}(C_\rho) \cong h^\bullet(\rho),$$

where (1) comes from the long exact sequence of reduced cohomology for cofibrations

$$\cdots \widetilde{h}(C_{\rho'}) \xrightarrow{\partial} \widetilde{h}^\bullet\left(\frac{C_\rho}{C_{\rho'}}\right) \xrightarrow{q^*} \widetilde{h}(C_\rho) \longrightarrow \cdots$$

by using the fact that $\widetilde{h}(C_{\rho'}) = 0$. $\square$

Lemma 4.3.6. *Let us consider the following data:*

- *an adjunction space $A \cup_{A'} B$, defined through a cofibration $A' \hookrightarrow A$ and a map $f : A' \to B$;*
- *a map $\rho_A : A \to X$ and a cofibration $\rho_B : B \hookrightarrow X$, such that $\rho_B \circ f = (\rho_A)|_{A'}$;*
- *the map $\rho : A \cup_A B \to X$ induced by ρ_A and ρ_B.*

as depicted in the diagram

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, hook] & B \arrow[d, hook] \arrow[ddr, bend left, "\rho_B"] & \\
A \arrow[r, "\pi"] \arrow[drr, bend right, "\rho_A"'] & A \cup_{A'} B \arrow[dr, "\rho"] & \\
& & X
\end{tikzcd}$$

Then we have a canonical isomorphism $h^\bullet(\rho : A \cup_{A'} B \to X) \cong h^\bullet(\rho_A : (A,A') \to (X,B))$.

Proof. It follows from the hypotheses that $B \hookrightarrow A \cup_{A'} B$ is a cofibration too (Proposition A.2.5). Identifying B with its image through ρ_B, it follows from Corollary 4.3.5 that

$h^\bullet(\rho : A \cup_{A'} B \to X)$ is canonically isomorphic to $h^\bullet(\rho : (A \cup_{A'} B, B) \to (X, B))$. Now, consider the natural morphism $(\mathrm{id}_X, \pi) : \rho_A \to \rho$, defined as follows:

$$\begin{tikzcd}
(A, A') \arrow[r, "\pi"] \arrow[d, "\rho_A"] & (A \cup_{A'} B, B) \arrow[d, "\rho"] \\
(X, B) \arrow[r, equal, "\mathrm{id}_X"] & (X, B).
\end{tikzcd}$$

Since both the domain and codomain of π are cofibrations, we have $h(\pi) \cong h(\overline{\pi})$ as in Lemma 4.3.3. Since

$$\overline{\pi} : \frac{A}{A'} \to \frac{A \cup_{A'} B}{B}$$

is a homeomorphism, Proposition 1.3.5 implies that $h(\overline{\pi}) = 0$ and consequently $h(\pi) = 0$. By the converse of the same proposition (adapted to the case of pairs), we conclude that the pull-back $\pi^* : h(A \cup_{A'} B, B) \to h(A, A')$ is an isomorphism. By an analogue of proposition 1.3.6, we conclude that $(\mathrm{id}_X, \pi)^* : h(\rho) \to h(\rho_A)$ is an isomorphism too. □

Corollary 4.3.7. *Let us consider a map $\rho : A \cup B \to X$ such that $A \cap B \hookrightarrow A$ and $\rho|_B : B \hookrightarrow X$ are cofibrations. We have the canonical isomorphism $h^\bullet(\rho : A \cup B \to X) \cong h^\bullet(\rho|_A : (A, A \cap B) \to (X, B))$.*

Proof. It is enough to choose $A' = A \cap B$ in the statement of Lemma 4.3.6, the function f being the embedding $A \cap B \hookrightarrow B$ and ρ_A and ρ_B being the restrictions of ρ. □

Remark 4.3.8. For a cofibration $i_B : B \hookrightarrow Y$, the last corollary gives us a particularly useful relation:

$$h(\mathrm{id}_Y \times \rho \cup i_B \times \mathrm{id}_X) \cong h(\mathrm{id}_Y \times \rho : (Y \times A, B \times A) \to (Y \times X, B \times X)), \tag{4.2}$$

Here, $\mathrm{id}_Y \times \rho \cup i_B \times \mathrm{id}_X : Y \times A \cup B \times X \to Y \times X$ is the natural map.

In particular, this relation enables us to define a new kind of product

$$h(Y, B) \times h(\rho) \to h(\mathrm{id}_Y \times \rho),$$

where $\mathrm{id}_Y \times \rho : (Y \times A, B \times A) \to (Y \times X, B \times X)$ whenever (Y, B) is a cofibration.

Indeed, given a multiplicative structure on a cohomology of maps as in Definition 1.5.4, we have

$$h(Y, B) \times h(\rho) \to h(i_B \wedge \rho).$$

Since $i_B : B \hookrightarrow Y$ is a cofibration, it holds that

$$h(i_B \wedge \rho) = h(\mathrm{id}_Y \times \rho \cup i_B \times \mathrm{id}_X).$$

Then, by applying (4.2), the product $h(Y, B) \times h(\rho)$ can be restated in our language as

$$h(Y, B) \times h(\rho) \to h(\mathrm{id}_Y \times \rho : (Y \times A, B \times A) \to (Y \times X, B \times X)). \tag{4.3}$$

This, in turn, means that we have a product between a cofibration and a map which assumes values in a cohomology of maps of pairs. This is an example of a product between sequences of cofibration and maps of sequences, which we will discuss in section 4.5.

4.3.2 Extension to $\mathtt{Top}^2_\omega$

Now we define a cohomology theory on maps of sequences.

Definition 4.3.9 (Cohomology on maps of sequences)**.** Given a cohomology theory on maps of pairs (h, ∂), we define a cohomology theory on sequences as the functor $h \circ J^2_{\omega,2} : \mathtt{Top}^2_\omega \to \mathtt{GrAb}$, where $J^2_{\omega,2}$ is the functor defined in (4.1).

Proposition 4.3.10. *The cohomology over $\mathtt{Top}^2_\omega$ satisfies the following properties:*

Long exact sequence: *Given $(\rho, \vec{\rho}) : (A, \vec{A}) \to (X, \vec{X})$, we have the following long exact sequence:*

$$\cdots \longrightarrow h^\bullet(\rho, \vec{\rho}) \xrightarrow{(id_X, \varnothing_A)^*} h(X, \vec{X}) \xrightarrow{(\rho, \varnothing)^*} h^\bullet(A, \vec{A}) \xrightarrow{\partial_\rho} h^{\bullet+1}(\rho, \vec{\rho}) \longrightarrow \cdots,$$

where $h(X, \vec{X})$ is shorthand for $h(\varnothing_X, \varnothing_{\vec{X}})$

Excision: *Let $\rho : (A, \vec{A}) \hookrightarrow (X, \vec{X})$ be an embedding. If $Z \subseteq A$ and $W \subseteq X_i$ are subspaces such that $\overline{Z} \subseteq \operatorname{int} A_i$, $\overline{W} \subseteq \operatorname{int} X_i$ for all $i = 1, \ldots, n$, and $\rho(Z) \subseteq W$, then the inclusion morphism $(i, i') : \rho \to \rho'$, where ρ' denotes the restriction of ρ, induces isomorphism in cohomology:*

$$\begin{array}{ccc}
(A \setminus Z, A_1 \setminus Z, \ldots, A_n \setminus Z) & \xhookrightarrow{\;i'\;} & (A, A_1, \ldots, A_n) \\
\big\downarrow{\scriptstyle \rho'} & & \big\downarrow{\scriptstyle \rho} \\
(X \setminus W, X_1 \setminus W, \ldots, X_n \setminus W) & \xhookrightarrow{\;i\;} & (X, X_1, \ldots, X_n)
\end{array}$$

Interchangeability *For cofibration sequences $(X, X_1, \ldots, X_n)$, one has*

$$h(X, X_1, \ldots, X_n) \cong h(X, X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

*for any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, in a **natural** way.*

Remark 4.3.11. It is important to understand that the relevant fact in exchangebility is **naturality** and not the isomorphism itself (which is obvious from its definition).

Proof. Both the long exact sequence and the excision follow immediately from the definition. So we only need to prove exchangebility.

By induction, it suffices to prove exchangebility for a transposition. We prove exchangebility for the triple (X, A, B). Consider the following ladder diagram:

$$\begin{tikzcd}
(\emptyset,\emptyset,\emptyset) \arrow[r] \arrow[d, hook] & (X,A,B) \arrow[d, hook] \\
(B,\emptyset,\emptyset) \arrow[r] & (X,A,B) \\
(B,\emptyset,\emptyset) \arrow[r] \arrow[u, hook] \arrow[d, hook] & (X,A,\emptyset) \arrow[u, hook] \arrow[d, hook] \\
(B,\emptyset,\emptyset) \arrow[r] & (X,A,A) \\
(B,\emptyset,\emptyset) \arrow[r] \arrow[u, hook] \arrow[d, hook] & (X,\emptyset,A) \arrow[u, hook] \arrow[d, hook] \\
(B,\emptyset,\emptyset) \arrow[r] & (X,B,A) \\
(\emptyset,\emptyset,\emptyset) \arrow[r] \arrow[u, hook] & (X,B,A) \arrow[u, hook]
\end{tikzcd} \tag{4.4}$$

Each vertical morphism induces an isomorphism in cohomology at each level. In order to see why, we observe that each row map has cohomology isomorphic to $\widetilde{h}(X/(A \cup B))$. □

Remark 4.3.12. This definition is narrower than we would like it to be from a practical perspective. For example, it cannot take into account the de Rham cohomology, since the union is not in the admissible category. Nevertheless, we believe that it is possible to give an axiomatization in the veins of Proposition 4.3.10. At the moment, we don't know if the interchangeability property is required to prove the equivalence of cohomology on $\mathtt{Top}^2_\omega$ with cohomology on $\mathtt{Top}$, as was the case with cohomology on $\mathtt{Top}^2_2$. I was not able to pursue this path since I could not find a proper axiomatization which would imply compatibility with Definition 4.3.1 or, at least, make Proposition 4.3.10 hold.

4.4 S^1 Integration

We now define the S^1-integration on relative cohomology on maps of pairs. Let $(\rho, \rho') : (A, A') \to (X, X')$ be a morphism and choose $1 \in S^1$ as marked point. Consider the morphisms of maps of pairs $(i_X, i'_A) : (\rho, \rho') \to \mathrm{id}_{S^1} \times (\rho, \rho')$ and $(\mathrm{pr}_X, \mathrm{pr}_A) : \mathrm{id}_{S^1} \times (\rho, \rho') \to (\rho, \rho')$. These maps induces the following maps

$$C(i) := C(i_X, i_A) = \frac{C_\rho}{C_{\rho'}} \to \frac{C_{\rho \times \mathrm{id}_{S^1}}}{C_{\rho' \times \mathrm{id}_{S^1}}} \text{ and } C(\mathrm{pr}) := C(\mathrm{pr}_X, \mathrm{pr}_A) : \frac{C_{\rho \times \mathrm{id}_{S^1}}}{C_{\rho' \times \mathrm{id}_{S^1}}} \to \frac{C_\rho}{C_{\rho'}}$$

which satisfies $C(\mathrm{pr}) \circ C(i) = \mathrm{id}_{\frac{C_\rho}{C_{\rho'}}}$ and fit in the following following split exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{h}^\bullet\left(\frac{\frac{C_{\mathrm{id}_{S^1}\times\rho}}{C_{\mathrm{id}_{S^1}\times\rho}}}{\frac{C_\rho}{C'_\rho}}\right) \arrow[r, "q"] & \widetilde{h}^\bullet\left(\frac{C_{\mathrm{id}_{S^1}\times\rho}}{C_{\mathrm{id}_{S^1}\times\rho'}}\right) \arrow[r, "C(i)^*"] \arrow[l, bend left, "h"] & \widetilde{h}^\bullet\left(\frac{C_\rho}{C_{\rho'}}\right) \arrow[r] \arrow[l, bend left, "C(\mathrm{pr})^*"] & 0, \end{tikzcd}$$

where $h(\alpha) = (C(\mathrm{pr})^*)^{-1}(\alpha - C(\mathrm{pr})^* \circ C(i)^*(\alpha))$. Using the following natural isomorphism

$$s : \widetilde{h}^n\left(\frac{C_{\rho\times\mathrm{id}_{S^1}}}{\frac{C_{\rho'\times\mathrm{id}}}{\frac{C_\rho}{C_{\rho'}}}}\right) \cong \widetilde{h}^n\left(\frac{\frac{C_{\rho\times\mathrm{id}_{S^1}}}{C_\rho}}{\frac{C_{\rho'\times\mathrm{id}}}{C_{\rho'}}}\right) \cong \widetilde{h}\left(\frac{\Sigma C_\rho}{\Sigma C_{\rho'}}\right) \cong \widetilde{h}^n\left(\Sigma\frac{C_\rho}{C_{\rho'}}\right) \cong \widetilde{h}^{n-1}\left(\frac{C_\rho}{C_{\rho'}}\right)$$

together with Lemma 4.3.4, which enables us to identify $h(\rho,\rho')$ and $h(\mathrm{id}_{S^1}\times\rho, \mathrm{id}_{S^1}\times\rho')$ with $\widetilde{h}(C_\rho/C_{\rho'})$ and $\widetilde{h}(C(\mathrm{id}_{S^1}\times\rho)/C(\mathrm{id}_{S^1}\times\rho'))$ respectively, we get sequence

$$\begin{tikzcd} s: \ 0 \arrow[r] & h^{\bullet-1}(\rho,\rho') \arrow[r, "q^*\circ s^{-1}"] & h^\bullet(\mathrm{id}_{S^1}\times\rho, \mathrm{id}_{S^1}\times\rho') \arrow[r, "(i_X,i_A)^*"] \arrow[l, bend left, "s\circ h"] & h^\bullet(\rho,\rho') \arrow[r] \arrow[l, bend left, "(\mathrm{pr}_X,\mathrm{pr}_A)^*"] & 0, \end{tikzcd} \tag{4.5}$$

We define the S^1-integration map as the composition $\int_{S^1} = s \circ h$.

4.5 Mixed multiplicative structure

Lemma 4.5.1. *Given a cofibration (Y,B) and a pair (X,A), the cohomology of the following maps:*

$$(\emptyset,\emptyset,\emptyset) \hookrightarrow (Y\times X, B\times X, Y\times A)$$

and

$$(Y\times A, B\times A, \emptyset) \hookrightarrow (Y\times X, B\times X, \emptyset),$$

are naturally isomorphic.

Proof. Consider the following diagram of inclusions

$$\begin{tikzcd} (\emptyset,\emptyset,\emptyset) \arrow[r] \arrow[d, hook] & (Y\times X, B\times X, Y\times A) \arrow[d, equal] \\ (Y\times A, B\times A, Y\times A) \arrow[r, hook] & (Y\times X, B\times X, Y\times A) \\ (Y\times A, B\times A, \emptyset) \arrow[r, hook] \arrow[u, hook] & (Y\times X, B\times X, \emptyset) \arrow[u, hook] \end{tikzcd} \tag{4.6}$$

The vertical morphisms induce isomorphisms, since we get the reduced cohomology of $X/A \wedge Y/B$ in each of the three rows (this can bee seen applying Lemma 4.3.3.) $\square$

For a map of sequences $(\rho, \vec{\rho}) : (A, \vec{A}) \to (X, \vec{X})$ and a sequence $(Y, \vec{B})$, we define $(Y, \vec{B}) \times (\rho, \vec{\rho})$ as the map of sequence

$$\mathrm{id}_Y \times \rho : (Y \times A, \vec{B} \times A, Y \times \vec{A}, B \times \vec{A}) \to (Y \times X, \vec{B} \times X, B \times \vec{X})$$

where

$$Y \times \vec{A} := (Y \times A_1, \ldots, Y \times A_n)$$

and analogously to the other products.

Remark 4.5.2. Given a pair (X, A) we can regard it as be either as $i_{(A,\emptyset)} : (A, \emptyset) \to (X, \emptyset)$ or as $\varnothing_{(X,A)} : (\emptyset, \emptyset) \to (X, A)$. Given another pair (Y, B), we can think of $(Y, B) \times i_{A,\emptyset}$ which is

$$\mathrm{id}_Y \times i_{A,\emptyset} : (Y \times A, B \times A, \emptyset) \to (Y \times X, B \times X, \emptyset)$$

or as $(Y, B) \times i_{(X,A)}$ which is

$$\mathrm{id}_Y \times \varnothing_{(X,A)} : (\emptyset, \emptyset, \emptyset) \to (Y \times X, Y \times A, A \times B)$$

By Lemma 4.5.1, there exists a natural isomorphism between $h(\mathrm{id}_Y \times \varnothing_{(X,A)})$ and $h(\mathrm{id}_Y \times i_{(A,\emptyset)})$. The same remark holds for arbitrary sequences $(X, \vec{A})$: any interpretation of of $h((Y, \vec{B}) \times (X, \vec{A}))$ is isomorphic to $h(Y \times X, \vec{B} \times X, Y \times \vec{A})$.

Remark 4.5.3. By the previous remark, $h((Y, B) \times (X, A))$ can be canonically identified with $h(Y \times X, B \times X, Y \times A)$. Composing with the isomorphism of tuples $t : Y \times X \to X \times A$ we get

$$h(Y \times X, B \times X, Y \times A) \cong h(X \times Y, X \times B, A \times Y)$$

Using the interchangeability property of Proposition 4.3.10, we get

$$h(X \times Y, X \times B, A \times Y) \cong h(X \times Y, A \times Y, X \times B)$$

Since $h(X \times Y, A \times Y, X \times B) \cong h((X, A) \times (Y, B))$, this shows that we have a canonical isomorphism

$$h((X, A) \times (Y, B)) \cong h((Y, B) \times (X, A)).$$

This observation hold for arbitrary sequences, that is,

$$h((X, \vec{A}) \times (Y, \vec{B})) \cong h((Y, \vec{B}) \times (X, \vec{A})).$$

Definition 4.5.4 (Module Structure over Sequences of Cofibrations)**.** A module structure over sequences of cofibrations on a cohomology theory on finite sequences of spaces is a natural transformation $\times : h(Y, \vec{B}) \otimes_{\mathbb{Z}} h(\rho, \vec{\rho}) \to h((Y, \vec{B}) \times (\rho, \vec{\rho}))$ satisfying the following properties

M1) (*Mixed associativity:*) Given a map $\rho : (A, \vec{A}) \to (X, \vec{X})$ and two cofibrations $(Y, \vec{B})$ and $(Z, \vec{C})$, we have that

$$(\gamma \times \beta) \times \alpha = \gamma \times (\beta \times \alpha)$$

for every $\alpha \in h(\rho, \vec{\rho})$, $\beta \in h(Y, \vec{B})$ and $\gamma \in h(Z, \vec{C})$. Here we are considering the product $\gamma \times \beta$ in the cohomology of $(Y \times Z, \vec{B} \times Z, Y \times \vec{Z})$ as in Remark 4.5.2.

M2) (*Graded-commutativity on sequences*) Given two cofibrations $(X, \vec{A})$ and $(Y, \vec{B})$, for every $\alpha \in h^n(X, \vec{A})$ and $\beta \in h^m(Y, \vec{B})$ we have that

$$\alpha \times \beta = (-1)^{nm} \beta \times \alpha,$$

up to the canonical identification $h((X, \vec{A}) \times (Y, \vec{B})) \cong h((Y, \vec{B}) \times (X, \vec{A}))$ as in Remark 4.5.3.

M3) (*Distributivity*)

$$(\alpha + \beta) \times \gamma = (\alpha \times \gamma) + (\beta \times \gamma)$$

and

$$\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$$

whenever these operations make sense.

M4) (*Unitarity*) There exists a unit element $1 \in h^0(pt)$.

M5) (*Stability*) Given $\alpha \in h(X, \vec{X})$ and $\gamma \in h(Y, \vec{B})$

$$\begin{tikzcd} h(Y, \vec{B}) \otimes h^n(A, \vec{A}) \arrow[r, "\times"] \arrow[d, "\mathrm{id}_{(Y,B)} \otimes \partial"] & h((Y, \vec{B}) \times (A, \vec{A})) \arrow[d, "\partial"] \\ h(Y, \vec{B}) \otimes h(\rho, \vec{\rho}) \arrow[r, "\times"] & h((Y, \vec{B}) \times (\rho, \vec{\rho})) \end{tikzcd}$$

where ∂ are the connecting morphisms.

Proposition 4.5.5. *Let (h, ∂) be a multiplicative cohomology on maps and let h_ω its extension to maps of sequences*[5]*. A multiplicative structure on h induces a multiplicative structure on cofibrations in h_ω.*

Proof. Fix a cofibration sequence $(Y, \vec{B})$ and a map of sequences $\rho : (A, \vec{A}) \to (X, \vec{X})$. Applying the functor $J^2_{\omega,2}$ to $h(Y, \vec{B}) \otimes h(\rho, \vec{\rho})$ we get

$$h(Y, B') \otimes h(\rho : (A, A') \to (X, X'))$$

5 First inducing a cohomology on maps of pairs through the cone functor and then using the definition of cohomology on sequences.

and applying the cone functor $C_2 : \mathtt{Top}_2^2 \to \mathtt{Top}^2$ it follows that

$$h(C(Y, B'), *) \otimes h(C(\rho))$$

Since B' is a cofibration, $(C(Y, B'), *)$ is a cofibration too. Applying the definition of product in $\mathtt{Top}^2$ for an excisive triple we get a product

$$h(i_*) \otimes h(C(\rho)) \to h(\mathrm{id}_* \times C(\rho) \cup_{* \times C(A,A')} i_* \times C(\rho))$$

where $i_* : * \hookrightarrow C(Y, B')$, By Remark 4.3.8, this is the same as

$$h(\mathrm{id}_{C(Y,B')} \times C(\rho)) : (C(Y, B') \times C(A, A), * \times C(A, A)) \hookrightarrow (C(Y, B') \times C(X, X'), * \times C(X, X'))$$

Since the pairs are cofibration, we apply Lemma 4.3.3 and get

$$h(\mathrm{id}_{C(Y,B')} \times C(\rho)) = h(\overline{\mathrm{id}_{C(Y,B')} \times C(\rho)})$$

where

$$\overline{\mathrm{id}_{C(Y,B')} \times C(\rho)} : \frac{C(Y, B') \times C(A, A')}{* \times C(A, A')} \to \frac{C(Y, B') \times C(X', X)}{* \times C(X, X')} \tag{4.7}$$

On the other hand, we have

$$h((Y, \vec{B}) \times h(\rho, \vec{\rho})) = h(\mathrm{id}_Y \times \rho : (Y \times A, \vec{B} \times A, Y \times \vec{A}) \to (Y \times X, \vec{B} \times X, Y \times \vec{X}))$$

applying the functor $J^2_{\omega,2}$ we get

$$h(\mathrm{id}_Y \times \rho : (Y \times A, B' \times A \cup Y \times A') \to (Y \times X, B' \times X \cup Y \times X'))$$

This is a cofibration pair. Applying lemma 4.3.3, we get

$$h(\mathrm{id}_Y \times \rho) = h(\overline{\mathrm{id}_Y \times \rho})$$

where

$$\overline{\mathrm{id}_Y \times \rho} : \frac{Y \times A}{B' \times A \cup Y \times A'} \to \frac{Y \times X}{B' \times X \cup Y \times X'} \tag{4.8}$$

Once we show that the cohomology of this two maps in (4.7) and (4.8) are the same we are done. Observe that the map in (4.8) is isomorphic to

$$\mathrm{id}_{\frac{Y}{B}} \wedge \overline{\rho} : \frac{Y}{B'} \wedge \frac{A}{A'} \to \frac{Y}{B'} \wedge \frac{X}{X'}$$

The naturally defined quotient $q_X : \frac{C(Y,B) \times C(X,X')}{* \times C(X,X')} \to \frac{Y}{B} \wedge \frac{X}{X'}$ induces isomorphism in cohomology. Therefore the morphism (q_X, q_A) give us the desired isomorphism. $\square$

Remark 4.5.6. Clearly, the mixed product of Definition 4.5.4 is not the most general product one can expect to define in a cohomology theory on maps of sequences, since we have a module structure over cofibrations only. But this product will be convenient for our needs.

4.6 Compactly-like Cohomology and Thom Isomorphism

Now, we review Section 1.6.2 under the light of the "new" product of the previous section. In this section, every manifold and map is assumed to be smooth.

4.6.1 Absolute case

The definition of compactly supported cohomology and vertically compact supported cohomology of a space remains the same as in Definition 1.6.3 and 1.6.10 we just rewrite the Thom isomorphism with the new product (Definition 4.5.4). The usual Thom isomorphism remains the same, so we deal only with the compact and doubly-vertically compact case.

Given a smooth fiber bundle, we wish to redefine the product between a vertically compact class and a compact class using the multiplicative structure over cofibration rather than the classical topological product as was done in (1.11). Clearly, this is not useful in the topological setting, but will repay our efforts in the differential framework.

To define the compactly supported Thom isomorphism $T_c : h_c^\bullet(X) \to h_c^{\bullet+n}(E)$ for a vector bundle $p : E \to X$ using the product defined in the last section, we need to multiply a class $u \in h_v^n(E)$ by the pullback of a class $\alpha \in h_c(X)$. Unfortunately, this is not so simple: we can only multiply by a pair which is a cofibration, but the complement V^c of a vertically compact set V is open by Proposition 1.6.9 and therefore is not a cofibration.

In order to solve this problem, we remark that it is possible to obtain a cofinal system $\mathcal{V}_e(p)$ of $\mathcal{V}(p)$ such that a vertically compact set $V \in \mathcal{V}_e(p)$ is a manifold of the same dimension as E (with its boundary in the fibers of E) and $(\operatorname{int} V)^c$ is also a manifold of the same dimension of E. Under this hypothesis they are the closure of their interior. The collaring theorem, tell us that $h(V^c) \cong h(\operatorname{int}(V)^c)$. To see this is true, one can think about the tubes described in section 1.6.

Compactly supported Thom isomorphism

Let $p : E \to X$ be a n-dimensional vector bundle with Thom class $u \in h_v^n(E)$. We are going to define the product

$$\begin{aligned} \cdot p : h_v(E) \otimes_{\mathbb{Z}} h_c(X) &\to h_c(E) \\ u \otimes \alpha &\mapsto \cdot u \cdot p^*\alpha, \end{aligned} \tag{4.9}$$

Choose a representative $u_V \in h(E, V^c) \cong h(E, \operatorname{int}(V)^c)$ with $V \in \mathcal{V}_e(p)$ of u and a representative $\alpha_K \in h(X, H^c)$, where $H := p^{-1}(K)$ with $K \in \mathcal{K}(X)$, of α. Applying the product of definition 4.5.4, which in this case is just the one that appears in (4.3), we have

$$u_V \times p^*(\alpha_K) \in h\left(\mathrm{id}_E \times i : (E \times H^c, \operatorname{int}(V)^c \times H^c) \hookrightarrow (E \times E, \operatorname{int}(V)^c \times E)\right)$$

Now, we use the diagonal morphism $(\Delta, \Delta|_{H^c}) : i \to \mathrm{id}_E \times i$ as in the diagram

$$\begin{tikzcd}
(H^c, \mathrm{int}(V)^c \cap H) \arrow[r, "\Delta|_{H^c}"] \arrow[d, "i"] & (E \times H^c, \mathrm{int}(V)^c \times H^c) \arrow[d, "\mathrm{id}_E \times i"] \\
(E, \mathrm{int}(V)^c) \arrow[r, "\Delta"] & (E \times E, \mathrm{int}(V)^c \times E)
\end{tikzcd} \tag{4.10}$$

where $\Delta(x) = (x, x)$, to pullback $u_V \times p^*(\alpha)$ obtaining

$$(\Delta, \Delta|_{H^c})^*(u_V \times p^*(\alpha_K)) \in h(i).$$

We want to apply Corollary 4.3.7 to $h(i)$ in such a way that we obtain a class in $h(k)$ where $k : (\mathrm{int}(\mathrm{int}(V)^c), \emptyset) \to (E, \emptyset)$. In order to do this, we consider the following auxiliary map

$$j : (H^c \cup \mathrm{int}(\mathrm{int}(V)^c), \mathrm{int}(V)^c \cap (H^c \cup \mathrm{int}(\mathrm{int}(V)^c))) \hookrightarrow (E, \mathrm{int}(V)^c)$$

Observe that we have two "inclusion morphisms" $i \hookrightarrow j$ and $k \hookrightarrow j$ as depicted in the following diagram

$$\begin{tikzcd}
(H^c, \mathrm{int}(V)^c \cap H^c) \arrow[r, "i"] \arrow[d, hook] & (E, \mathrm{int}(V)^c) \arrow[d, hook] \\
(H^c \cup \mathrm{int}(\mathrm{int}(V)^c), \mathrm{int}(V)^c \cap (H^c \cup \mathrm{int}(\mathrm{int}(V)^c))) \arrow[r, "j"] & (E, \mathrm{int}(V)^c) \\
(\mathrm{int}(\mathrm{int}(V)^c) \cup H^c, \emptyset) \arrow[r, hook, "k"] \arrow[u, hook] & (E, \emptyset) \arrow[u, hook]
\end{tikzcd}$$

Excision tell us that

$$(H^c \cup \mathrm{int}(\mathrm{int}(V)^c), \mathrm{int}(V)^c \cap (H^c \cup \mathrm{int}(\mathrm{int}(V)^c))) \cong (H^c, \mathrm{int}(V)^c \cap H^c)$$

and a analogue of Proposition 1.3.6 implies that the "inclusion" morphism $i \hookrightarrow j$ is a isomorphism. The morphism j satisfies the conditions of Corollary 4.3.7 which implies that $k \hookrightarrow j$ is a isomorphism. We conclude that we have isomorphisms

$$h(i) \xleftarrow{\sim} h(j) \xrightarrow{\sim} h(k). \tag{4.11}$$

Since $(\mathrm{int}(\mathrm{int}(V)^c)) = V^c$ by definition of $V \in \mathcal{V}_e(p)$, we have

$$(\Delta, \Delta|_{H^c})^*(u_V \times p^*(\alpha)) \in h(k : \mathrm{int}(\mathrm{int}(V)^c) \cup H^c \hookrightarrow E) = h(E, (V \cap H)^c).$$

up to implicitly isomorphisms. Notice that $V \cap H \in \mathcal{K}(E)$. Passing the colimit where we use the fact that $\mathcal{V}_e(p)$ is cofinal, we obtain product 4.9.

Since this construction is very important to us, we highlight its construction in the following composition:

$$\begin{aligned}\cdot p^* : h(E, V^c) \otimes_{\mathbb{Z}} h(E, H^c) \xrightarrow{(1)} h(E, \mathrm{int}(V)^c) \otimes_{\mathbb{Z}} h(E, H^c) \xrightarrow[(4.3)]{\times} \\ h((E \times H^c, \mathrm{int}(V)^c \times H^c) \overset{i'}{\hookrightarrow} (E \times E, \mathrm{int}(V)^c \times E)) \xrightarrow[4.10]{(\Delta, \Delta|_{H^c})^*} \\ h((H^c, \mathrm{int}(V)^c \cap H^c) \overset{i}{\hookrightarrow} (E, \mathrm{int}\, V^c)) \overset{(4.11)}{\cong} \\ h(\mathrm{int}(\mathrm{int}(V)^c) \cup H^c \overset{k}{\hookrightarrow} E) \overset{(2)}{=} h((V \cap H)^c \overset{k}{\hookrightarrow} E) \quad (4.12)\end{aligned}$$

where in (1) we use that $V^c \simeq \mathrm{int}(V)^c$ and in (2) we use that $\mathrm{int}(\mathrm{int}(V)^c) = V^c$ for $V \in \mathcal{V}_e(p)$.

Remark 4.6.1. The doubly-vertically compact case can be described in the same way. In this case, we get a product $\cdot\,\mathrm{pr} : h_v(E) \otimes_{\mathbb{Z}} h_v(X) \to h_{vv}(E)$.

The differential version of the next proposition will play an important role in the characterization of the the differential integration.

Proposition 4.6.2. *Fix a oriented vector bundle embedding $(p, p') : \bar{i} \to i$ as in the diagram*

$$\begin{array}{ccc} E' & \overset{\bar{i}}{\hookrightarrow} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \overset{i}{\hookrightarrow} & X \end{array}$$

where $X' \hookrightarrow X$ is an open embedding. Let $T_c : h_c^\bullet(X) \to h_c^{\bullet+n}(E)$ be the compactly supported Thom isomorphism associated to p and $T'_c : h_c^\bullet(X') \to h_c^{\bullet+n}(E')$ be the compactly supported Thom isomorphism associated to $p' : E' \to X'$ with its induced Thom class. It follows that

$$T_c \circ i_* = \bar{i}_* \circ T'_c$$

Proof. The proof is carried at representative level. First, we remark that the Thom classes $u \in h_v(E)$ and $u' \in h_v(E')$ are relate at representative level through $\bar{i}$ as $u'_V = \bar{i}^* u_{\bar{i}(V)}$. In the same way, given a class $\alpha' \in h_c(X)$ we have its image $\alpha = i_* \alpha'$ at representative level as $\alpha'_K = i^* \alpha_{i(K)}$ We wish to verify that

$$u_V \cdot p'^* \alpha_K = \bar{i}^* (u_{\bar{i}(V)} \cdot p^* \alpha_{i(K)})$$

which we do applying the construction of the product as pullback by diagonal.

$$\begin{aligned}\bar{i}^*(u_{\bar{i}(V)}\cdot p^*\alpha_{i(K)}) =&\bar{i}^*(\Delta,\Delta')^*(u_{\bar{i}(V)}\times p^*\alpha_{i(K)})\\ =&(\Delta,\Delta')\circ(\bar{i}^*\times\bar{i}^*)(u_{\bar{i}(V)}\times p^*\alpha_{i(K)})\\ =&(\Delta,\Delta')\circ(\bar{i}^*u_{\bar{i}(V)}\times(\bar{i}^*\circ p^*)\alpha_{i(K)})\\ =&(\Delta,\Delta')\circ\left(u'_V\times p'^*\circ i^*\alpha_{i(K)}\right)\\ =&(\Delta,\Delta')\circ\left(u'_V\times p'^*\alpha'_K\right)\\ =&u'_V\cdot p'^*\alpha'_K\end{aligned}$$

which shows the compatibility. □

Remark 4.6.3. The same holds in the vertically compact case exchanging T_c by T_v and reinterpreting the map $\bar{i}_*: h_v(E')\to h_v(E)$ as the one in Proposition 1.6.22.

4.6.2 The relative case

In section 1.6.5, we commented on the relative version of the vertically supported compact cohomology but refrain ourselves to give a proper definition for the relative cohomology with compact supports. In this section we resume the subject.

Let us consider a pair of spaces (X,A), with $A\subseteq X$ closed. We are going to define $h_c^\bullet(X,A)$ adapting the construction used in (RUFFINO; BARRIGA, 2021, sec. 5.1).

We think of $\mathcal{K}(X)$ as a category, whose objects are the compact subsets of X and such that the set of morphisms from K to L contains one element if $K\subseteq L$ and is empty otherwise. There is a natural functor $F_{(X,A)}$ assigning to an object K the embedding $i_K: K^c\cup A\to X$ and to a morphism $K\subseteq L$ the natural morphism $i_{KL}: i_L\to i_K$ defined by the following diagram:

$$\begin{tikzcd} A\cup L^c \arrow[r, hook] \arrow[d, hook, "i_L"] & A\cup K^c \arrow[d, hook, "i_K"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

The compactly-supported groups $h^\bullet(X,A)$ are defined as the colimit of the composition functor $h\circ F_{(X,A)}:\mathcal{K}(X)\to \texttt{GrAb}$ and they are functorial with respect to open embeddings of the form $\iota := (\iota,\iota|_B):(Y,B)\to(X,A)$, such that $B=\iota^{-1}(A)$. In fact, for any compact subset $K\subset Y$, from the embedding of pairs $\iota_K: i_K\to i_{\iota(K)}$, represented by the following diagram

$$\begin{tikzcd} A\cup K^c \arrow[r, hook] \arrow[d, hook, "i_K"] & Y\cup \iota(K)^c \arrow[d, hook, "i_{\iota(K)}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

we get the induced excision isomorphism $i_K^* : h_c^\bullet(i_{\iota(K)}) \to h_c^\bullet(i_K)$. If $K \subseteq L$, the following diagram commutes:

$$\begin{tikzcd} h(i_K) \arrow[r, "i^*_{KL}"] \arrow[d, "(\iota_K^*)^{-1}"] & h(i_L) \arrow[d, "(\iota_L^*)^{-1}"] \\ h(i_{\iota(K)}) \arrow[r, "i^*_{\iota(K)\iota(L)}"] & h(i_{\iota(L)}) \end{tikzcd}$$

therefore we get the induced morphism between the colimits $\iota_* : h_c(Y, B) \to h_c(X, A)$.

It is possible to generalize this construction to a proper map $\rho : A \to X$. We define the relative groups $h_c(\rho)$ as above, replacing the embeddings $i_K : K^c \cup A \hookrightarrow X$ by the maps of pairs $\rho_K : (A, \rho^{-1}(K)^c) \to (X, K^c)$.[6] The corresponding groups are canonically isomorphic to the cohomology group of $\rho^+ : A^+ \to X^+$, the induced map between the one-point compactifications. Moreover, in the case of a pair (X, A), the correspondence with the definition given above (through the embeddings $i_K : K^c \cup A \hookrightarrow X$) is provided by Corollary 4.3.7. The groups $h_c(\rho)$ are functorial with respect to open embeddings of the form $\iota := (\iota, \iota') : \eta \to \rho$, i.e

$$\begin{tikzcd} B \arrow[r, hook, "\iota'"] \arrow[d, "\eta"] & A \arrow[d, hook, "\rho"] \\ Y \arrow[r, "\iota"] & X \end{tikzcd} \tag{4.13}$$

such that $\iota'(B) = \rho^{-1}(\iota(Y))$.

Since we are aiming at the differential case we will prefer to give another definition which will work in the differential case as well. In the absolute case, we used a special cofinal system $\mathcal{V}_e(p)$ which ensures that the pairs have the same homotopy type of cofibrations while still being manifolds. We modify the definition above in order to have the same properties in the relative case.

We denote the directed set $\mathcal{K}(\rho)$ whose elements are pairs $(K, \widehat{K})$ such that

- $K \subseteq X$ and $\widehat{K} \subseteq A$ are compact;
- K, $\mathrm{int}(K)^c$, $\widehat{K}$, and $\mathrm{int}(\widehat{K})^c$ are manifolds;
- $\rho^{-1}(K) \subseteq \mathrm{int}(\widehat{K})$

and the partial ordering is given by set inclusion in both components, that is, $(K, \widehat{K}) \preceq (L, \widehat{L})$ if and only if $K \subseteq L$ and $\widehat{K} \subseteq \widehat{L}$. We think of $\mathcal{K}(\rho)$ as a category, such that the set of morphisms from $(K, \widehat{K})$ to $(L, \widehat{L})$ contains one element if $(K, \widehat{K}) \preceq (L, \widehat{L})$ and is empty otherwise. Starting from a proper smooth map $\rho : A \to X$, there is a natural functor $F_\rho : \mathcal{K}(\rho) \to \mathtt{Top}_2^2$ assigning to an object $(K, \widehat{K})$ the map $\rho_{(K,\widehat{K})} : (A, \mathrm{int}(\widehat{K})^c) \to (X, \mathrm{int}(K)^c)$

[6] Observe that the domain and the codomain are not cofibrations, but this is not a problem in the topological setting.

and to a morphism $(K,\widehat{K}) \preceq (L,\widehat{L})$ the natural morphism $(i_{(K,L)}, i_{(\widehat{K},\widehat{L})}) : \rho_{(L,\widehat{L})} \to \rho_{(K,\widehat{K})}$ defined by the following diagram:

$$\begin{tikzcd}
(A, \operatorname{int}(\widehat{L})^c) \arrow[r, hook, "i_{\widehat{K}\widehat{L}}"] \arrow[d, "\rho_{L,\widehat{L}}"] & (A, \operatorname{int}(\widehat{K})^c) \arrow[d, "\rho_{K,\widehat{K}}"] \\
(X, \operatorname{int}(L)^c) \arrow[r, hook, "i_{KL}"] & (X, \operatorname{int}(K)^c)
\end{tikzcd}$$

Definition 4.6.4. The compactly-supported groups $h_c(\rho)$ are defined as the colimit of the composition functor

$$h_c(\rho) := h \circ F_\rho : \mathcal{K}(\rho) \to \texttt{GrAb} \tag{4.14}$$

These groups are functorial with respect to open embeddings of the form (4.13) such that $\iota'(B) = \rho^{-1}(\iota(Y))$. Moreover, we have a natural module structure on $h_c(\rho)$ over $h(X)$, such that the forgetful transformation $h_c(\rho) \to h(\rho)$ is a morphism of $h(X)$-modules.

The case of the vertically compact sets is similar. Given a relative fiber bundle $(F, f) : \overline{\rho} \to \rho$ as in the diagram

$$\begin{tikzcd}
B \arrow[r, "\overline{\rho}"] \arrow[d, "f"] & Y \arrow[d, "F"] \\
A \arrow[r, "\rho"] & X
\end{tikzcd},$$

we define the directed set $\mathcal{V}(F, f)$ whose elements are pairs $(V, \widehat{V})$ such that

- $V \in \mathcal{V}(F)$ and $\widehat{V} \in \mathcal{V}(f)$;
- V, $\operatorname{int}(V)^c$, $\widehat{V}$, and $\operatorname{int}(\widehat{V})^c$ are manifolds;
- $\overline{\rho}^{-1}(V) \subseteq \operatorname{int}(\widehat{V})$

with order defined componentwise, that is, $(V, \widehat{V}) \preceq (W, \widehat{W})$ if and only if $V \subseteq W$ and $\widehat{V} \subseteq \widehat{W}$. We can see $\mathcal{V}(F, f)$ as a category such that the set of morphism from $(V, \widehat{V})$ to $(W, \widehat{W})$ has one element if $(V, \widehat{V}) \preceq (W, \widehat{W})$ and is empty otherwise. To each relative fiber bundle (F, f), we associate the functor $G_{(F,f)} : \mathcal{V}(F, f) \to \texttt{Top}_2^2$ which assigns to a object $(V, \widehat{V})$ the map $\overline{\rho}_{V,\widehat{V}} : (B, \operatorname{int}(\widehat{V})^c) \to (Y, \operatorname{int} V^c)$ and to a morphism $(V, \widehat{V}) \preceq (W, \widehat{W})$ the natural morphism $(i_{(V,W)}, i_{(\widehat{V},\widehat{W})}) : \overline{\rho}_{(W,\widehat{W})} \to \overline{\rho}_{(V,\widehat{V})}$ defined by the following diagram

$$\begin{tikzcd}
(B, \operatorname{int}(\widehat{W})^c) \arrow[r, hook, "i_{\widehat{V}\widehat{W}}"] \arrow[d, "\overline{\rho}_{W,\widehat{W}}"] & (B, \operatorname{int}(\widehat{V})^c) \arrow[d, "\overline{\rho}_{V,\widehat{V}}"] \\
(Y, \operatorname{int}(W)^c) \arrow[r, hook, "i_{VW}"] & (Y, \operatorname{int}(V)^c)
\end{tikzcd}$$

Definition 4.6.5. With the notation above, we define the relative vertically compact cohomology group $h_v(\overline{\rho})$ as the colimit

$$h_v(\overline{\rho}) = \operatorname{colim} h \circ G_{(F,f)} : \mathcal{V}(F, f) \to \texttt{GrAb}$$

Again, these groups are functorial with respect to open embeddings of relative fibre bundles over the same map as in the following diagram

$$\begin{tikzcd}
& B \arrow[r, "\overline{\rho}"] \arrow[ddl, "\jmath"'] & Y \arrow[ddl, "F"] \\
B' \arrow[ur, hook, "i'"] \arrow[r, "\overline{\rho}'"] \arrow[d, "f"] & Y' \arrow[ur, hook, "i"] \arrow[d, "F"] & \\
A \arrow[r, "\rho"] & X &
\end{tikzcd}$$

Remark 4.6.6. We observe that, when $\rho : A \hookrightarrow X$ is a closed embedding, so that $B = Y|_A$, we can choose $\widehat{V} = \overline{\rho}^{-1}(V) = V \cap (Y|_A)$ and take the direct limit over K, since $\overline{\rho}^{-1}(K)$ and $(Y|_A) \setminus \mathrm{int}(\rho^{-1}(K))$ are manifolds in this case.

Remark 4.6.7. It is important to bear in mind that, in the **topological** context, we did not need to define $\mathcal{K}(\rho)$ nor $\mathcal{V}(F, f)$ requiring that the sets involved are manifolds satisfying the conditions on the interiors and closures. But, in the differential setting, this will be essential.

Classical relative Thom isomorphism

Fix a $h^\bullet$-oriented relative vector bundle $(P, p) : \overline{\rho} \to \rho$ as in the diagram

$$\begin{tikzcd}
F \arrow[r, "\overline{\rho}"] \arrow[d, "p"] & E \arrow[d, "P"] \\
A \arrow[r, "\rho"] & X
\end{tikzcd}$$

such that $P : E \to X$ is h-oriented by a Thom class $u \in h^n(E)$ and $p : F \to X$ is oriented with Thom class u' induced by u. Recall that the induced Thom class on F is given at representative level by $\overline{\rho}^* u$. We construct the relative Thom isomorphism $T : h^\bullet(\rho) \to h_v^{\bullet+n}(\overline{\rho})$ as follows: we represent the Thom class $u \in h_v(E)$ by $u_V \in h(E, \mathrm{int}(V)^c)$ and consider the product

$$u_V \times (P, p)^*(\alpha) \in h(\mathrm{id}_E \times \overline{\rho} : (E \times F) \to (E \times E, \mathrm{int}(V)^c \times E)).$$

for a class $\alpha \in h(\rho)$. Next, we pull-back through the following diagonal morphism, that we call $(\Delta, \Delta_{\overline{\rho}})$:

$$\begin{tikzcd}
(E \times F, \mathrm{int}(V)^c \times F) \arrow[r, "\mathrm{id}_E \times \overline{\rho}"] & (E \times E, \mathrm{int}(V)^c \times E) \\
(F, \mathrm{int}(\widehat{V})^c) \arrow[u, "\Delta_\rho"] \arrow[r, "\overline{\rho}"] & (E, \mathrm{int}(V)^c) \arrow[u, hook, "\Delta"']
\end{tikzcd}$$

where $\Delta_{\overline{\rho}}(x) := (x, \overline{\rho}(x))$ and we have choose $\widehat{V}$ in such way that $\overline{\rho}^{-1}(C) \subseteq \mathrm{int}(\widehat{K})$. Hence, we set $u_V \cdot (P, p)^*(\alpha) = (\Delta, \Delta_\rho)^*(u_V \times (P, p)^*(\alpha)) \in h(\overline{\rho}_{\widehat{V}, V})$. We define $T(\alpha)$ as the colimit of this last expression.

Relative compactly Thom isomorphism

To finish this section, we show how to define the compactly supported relative Thom isomorphism. First, we need to define a product of the form

$$\cdot(P,p)^* : h_v(E) \otimes h_c(\overline{\rho}) \to h_c(\overline{\rho})$$

The product is constructed in an analogue way as the absolute version in (4.12). Let $u \in h_v^n(E)$ be a Thom class on E represented by a class $u_V \in h^n(E, \mathrm{int}(V)^c) \cong h^n(E, V^c)$ with $V \in \mathcal{V}_e(p)$. For a class $\alpha \in h_c(\rho)$ represented by a class $\alpha_{K,\widehat{K}} \in h(\rho_{\widehat{K},K} : (A, \mathrm{int}(\widehat{K})^c) \to (A, \mathrm{int}(K)^c)$, we denote by $\beta_{H,\widehat{H}} = (P,p)^*(\alpha_{K,\widehat{K}}) \in h(\overline{\rho}_{H,\widehat{H}} : (E, \mathrm{int}(\widehat{H})^c) \to (E, \mathrm{int}(E)^c)$ where $\widehat{H} := p^{-1}(\widehat{K})$ and $H := P^{-1}(K)$ which is assumed to be transverse to V. Using the product 4.5.4, we get

$$u_V \times \beta_{H,\widehat{H}} \in h(\mathrm{id}_E \times \overline{\rho} : (E\times F, \mathrm{int}(V)^c \times F, E \times \mathrm{int}(\widehat{H})^c) \to (E\times E, \mathrm{int}(V)^c \times E, E\times \mathrm{int}(H)^c))$$

Pulling along the diagonal $(\Delta, \Delta_{\overline{\rho}}) : \overline{\rho} \to \mathrm{id}_E \times \overline{\rho}$

$$\begin{tikzcd}
(E \times F, \mathrm{int}(V)^c \times F, E \times \mathrm{int}(\widehat{H})^c) \arrow[r, "\mathrm{id}_E \times \overline{\rho}"] & (E \times E, \mathrm{int}(V)^c \times E, E \times \mathrm{int}(H)^c) \\
(F, \mathrm{int}(V)^c \cup \mathrm{int}(\widehat{H})^c, \emptyset) \arrow[u, "\Delta_\rho"] \arrow[r, "\overline{\rho}'"] & (E, \mathrm{int}(\widehat{V})^c \cup \mathrm{int}(H)^c, \emptyset) \arrow[u, "\Delta"']
\end{tikzcd}$$

where $\widehat{V} \in \mathcal{V}_e(p)$ is any special vertical set such that $\overline{\rho}^{-1}(V) \subseteq \mathrm{int}(\widehat{V})$ and $\widehat{V}$ is transverse to $\widehat{H}$. Then, we get a class

$$(\Delta, \Delta_{\overline{\rho}})^*(u_V \times \beta_{H,\widehat{H}}) \in h(\overline{\rho} : (F, (\mathrm{int}(\widehat{V} \cap \widehat{H}))^c) \to (E, (\mathrm{int}(V \cap H))^c)$$

with

- $\widehat{V} \cap \widehat{H} \in \mathcal{K}(F)$, $V \cap H \in \mathcal{K}(E)$;
- $\widehat{V} \cap \widehat{H}$, $\mathrm{int}(\widehat{V} \cap \widehat{H})^c$, $V \cap H$ and $\mathrm{int}(V \cap H)^c$ are manifolds.
- $\overline{\rho}^{-1}(V \cap H) \subseteq \widehat{V} \cap \widehat{H}$

which shows that we can define the Thom morphism passing to a colimit.

4.7 Umkher Maps

Once we have (re)defined the Thom isomorphism, we can define the umkehr maps in the exact same way as in Section 1.7. Since there is no change in the absolute case neither in relative case with compact fibers, we will focus on the relative compactly supported umkehr map.

Recall from section 1.7.2, one has a natural notion of relative fiber bundle orientation: for a fiber bundle $(F, f) : \overline{\rho} \to \rho$ such that F is h-oriented by $[\iota, u, \phi]$, we have naturally induced a h-orientation on f. In this section, every relative vector bundle is assumed to be endowed with a natural orientation.

Recall the definition of the relative $\mathbb{R}^N$ integration map $\int_{\mathbb{R}^n}^{v} : h_v(\overline{\rho} \times \mathrm{id}_{\mathbb{R}^n}) \to h(\rho)$ in 1.18. Although we have modified the definition of $h_v(\overline{\rho})$, the same construction of $\int_{\mathbb{R}^n}^{v}$ still works in this case with the proper interpretation of the pushfoward $(\mathrm{id}_X \times j, \mathrm{id}_A \times j)_* :$ $h_v(\rho \times \mathbb{R}) \to h(\rho \times \mathrm{id}_{S^1})$.

As in the absolute case, we can construct a compact version of the $\mathbb{R}^n$ integration map $\int_{\mathbb{R}^n}^{c} h_c(\rho \times \mathrm{id}_{\mathbb{R}^n}) \to h_c(\rho)$ provided that ρ is proper. We define it as

$$\int_{\mathbb{R}}^{c} = \int_{S^1}^{\mathbb{R}} \circ (\mathrm{id}_X \times j, \mathrm{id}_A \times j)_*$$

where $\int_{S^1}^{c} : h_c(\rho \times \mathrm{id}_{S^1}) \to h_c(\rho)$ is the relative compact S^1 integration. The compact S^1 is defined in the following way: the cohomology of maps of the form $\rho_{K,\widehat{K}} \times \mathrm{id}_{S^1}$: $(X \times S^1, K \times K) \hookrightarrow (X \times S^1, K\widehat{K} \times S^1)$ forms a cofinal system in $\mathcal{K}(\rho \times \mathrm{id}_{S^1})$. For a class $\alpha \in h_c(\rho \times \mathrm{id}_{S^1})$ represented by $\alpha_{K \times S^1, \widehat{K} \times S^1} \in h(\rho_{K,\widehat{K}} \times \mathrm{id}_{S^1})$ we define $\int_{S^1}^{c} \alpha$ by taking the colimit on the

$$\int_{S^1} \alpha_{K \times S^1, \widehat{K} \times S^1}$$

where $i_{K \times S^1, \widehat{K} \times S^1} : h((\rho \times \mathrm{id}_{S^1})_{K \times S^1, \widehat{K} \times S^1} \to h_c(\rho \times \mathrm{id}_{S^1})$ is the natural morphism.

Definition 4.7.1 (Relative Compactly Supported Differential Integration Map)**.** Let $(F, f) : \rho \to \overline{\rho}$ be a relative bundle over a proper map ρ such that F has an h-orientation $[\iota, u, \phi]$ and f is endowed with the induced orientation $[\iota', u', \phi']$. The relative compactly supported umkehr map $(F, f)_{c!} : h_c^{\bullet}(\overline{\rho}) \to h_c^{\bullet - (n-m)}(\rho)$ is defined as

$$(F, f)_{c!}(\alpha) = \int_{\mathbb{R}^n}^{c} (i, i')_* \circ ((\phi, \phi')^{-1})^* \circ T(\alpha)$$

4.8 Conclusion

In this chapter, we have presented the Thom isomorphisms and umkehr maps using a new notion of product. The umkher maps obtained here are summarized in Table 4.

Umkher \ Type	Absolute	Relative
Compact Fiber	□	□
Compact	✓	✓
Vertical	✓	□

Table 4 – Umkehr maps in cohomology using the new product. The ✓ denotes the existence of the umkehr map.The □ denotes existence, although it was not constructed here.

Observe that, in contrast with Chapter 1, we construct a relative compactly supported integration.

5 Differential Cohomology on Finite Sequences and General Integration Maps

5.1 Introduction

At last, we have all the topological tools we need to define differential integration maps at hand. In this chapter, we complete the set of differential tools and construct these maps.

We start by defining the differential refinement of a cohomology theory on maps of sequences. Then, we define the differential refinement of the cofibration-relative product, which we call the parallel-relative product.

Using this product, we define the compactly supported differential Thom isomorphism and the doubly-vertically-compactly Thom isomorphism, which are used to define compactly and vertically-compactly supported integration, respectively. We prove some properties of these integration maps and show that these properties completely characterize them.

We end the discussion by showing how to define relative versions of the integration maps. We start with the classical[1] differential relative Thom morphism, which is used to construct the differential integration map for relative bundles with compact fibers. While the absolute versions used the parallel-relative product only between parallel classes, this one needs a multiplication between a parallel and a relative one.

Next, we present the compactly supported relative differential Thom morphism. In contrast with the previous cases, which only required the product between a parallel class and a map, resulting in a map of pairs, this one needs maps between sequences (more precisely triples).

5.2 Differential cohomology on maps of sequences

We have already defined the category of maps of finite sequences of smooth manifold in Section 2.6, but it is convenient to restate the definitions now.

We define the category of finite sequences of manifolds, $\mathtt{Man}_\omega$ as the subcategory of $\mathtt{Top}_\omega$, defined in Section 4.2 of the previous chapter, whose elements are sequences of manifolds and whose morphisms are smooth maps. As in the topological case, we define the category $\mathtt{Man}^2_\omega$, as the category of arrows of $\mathtt{Man}_\omega$.

[1] Which is a bad name, but is used to keep the syntactic parallelism

In this chapter, whenever we talk about sequences and maps of sequences (or pairs and maps of pairs), we are referring to the smooth case.

A differential form over a sequence $(X,\vec{X})=(X,X_1,\dots,X_n)$ is a differential form $\omega\in\Omega(X)$ such that $\omega|_{X_i}=0$ for $i=1,\dots,n$. We denote the set of forms over a sequence by $\Omega(X,\vec{X})$ and remark that they are meant to be seen as an analogue of parallel forms[2]. A map of sequences $(\rho,\vec{\rho}):(A,\vec{A})\to(X,\vec{X})$ induces a map $(\rho,\vec{\rho})^*:\Omega(X,\vec{X})\to\Omega(A,\vec{A})$ through the pullback. A *relative form* $(\omega,\theta)\in\Omega(\rho,\vec{\rho})$ is an element of the mapping cone complex of the morphism $(\rho,\vec{\rho})^*$ and a *parallel form* $\omega\in\Omega_{\mathrm{par}}(\rho,\vec{\rho})$ is a differential form $\omega\in\Omega(X,\vec{X})$ in the kernel of $(\rho,\vec{\rho})^*$.

Although we have not mentioned in the previous chapter, the Chern-Dold character extends naturally to the sequence setting via the isomorphism with the usual cohomology. We use the same notation $ch:h\to H\mathfrak{h}_{\mathbb{R}}$ for it in the setting of cohomology on maps of (topological) sequences.

Definition 5.2.1. A differential refinement of a cohomology theory (h,∂) on maps of sequences is a (contravariant) functor $\widehat{h}:\mathtt{Man}^{2,\mathrm{op}}_{\omega}\to\mathtt{GrAb}$ along with three natural transformations

$$R:\widehat{h}\to\Omega_{\mathrm{cl}}\mathfrak{h}_{\mathbb{R}}\ \text{(curvature)}\qquad I:\widehat{h}\to h\ \text{(forgetful map)}\qquad a:\frac{\Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}}{\mathrm{Im}(d)}\to\widehat{h}^{\bullet}\ \text{(trivialization)}$$

satisfying the following two axioms:

A1) Let $(\rho,\vec{\rho}):(A,\vec{A})\to(X,\vec{X})$ be a smooth map of smooth sequences, $q_{\mathrm{dR}}:\Omega_{cl}(\rho,\vec{\rho})\to H_{\mathrm{dR}}(\rho,\vec{\rho})$ the projection to de Rham cohomology, $r:H_{\mathrm{dR}}(\rho,\vec{\rho})\to H\mathbb{R}(\rho,\vec{\rho})$ the de Rham isomorphism and $ch:h\to H\mathfrak{h}_{\mathbb{R}}$ the Chern-Dold character. Then the following diagram is commutative and the first line is exact:

$$\begin{tikzcd}
h^{\bullet}(\rho,\vec{\rho}) \arrow[r, "r\circ ch"] & \dfrac{\Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}(\rho,\vec{\rho})}{\mathrm{Im}(d)} \arrow[r, "a"] \arrow[dr, "d"'] & \widehat{h}^{\bullet}(\rho,\vec{\rho}) \arrow[r, "I"] \arrow[d, "R"] & h^{\bullet}(\rho,\vec{\rho}) \arrow[r] \arrow[d, "ch"] & 0 \\
 & & \Omega^{\bullet}_{\mathrm{cl}}\mathfrak{h}_{\mathbb{R}}(\rho,\vec{\rho}) \arrow[r, "r\circ q_{\mathrm{dR}}"] & H^{\bullet}_{\mathbb{R}}(\rho,\vec{\rho}) &
\end{tikzcd}$$

A2) For a map $\rho:(A,\vec{A})\to(X,\vec{X})$, the following diagram is commutative:

$$\begin{tikzcd}
\widehat{h}(\rho,\vec{\rho}) \arrow[r, "(id_{(X,\vec{X})},\varnothing_{(A,\vec{A})})^*"] \arrow[d, "cov"] & \widehat{h}(X,\vec{X}) \arrow[d, "\rho^*"] \\
\Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}(A,\vec{A}) \arrow[r, "a"] & \widehat{h}(A,\vec{A})
\end{tikzcd}$$

where cov is the second component of the curvature, that is, if $R(\widehat{\alpha})=(\omega,\theta)$, then $cov(\widehat{\alpha})=\theta$.

[2] In Section 2.6, we used the notation $\Omega_{\mathrm{par}}(X,\vec{X})$ for these, but in this chapter we drop the subscript.

Definition 5.2.2 (Flat and Parallel Classes)**.** A differential class $\widehat{\alpha} \in \widehat{h}(\rho, \vec{\rho})$ is called *flat* if $R(\widehat{\alpha}) = 0$ and *parallel* if $cov(\widehat{\alpha}) = 0$.

We keep the same notation for the subgroups of flat $\widehat{h}_{\text{flat}}$ and parallel classes $\widehat{h}_{\text{par}}$. The flat group is a topological cohomology theory on maps of sequences provided that the differential cohomology has a differential S^1-integration[3]. The group of parallel classes satisfies the axioms of a parallel cohomology 3.3.11 adapted to the the setting of sequences as well. We denote by $\Omega_{ch}\mathfrak{h}_{\mathbb{R}}$ the set of closed forms whose de Rham cohomology class is in the image of ch. As before, we call them $\mathfrak{h}$-forms.

The following analogue of Proposition 3.3.12 will be useful.

Proposition 5.2.3. *For any map of sequences $\rho : (A, \vec{A}) \to (X, \vec{X})$, the following sequence is exact:*

$$0 \longrightarrow \widehat{h}_{flat}(\rho, \vec{\rho}) \hookrightarrow \widehat{h}_{par}(\rho, \vec{\rho}) \xrightarrow{R_{par}} \Omega^{\bullet}_{par,ch}\mathfrak{h}_{\mathbb{R}}(\rho, \vec{\rho}) \longrightarrow 0$$

Proof. Exactness holds at $\widehat{h}_{\text{flat}}$ and $\widehat{h}_{\text{par}}$ by definition, hence we only need to verify that R_{par} is surjective. If $\omega \in \Omega^{\bullet}_{\text{par},ch}\mathfrak{h}_{\mathbb{R}}(\rho)$, then $(\omega, 0) \in \Omega^{\bullet}_{ch}\mathfrak{h}_{\mathbb{R}}(\rho)$. By definition of $\Omega^{\bullet}_{ch}\mathfrak{h}_{\mathbb{R}}(\rho)$, there exists $\alpha \in h(\rho)$ such that $ch(\alpha) = q_{\text{dR}}(\omega, 0)$. Axiom $A1$ asserts the surjectivity of I, thus there exists a differential class $\widehat{\alpha} \in \widehat{h}(\rho)$ such that $I(\widehat{\alpha}) = \alpha$. By the commutative square in axiom A1, we have $q_{\text{dR}} \circ R(\widehat{\alpha}) = ch \circ I[\widehat{\alpha}] = q_{\text{dR}}(\omega, 0)$, which implies that there $R(\widehat{\alpha}) = (\omega, 0) + d(\mu, \nu)$ for some $(\mu, \nu) \in \Omega^{\bullet-1}\mathfrak{h}_{\mathbb{R}}(\rho)$. Using Axiom A1 once more, where $d = R \circ a$, we can write

$$R(\widehat{\alpha}) = (\omega, 0) + R \circ a(\mu, \nu) \implies R(\widehat{\alpha} - a(\mu, \nu)) = (\omega, 0)$$

So the differential class $\widehat{\beta} = \widehat{\alpha} - a(\mu, \nu)$ is parallel and satisfies the desired condition. □

Remark 5.2.4. When we presented Proposition 3.3.13, we commented that the proof relied on Proposition 3.3.12 and on a compatibility between the Chern character and the induced morphisms. For the sake of completeness, we mention that this compatibility is to ensure that the pullback of an $\mathfrak{h}$-form is an $\mathfrak{h}$-form. The result is clear once one considers the following commutative diagram:

$$\begin{tikzcd}
h(\rho) \arrow[r, "r^{-1}\circ ch"] & H_{\text{dR}}(\rho) & \Omega_{ch}\mathfrak{h}_{\mathbb{R}}(\rho) \arrow[l, "q_{\text{dR}}"'] \\
h(\eta) \arrow[u, "(f,g)^*"] \arrow[r, "r^{-1}\circ ch"] & H_{\text{dR}}(\eta) \arrow[u, "(f,g)^*"] & \Omega_{ch}\mathfrak{h}_{\mathbb{R}}(\eta) \arrow[l, "q_{\text{dR}}"'] \arrow[u, "(f,g)^*"]
\end{tikzcd}$$

We also have the differential version of the interchangeability property of Proposition 4.3.1 for parallel differential cohomology:

[3] S^1 integration is defined in the same way as in 3.4.1 by replacing maps with maps of pairs.

Lemma 5.2.5 (Interchangeability for Parallel Classes)**.** *There is a natural isomorphism between $\widehat{h}_{par}(X, X_1, \ldots, X_n)$ and $\widehat{h}_{par}(X, X_{\sigma(1)}, \ldots, X_{\sigma(n)})$.*

Proof. This results follows from applying the Five Lemma and Proposition 5.2.3 at each stage of the (4.4), *e.g.*, applying it at the top row

$$\begin{tikzcd}
(\emptyset,\emptyset,\emptyset) \arrow[r, hook, "i_1"] \arrow[d, "j'"] & (X,A,B) \arrow[d, "j"] \\
(B,\emptyset,\emptyset) \arrow[r, hook, "i_2"] & (X,A,B)
\end{tikzcd}$$

results in

$$\begin{tikzcd}
0 \arrow[r] & \widehat{h}_{\text{flat}}(i_1) \arrow[r, hook] & \widehat{h}_{\text{par}}(i_1) \arrow[r, "R_{\text{par}}"] & \Omega_{ch}\mathfrak{h}_{\mathbb{R}}(i_1) \arrow[r] & 0 \\
0 \arrow[r] & \widehat{h}_{\text{flat}}(i_2) \arrow[r, hook] \arrow[u, "(j,j')^*"] & \widehat{h}_{\text{par}}(i_2) \arrow[r, "R_{\text{par}}"] \arrow[u, "(j,j')^*"] & \Omega_{ch}\mathfrak{h}_{\mathbb{R}}(i_2) \arrow[r] \arrow[u, "(j,j')^*"] & 0
\end{tikzcd}$$

Topologically, each vertical pair induces an isomorphism in cohomology, as already addressed in Proposition 4.3.10. This is also true for the closed parallel differential forms: the vertical maps are also isomorphisms at each stage due to the fact that these parallel differential forms are null in both A and B. The compatibility with ch ensures that the same holds for $\mathfrak{h}$-forms. □

5.3 Parallel-Relative Product

In this section, we axiomatize the relative-parallel product, but we need to establish some results first in order to make the definition sound. The following lemma is the differential analogue of Lemma 4.5.1:

Lemma 5.3.1. *There is a natural isomorphism between $\widehat{h}_{par}((\emptyset,\emptyset,\emptyset) \hookrightarrow (Y \times X, B \times X, Y \times A))$ and $\widehat{h}_{par}((Y \times A, B \times A, \emptyset) \hookrightarrow (Y \times X, B \times X, \emptyset))$*

Proof. This is another instance of application of proposition 5.2.3. This time we use it on Lemma 4.5.1. Consider the two pairs of vertical morphisms on (4.6). Topologically, both induce isomorphisms in cohomology according to Proposition 4.5.1. The curvature of each row is a form on $X \times Y$ that vanishes on $X \times B$ and on $A \times Y$, representing the image of the same Chern character, hence we get two isomorphisms in this case too. Applying the five lemma to the corresponding exact sequences described in Lemma 5.2.3, we see that the induced pullbacks on parallel classes are isomorphisms too. □

Remark 5.3.2. Due to Lemmas 5.2.5 and 5.3.1 we have an identification

$$\widehat{h}_{\text{par}}((X,\vec{A}) \times (Y,\vec{B}) \cong \widehat{h}_{\text{par}}((Y,\vec{B}) \times (X,\vec{A})) \cong h((Y,\vec{B})) \cong \widehat{h}_{\text{par}}((Y \times Z, \vec{B} \times Z, Y \times \vec{Z}))$$

as in Remarks 4.5.2 and 4.5.3.

Definition 5.3.3 (Parallel-Relative Product)**.** A *parallel-relative product* in a differential refinement $(\widehat{h}^\bullet, R, I, a)$ of a multiplicative relative cohomology theory on finite sequences of spaces $(h^\bullet, \partial)$ is a natural transformation on $\times : h_{\text{par}}(Y, \vec{B}) \otimes_{\mathbb{Z}} h(\rho, \vec{\rho}) \to h((Y, \vec{B}) \times (\rho, \vec{\rho}))$, satisfying the following axioms

M1) (*Mixed associativity*): for every $\widehat{\gamma} \in \widehat{h}^\bullet(\rho, \vec{\rho}_n)$, $\widehat{\beta} \in \widehat{h}^\bullet_{\text{par}}(Y, \vec{B})$ and $\widehat{\alpha} \in \widehat{h}^\bullet_{\text{par}}(Z, \vec{C})$ we have that

$$(\widehat{\alpha} \times \widehat{\beta}) \times \widehat{\gamma} = \widehat{\alpha} \times (\widehat{\beta} \times \widehat{\gamma}).$$

Here we are considering the product $\gamma \times \beta$ in the parallel cohomology of $(Y \times Z, \vec{B} \times Z, Y \times \vec{Z})$ as in Remark 5.3.2.

M2) (*Graded-commutativity on parallel classes*): for every $\widehat{\alpha} \in \widehat{h}^n_{\text{par}}(X, \vec{A})$ and $\widehat{\beta} \in \widehat{h}^m_{\text{par}}(Y, \vec{B})$ we have

$$\widehat{\beta} \times \widehat{\alpha} = (-1)^{nm} \widehat{\alpha} \times \widehat{\beta},$$

up to the canonical identification $h((X, \vec{A}) \times (Y, \vec{B})) \cong h((Y, \vec{B}) \times (X, \vec{A}))$ as in Remark 5.3.1

M3) (*Distributivity*) : $(\widehat{\alpha} + \widehat{\beta}) \times \widehat{\gamma} = (\widehat{\alpha} \times \widehat{\gamma}) + (\widehat{\beta} \times \widehat{\gamma})$ and $\widehat{\alpha} \times (\widehat{\beta} + \widehat{\gamma}) = (\widehat{\alpha} \times \widehat{\beta}) + (\widehat{\alpha} \times \widehat{\gamma})$ whenever these operations make sense.

M4) (*Unitarity*) : There exists the unit element $1 \in \widehat{h}^0(*)$.

M5) (*Compatibility with the natural transformations of* $\widehat{h}$) The following identities hold:

- (*curvature*) $R(\widehat{\alpha} \times \widehat{\beta}) = R_{\text{par}}(\widehat{\beta}) \times R(\widehat{\alpha})$;
- (*forgetful map*) $I(\alpha \times \beta) = I_{\text{par}}(\widehat{\alpha}) \times I(\widehat{\beta})$;
- (*trivialization 1*) $a_{\text{par}}(\omega) \times \widehat{\beta} = a(\omega' \times R(\widehat{\beta}))$;
- (*trivialization 2*) $\widehat{\alpha} \times a(\omega, \theta) = a(R_{\text{par}}(\widehat{\alpha}) \times (\omega, \theta))$.

5.4 Differential integration: absolute case

From now on, we suppose that $(\widehat{h}, R, I, a)$ is differential cohomology over $\mathtt{Top}^2_\omega$ endowed with a parallel-relative product.

5.4.1 Compactly supported Thom morphism

Once we get a parallel-relative product, we are able to define the product between a vertical differential class and compact differential class

$$\begin{aligned} \cdot p^* : \widehat{h}_v(E) \otimes_{\mathbb{Z}} \widehat{h}_c(X) &\to h_c^{\bullet+n}(E) \\ \alpha \otimes \beta &\mapsto \alpha \cdot p^* \beta. \end{aligned} \tag{5.1}$$

This is done using the same composition as in (4.9):

$$\begin{aligned}
\cdot\, p^* : \widehat{h}_{\mathrm{par}}(E, V^c) \otimes_{\mathbb{Z}} \widehat{h}_{\mathrm{par}}(E, H^c) &\xrightarrow{(1)} \widehat{h}_{\mathrm{par}}(E, \mathrm{int}(V)^c) \otimes_{\mathbb{Z}} \widehat{h}_{\mathrm{par}}(E, H^c) \xrightarrow[Def.(5.3.3)]{\times} \\
\widehat{h}_{\mathrm{par}}((E \times H^c, \mathrm{int}(V)^c \times H^c) &\overset{i'}{\hookrightarrow} (E \times E, \mathrm{int}(V)^c \times E)) \xrightarrow[4.10]{(\Delta,\Delta')^*} \\
\widehat{h}_{\mathrm{par}}((H^c, \mathrm{int}(V)^c \cap H^c) &\overset{i}{\hookrightarrow} (E, \mathrm{int}\, V^c)) \overset{(*)}{\cong} \\
\widehat{h}_{\mathrm{par}}(\mathrm{int}(\mathrm{int}(V)^c) \cup H^c &\overset{k}{\hookrightarrow} E) \overset{(2)}{=} \widehat{h}_{\mathrm{par}}((V \cap H)^c \overset{k}{\hookrightarrow} E)
\end{aligned} \tag{5.2}$$

except that the isomorphism at $(*)$ now becomes

$$\widehat{h}_{\mathrm{par}}(i) \xleftarrow{\sim} \widehat{h}_{\mathrm{par}}(j) \xrightarrow{\sim} \widehat{h}_{\mathrm{par}}(k), \tag{5.3}$$

where

$$\begin{aligned}
i &: (H^c, \mathrm{int}(V)^c \cap H^c) \hookrightarrow (E, \mathrm{int}(V)^c) \\
j &: (H^c \cup \mathrm{int}(\mathrm{int}(V)^c), \mathrm{int}(V)^c \cap (H^c \cup \mathrm{int}(\mathrm{int}(V)^c))) \hookrightarrow (E, \mathrm{int}(V)^c) \\
k &: (\mathrm{int}(\mathrm{int}(V)^c) \cup H^c, \emptyset) \hookrightarrow (E, \emptyset).
\end{aligned}$$

To prove that the arrows in (5.3) are isomorphisms we use Proposition 5.2.3 again. For example, let us illustrate the case $k \hookrightarrow j$:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{h}_{\mathrm{flat}}(j) \arrow[r, hook] \arrow[d] & \widehat{h}_{\mathrm{par}}(j) \arrow[r, "R_{\mathrm{par}}"] \arrow[d] & \Omega_{\mathrm{par},ch}\mathfrak{h}_{\mathbb{R}}(j) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{h}_{\mathrm{flat}}(k) \arrow[r, hook] & \widehat{h}_{\mathrm{par}}(k) \arrow[r, "R_{\mathrm{par}}"] & \Omega_{\mathrm{par},ch}\mathfrak{h}_{\mathbb{R}}(k) \arrow[r] & 0
\end{tikzcd}$$

The left arrow is an isomorphism, as we have already discussed in the topological case. In order to see that the right arrow is an isomorphism, observe that a form $\omega \in \Omega_{\mathrm{par}}(k)$ is a form $\omega \in \Omega(E)$ such that $\omega|_{H^c} = 0$ and $\omega_{\mathrm{int}(V)^c} = 0$, where the last equality holds by continuity. This means that ω defines a form in $\Omega(j)$ and the map is surjective. It is also clearly injective, and thus an isomorphism. Applying the five lemma, we conclude that the central arrow is an isomorphism.

Using this product, we can finally give a proper definition of the compactly supported Thom morphism.

Definition 5.4.1 (Compactly supported differential Thom morphism)**.** Let $p : E \to X$ be a $\widehat{h}$-oriented smooth vector bundle with a differential Thom class $\widehat{u}$. We define the *compactly supported differential Thom morphism* by

$$\begin{aligned}
T : \widehat{h}_c(X) &\to \widehat{h}_c(E) \\
\widehat{\alpha} &\mapsto \widehat{u} \cdot p^* \widehat{\alpha}.
\end{aligned}$$

The same proof carried in the topological case (Proposition 4.6.2) also proves the following result.

Proposition 5.4.2. *Fix an oriented vector bundle embedding $(p, p') : \bar{i} \to i$ as in the diagram*

$$\begin{tikzcd} E' \arrow[r, hook, "\bar{i}"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ X' \arrow[r, hook, "i"] & X \end{tikzcd}$$

where $X' \hookrightarrow X$ is an open embedding.

Let $T_c : h_c^\bullet(X) \to h_c^{\bullet+n}(E)$ be the compactly supported Thom isomorphism associated with p and $T'_c : h_c^\bullet(X') \to h_c^{\bullet+n}(E')$ the compactly supported Thom isomorphism associated with $p' : E' \to X'$ with its induced Thom class. Then

$$T_c \circ i_* = \bar{i}_* \circ T'_c$$

The same holds in the vertically compact case by exchanging T_c with T_v and reinterpreting $\bar{i}$ in the appropriate sense.

5.4.2 Compactly Supported Integration

Using the Thom morphism, we can finally define the compactly supported differential integration:

Definition 5.4.3 (Compactly supported differential integration)**.** Given an $\widehat{h}$-oriented neat fibered manifold $f : Y \to X$, where Y has dimension n and X has dimension m, with orientation $[\iota, \widehat{u}, \phi]$, we define its *compactly supported differential integration map* by

$$\widehat{f}_{c!}(\widehat{\alpha}) = \int_{\mathbb{R}^L}^{c} i_* \circ (\phi^{-1})^* \widehat{T}_{c,N}((\iota^{-1})^* \widehat{\alpha}). \tag{5.4}$$

Proposition 5.4.4 (Property I1)**.** *Under the same notation of Definition 5.4.3, the following diagram is commutative:*

$$\begin{tikzcd} \dfrac{\Omega_c^{\bullet-1}\mathfrak{h}_{\mathbb{R}}(Y)}{\mathrm{Im}(d)} \arrow[r, "a_c"] \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] & \widehat{h}_c^\bullet(Y) \arrow[r, "I_c"] \arrow[d, "\widehat{f}_{c!}"] \arrow[rr, bend left, "R_c"] & h_c^\bullet(Y) \arrow[d, "f_{c!}"] & \Omega_{c,cl}^\bullet\mathfrak{h}_{\mathbb{R}}(Y) \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] \\ \dfrac{\Omega_c^{\bullet-1-(n-m)}\mathfrak{h}_{\mathbb{R}}(X)}{\mathrm{Im}(d)} \arrow[r, "a_c"] & \widehat{h}_c^{\bullet-(n-m)}(X) \arrow[r, "I_c"] \arrow[rr, bend right, "R_c"] & h_c^{\bullet-(n-m)}(X) & \Omega_{c,cl}^{\bullet-(n-m)}\mathfrak{h}_{\mathbb{R}}(X) \end{tikzcd}$$

Proof. We prove the result for a_c. The other compatibilities are similar. Let $\omega \in \Omega_c(X)$. Then

$$\begin{aligned} a_c \circ R_{[\iota,\widehat{u},\phi]}(\omega) &= a_c \int_{\mathbb{R}^L}^{c} (i_*)(\phi^{-1})^*(R_v(\widehat{u}) \wedge p^*\omega) \\ &= \int_{\mathbb{R}^L}^{c} (i_*)(\phi_{-1})^* a_c(R_v(\widehat{u}) \wedge p^*\omega) \\ &= \int_{\mathbb{R}^L}^{c} (i_*)(\phi^{-1})^*\widehat{u} \cdot p^*(a_c(\omega)) \end{aligned}$$

We remark that we have used the compatibility of $\int_{\mathbb{R}^L}^{c}$ and a_c which comes from the compatibility of $\int_{S^1}^{c}$ with a_c (see Remark 3.7.8), as well as the naturality of a_c with i_*. The last passage can be verified at the representative level using the compatibility of the parallel-relative[4] product with a. □

Proposition 5.4.5 (Property I2). *Consider two $\widehat{h}$-oriented neat fibered manifolds $f : Y \to X$ and $g : Z \to Y$. One has $(f \circ g)_! = f_! \circ g_!$, provided that $f \circ g$ is endowed with the product orientation.*

We omit the proof since it is exactly the same as that in (RUFFINO, 2017, Lemma 3.27).

Proposition 5.4.6 (Property I3). *Let $f : Y \to X$ be an (n, m)-fibered manifold endowed with an $\widehat{h}$-orientation $[\iota, \widehat{u}, \phi]$ and $i : X' \hookrightarrow X$ be an open embedding. Consider the fibered map $(f', f) : j \to i$ over i as described in the diagram*

$$\begin{tikzcd} Y' \arrow[r, hook, "j"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, hook, "i"] & X \end{tikzcd}$$

and let f' be endowed with the orientation (see section 5.5.1 below). Then $\widehat{f}_{c!} \circ j_* = i_* \circ \widehat{f}'_{c!}$.

Proof. We need to prove the border of the following diagram is commutative:

$$\begin{tikzcd} \widehat{h}_c(X') \arrow[r, "i_*"] & \widehat{h}_c(X) \\ \widehat{h}_c(X' \times \mathbb{R}^N) \arrow[u, "\int_{\mathbb{R}^N}^{c}"] \arrow[r, "(i\times \mathrm{id})_*"] & \widehat{h}_c(X \times \mathbb{R}^N) \arrow[u, "\int_{\mathbb{R}^N}^{c}"] \\ \widehat{u} \cdot \pi^*\widehat{\alpha} \in \widehat{h}_c(N') \arrow[u, "\iota'_*\varphi'_*"] \arrow[r, "k_*"] & \widehat{h}_c(N) \ni \widehat{u}' \cdot \pi'^* i_*\widehat{\alpha} \arrow[u, "\iota_*\varphi_*"] \\ \widehat{\alpha} \in \widehat{h}_c(Y') \arrow[u, "\widehat{T}'_c"] \arrow[r, "j_*"] & \widehat{h}_c(Y) \ni j_*\widehat{\alpha} \arrow[u, "\widehat{T}_c"] \end{tikzcd}$$

[4] In fact, parallel-parallel product.

i.e.,

$$i_* \int_{\mathbb{R}^N}^c \iota_* \varphi_*(\widehat{u} \cdot \pi^* \widehat{\alpha}) = \int_{\mathbb{R}^N}^c \iota'_* \varphi'_*(\widehat{u}' \cdot \pi'^* j_* \widehat{\alpha})$$

The bottom square is commutative due to Proposition 5.4.2. The commutativity of the second square stems from the functoriality of differential cohomology with compact supports. Therefore we need to verify only the commutativity of the top square.

Recall the definition of the integration map $\int_{\mathbb{R}^N}^c : h(X \times \mathbb{R}^n) \to h^{\bullet-1}(X)$, as defined [5] in (1.7.1)

$$\int_{\mathbb{R}}^c := \int_{S^1}^c \circ (\mathrm{id}_X \times j)_*$$

iterated N times. Observe that

$$\begin{aligned}
\int_{\mathbb{R}}^c \circ (i \times \mathrm{id}_{\mathbb{R}^n})_* &= \int_{S^1}^c \circ (\mathrm{id}_X \times j)_* \circ (i \times \mathrm{id}_{\mathbb{R}^n})_* \\
&= \int_{S^1}^c \circ ((i \times \mathrm{id}_{\mathbb{R}^{n-1}}) \times j)_* \\
&= \int_{S^1}^c \circ ((i \times \mathrm{id}_{\mathbb{R}^{n-1}})_* \times \mathrm{id}_{S^1}) \circ (\mathrm{id}_X \times j)_* \\
&\overset{(*)}{=} (i \times \mathrm{id}_{\mathbb{R}^{n-1}})_* \circ \int_{S^1}^c (\mathrm{id}_X \times j_*) \\
&= (i \times \mathrm{id}_{\mathbb{R}^{n-1}})_* \circ \int_{\mathbb{R}}^c,
\end{aligned}$$

where in $(*)$ we used that the compactly supported differential S^1-integration is natural with respect to the pushfoward (see Remark 3.7.8). $\square$

Proposition 5.4.7 (Property I4). *If $p : E \to X$ is an $\widehat{h}$-oriented real vector bundle with Thom class $\widehat{u}$. Consider its orientation as in Remark 3.7.12. Then $\widehat{p}_{c,!}$ is a left inverse of the Thom morphism,* i.e., *$\widehat{p}_{c,!} \circ \widehat{T}_c(\widehat{\alpha}) = \widehat{\alpha}$ for every $\widehat{\alpha} \in \widehat{h}_c(X)$.*

Proof. This proof is identical to the topological one presented in Proposition (1.7.17). We reproduce it for completeness. Observe that[6]

$$\begin{aligned}
\widehat{T}_{c,N}((i_E^{-1})^* \circ \widehat{T}_{E,c}(\widehat{\alpha})) &= \mathrm{pr}_F^*(\widehat{u}_F) \cdot \mathrm{pr}_E'^* \circ (i_E^{-1})^* (\widehat{u}_E \cdot p_E^*(\widehat{\alpha})) \\
&= \mathrm{pr}_F^*(\widehat{u}_F) \cdot \mathrm{pr}_E^*((\widehat{u}_E) \cdot p_E^*(\widehat{\alpha})) \\
&= \mathrm{pr}_F^*(\widehat{u}_F) \cdot \mathrm{pr}_E^*(\widehat{u}_E) \cdot \mathrm{pr}_E^* \circ p_E^*(\widehat{\alpha}) \\
&= (\widehat{u}_F \times \widehat{u}_F) \cdot (p_E \circ \mathrm{pr}_E)^*(\widehat{\alpha}) \\
&= \widehat{u}_{E \oplus F} \cdot p_{E \oplus F}^*(\widehat{\alpha})
\end{aligned}$$

where we used $p_{E \oplus F} = p_E \circ \mathrm{pr}_E$. Using the isomorphism to $E \oplus F \xrightarrow{\sim} X \times \mathbb{R}^N$ and integrating in $\mathbb{R}^n$ we get

$$\int_{\mathbb{R}^n} \widehat{u}_{X \times \mathbb{R}^N} \cdot \mathrm{pr}_X^*(\widehat{\alpha}) = \widehat{\alpha}$$

where the equality is due to Lemma 3.7.7. $\square$

[5] Remember that the definition is the same for the differential case.

[6] Some care should be taken here. The reader should be aware that the product is denoted by $\cdot\,\mathrm{pr}^*$ and that the properties are proved at representative level.

Now, we can state and prove the main theorem of this work:

Theorem 5.4.8 (Axiomatic characterization of the compactly supported differential Integration). *Fix a multiplicative differential cohomology theory with S^1-integration. The compactly supported differential integration map $\widehat{f}_{c!} : \widehat{h}_c^{\bullet}(Y) \to \widehat{h}_c^{\bullet-(n-m)}(X)$ defined for each $\widehat{h}$-oriented neat submersion $f : Y \to X$ between differential manifolds of dimension n and m, respectively, is the unique map satisfying the following properties:*

I1) The following diagram commutes:

$$\begin{tikzcd} \dfrac{\Omega_c^{\bullet-1}\mathfrak{h}_{\mathbb{R}}(Y)}{\mathrm{Im}(d)} \arrow[r, "a_c"] \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] & \widehat{h}_c^{\bullet}(Y) \arrow[r, "I_c"] \arrow[d, "\widehat{f}_{c!}"] \arrow[rr, bend left, "R_c"] & h_c^{\bullet}(Y) \arrow[d, "f_{c!}"] & \Omega_{c,cl}^{\bullet}\mathfrak{h}_{\mathbb{R}}(Y) \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] \\ \dfrac{\Omega_c^{\bullet-1-(n-m)}\mathfrak{h}_{\mathbb{R}}(X)}{\mathrm{Im}(d)} \arrow[r, "a_c"] & \widehat{h}_c^{\bullet-(n-m)}(X) \arrow[r, "I_c"] \arrow[rr, bend right, "R_c"] & h_c^{\bullet-(n-m)}(X) & \Omega_{c,cl}^{\bullet-(n-m)}\mathfrak{h}_{\mathbb{R}}(X) \end{tikzcd}$$

I2) It is natural with respect to composition, i.e., *given two $\widehat{h}$-oriented neat submersions, $f : Y \to X$ and $g : Z \to Y$, $\widehat{(f \circ g)}_{c!} = \widehat{f}_{c!} \circ \widehat{g}_{c!}$, where $f \circ g$ is endowed with the product orientation.*

I3) It is natural with respect to open embedding, i.e., given the diagram

$$\begin{tikzcd} Y' \arrow[r, hook, "i"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, hook, "j"] & X \end{tikzcd}$$

where $(i,j) : f' \to f$ is an open embedding and the orientation of f' é obtained by restriction of the orientation of f, we have $\widehat{f}_{c!} \circ i_* = j_* \circ \widehat{f'}_{c!}$.

I4) If $p : E \to X$ is an $\widehat{h}$-oriented real vector bundle, then $\widehat{p}_{c!}$ is a left inverse of the Thom morphism, i.e., *$\widehat{p}_{c!} \circ \widehat{T}_c(\widehat{\alpha}) = \widehat{\alpha}$ for every $\widehat{\alpha} \in \widehat{h}_c(X)$, whenever $p : E \to X$ is endowed with its induced $\widehat{h}$-orientation as a map.*

Proof. In propositions 5.4.4, 5.4.5, 5.4.6 and 5.4.7 we have already proved that the map defined in (5.4.3) satisfies these properties. It remains to prove the uniqueness.

Let us suppose that $\widehat{f}_{c!}$ is any integration map satisfying *I1)* to *I4)*. We first observe that, given an $\widehat{h}$-oriented real vector bundle $p : E \to X$, axioms *I1)* and *I4)* completely determine $\widehat{p}_{c!}$. In fact, given $\widehat{\alpha} \in \widehat{h}_c(E)$, we set $\alpha := I_c(\widehat{\alpha})$. Since the Thom morphism is topologically an isomorphism, we set $\beta := T_c^{-1}(\alpha)$. Refining β to any differential class $\widehat{\beta}$, we get $\widehat{\alpha} = \widehat{T}_c(\widehat{\beta}) + a_c(\omega)$, for a suitable $\omega \in \Omega_c^{\bullet-1}\mathfrak{h}_{\mathbb{R}}(E)$.

Therefore, applying *I1)* and *I4)*, we get $\widehat{p}_{c!}(\widehat{\alpha}) = \widehat{\beta} + a_c(R_{[i,\widehat{u},\phi]}(\omega))$. In particular, if $\mathrm{pr}_X : X \times \mathbb{R}^N \to X$ is the product bundle endowed with its natural orientation, as in Remark 3.6.4, then $\widehat{\mathrm{pr}_{X}}_{c!} = \int_{\mathbb{R}^N}$, since $\int_{\mathbb{R}^N}$ verifies *I1)* and *I4)*, as it is easy to verify applying the axiom of S^1-integration (Remark 3.7.7). Another particular case is the zero-bundle $\mathrm{id}_X : X \to X$, with the trivial orientation 1. In this case we have $\widehat{\mathrm{id}_X}_{c!} = \mathrm{id}_X$.

Given a $\widehat{h}$-oriented neat submersion $f : Y \to X$, we consider the following diagram:

$$\begin{tikzcd} N_{\iota(Y)} \arrow[r, "i\circ\phi"] \arrow[d, "f\circ\iota^{-1}\circ\pi_N"] & X \times \mathbb{R}^N \arrow[d, "\mathrm{pr}_X"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

Using axiom I3), we get the following commutative diagram:

$$\begin{tikzcd} \widehat{h}_c^{\bullet}(N_{\iota(Y)}) \arrow[r, "(i\circ\phi)_*"] \arrow[d, "\widehat{f\circ\iota^{-1}\circ\pi_N}_{c!}"] & \widehat{h}_c^{\bullet}(X \times \mathbb{R}^N) \arrow[d, "\widehat{\mathrm{pr}_X}_{c!}"] \\ \widehat{h}_c^{\bullet-N}(X) \arrow[r, equal] & \widehat{h}_c^{\bullet-N}(X) \end{tikzcd}$$

Given $\widehat{\beta} \in \widehat{h}_c(N_{\iota(Y)})$, it follows from the previous discussion that $(\widehat{f \circ \iota^{-1} \circ \pi_N})_{c!}(\widehat{\beta}) = \int_{\mathbb{R}^N}^c i_*(\phi^{-1})^*\widehat{\beta}$. For a given $\widehat{\alpha} \in \widehat{h}_c(Y)$, we get

$$(\widehat{f \circ \iota^{-1} \circ \pi_N})_{c!}(\widehat{T}_{Nc}\widehat{\alpha}) = \int_{\mathbb{R}^N}^c i_*(\phi^{-1})^* T_{Nc}(\iota^{-1})^*\widehat{\alpha}$$

Applying axioms *I2)* followed by axiom *I4)* we get[7] $\widehat{f}_{c!}(\widehat{\alpha}) = \int_{\mathbb{R}^N} i_*(\phi^{-1})^* T(\widehat{(\iota^{-1})^*\alpha})$, i.e. $\widehat{f}_{c!}$ coincides with (5.4). This proves the uniqueness of the integration map. □

Remark 5.4.9. Axiom *I1)* could be stated only with respect to a_c, hence the naturality with respect to R_c and I_c could be seen as a consequence of the other axioms. In fact, in order to prove uniqueness, we used the commutativity with a_c to deduce that $\widehat{p}_{c!}$ is completely determined by axioms *I1)* and *I4)* for any vector bundle $p : E \to X$. The rest of the proof does not rely on any of the compatibilities of axiom *I1)*.

5.4.3 Doubly-vertically compact Thom morphism

Fix a vector bundle $p : E \to Y$ and a fiber bundle $f : Y \to X$ such that $f \circ p$ is a vector bundle. The same considerations which we have used to construct the compactly supported differential Thom morphisms work here: choose some representative $\alpha_U \in \widehat{h}_{\mathrm{par}}(Y, U^c)$ for some $U \in \mathcal{V}(f)$ representing a class $\alpha \in \widehat{h}_v(Y)$. Define $H := p^{-1}(V)$. The same expression as in (5.2) can be used to define the product $\cdot p : \widehat{h}_v(E) \otimes \widehat{h}_v(Y) \to \widehat{h}_{vv}(E)$. Just observe that $V \cap H \in \mathcal{VV}(p, f)$.

[7] We really get $(\widehat{f \circ \iota^{-1}})((\iota^{-1})^*\widehat{\alpha})$ but this is the same as $\widehat{f}_{c!}(\widehat{\alpha})$ since ι is a isomorphism.

Definition 5.4.10 (Doubly-Vertically Compactly Supported Thom Morphism)**.** Let $p : E \to Y$ be a $\widehat{h}$-oriented smooth vector bundle with a differential Thom class $\widehat{u}$ and $f : Y \to X$ be smooth fiber bundle such that $f \circ p$ is a smooth vector bundle. We define the compactly supported differential Thom morphism by

$$\begin{aligned} T : \widehat{h}_v(Y) &\to \widehat{h}_{vv}(E) \\ \widehat{\alpha} &\mapsto \widehat{u} \cdot p^* \widehat{\alpha} \end{aligned}$$

5.4.4 Vertically Compact Integration

Definition 5.4.11 (Vertically-compactly supported differential integration)**.** Given a smooth fiber bundle $f : Y \to X$, where Y has dimension n and X has dimension m, and f is an $\widehat{h}$-oriented map with orientation $[\iota, \widehat{u}, \phi]$, we define its *vertically-compactly supported differential integration map* as the map

$$\widehat{f}_{v!}(\widehat{\alpha}) = \int_{\mathbb{R}^n} i_* \circ (\phi^{-1})^* T_{v,N}((\iota^{-1})^* \widehat{\alpha})$$

Propositions 5.4.4,5.4.5 and 5.4.7, also hold for the vertically-compactly supported differential integration doing the proper changes, but proposition 5.4.6 has to be modified since the map $i_* : h_v(Y') \to h_v(Y)$ has no meaning in the context of vertical cohomology, because Y' and Y are not defined over the same base. The modification is to require that $X' = X$.

Proposition 5.4.12. *Let $i : Y' \hookrightarrow Y$ be an open embedding of fiber bundles $f : Y \to X$ and $f' : Y' \to X$ over X as in the following diagram*

$$\begin{tikzcd} Y' \arrow[rr, hook, "i"] \arrow[dr, "f'"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

Assume further that f is $\widehat{h}$-oriented and f' is endowed with the induced orientation. In this case we have $\widehat{f}_{v!} \circ i_* = \widehat{f'}_{v!}$.

The characterization theorem also holds in this case with virtually the same proof.

Theorem 5.4.13 (Axiomatic characterization of the vertically-compactly supported differential Integration)**.** *Fix a multiplicative differential co-homology theory with S^1-integration. The vertically-compactly supported differential integration map $\widehat{f}_{v!} : \widehat{h}^{\bullet}_v(Y) \to \widehat{h}^{\bullet-k}_v(X)$, defined for any $\widehat{h}$-oriented fiber bundle $f : Y \to X$ with k-dimensional fiber, is the unique map satisfying the following properties:*

I1) The following diagram commutes:

$$\begin{tikzcd}
\dfrac{\Omega_v^{\bullet-1}(Y;\mathfrak{h}_{\mathbb{R}})}{\mathrm{Im}(d)} \arrow[r, "a"] \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] & \widehat{h_v}^{\bullet}(Y) \arrow[r, "I_v"] \arrow[d, "\widehat{f}_{v!}"] \arrow[rr, bend left, "R_v"] & h_v^{\bullet}(Y) \arrow[d, "f_{v!}"] & \Omega^{\bullet}_{v,cl}(Y;\mathfrak{h}_{\mathfrak{v}\mathbb{R}}) \arrow[d, "R_{[\iota,\widehat{u},\phi]}"] \\
\dfrac{\Omega_v^{\bullet-1-k}(X;\mathfrak{h}_{\mathbb{R}})}{\mathrm{Im}(d)} \arrow[r, "a"] & \widehat{h}_v^{\bullet-k}(X) \arrow[r, "I_v"] \arrow[rr, bend right, "R_v"'] & h_v^{\bullet-k}(X) & \Omega^{\bullet-k}_{v,cl}(X;\mathfrak{h}_{\mathbb{R}})
\end{tikzcd}$$

I2) It is natural with respect to composition, i.e., *given two $\widehat{h}$-oriented neat submersions, $f : Y \to X$ and $g : Z \to Y$, $\widehat{(f \circ g)}_{v!} = \widehat{f}_{v!} \circ \widehat{g}_{v!}$, where $f \circ g$ is endowed with the product orientation.*

I3') It is natural with respect to open embedding, i.e., given the diagram

$$\begin{tikzcd}
Y' \arrow[rr, hook, "i"] \arrow[dr, "f'"'] & & Y \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

where $i : Y' \hookrightarrow Y$ is an open embedding of fiber bundles and the orientation of f' is induced by the orientation of f, we have $\widehat{f}_{v!} \circ i_* = \widehat{f}'_{v!}$.

I4) If $p : E \to X$ is a $\widehat{h}$-oriented real vector bundle, then $\widehat{p}_{v!}$ is left inverse of the Thom morphism, i.e., *$\widehat{p}_{v!} \circ \widehat{T}_v(\widehat{\alpha}) = \widehat{\alpha}$ for every $\widehat{\alpha} \in \widehat{h}_v(X)$.*

5.5 Relative Differential Integration Maps

Recall the functor $F_\rho : \mathcal{K}(\rho) \to \mathtt{Man}_2^2$ and $G_{(F,f)} : \mathcal{V}(F, f) \to \mathtt{Man}_2^2$ used in Definitions 4.6.4 and 4.6.5. With the same notation we have

Definition 5.5.1 (Compactly and Vertically-compactly supported differential cohomology)**.** We define the compactly supported differential cohomology $\widehat{h}_c(\rho)$ for a map $\overline{\rho}$ as the colimit

$$\widehat{h}_c(\rho) = \operatorname{colim} \widehat{h}_{\mathrm{par}} \circ F_\rho$$

and the vertically-compactly supported differential cohomology $\widehat{h}_v(\overline{\rho})$ with respect to a relative vector bundle $(F, f) : \overline{\rho} \to \rho$ ad the colimit

$$\widehat{h}_v(\rho) = \operatorname{colim} \widehat{h}_{\mathrm{par}} \circ G_{(F,f)}$$

These constructions are functorial with respect to open embeddings as their topological analogues. Moreover, it is possible to define compactly and vertically-compactly analogues of the natural transformations R, I, a which we denote by R_c, I_c, a_c and R_v, I_v, a_v respectively.

5.5.1 Relative differential Thom morphism and integration

The definition of the classical relative differential Thom morphisms mirrors the one we have done in section 4.6.2. Before presenting it, let us briefly review some aspects of relative vector bundle over a map ρ.

Recall that a relative vector bundle $(P, p) : \overline{\rho} \to \rho$, as depicted in the following diagram,

$$\begin{array}{ccc} F & \xrightarrow{\overline{\rho}} & E \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle P} \\ A & \xrightarrow{\rho} & X \end{array} \tag{5.5}$$

is a morphism of vector bundles such that $\overline{\rho} : F \to E$ is fiberwise an isomorphism. Given a differential Thom class $\widehat{u} \in h^n_v(E)$ we can pull it back to differential Thom class $\widehat{u}' \in h^n_v(F)$ via $\overline{\rho}$. We say that a relative vector bundle is $\widehat{h}$-oriented if it is endowed with a pair of differential Thom classes $(\widehat{u}, \widehat{u}')$ where $\widehat{u}'$ is induced by $\widehat{u}$.

Fix a vector bundle $(P, p) : \overline{\rho} \to \rho$ as above. Given a class $\widehat{\alpha} \in \widehat{h}(\rho)$ we define the product $\widehat{u} \cdot (P, p)^*\widehat{\alpha} \in h_v(\overline{\rho})$ as follows: choose a representative $\widehat{u}_V \in \widehat{h}_{\mathrm{par}}(E, \mathrm{int}(V)^c) \cong \widehat{h}_{\mathrm{par}}(E, V^c)$, with $V \in \mathcal{V}_e(P)$, of $\widehat{u}$. Now, using the paralell-relative product to multiply $\widehat{u}_V$ and $\widehat{\alpha}$ we get

$$\widehat{u}_V \times (P, p)^*\widehat{\alpha} \in h(\mathrm{id}_E \times \overline{\rho} : (E \times F, \mathrm{int}(V)^c \times F) \to (E \times E, \mathrm{int}(V)^c) \times E)$$

Next, we consider the pull back of $\widehat{u}_V \times (P, p)^*\widehat{\alpha}$ by the following diagonal map

$$\begin{array}{ccc} (E \times F, \mathrm{int}(V)^c \times F) & \xrightarrow{\mathrm{id}_E \times \overline{\rho}} & (E \times E, \mathrm{int}(V)^c \times E) \\ {\scriptstyle \Delta_{\overline{\rho}}}\big\uparrow & & {\scriptstyle \Delta}\big\uparrow \\ (F, \mathrm{int}(\widehat{V})^c) & \xrightarrow{\overline{\rho}_{V,\widehat{V}}} & (E, \mathrm{int}(V)^c) \end{array}$$

where $\widehat{V} \in \mathcal{V}_e(p)$ is such that $\overline{\rho}^{-1}(V) \subseteq \mathrm{int}\,\widehat{V}$. Taking the colimit over

$$\widehat{u}_V \cdot \alpha := (\Delta, \Delta_\rho)^*(\widehat{u}_V \times \alpha)$$

give the desired product $\widehat{u} \cdot (P, p)^*\widehat{\alpha}$.

Now we can define the most general relative differential integration map with compact fibres. Let us fix a bundle map $(F, f) : \overline{\rho} \to \rho$ as in the following diagram.

$$\begin{array}{ccc} B & \xrightarrow{\overline{\rho}} & Y \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle F} \\ A & \xrightarrow{\rho} & X \end{array}$$

In section 1.7.2, we discussed how a topological orientation of the map $F : Y \to X$ induces a natural orientation on $f : B \to A$. The same holds in the differential framework as we pass to describe.

Given an $\widehat{h}$-orientation $[\iota, \widehat{u}, \phi]$ of $F : Y \to X$, we endow $f : B \to A$ with a $\widehat{h}$-orientation $[\iota', \widehat{u}', \phi']$ in $f : B \to A$ defined in the following way:

- If $\iota(y) = (F(y), j(y))$, define $\iota' : B \to A \times \mathbb{R}^n$ by $\iota'(b) := (f(b), j(\overline{\rho}(b))$. This map makes the following diagram commutative:

$$\begin{tikzcd}
& A \times \mathbb{R}^n \arrow[rr, "\rho\times\mathrm{id}_{\mathbb{R}^n}"] \arrow[ddl, "\mathrm{pr}_A"] & & X \times \mathbb{R}^n \arrow[ddl, "\mathrm{pr}_X"] \\
B \arrow[ur, hook, "\iota'"] \arrow[rr, "\overline{\rho}"] \arrow[d, "f"] & & Y \arrow[ur, hook, "\iota"'] \arrow[d, "F"] & \\
A \arrow[rr, "\rho"] & & X &
\end{tikzcd}$$

- Denote by $\rho_\iota : \iota'(B) \to \iota(Y)$ the restriction of $\rho \times \mathrm{id}_{\mathbb{R}^n}$ to $\iota'(B)$. Its differential induces a map $\widetilde{\rho} : N(\iota'(A)) \to N(\iota(X))$ by the diagram

$$\begin{tikzcd}
T\iota(Y) \arrow[r, "q_Y"] \arrow[d, "d\rho_\iota"] & \frac{i^*T(X\times\mathbb{R}^n)}{i(TY)} \arrow[r, "\pi_N"] \arrow[d, "\widetilde{\rho}"] & \iota(Y) \arrow[d, "\rho_\iota"] \\
T\iota'(B) \arrow[r, "q_B"] & \frac{i'^*T(A\times\mathbb{R}^n)}{i'(TB)} \arrow[r, "\pi'_N"] & \iota(B)
\end{tikzcd} .$$

  We define $\widehat{u}' := \widetilde{\rho}^*\widehat{u}$.

- Calling $\phi(n) = (F \circ \pi_N(n), \varphi(n))$, we set $\phi'(n') := (g \circ \pi'_N(n'), \varphi(\widetilde{\rho}(n')))$ and we call U' the image of ϕ'. Then the pair (U', ϕ') is a tubular neighbourhood of $\iota'(B) \in A \times \mathbb{R}^n$.

Summing up, we have the following commutative diagram

$$\begin{tikzcd}
A \times \mathbb{R}^n \arrow[rrr, bend left, "\rho\times\mathrm{id}_{\mathbb{R}^n}"] \arrow[ddddr, bend right, "\mathrm{pr}_A"'] & U' \arrow[l, hook', "\subseteq"] \arrow[r, hook, "\rho\times\mathrm{id}_{\mathbb{R}^n}"] & U \arrow[r, hook, "\subseteq"'] & X \times \mathbb{R}^n \arrow[ddddl, bend left, "\mathrm{pr}_X"] \\
& N(\iota'(B)) \arrow[r, "\widetilde{\rho}"] \arrow[u, "\phi'"] \arrow[d, "\pi'_N"] & N(\iota(Y)) \arrow[u, "\phi"] \arrow[d, "\pi_N"] & \\
& \iota'(B) \arrow[r, "\rho_\iota"] \arrow[uul, hook, bend left, "\subseteq"] & \iota(Y) \arrow[uur, hook, bend right, "\subseteq"'] & \\
& B \arrow[r, "\overline{\rho}"] \arrow[u, "\iota'"] \arrow[d, "f"] & Y \arrow[u, "\iota"] \arrow[d, "F'"] & \\
& A \arrow[r, "\rho"] & X &
\end{tikzcd}$$

We say that a relative fiber bundle (F, f) is $\widehat{h}$-oriented if the map F is $\widehat{h}$-oriented and f is endowed with the induced orientation (at representative level).

Remark 5.5.2. We constructed a orientation at representative level, nevertheless it is possible to show that the induced orientation class of the representative $(\iota', \widehat{u}', \phi')$ only depends on orientation class of $(\iota, \widehat{u}, \phi)$.

It only remains to discuss the $\mathbb{R}^n$ integration map. It is completely analogous to the relative topological case. Given a class $\alpha \in h_v(\rho \times \mathrm{id}_{\mathbb{R}})$ we have the open embedding $(\mathrm{id}_X \times j, \mathrm{id}_A \times j) : (\rho \times \mathrm{id}_{\mathbb{R}}) \hookrightarrow (\rho \times \mathrm{id}_{S^1})$ of relative fiber bundle over ρ. We define

$$\int_{\mathbb{R}} := \int_{S^1} \circ (\mathrm{id}_X \times j, \mathrm{id}_A \times j)_*$$

and

$$\int_{\mathbb{R}^n} := \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{n}.$$

Definition 5.5.3 (Relative differential Integration for relative fiber bundles with compact fibers morphism)**.** Fix a $\widehat{h}$-oriented relative fiber bundle $(F, f) : \overline{\rho} \to \rho$ with compact fibers. Let $(\iota, \widehat{u}, \phi)$ be a representative of the orientation of F and $(\iota', \widehat{u}', \phi')$ the induced representative of the orientation of f. We define the relative differential integration map $\widehat{(F, f)_!} : \widehat{h}^\bullet(\overline{\rho}) \to \widehat{h}^{\bullet - n}(\rho)$ by

$$\widehat{(F, f)_!}(\widehat{\alpha}) = \int_{\mathbb{R}^n} (i, i')_* \circ ((\phi, \phi)^{-1})^* \circ \widehat{T}_N(((\iota, \iota')^{-1})^*(\widehat{\alpha})$$

5.5.2 Compactly supported relative differential Thom morphism and integration

The construction of the Thom isomorphism is completly analogous to the topological one. Fix a oriented relative vector bundle $(P, p) : \overline{\rho} \to \rho$ over a proper map ρ with Thom class $(\widehat{u}, \widehat{u}')$ as in (5.5). In order to define its compactly supported relative differential Thom morphism, we need to make sense of the following product:

$$\widehat{u} \cdot (P, p)^* \widehat{\alpha} \in h_c(\overline{\rho})$$

where $\widehat{\alpha} \in \widehat{h}_c(\rho)$. Fix a representative $\widehat{u}_V \in \widehat{h}_{\mathrm{par}}(E, \mathrm{int}(V)^c)$ of $\widehat{u}$. Choose and representative $\widehat{\alpha} \in \widehat{h}_{\mathrm{par}}(\rho_{K,\widehat{K}} : (A, \mathrm{int}(\widehat{K})^c) \to (X, \mathrm{int}(K)^c))$ of $\widehat{\alpha}$. Define $\widehat{\beta}_{H,\widehat{H}} := (P, p)^*(\widehat{\alpha}_{K,\widehat{K}}) \in \widehat{h}_{\mathrm{par}}(\overline{\rho}_{H,\widehat{H}} : (E, \mathrm{int}(\widehat{H})^c) \to (E, \mathrm{int}(E)^c)$ where the where $\widehat{H} := p^{-1}(\widehat{K})$ and $H := P^{-1}(K)$ are manifolds assumed to be transverse to V.

Now, we multiply a class on a map of pairs by a parallel class, obtaining a class on a sequence with two entries using the product in Definition 5.3.3:

$$\widehat{u}_V \times \widehat{\beta}_{H,\widehat{H}} \in \widehat{h}_{\mathrm{par}}(\mathrm{id}_E \times \overline{\rho} : (E \times F, \mathrm{int}(V)^c \times F, E \times \mathrm{int}(\widehat{H})^c) \to (E \times E, \mathrm{int}(V)^c \times E, E \times \mathrm{int}(H)^c))$$

Pulling along the diagonal $(\Delta, \Delta_\rho) : \overline{\rho} \to \mathrm{id}_E \times \overline{\rho}$

$$\begin{tikzcd}
(E \times F, \mathrm{int}(V)^c \times F, E \times \mathrm{int}(\widehat{H})^c) \arrow[r, "\mathrm{id}_E \times \overline{\rho}"] & (E \times E, \mathrm{int}(V)^c \times E, E \times \mathrm{int}(H)^c) \\
(F, \mathrm{int}(\widehat{V})^c \cup \mathrm{int}(\widehat{H})^c, \emptyset) \arrow[u, "\Delta_\rho"] \arrow[r, "\overline{\rho}'"] & (E, \mathrm{int}(V)^c \cup \mathrm{int}(H)^c, \emptyset) \arrow[u, "\Delta"']
\end{tikzcd}$$

where $\widehat{V} \in \mathcal{V}_e(p)$ is any special vertical set such that $\overline{\rho}^{-1}(V) \subseteq \mathrm{int}(\widehat{V})$ and $\widehat{V}$ is transverse to $\widehat{H}$, we get a class

$$(\Delta, \Delta_\rho)^*(\widehat{u}_V \times \widehat{\beta}_{H,\widehat{H}}) \in \widehat{h}_{\mathrm{par}}(\overline{\rho} : (F, (\mathrm{int}(\widehat{V} \cap \widehat{H}))^c) \to (E, (\mathrm{int}(V \cap H))^c)$$

with

- $\widehat{V} \cap \widehat{H} \in \mathcal{K}(F)$, $V \cap H \in \mathcal{K}(E)$;
- $\widehat{V} \cap \widehat{H}$, $\mathrm{int}(\widehat{V} \cap \widehat{H})^c$, $V \cap H$ and $\mathrm{int}(V \cap H)^c$ are manifolds.
- $\overline{\rho}^{-1}(V \cap H) \subseteq \widehat{V} \cap \widehat{H}$

which give us the Thom morphism in the colimit. Using it we are finally able to define the last integration map we will present here.

Definition 5.5.4 (Compactly supported relative differential integration map)**.** Given an $\widehat{h}$-oriented fiber bundle over a proper map $(F, f) : \overline{\rho} \to \rho$ we define its compactly-supported relative differential integration map by $\widehat{(F, f)}_{c!} : \widehat{h}_c^\bullet(\overline{\rho}) \to \widehat{h}_c^{\bullet-(n-m)}(\rho)$ by

$$\int_{\mathbb{R}^L}^{c} (i, i')_*((\phi, \phi')^{-1})^* T_{Nc}(((\iota, \iota')^{-1})^* \alpha)$$

5.5.3 Other possible integration maps

We could also have constructed the doubly-vertically-compactly supported Thom isomorphism $\widehat{h}_v \to \widehat{h}_{vv}$ and the vertically-compactly supported differential integration map.

Another direction which we could follow, but it was not pursued here, is that we could consider higher versions of the doubly-vertically compact maps. For example, we could consider a version with is triply vertically-compactly supported on maps $\widehat{h}_{vvv}$. With this kind of construction, we could have chains of integration in the form

$$\widehat{h}_{vv} \xrightarrow{\int} \widehat{h}_v \xrightarrow{\int} \widehat{h}$$

We could also extend the definition of compactly-supported cohomology to proper maps of sequences (in particular, of pairs) and construct the compactly-supported version of the products in the form respectively $\widehat{h}_c(X, A) \otimes \widehat{h}_{\mathrm{par}}(X, B) \to \widehat{h}_c(\iota : (A, A \cap B) \to (X, B))$ and $\widehat{h}_c(\rho : A \to X) \otimes \widehat{h}_{\mathrm{par}}(X, B) \to \widehat{h}_c(\rho : (A, \rho^{-1}(B)) \to (X, B))$

5.6 Conclusion

We have completed the construction of almost all of the missing versions of integration maps as displayed in Table 5. Moreover, we have given a set of axioms for the compactly supported and vertically compactly supported versions. The only thing which

Umkher \ Type	Absolute	Relative
Compact Fiber	✓	✓
Compact	✓	✓
Vertical	✓	□

Table 5 – Differential integration maps in differential cohomology constructed using the new parallel-relative product. The ✓ denotes the existence of the integration map. The □ denotes the possibility to define it, but was not done here.

remains to show is that there exists differential cohomology theories on finite sequences of manifolds in the sense of Definition 5.2.1 and that these theories possess a parallel-relative product in the sense of Definition 5.3.3. We do this in the next chapter.

6 Models of Differential Cohomology on Maps of Pairs

6.1 Introduction

In the last chapter, we have shown how to define the integration maps and have characterized them axiomatically. But we did not exhibit any model of a differential cohomology on maps of sequences and thus we have not show the existence of the the parallel-relative product in any theory beyond the trivial de Rham toy model.

In this chapter, we present three models of differential cohomology theories on sequences, all of them endowed with a parallel-relative product:

- The Cheeger-Simons differential characters model, which refines ordinary cohomology with integer coefficients;
- The Freed-Lott model refining topological complex K-theort;
- The Hopkins-Singer model, which is able to refine every sufficiently regular cohomology theory.

The Hopkins-Singer model is particularly interesting: since it can be used to refine any cohomology theory to a differential cohomology on maps of sequences, it ensures the existence of the differential integration maps in any theory which can be endowed with a differential S^1-integration.

6.2 Cheeger-Simmons models on maps of sequences

This section extends the model of Cheeger-Simons differential characters (see section B.3 in the Appendix B for a review) to the case of maps of sequences. For a smooth sequence $(X, \vec{X}) = (X, X_1, \dots, X_n)$, we denote by $S^{\mathrm{sm}}_{\mathrm{par}}(X, \vec{X})$ the group of singular chain complex $S^{\mathrm{sm}}_{\mathrm{par}}(X, \bigcup_n X_n)$ as defined in (B.4). Let $\rho : (A, \vec{A}) \to (X, \vec{X})$ be a map of sequences. We define the group of relative singular chains $S^{\mathrm{sm}}(\rho)$ as the mapping cone complex of the chain morphism

$$\rho_\# : S^{\mathrm{sm}}_{\mathrm{par}}(A, \vec{A}) \to S^{\mathrm{sm}}_{\mathrm{par}}(X, \vec{X}).$$

For a relative form $\Omega^\bullet(\rho)$ and a relative chain $(\sigma, \tau) \in S^{\mathrm{sm}}_\bullet(\rho)$ we define

$$\int_{(\sigma,\tau)} (\omega, \theta) := \int_\sigma \omega + \int_\tau \theta.$$

Definition 6.2.1. A differential character of degree n on a map of sequences $\rho : (A, \vec{A}) \to (X, \vec{X})$ is a pair $(\chi, (\omega, \theta))$ where $\chi : Z^{\mathrm{sm}}_{n-1}(\rho) \to \frac{\mathbb{R}}{\mathbb{Z}}$ is a homomorphism and $(\omega, \theta) \in \Omega^n(\rho)$ is a relative n-form such that

$$\chi(\partial c) = \int_c (\omega, \theta) \mod \mathbb{Z}$$

for every $c \in S^{sm}_n(\rho)$.

The set of differential character will be denote by $\widehat{H}(\rho)$. This is indeed an abelian group where the sum is defined by

$$(\chi, (\omega, \theta)) + (\chi', (\omega', \theta')) := ((\mathrm{ch}\,/ + \chi'), (\omega + \omega', \theta + \theta')$$

with $(\chi + \chi)'(z) = \chi(z) + \chi'(z)$.

Remark 6.2.2. The relative form (ω, θ) has integral periods since

$$\int_{\partial c} (\omega, \theta) = \chi(\partial(\partial c)) = 0 \mod \mathbb{Z}$$

By an analogue of Proposition A.3.6 to this setting, we conclude that the form is also closed.

Remark 6.2.3. Actually, the form (ω, θ) is completely determined by χ in the sense that, if $(\chi, (\omega, \theta)) = (\chi, (\omega', \theta'))$ than $(\omega, \theta) = (\omega', \theta')$ as in the absolute case (see Section B.3).

We denote the groups of differential characters of degree n by $\widehat{H}^n(\rho)$. In fact, we have a (contravariant) functor $\widehat{H} : \mathtt{Man}^{2,\mathrm{op}}_{\omega} \to \mathtt{GrAb}$ which acts on morphisms $(f, g) : \rho \to \eta$ as

$$(f, g)^*(\chi, (\omega, \theta)) = (\chi \circ (f, g)_\#, (f, g)^*(\omega, \theta))$$

where $(f, g)_\# : S(\rho) \to S(\eta)$ is the induced map at chain level.

The map $R : \widehat{H}(\rho) \to \Omega_{\mathrm{cl}}(\rho)$ defined by $R((\chi, (\omega, \theta)) = (\omega, \theta)$ is a group homomorphism and will be our **curvature** in this model.

Now, we observe that $Z^{\mathrm{sm}}_{\bullet-1}(\rho)$ is a free group and therefore is projective. This enable us to lift the homomorphism $\chi : Z^{\mathrm{sm}}_{\bullet-1}(\rho) \to \frac{\mathbb{R}}{\mathbb{Z}}$ to a homomorphism $\widetilde{\chi} : Z^{\mathrm{sm}}_{\bullet-1}(\rho) \to \mathbb{R}$ as depicted in diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "q"] \\ Z^{\mathrm{sm}}_{\bullet-1}(\rho) \arrow[ur, "\widetilde{\chi}"] \arrow[r, "\chi"] & \frac{\mathbb{R}}{\mathbb{Z}} \end{tikzcd}$$

Consider the map $I(\widetilde{\chi}) : Z^{\mathrm{sm}}_{n-1}(\rho) \to \mathbb{R}$ defined by

$$I(\widetilde{x})(c) = \int_c R(\chi) - \widetilde{\chi}(\partial c). \tag{6.1}$$

It can be verified that this map is a integral cocyle and that its cohomology class is independent of the lift $\widetilde{\chi}$. We denote the cohomology class of $I(\widetilde{\chi})$ by $I(\chi)$. The map $I(\chi) : \widehat{H}(\rho) \to H(\rho)$ is a homomorphism and will play the role of **forgetful map**.

Given a form $(\omega, \theta) \in \Omega^{\bullet-1}(\rho)$ we define the differential character

$$a(z) := \int_z (\omega, \theta) \mod \mathbb{Z}$$

If $(\omega', \theta') = (\omega, \theta) + d(\mu, \nu)$, then $a(\omega, \theta) = a(\omega', \theta')$. This shows that we have a homomorphism $a : \frac{\Omega^{\bullet-1}(\rho)}{\mathrm{Im}(d)} \to \widehat{H}^{\bullet}(\rho)$ which is the **trivialization**.

Proposition 6.2.4. *The data $(\widehat{H}, R, I, a)$ forms a differential cohomology theory.*

The proof of this results is analogue to the proof of the relative version on maps (see Proposition B.3.3), the only change is that it relies on the de Rham isomorphism for sequences.

With the same notation of definition 5.2.2, we write $(\widehat{H}_{\mathrm{par}}, R_{\mathrm{par}}, I_{\mathrm{par}}, a_{\mathrm{par}})$ for the parallel character obtained from the differential cohomology and restricted to closed embeddings.

6.2.1 Eilenberg-Zilber Maps for Sequences

In the appendix, we present the Eilenberg-Zilber morphisms (section B.2.3) and extend them to relative setting of maps. In the present setting, we will generalize these morphisms to the case of sequences:

$$S_{\bullet}((Y, \vec{B}) \times (\rho, \vec{\rho})) \underset{E}{\overset{A}{\rightleftarrows}} S_{\mathrm{par},\bullet}(Y, \vec{B}) \otimes S_{\bullet}(\rho, \vec{\rho}); \tag{6.2}$$

These morphisms are construct in analogous way as we have done in Section B.2.3. More specifically, for a map $\rho : (A, \vec{A}) \to (X, \vec{X})$, the left-hand side of (6.2) can be written as the mapping cone of

$$(\mathrm{id}_Y \times \rho)_{\#} : S_{\mathrm{par}}(Y \times A, B' \times A, Y \times A') \to S_{\mathrm{par}}(Y \times X, B' \times X, Y \times X') \tag{6.3}$$

where $A' = \bigcup_i A_i$, $B' = \bigcup_i B_i$ and $X' = \bigcup_i X_i$. On the other hand, we have

$$S(Y, \vec{B}) \otimes S(\rho) = (S_{\mathrm{par}}(Y, B') \otimes S_{\mathrm{par}}(A, A')) \oplus (S_{\mathrm{par}}(Y, B') \otimes S(X, X')). \tag{6.4}$$

Observe that

$$S_{\mathrm{par}}(Y \times X, B' \times X, Y \times X') = S_{\mathrm{par}}(Y \times X, B' \times X \cup Y \times A')$$

and

$$S_{\mathrm{par}}(Y \times X, B' \times X, Y \times X') = S_{\mathrm{par}}(Y \times X, B' \times X \cup Y \times A')$$

It is possible to prove (DIECK, 2008, 9.7.3, p.239) that, when dealing with cofibrations, the Eilenberg-Zilber morphisms descend to quotient and give us chain homotopies

$$S_{\mathrm{par},\bullet}(Y \times X, B' \times X \cup Y \times X') \underset{E}{\overset{A}{\rightleftarrows}} S_{\mathrm{par},\bullet}(Y, B') \otimes S_{\mathrm{par},\bullet}(X, X');$$

Using this fact on equation (6.3) and (6.4) give us the desired map (6.2).

6.2.2 Kunneth Theorem and Splitting Cycles

By using the Eilenberg-Zilber maps (6.2) in the algebraic Künneth theorem A.4.8, we get another exact sequence for a cofibration (Y, B) and a map $\rho : A \to X$:

$$\begin{tikzcd} 0 \arrow[r] & H(Y,B) \otimes_{\mathbb{Z}} H(\rho, \vec{\rho}) \arrow[r, "\times"] & H((Y,\vec{B}) \times (\rho, \vec{\rho})) \arrow[r] & \mathrm{Tor}(H(Y,\vec{B}), H(\rho,\vec{\rho})) \arrow[r] \arrow[l, bend left, "\phi"] & 0. \end{tikzcd} \tag{6.5}$$

Here $\times : H(Y, \vec{B}) \otimes H(\rho, \vec{\rho}) \to H((Y, \vec{B}) \times (\rho, \vec{\rho})$ is induced by the composition $E \circ \otimes$. The sequence splits, although not in a canonically way.

We will use Künneth sequence to obtain a useful decomposition of cycles. Consider the following sequences split exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_\bullet(\rho,\vec{\rho}) \arrow[r, hook, "i_\rho"] & S_\bullet(\rho,\vec{\rho}) \arrow[r, "\partial_\rho"] \arrow[l, dashed, bend left, "s_\rho"] & B_{\bullet-1}(\rho,\vec{\rho}) \arrow[r] & 0; \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z_\bullet(Y,B) \arrow[r, hook, "i_Y"] & S_{\mathrm{par},\bullet}(Y,\vec{B}) \arrow[r, "\partial_Y"] \arrow[l, dashed, bend left, "s_Y"] & B_{\bullet-1}(Y,\vec{B}) \arrow[r] & 0. \end{tikzcd}$$

where we choose splittings s_ρ and s_Y. Together, these sequences yield

$$\begin{tikzcd} Z_{\mathrm{par}}(Y,\vec{B}) \otimes_{\mathbb{Z}} Z(\rho,\vec{\rho}) \arrow[r, shift left, "i_Y \otimes i_\rho"] \arrow[dr, shift left, "\times"] & Z(S_{\mathrm{par}}(Y,\vec{Y}) \otimes_{\mathbb{Z}} S_\bullet(\rho,\vec{\rho})) \arrow[l, dashed, shift left, "s_Y \otimes s_\rho"] \arrow[d, shift left, dashed, "A^{\rho,Y}"'] \\ & Z((Y,\vec{B}) \times (\rho,\vec{\rho})) \arrow[u, shift left, "E^{\rho,Y}"'] \arrow[ul, dashed, shift left, "S"] \end{tikzcd} \tag{6.6}$$

where $S = (s_Y \otimes s_\rho) \circ A$. In particular, since $A \circ E = \mathrm{id}$, it follows that $S \circ \times = \mathrm{id}$.

We call a cycle $t \in S_\bullet(\rho)$ a torsion cycle if there exists a $n \in \mathbb{N}$ such that $nt = \partial c$ for some $c \in S_{\bullet+1}(\rho)$.

Lemma 6.2.5. *Given a cycle $z \in Z_n^{sm}((Y, \vec{B}) \times (\rho, \vec{\rho}))$, we can write it as*

$$z = \sum_{p+q=n} y_p \times x_q + t$$

where $y_p \in Z^{sm}_{par,p}(Y, \vec{B})$, $x_q \in Z^{sm}_q(\rho, \vec{\rho})$, and t is torsion cycle.

Proof. Write $S(z) = \sum_{p+q=n} y_p \otimes x_q$. From this follows that $\times \circ S(z) = \sum_{p+q=n} y_p \times x_q$. Define

$$t := z - \sum_{p+q=n} y_p \times x_q$$

Passing to cohomology, we get

$$[t] := [z] - \sum_{p+q=n} [y_p] \times [x_q].$$

Since $S(t) = 0$, it follows that $[t]$ is in the image of ϕ as in (6.5). Since $\mathrm{Tor}(H(Y, \vec{B}), H(\rho, \vec{\rho}))$ is a torsion grou, we conclude that $[t]$ is torsion group, which means that t is a torsion cycle. □

6.2.3 Parallel-relative product

Lets evaluate a differential character over a torsion cycle

Proposition 6.2.6. *Let $t \in S_{n-1}(\rho, \vec{\rho})$ be a torsion cycle such that $nt = \partial c$ and $\chi \in \widehat{H}(\rho, \vec{\rho})$ a differential character. We have*

$$\chi(t) = \frac{1}{n} \int_c R(\chi) - \widetilde{I}(\chi)(c)$$

where $\widetilde{I}(\chi)$ is any lift of $I(\chi)$.

The proof is similar to Proposition B.3.4

The multiplicative structure is very similar to the absolute-relative product of Definition B.3.6.

Definition 6.2.7 (Parallel Relative Product)**.** Let $z \in Z((Y, \vec{B}) \times (\rho, \vec{\rho}))$ and write it as $z = y \times x + t$ as in Lemma 6.2.5. We define the product $\times : \widehat{H}^p_{\mathrm{par}}(Y, \vec{B}) \otimes \widehat{H}^q(\rho, \vec{\rho}) \to \widehat{H}^{p+q}((Y, \vec{B}) \times (\rho, \vec{\rho}))$ as

$$(\chi' \times \chi)(z) = (\chi' \times \chi)(y \times z) + (\chi' \times \chi)(t) \tag{6.7}$$

where, for products,

$$(\chi \times \chi')(y \times x) = \chi'(y) \cdot \int_x R(\chi) + \int_y R_{\mathrm{par}}(\chi') \cdot \chi(x) \mod \mathbb{Z}$$

with the conventions

- $\chi(\sigma) = 0$ if $\sigma' \notin Z^{\mathrm{sm}}_{\mathrm{par},p-1}(Y, \vec{B})$ and $\chi(\sigma, \tau) = 0$ if $(\sigma, \tau) \notin Z^{\mathrm{sm}}_{\mathrm{par},q-1}(\rho)$ and
- $R_{\mathrm{par}}(\chi)(\sigma') = 0$ if $\sigma' \notin \Omega^p(Y, \vec{B})$ $R(\chi)((\sigma, \tau)) = 0$ if $(\sigma, \tau) \notin \Omega^q(\rho, \vec{\rho})$ and ,

and, for cycles,

$$(\chi' \times \chi)(t) = \frac{1}{n}\int_c R_{\mathrm{par}}(\chi') \times R(\chi) - \left(\widetilde{I(\chi)} \times \widetilde{I_{\mathrm{par}}(\chi')}\right)(c) \mod \mathbb{Z}$$

where $nt = \partial c$.

The fact that this is a relative character can be verified in the exactly same way as in the absolute-relative case (see Proposition B.3.7).

Proposition 6.2.8. *The product defined in* (6.7) *is a parallel-relative product in the sense of Definition* (5.3.3)*.*

The proof in (BäR; BECKER, 2014, Theorem 26, 147) can be promptly adapted to this setting.

6.3 K-Theory on maps of pairs

Despite being a relevant topic to this work, we will only present the main definitions and quote the theorems without proof. A full account can be found in Nuñez's thesis.

We will only present differential K-theory on maps of pairs. The generalization to the finite sequence of manifolds is similar and the reader can found detail in the mentioned thesis.

6.3.1 Vector Triples on Maps of Pairs

In the topological framework, given a map of pairs $\rho : (A, A') \to (X, X')$, such that all the spaces involved have the homotopy type of a finite CW-complex, we call vector bundle triple a triple of the form $(\mathcal{E}, \mathcal{F}, \alpha)$, where:

- $\mathcal{E} = (E_1, E_2, \beta)$ and $\mathcal{F} = (F_1, F_2, \gamma)$ are vector bundle triples on (X, X');
- $\alpha = (\alpha_1, \alpha_2) : \rho^*\mathcal{E} \xrightarrow{\sim} \rho^*\mathcal{F}$ is an isomorphism of triples on (A, A').

Explicitly, the second item states that $\alpha_1 : \rho^* E_1 \xrightarrow{\sim} \rho^* F_1$ and $\alpha_2 : \rho^* E_2 \xrightarrow{\sim} \rho^* F_2$ are isomorphisms such that the following diagram commutes:

$$\begin{tikzcd}
(\rho^* E_1)|_{A'} \arrow[r, "\rho^*\beta"] \arrow[d, "\alpha_1|_{A'}"] & (\rho^* E_2)|_{A'} \arrow[d, "\alpha_2|_{A'}"] \\
(\rho^* F_1)|_{A'} \arrow[r, "\rho^*\gamma"] & (\rho^* F_2)|_{A'}
\end{tikzcd} \tag{6.8}$$

A triple of the form $(\mathcal{E}, \mathcal{E}, \mathrm{id})$ is called elementary. The direct sum of vector bundle triples is defined componentwise and an isomorphism from $(\mathcal{E}, \mathcal{F}, \alpha)$ to $(\mathcal{E}', \mathcal{F}', \alpha')$ is a pair (Φ, Ψ) such that:

- $\Phi = (\phi_1, \phi_2) : \mathcal{E} \to \mathcal{E}'$ and $\Psi = (\psi_1, \psi_2) : \mathcal{F} \xrightarrow{\sim} \mathcal{F}'$ are isomorphisms of triples;
- the following diagram commutes:

$$\begin{array}{ccc} \rho^*\mathcal{E} & \xrightarrow{\alpha} & \rho^*\mathcal{F} \\ \downarrow{\scriptstyle \rho^*\Phi} & & \downarrow{\scriptstyle \rho^*\Psi} \\ \rho^*\mathcal{E}' & \xrightarrow{\alpha'} & \rho^*\mathcal{F}' \end{array}$$

Concretely, we have the isomorphisms $\phi_1 : E_1 \xrightarrow{\sim} E_1'$, $\phi_2 : E_2 \xrightarrow{\sim} E_2'$, $\psi_1 : F_1 \xrightarrow{\sim} F_1'$ and $\psi_2 : F_2 \xrightarrow{\sim} F_2'$, such that the following diagrams commute (because Φ and Ψ are morphisms of triples):

$$\begin{array}{ccc} E_1|_{X'} & \xrightarrow{\beta} & E_2|_{X'} \\ \downarrow{\scriptstyle \phi_1|_{X'}} & & \downarrow{\scriptstyle \phi_2|_{X'}} \\ E_1'|_{X'} & \xrightarrow{\beta'} & E_2'|_{X'} \end{array} \qquad \begin{array}{ccc} F_1|_{X'} & \xrightarrow{\gamma} & F_2|_{X'} \\ \downarrow{\scriptstyle \psi_1|_{X'}} & & \downarrow{\scriptstyle \psi_2|_{X'}} \\ F_1'|_{X'} & \xrightarrow{\gamma'} & F_2'|_{X'} \end{array}$$

and the following diagram commutes for $i = 1, 2$

$$\begin{array}{ccc} \rho^*E_i & \xrightarrow{\alpha_i} & \rho^*F_i \\ \downarrow{\scriptstyle \rho^*\phi_1} & & \downarrow{\scriptstyle \rho^*\psi_i} \\ \rho^*E_i & \xrightarrow{\alpha_i'} & \rho^*F_i' \end{array}$$

We call $\mathrm{Vec}(\rho)$ the semi-group of isomorphism classes of vector bundle triples with the operation of direct sum. Moreover, we introduce in $\mathrm{Vec}(\rho)$ the equivalence relation '$\simeq$', analogous to the one defined above, and we set $K(\rho) := \mathrm{Vec}(\rho)/\simeq$. We now consider the differential extension of $K(\rho)$, assuming that $\rho : (A, A') \to (X, X')$ is a smooth map between compact manifold pairs. From now on we assume that every vector bundle is endowed with an Hermitian metric, every isomorphism is unitary and every connection is compatible with the corresponding metric.

Definition 6.3.1. A connection on $(\mathcal{E}, \mathcal{F}, \alpha)$ is a triple $\nabla := (\nabla^E, \nabla^F, \widetilde{\nabla})$ such that:

- $\nabla^{\mathcal{E}} = (\nabla^{E_1}, \nabla^{E_2})$ and $\nabla^{\mathcal{F}} = (\nabla^{F_1}, \nabla^{F_2})$ are parallel connections respectively on $\mathcal{E}$ and $\mathcal{F}$;
- calling $\mathrm{pr}_A : A \times I \to A$ the projection, $\widetilde{\nabla} = (\overline{\nabla}, \overline{\nabla}')$ is a parallel connection on $\mathrm{pr}_A^* \rho^* E$ that interpolates between $\rho^*\nabla^E$ and $\alpha^*\rho^*\nabla^F$.

Explicitly, in the second item, $\overline{\nabla}$ interpolates between $\rho^*\nabla^{E_1}$ and $\alpha_1^*\rho^*\nabla^{F_1}$ and $\overline{\nabla}'$ interpolates between $\rho^*\nabla^{E_2}$ and $\alpha_2^*\rho^*\nabla^{F_2}$, in such a way that $(\mathrm{pr}_A^* \rho^*\beta)^*(\overline{\nabla}'|_{A'\times I}) = \overline{\nabla}|_{A'\times I}$. The latter identity restricts on $A' \times \{0\}$ to $(\rho^*\beta)^*((\rho^*\nabla^{E_2})|_{A'}) = (\rho^*\nabla^{E_1})|_{A'}$, that is correct, since it is the pull-back through ρ of $\beta^*(\nabla^{E_2}|_{X'}) = \nabla^{E_1}|_{X'}$, that is part of the definition of parallel triple. Similarly, it restricts on $A' \times \{1\}$ to $(\rho^*\beta)^*((\alpha_2^*\rho^*\nabla^{F_2})|_{A'}) = (\alpha_1^*\rho^*\nabla^{F_1})|_{A'}$;

since $\rho^*\beta = \alpha^*\rho^*\gamma$ because of the commutativity of diagram (6.8), this is equivalent to the pull-back through ρ and α of $\gamma^*(\nabla^{F_2}|_{X'}) = \nabla^{F_1}|_{X'}$, that again is part of the definition of parallel triple

The definitions of relative Chern character $ch(\nabla) \in \Omega^{\mathrm{ev}}_{\mathrm{cl}}(\rho)$ and relative Chern-Simons class $CS(\nabla, \nabla') \in \Omega^{\mathrm{odd}}(\rho)/\operatorname{Im}(d)$ are identical to that given in Definition C.2.11, replacing E and F respectively by $\mathcal{E}$ and $\mathcal{F}$. It shares the same properties as the relative case too. A *differential vector bundle triple* is defined as in section C.3, leading to definition C.3.2 in the framework of maps of pairs. There is a natural right-module structure on $\widehat{K}(\rho)$ over $\widehat{K}(X)$ defined by the same expression in Definition C.3.5 (we stress that we are not considering $\widehat{K}(X, X')$, but only $\widehat{K}(X)$, as the ring of scalars), leading to the analogous definition of exterior product. The extension to any degree and the definition of S^1-integration follow as we have seen above.

6.3.2 Parallel-Relative Product

Now we have all the tools to construct the parallel-relative product through the Freed-Lott model. Given $(\mathcal{T}, \nabla, \Theta) \in \widehat{K}(\rho : A \to X)$ and $(\mathcal{T}', \nabla', \omega') \in \widehat{K}_{\mathrm{par}}(Y, B)$, we have to construct the corresponding product $(\mathcal{T}'', \nabla'', \Theta'') \in K(\rho \times \mathrm{id}_Y : (A \times Y, A \times B) \to (X \times Y, X \times B))$ We set:

$$\mathcal{T} = (E, F, \alpha) \qquad \nabla = (\nabla^E, \nabla^F, \widetilde{\nabla}) \qquad \Theta = (\omega, \theta)$$
$$\mathcal{T}' = (E', F', \alpha') \qquad \nabla' = (\nabla^{E'}, \nabla^{F'}) \qquad .$$

We have to define

$$\mathcal{T}'' = (\mathcal{E}, \mathcal{F}, \alpha'') \quad \nabla'' = (\nabla^{\mathcal{E}}, \nabla^{\mathcal{F}}, \widetilde{\widetilde{\nabla}}) \quad \Theta'' = (\omega'', \theta''),$$

where

$$\mathcal{E} = (E_1'', E_2'', \beta'') \quad \mathcal{F} = (F_1'', F_2'', \gamma'') \quad \alpha'' = (\alpha_1'', \alpha_2'')$$
$$\nabla^{\mathcal{E}} = (\nabla^{E_1''}, \nabla^{E_2''}) \quad \nabla^{\mathcal{F}} = (\nabla^{F_1''}, \nabla^{F_2''}) \quad \widetilde{\widetilde{\nabla}} = (\overline{\nabla}, \overline{\nabla}').$$

Since we informally think of the product $\mathcal{T}\mathcal{T}'$ on $X \times Y$ as $(E-F)(E'-F') = (EE'+FF') - (EF' + FE')$ and we informally think of $\mathcal{T}''$ on $X \times Y$ as $E - F = (E_1'' - E_2'') - (F_1'' - F_2'')$, we define

$$E_1'' := E \boxtimes E' \quad E_2'' := E \boxtimes F' \quad \beta'' := \mathrm{id}_E \boxtimes \alpha'$$
$$F_1'' := F \boxtimes E' \quad F_2'' := F \boxtimes F' \quad \gamma'' := \mathrm{id}_F \boxtimes \alpha'.$$

Then, it is natural to define

$$\alpha_1'' := \alpha \boxtimes \mathrm{id}_E' \quad \alpha_2'' := \alpha \boxtimes \mathrm{id}_F'' .$$

The connections $\nabla^{E_1''}$, $\nabla^{E_2''}$, $\nabla^{F_1''}$ and $\nabla^{F_2''}$ are defined as the corresponding tensor products of ∇^E, ∇^F, $\nabla^{E'}$ and $\nabla^{F'}$. We conclude by setting $\overline{\nabla} := \widetilde{\nabla} \boxtimes \nabla^{E'}$ and $\overline{\nabla}' := \widetilde{\nabla} \boxtimes \nabla^{F'}$. The parallel-relative product between two classes of any degree can be easily defined through the one in degree 0.

6.4 Hopkins-Singer model for maps of pairs

6.4.1 Differential functions for maps of pairs

The basic notions discussed here can be found in Appendix D. We start by fixing an Ω-spectrum $((E_n, e_n), \epsilon_n)$ and consider its associated cohomology theory $E^\bullet$. We extend it to maps of pairs.

According to Lemma 4.3.4, one can identify $h(\rho, \rho')$ with $h(C_\rho, C'_\rho)$ for any map of pairs $(\rho, \rho') : (A, A') \to (X, X')$. In the cohomology induced by the spectrum, the group $E^\bullet(C_\rho, C_{\rho'})$ can be identified with the classes of homotopy $[(C_\rho, C_{\rho'}), (E_n, e_n)]$. Another way view this group is to consider homotopy classes $[(M_\rho, M_{\rho'}), (E_n, e_n)]$ with the additional requirement that $f(A \times 1) = \{e_n\}$ for a $[f] \in [(M_\rho, M_{\rho'}), (E_n, e_n)]$, in other words, we are considering the group

$$[(M_\rho, M_\rho \cup A \times \{1\}), (E_n, e_n)]$$

with its abelian structure induced by the isomorphism with $[(C_\rho, C_{\rho'}), (E_n, e_n)]$.

We fix the same notion of a smooths structure on M_ρ as we have done in the Appendix (section D.4.1.) Fix a graded real vector space V. Let $(Y, *)$ be a space with marked point and $\kappa_n \in S^n_{\mathrm{par}}(Y, y; V)$ be a cocyle.

Definition 6.4.1 (Relative Differential Function of Maps of Pairs)**.** A differential function $f : (\rho, \rho') \to (Y, y_*, \kappa_n)$ is a triple (f, h, ω), where

- $f : (M_\rho, M_{\rho'}) \to (Y, y_*)$ is a map of pairs;
- $h \in S^{n-1}_{\mathrm{sm}}(M_\rho, M_{\rho'}; V)$ is a smooth cochain;
- $\omega \in \Omega^n_{\mathrm{cl}} V(X, X')$ is differential for;

satisfying the following condition

$$\delta h(c) = \chi^m_\rho(\omega)(c) + \rho^* \kappa(c).$$

for any chain $c \in S^{\mathrm{sm}}_n(M_\rho, M_{\rho'}; V)$

Here, the morphism $\chi^\bullet_{\rho,\rho'} : \Omega^\bullet V(X, X') \to S^\bullet_{\mathrm{par,sm}}(M_\rho, M_{\rho'}; V)$ is defined by

$$\chi_{\rho,\rho'}(\omega)(c) := c^*_\rho \chi_{(X,X')}$$

where c_ρ is the collapse map and $\chi_{(X,X')}$ is the de Rham isomorphism map at chain level.

A pair of differential functions (f_0, h_0, ω) and (f_1, h_1, ω') from (ρ, ρ') to (Y, y_*) is to be homotopic if there exists a differential function $(F, H, \pi_X^*\omega) : (\rho \times \mathrm{id}_I, \rho' \times \mathrm{id}_I) \to (Y, *, \kappa_n)$, called homotopy, such that

- $\omega = \omega'$
- F is a homotopy of pairs between f_0 and f_1;
- H is a cochain such that $H|_0 = h_0$ and $H|_1 = h_1$
- $(F, H, \mathrm{pr}_X^* \omega)|_{X' \times I} = (k_*, 0, 0)$, where k_{y_*} is the constant map.

A *strong trivialization* with respect to $\eta \in \Omega^{n-1}(A, A')$, is a homotopy between a differential function (f, h, ω) from (ρ, ρ') to $(Y, *, \kappa_n)$ and the differential function $(k_*, \chi_{\rho,\rho'}(\eta), d\eta)$.

6.4.2 Hopkins-Singer Model for Pairs

Actually, the maps introduced in section D.4.2 are also maps of pairs

- $\iota_{M(A)} : (M(A), M(A')) \to (M_\rho, M_{\rho'})$ given by $\iota(a, t) = [(a, t)]$;
- $\mathrm{pr}_A : (M(A), M(A')) \to (A, A')$ the projection;

Under the same notation as in the aforementioned section of the appendix, we fix a rationally-even cohomology theory $E^\bullet$, represented by an Ω-spectrum $(E_n, e_n, \varepsilon_n)$ and denote by $\iota_n \in S^n(E_n, e_n; \mathfrak{C}_\mathbb{R})$, for $n \in \mathbb{Z}$, the especial cocyles representing the Chern-Dold character and satisfying the condition $\iota_{n-1} = \int_{S^1} \varepsilon_n^* \iota_n$ as in Proposition D.3.1.

We define the pullbacks

$$(\iota_{(X,X')} \circ \rho)^*(f, h, \omega) := (f \circ \iota_X \circ \rho, (\iota_X \circ \rho)^* h, \rho^* \omega)$$

and

$$\iota_{M(A)}^*(f, h, \omega) := (f \circ \iota_{M(A)}, \iota_{M(A)}^* h, (\rho \circ \mathrm{pr}_A)^* \omega)$$

Definition 6.4.2. A differential n-class over $\rho : (A, A') \to (X, X')$ is an equivalence class $[(f, h, \omega, \eta)]$, where

- $(f, h, \omega) : (\rho, \rho') \to (E_n, e_n, \iota_n)$ is a differential function such that

  $$\iota_{M(A)}^*(f, h, \omega) : (\iota_X \circ \rho)^*(f, h, \omega) \simeq (k_{e_n}, \chi(\eta), d\eta)$$

  is a strong trivialization induces by η.

- (f, h, ω, η) is equivalent to (f', h', ω', η') if and only $(f, h, \omega) \simeq (f', h', \omega)$.

The set of differential n-classes will be denoted by $\widehat{E}^n(\rho, \rho')$.

This definition implies that $\rho^*\omega = d\eta$ and together with $d\omega = 0$, implies that $(\omega, \theta) \in \Omega_{\text{cl}}(\rho, \rho')$.

Given a morphism between two smooth maps of pairs $(\phi, \psi) : (\rho, \rho') \to (\eta, \eta')$, the morphism $(\phi, \psi)_\sharp : M_\rho \to M_\eta$ defined in section D.4.2, is also a map of pairs. We define the pullback of differential classes as

$$(\phi, \psi)^*[(f, h, \omega, \theta)] := [(f \circ (\phi, \psi)_\sharp, (\phi, \psi)_\sharp^* h, \phi^*\omega, \psi^*\eta]$$

In a similar fashion as the relative case, we can endow the set $\widehat{E}(\rho, \rho')$ with a abelian group structure. In fact, the definition is precisely the same as the relative case, namely

$$\begin{aligned} &[(f_0, h_0, \omega_0, \theta_0)] + [(f_1, h_1, \omega_1, \theta_1)] := \\ &\qquad\qquad [(\alpha \circ (f_0, f_1), h_0 + h_1 + (f_0, f_1)^* A_{n-1}, \omega_0 + \omega_1, \theta_0 + \eta_1)] \end{aligned}$$

where the A_n cochains were defined in (D.3). Also, it can be easily seem that the map $\chi_\rho : \Omega^n(\rho; \mathfrak{E}_\mathbb{R}) \to S^n_{\text{sm}}(M_\rho; \mathfrak{E}_\mathbb{R})$ as defined in (D.4) (up to coboundaries) can be seen as map from $\Omega^n(\rho, \rho') \to S^n_{\text{sm}}(M_\rho, M_{\rho'})$ just by replacing the natural morphisms by its relative version.

As in the relative session, we define the natural transformations as

- $R[(f, h, \omega, \theta)] : (\omega, \theta)$;
- $I[(f, h, \omega, \theta)] := [f]$;
- $a(\omega, \theta) := [(k_{e_n}, \chi_{\rho,\rho'}(\omega, \theta), d(\omega, \theta))]$

6.4.3 S^1-Integration

We wish to define the S^1-integration for maps of pairs as in the relative case (see section D.4.3). This can be accomplished in the exact same way. As in the relative case, we have a non-canonical splitting

$$E^{n+1}(\rho \times id_{S^1}, \rho' \times id_{S^1}) = E^{n+1}(\rho, \rho') \oplus E^n(\rho, \rho').$$

Again, we identify $M_{\rho \times \text{id}_{S^1}}$ with $M_\rho \times S^1$. The term $E^{n+1}(\rho, \rho')$ is the image of the morphism, $\text{pr}_1 : (M_\rho \times S^1, M_{\rho'} \times S^1) \to (M_\rho, M_{\rho'})$ and $h^n(\rho, \rho')$ can be identify with the kernel of the map induced by the embedding $i_1 : (M_\rho, M_{\rho'}) \hookrightarrow (M_\rho \times S^1, M_{\rho'} \times S^1)$.

Given a class $\widehat{\alpha} \in \widehat{E}^n(\rho, \rho')$, we define $\alpha' := \alpha - \pi_1^* i_1^* \alpha$ and choose some representative for it $\alpha' := [(f, h, \omega, \theta)]$. This implies, that $i_1^* \alpha' = 0$ and thus $i^* I(\alpha') = 0$. This enable us to choose $f : (M_\rho \times S^1, M_{\rho'} \times S^1) \to (E_n, e_n)$ in such a way that $f \circ i_1(M_\rho, M_{\rho'}) = \{e_n\}$. Hence, it induces a map $\widetilde{f} : (M_\rho, M_{\rho'}) \to (\Omega E_n, k_{e_n})$ where k_{e_n} is the constant loop at e_n. We define $\int_{S^1} f := (\epsilon_n^\perp)^{-1} \circ f^\perp$ and the differential S^1 integration

$$\int_{S^1} \widehat{\alpha} := \left[\int_{S^1} f, \int_{S^1} h, \int_{S^1} \omega, \int_{S^1} \eta\right]$$

6.4.4 Relative-Parallel Product

Now, we wish to define the relative-product but first we recall some aspects of the product discussed in section D.4.3. Let h is a multiplicative cohomology theory and $\mu_{n,m} : E_n \wedge E_m \to E_{n+m}$ represents the multiplication at level of spectrum.

As in the relative case, there exists maps $P, Q : \Omega^n(\rho; \mathfrak{E}_\mathbb{R}) \otimes \Omega^m(Y, B; \mathfrak{E}_\mathbb{R}) \to S^{n+m}(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B; \mathfrak{E}_\mathbb{R})$ defined by $P((\omega, \theta) \otimes \omega') := \chi_{\rho\times\mathrm{id}_Y, \rho\times\mathrm{id}_B}(\omega \times \omega', \eta \times \omega')$ and $Q(\omega \otimes \omega_1) := \chi_\rho(\omega, \theta) \times \chi_{Y,B}(\omega')$, where $\chi_{(Y,B)}$ is just χ_Y seem a pair. Again, the method of acyclic models give us a chain homotopy $B : \Omega(\rho) \otimes \Omega(Y, B) \to S^n_{\mathrm{sm}}(\rho \times \mathrm{id}_Y, \rho \times \mathrm{i}_B)$ between P and Q:

$$\chi_{\rho\times\mathrm{id}_Y, \rho\times\mathrm{id}_B}((\omega, \theta) \times \omega') - \chi_\rho(\omega, \theta) \times \chi_{(Y,B)}(\omega') = \delta B((\omega, \theta) \otimes \omega') - Bd((\omega, \eta) \otimes \omega')$$

Recall by —, there exists cocycles $M_{n,m} \in S^{n+m-1}(E_n \wedge E_m, e_n \wedge e_m; \mathfrak{E}_\mathbb{R})$ such that

$$\delta M_{n,m} = \iota_n \times \iota_m + \mu^*_{n,m} \iota_{n+m}$$

Given a map $\rho\, A \to X$ and two maps $f : (M_\rho, A \times \{1\}) \to (Z, z)$ and $g : (Y, B) \to (Z, z)$ we can define a map of pairs $(f, g) : (M_{\rho\times\mathrm{id}_Y}, M_{\rho\times\mathrm{id}_B}) \to (Z \times Z, z \times z)$ as in composition[1] (D.7). This is indeed a map of pairs[2] since the second map $g|B = e_m$ and satisfies the following properties

$$\Phi(f, g)^* \partial M_{n,.m} = f^* \iota_n \times g^* \iota_m + (f, g)^* \mu_{n,m}$$

Moreover, this map defines a product of pairs in (topological) cohomology in the framework of spectra. In order words, we define the product of $[f] \in E^n(\rho)$ and $[f'] \in E^m(Y, B)$ as $[f] \times [g] := [(f, f')]$.

Definition 6.4.3. We define the product of a relative class by a parallel one $\times : \widehat{E}^n(\rho) \otimes_\mathbb{Z} \widehat{E}^m_{\mathrm{par}}(Y, B) \to \widehat{E}^{m+m}(\rho \times \mathrm{id}_Y, \rho \times \mathrm{id}_B)$ where n, m are even as

$$\begin{aligned}
&[(f, h, \omega, \eta)] \times [(f', h', \omega')] = \\
&\qquad [(\mu_{n,m} \circ (f, f'), \\
&\qquad h \times \chi(\omega') + (-1)^{nm} \chi(\omega, \eta) \times h' + h \times \partial h' + B((\omega, \eta) \otimes \omega') + (f, f')^* M_{n,m}, \\
&\qquad \omega \times \omega', \eta \times \omega')].
\end{aligned}$$

[1] Ommiting the isomorphism $M_{\rho\times\mathrm{id}_Y} \cong M_\rho \times Y$.

[2] This is not true for each step in the composition, but it is indeed the case for the composition.

For the other cases we have, still assuming n, m even, we have

- for $\widehat{\alpha} \in \widehat{E}^{n-1}(\rho)$ and $\widehat{\beta} \in \widehat{E}^{m}(Y)$, $\widehat{\alpha} \times \widehat{\beta} := \int_{S^1} \pi_{1,\rho}^* \widehat{\alpha} \times \widehat{\beta}'$;
- for $\widehat{\alpha} \in \widehat{E}^{n}(\rho)$ and $\widehat{\beta} \in \widehat{E}^{m-1}(Y)$, $\widehat{\alpha} \times \widehat{\beta} := \int_{S^1} \widehat{\alpha}' \times \pi_{1,Y}^* \widehat{\beta}$;
- for $\widehat{\alpha} \in \widehat{E}^{n-1}(\rho)$ and $\widehat{\beta} \in \widehat{E}^{m-1}(Y)$, $\widehat{\alpha} \times \widehat{\beta} := -\int_{S^1} \int_{S^1} \pi_{1,\rho}^* \widehat{\alpha'} \times \pi_{1,Y}^* \widehat{\beta'}$.

where $\pi_{1,\rho} : \rho \times \mathrm{id}_{S^1} \to \rho$ is the natural projection, $\widehat{\alpha}'$ is the unique differential class in $\widehat{E}^n(\rho \times \mathrm{id}_{S^1})$ such that $\int_{S^1} \widehat{\alpha}' = \alpha$ and $R(\widehat{\alpha}') = dt \times \pi_{1,\rho}^*(R(\widehat{\alpha})$ and $\widehat{\beta}'$ is defined analogously.

6.5 Conclusion

In this chapter we have construct model of differential refinement theory on finite sequences of manifolds. All of these theories displays the parallel-relative product which enables the construction of the differential integration maps contruct in the previous chapter.

Bibliography

ABRAHAM, R.; MARSDEN, J. E.; RATIU, T. S. *Manifolds, tensor analysis, and applications*. 2. ed. ed. New York Heidelberg: Springer, 1988. (Applied mathematical sciences, 75). ISBN 978-1-4612-6990-8 978-0-387-96790-5 978-3-540-96790-3. Cited 2 times on pages 91 and 219.

ADAMS, J. F. *Stable Homotopy Theory*. 3. ed. Berlin Heidelberg: Springer-Verlag, 1969. (Lecture Notes in Mathematics). ISBN 978-3-540-04598-4. Available from: <https://www.springer.com/gp/book/9783540045984>. Cited on page 259.

AGUILAR, M. et al. *Algebraic topology from a homotopical viewpoint*. New York Berlin Heidelberg: Springer, 2002. (Universitext). ISBN 978-1-4419-3005-7 978-0-387-95450-9. Cited 2 times on pages 71 and 111.

AMABEL, A.; DEBRAY, A.; HAINE, P. J. *Differential Cohomology: Categories, Characteristic Classes, and Connections*. [s.n.], 2021. ArXiv: 2109.12250. Available from: <http://arxiv.org/abs/2109.12250>. Cited on page 29.

ARKOWITZ, M. *Introduction to Homotopy Theory*. New York: Springer-Verlag, 2011. (Universitext). ISBN 978-1-4419-7328-3. Available from: <https://www.springer.com/gp/book/9781441973283>. Cited 2 times on pages 212 and 213.

ATIYAH, M. F.; ANDERSON, D. W. *K-theory*. [s.n.], 2018. OCLC: 1028553031. ISBN 978-0-429-49354-6. Available from: <https://search.ebscohost.com/login.aspx?direct=true&scope=site&db=nlebk&db=nlabk&AN=1728843>. Cited 2 times on pages 247 and 249.

BECKER, J. C.; GOTTLIEB, D. H. CHAPTER 25 - A History of Duality in Algebraic Topology. In: JAMES, I. M. (Ed.). *History of Topology*. Amsterdam: North-Holland, 1999. p. 725–745. ISBN 978-0-444-82375-5. Available from: <https://www.sciencedirect.com/science/article/pii/B9780444823755500262>. Cited 2 times on pages 30 and 73.

BOTT, R.; TU, L. W. *Differential Forms in Algebraic Topology*. New York: Springer-Verlag, 1982. (Graduate Texts in Mathematics). ISBN 978-0-387-90613-3. Available from: <https://www.springer.com/gp/book/9780387906133>. Cited 4 times on pages 85, 89, 100, and 111.

BROWN, R.; BROWN, R. *Topology and groupoids*. Rev., updated, and expanded version. Deganwy: www.groupoids.org, 2006. ISBN 978-1-4196-2722-4. Cited 2 times on pages 41 and 210.

BRYLINSKI, J.-L. *Loop Spaces, Characteristic Classes and Geometric Quantization*. Birkhäuser Basel, 1993. (Modern Birkhäuser Classics). ISBN 978-0-8176-4730-8. Available from: <https://www.springer.com/gp/book/9780817647308>. Cited on page 120.

BUNKE, U. Differential cohomology. *arXiv:1208.3961 [math]*, aug. 2013. ArXiv: 1208.3961. Available from: <http://arxiv.org/abs/1208.3961>. Cited 4 times on pages 29, 30, 135, and 136.

BUNKE, U.; SCHICK, T. Uniqueness of smooth extensions of generalized cohomology theories. *Journal of Topology*, v. 3, n. 1, p. 110–156, 2010. ISSN 17538416. ArXiv: 0901.4423. Available from: <http://arxiv.org/abs/0901.4423>. Cited 5 times on pages 29, 56, 78, 129, and 131.

BUNKE, U.; SCHICK, T. Differential K-theory. A survey. *arXiv:1011.6663 [hep-th]*, v. 17, 2012. ArXiv: 1011.6663. Available from: <http://arxiv.org/abs/1011.6663>. Cited on page 29.

BäR, C.; BECKER, C. *Differential Characters*. Cham: Springer International Publishing, 2014. v. 2112. (Lecture Notes in Mathematics, v. 2112). ISBN 978-3-319-07033-9 978-3-319-07034-6. Available from: <http://link.springer.com/10.1007/978-3-319-07034-6>. Cited 6 times on pages 29, 34, 190, 239, 245, and 246.

CHEEGER, J.; SIMONS, J. Differential characters and geometric invariants. In: ALEXANDER, J. C.; HARER, J. L. (Ed.). *Geometry and Topology*. Berlin, Heidelberg: Springer, 1985. (Lecture Notes in Mathematics), p. 50–80. ISBN 978-3-540-39738-0. Cited on page 29.

CHERN, S.-S.; SIMONS, J. Characteristic Forms and Geometric Invariants. *Annals of Mathematics*, v. 99, n. 1, p. 48–69, 1974. ISSN 0003-486X. Publisher: Annals of Mathematics. Available from: <https://www.jstor.org/stable/1971013>. Cited on page 29.

CLEMENTE, G. L. *Tópicos de Topologia Algébrica*. Tese (Monografia de Conclusão de Curso) — Universidade Federal de São Carlos, São Carlos, nov. 2018. Available from: <https://www.dm.ufscar.br/graduacao/attachments/article/340/Gabriel_Longatto_Clemente.pdf>. Cited 2 times on pages 228 and 236.

CLEMENTE, G. L. *Ordinary and Twisted K-Theory*. Tese (Dissertação de Mestrado) — Universidade Federal de São Carlos, São Carlos, mar. 2022. Cited on page 63.

COHEN, R. L.; KLEIN, J. R. Umkehr maps. *Homology, Homotopy and Applications*, v. 11, n. 1, p. 17–33, jan. 2009. ISSN 1532-0073, 1532-0081. Publisher: International Press of Boston. Available from: <https://projecteuclid.org/journals/homology-homotopy-and-applications/volume-11/issue-1/Umkehr-maps/hha/1251832558.full>. Cited on page 73.

DIECK, T. t. *Algebraic topology*. Zürich: European Mathematical Society, 2008. (EMS textbooks in mathematics). OCLC: ocn261176011. ISBN 978-3-03719-048-7. Cited 10 times on pages 56, 62, 146, 188, 208, 210, 214, 228, 229, and 237.

DUNDAS, B. I. *A short course in differential topology*. Cambridge, United Kingdom ; New York, NY: Cambridge University Press, 2018. (Cambridge mathematical textbooks). ISBN 978-1-108-42579-7. Cited 2 times on pages 90 and 103.

FREED, D. S.; LOTT, J. An index theorem in differential K-theory. *Geometry & Topology*, v. 14, n. 2, p. 903–966, mar. 2010. ISSN 1364-0380, 1465-3060. ArXiv: 0907.3508 version: 2. Available from: <http://arxiv.org/abs/0907.3508>. Cited 2 times on pages 30 and 254.

GREUB, W. H.; HALPERIN, S.; VANSTONE, R. *Connections, curvature, and cohomology*. New York: Academic Press, 1972. (Pure and applied mathematics; a series of monographs and textbooks, v. 47). ISBN 978-0-12-302701-6. Cited 3 times on pages 85, 97, and 221.

HATCHER, A. *Algebraic topology*. Cambridge ; New York: Cambridge University Press, 2002. ISBN 978-0-521-79160-1 978-0-521-79540-1. Cited 3 times on pages 63, 234, and 238.

HIRSCH, M. W. *Differential Topology*. New York, NY: Springer New York, 1976. OCLC: 853264013. ISBN 978-1-4684-9449-5. Available from: <http://public.ebookcentral.proquest.com/choice/publicfullrecord.aspx?p=3085149>. Cited 2 times on pages 75 and 79.

HOPKINS, M. J.; SINGER, I. M. Quadratic functions in geometry, topology, and M-theory. *Journal of Differential Geometry*, v. 70, n. 3, p. 329–452, jul. 2005. ISSN 0022-040X. Publisher: Lehigh University. Available from: <https://projecteuclid.org/euclid.jdg/1143642908>. Cited 6 times on pages 29, 30, 74, 104, 259, and 261.

HUSEMöLLER, D. *Fibre bundles*. 3rd ed. ed. New York: Springer-Verlag, 1994. (Graduate texts in mathematics, 20). ISBN 978-0-387-94087-8. Cited 4 times on pages 66, 85, 247, and 249.

HUYBRECHTS, D. *Complex geometry: an introduction*. Berlin New York: Springer, 2005. (Universitext). ISBN 978-3-540-26687-7. Cited on page 120.

JAMES, I. M. *Fibrewise Topology*. Cambridge: Cambridge University Press, 1989. (Cambridge Tracts in Mathematics). ISBN 978-0-521-08925-8. Available from: <https://www.cambridge.org/core/books/fibrewise-topology/FC885A25C877B62C4D40B709AA9B0240>. Cited on page 64.

KAROUBI, M. *K-Theory: An Introduction*. Berlin Heidelberg: Springer-Verlag, 1978. (Classics in Mathematics). ISBN 978-3-540-79889-7. Available from: <https://www.springer.com/gp/book/9783540798897>. Cited 5 times on pages 69, 78, 79, 247, and 249.

KLONOFF, K. R. *An index theorem in differential K-theory*. Tese (Thesis) — University of Texas, may 2008. Accepted: 2008-08-29T00:18:46Z Artwork Medium: electronic Interview Medium: electronic. Available from: <https://repositories.lib.utexas.edu/handle/2152/3912>. Cited on page 254.

KOSINSKI, A. A. *Differential manifolds*. Dover ed. Mineola, N.Y: Dover Publications, 2007. OCLC: ocn123912704. ISBN 978-0-486-46244-8. Cited on page 74.

LAWSON, H. B.; MICHELSOHN, M.-L. *Spin geometry*. Princeton, N.J: Princeton University Press, 1989. (Princeton mathematical series, 38). ISBN 978-0-691-08542-5. Cited on page 135.

LEE, J. M. *Introduction to smooth manifolds*. 2nd ed. ed. New York ; London: Springer, 2013. (Graduate texts in mathematics, 218). OCLC: ocn800646950. ISBN 978-1-4419-9981-8 978-1-4419-9982-5. Cited on page 215.

MACK, J. On a class of countably paracompact spaces. *Proceedings of the American Mathematical Society*, v. 16, n. 3, p. 467–472, 1965. ISSN 0002-9939, 1088-6826. Available from: <https://www.ams.org/proc/1965-016-03/S0002-9939-1965-0177388-1/>. Cited on page 66.

MASSEY, W. S. *Homology and cohomology theory: an approach based on Alexander-Spanier cochains*. New York: M. Dekker, 1978. (Monographs and textbooks in pure and applied mathematics, 46). ISBN 978-0-8247-6662-7. Cited on page 63.

MASSEY, W. S. *A Basic Course in Algebraic Topology*. [s.n.], 2019. OCLC: 1190416136. ISBN 978-1-4939-9063-4. Available from: <https://doi.org/10.1007/978-1-4939-9063-4>. Cited on page 63.

MAY, J. P. *A concise course in algebraic topology*. Chicago: University of Chicago Press, 1999. (Chicago lectures in mathematics). ISBN 978-0-226-51182-5 978-0-226-51183-2. Cited 2 times on pages 41 and 210.

MELO, W. d. *Topologia das Variedades*. 1. ed. Rio de Janeiro: SBM, 2019. (Coleção Fronteiras da Matemática, 04). ISBN 978-85-8337-147-2. Available from: <https://loja.sbm.org.br/topologia-das-variedades.html>. Cited 3 times on pages 85, 89, and 111.

MITS314. Forum, *Realizing a vector bundle as a summand of a trivial bundle*. 2021. Available from: <https://math.stackexchange.com/questions/4056648/realizing-a-vector-bundle-as-a-summand-of-a-trivial-bundle>. Cited on page 79.

MUKHERJEE, A. Forum, *Is every vector bundle over a noncompact finite-dimensional manifold a summand of a trivial bundle?* 2015. Available from: <https://mathoverflow.net/questions/222127/is-every-vector-bundle-over-a-noncompact-finite-dimensional-manifold-a-summand-o>. Cited on page 79.

MUNKRES, J. R. *Topology*. 2nd ed. ed. Upper Saddle River, NJ: Prentice Hall, Inc, 2000. ISBN 978-0-13-181629-9. Cited on page 223.

NICOLAESCU, L. I. *Lectures on the Geometry of Manifolds*. 3. ed. WORLD SCIENTIFIC, 2020. ISBN 9789811214813 9789811214820. Available from: <https://www.worldscientific.com/worldscibooks/10.1142/11680>. Cited 5 times on pages 85, 89, 90, 100, and 111.

NUñEZ, J. C. *Differential cohomology on maps of pairs and relative-parallel product*. Tese (PhD Thesis) — Universidade Federal de São Carlos, oct. 2021. Accepted: 2021-11-29T12:24:30Z Publisher: Universidade Federal de São Carlos. Available from: <https://repositorio.ufscar.br/handle/ufscar/15183>. Cited 5 times on pages 34, 35, 190, 247, and 249.

Roman Sauer. *45 Acyclic models theorem*. 2021. Available from: <https://www.youtube.com/watch?v=9NLZihmOAA0>. Cited on page 228.

ROTMAN, J. J. *An introduction to algebraic topology*. New York Heidelberg: Springer, 1988. (Graduate texts in mathematics, 119). ISBN 978-1-4612-8930-2 978-0-387-96678-6 978-3-540-96678-4. Cited on page 211.

ROTMAN, J. J. *An introduction to homological algebra*. 2nd ed. ed. New York: Springer, 2009. (Universitext). ISBN 978-0-387-68324-9. Cited 3 times on pages 227, 228, and 229.

RUDYAK, Y. B. *On Thom Spectra, Orientability, and Cobordism*. Berlin Heidelberg: Springer-Verlag, 1998. (Springer Monographs in Mathematics). ISBN 978-3-540-62043-3. Available from: <https://www.springer.com/gp/book/9783540620433>. Cited 3 times on pages 62, 259, and 260.

RUFFINO, F. F. Relative Deligne cohomology and Cheeger-Simons characters. *arXiv:1401.0631 [math]*, jan. 2014. ArXiv: 1401.0631. Available from: <http://arxiv.org/abs/1401.0631>. Cited 2 times on pages 127 and 261.

RUFFINO, F. F. Relative differential cohomology. *Journal of Homotopy and Related Structures*, v. 10, n. 4, p. 923–937, dec. 2015. ISSN 1512-2891. Available from: <https://doi.org/10.1007/s40062-014-0088-1>. Cited on page 259.

RUFFINO, F. F. Flat pairing and generalized Cheeger–Simons characters. *Journal of Homotopy and Related Structures*, v. 12, n. 1, p. 143–168, mar. 2017. ISSN 1512-2891. Available from: <https://doi.org/10.1007/s40062-015-0124-9>. Cited 9 times on pages 30, 73, 76, 103, 136, 138, 139, 140, and 174.

RUFFINO, F. F. *Introdução à Topologia Diferencial and Algébrica - Vol 1*. 2020. Available from: <https://www.dm.ufscar.br/profs/ferrariruffino/Top%20dif%204.pdf>. Cited 3 times on pages 66, 85, and 215.

RUFFINO, F. F.; BARRIGA, J. C. R. Relative differential cohomology and generalized Cheeger–Simons characters. *Communications in Analysis and Geometry*, v. 29, n. 4, p. 921–1005, jul. 2021. ISSN 1944-9992. Publisher: International Press of Boston. Available from: <https://www.intlpress.com/site/pub/pages/journals/items/cag/content/vols/0029/0004/a004/abstract.php>. Cited 13 times on pages 30, 34, 80, 119, 128, 129, 130, 132, 139, 159, 259, 261, and 268.

SHAHBAZI, Z. *Differential Geometry of Relative Gerbes*. Tese (Doutorado) — University of Toronto, apr. 2004. ArXiv: math/0505237. Available from: <http://arxiv.org/abs/math/0505237>. Cited 3 times on pages 119, 122, and 236.

SIMONS, J.; SULLIVAN, D. Axiomatic characterization of ordinary differential cohomology. *Journal of Topology*, v. 1, n. 1, p. 45–56, jan. 2008. ISSN 1753-8416. Available from: <https://doi.org/10.1112/jtopol/jtm006>. Cited 2 times on pages 29 and 125.

SPANIER, E. H. *Algebraic topology*. [s.n.], 1966. OCLC: 1292074625. ISBN 978-1-4684-9322-1. Available from: <https://search.ebscohost.com/login.aspx?direct=true&scope=site&db=nlebk&db=nlabk&AN=2767651>. Cited on page 228.

STOLZ, S. *Index theory*. 2020. Available from: <https://www3.nd.edu/~stolz/2020S_Math80440/Index_theory_S2020.pdf>. Cited on page 30.

STRICKLAND, N. P. *THE CATEGORY OF CGWH SPACES*. 2009. Available from: <https://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf>. Cited on page 39.

SWITZER, R. M. *Algebraic Topology - Homotopy and Homology*. Berlin Heidelberg: Springer-Verlag, 2002. (Classics in Mathematics). ISBN 978-3-540-42750-6. Available from: <https://www.springer.com/gp/book/9783540427506>. Cited 2 times on pages 46 and 259.

UPMEIER, M. *Algebraic Structure and Integration in Generalized Differential Cohomology*. Tese (Doutorado) — Georg-August-Universität Göttingen, jan. 2014. Accepted: 2014-01-22T10:47:48Z. Available from: <https://ediss.uni-goettingen.de/handle/11858/00-1735-0000-0022-5E03-9>. Cited 4 times on pages 214, 259, 260, and 261.

Appendix

APPENDIX A – Complements of Topology Geometry and Topology

A.1 Introduction

This appendix is a collection of disconnected facts and proofs which were mentioned but omitted in the main text.

A.2 Complements of Topology

A.2.1 Complements of General Topology

Sometimes in this text, we will use the *collapse map* of a mapping cylinder over $\rho : A \to X$ to its base, $c : M_\rho \to X$ defined by

$$c(m) := \begin{cases} x, & m = j_X(x) \\ \rho(a), & m = j_{A\times I}(a,t) \end{cases}. \tag{A.1}$$

This map makes the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, hook, "i_0"] & X \arrow[d, "i_X"] \arrow[ddr, equal, bend left, "id_X"] & \\
A \times I \arrow[r, "i_{A\times I}"] \arrow[drr, bend right, "\rho\circ \mathrm{pr}_A"'] & M_\rho \arrow[dr, dashed, "c"] & \\
& & X
\end{tikzcd}$$

where $i_0 \colon A \to A \times \{0\}$ is the inclusion on the base of the cylinder and $\mathrm{pr}_A \colon A \times I \to A$ is the projection. This map is a homotopy inverse of i_X, that is, $i_X \circ c \simeq_H id_{M_\rho}$ via the homotopy

$$H(m,s) := \begin{cases} j_{A\times I}(a, ts), & m = j_{A\times I}(a,t) \\ j_X(x), & m = j_X(x) \end{cases}$$

where $H(m,0) = j_X \circ c(m)$ and $H(m,1) = id_{M_\rho}(m)$.

We also use some times the following identity concerning smash products (A, a) a

$$\frac{X \times Y}{A \times Y \cup X \times B} = \frac{X}{A} \wedge XB$$

for a pointed space (X, x) and (Y, y) and subspaces $A \subset X$ and Now we address another point which was mentioned at end of Section 1.5. Given continuous maps $\rho : A \to X$ and

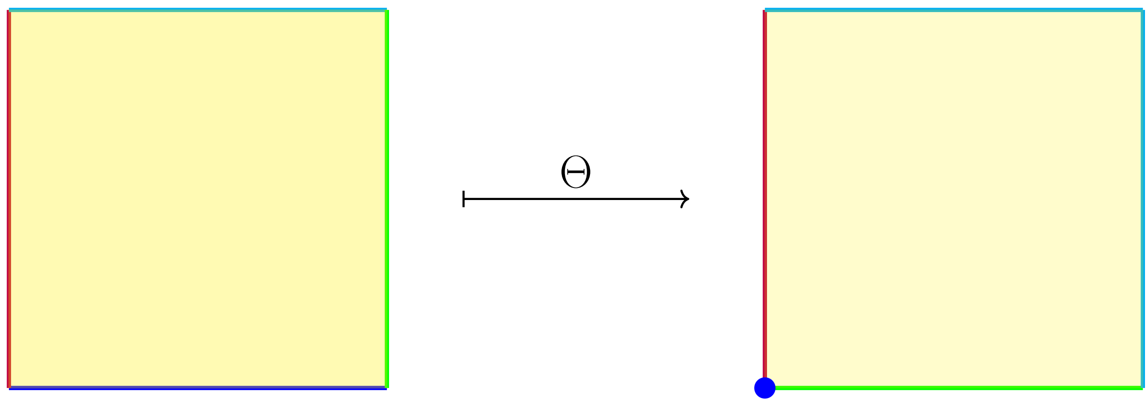

Figure 9 – Properties of the required morphism Θ. Each color of the left square goes to the the same color in right square.

$\eta : B \to Y$, consider the $\rho \wedge \eta : M_{\rho,\eta} \to X \times Y$ defined as

$$(\rho \wedge \eta)(p) = \begin{cases} (\rho(a), y), & \text{if } p = [a, y], \text{ where } (a, y) \in A \times Y \\ (\rho(a), \eta(b)), & \text{if } p = [a, b, t], \text{ where } (a, b, t) \in A \times B \times I \\ (x, \eta(b)), & \text{if } p = [x, b], \text{ where } (x, b) \in X \times B \end{cases}$$

Proposition A.2.1. *There exists a canonical (pointed) homeomorphism* $(C(\rho \wedge \eta), *) \approx (C(\rho), *) \wedge (C(\eta), *)$*, where the* $*$ *denotes the point of the cone, uniquely determined up to homotopy equivalence.*

A proof can be found in embbeded in a proof in (DIECK, 2008, Proprosition 7.2.7,p.166). For sake of completeness, we give a proof

Proof. We fix map $\Theta : I \times I \to I \times I$ satisfying the following four properties

- $\Theta(I \times \{0\}) = \{(0, 0)\}$;
- the restriction $\Theta : I \times I \backslash I \times \{0\} \to I \times I \backslash \{(0, 0)\}$ is an orientation-preserving homeomorphism;
- $\Theta(\{0\} \times I) = \{0\} \times I$;
- $\Theta(I \times \{1\}) = I \times \{1\} \cup \{1\} \times I$;
- $\Theta(\{1\} \times I) = I \times \{0\}$

The properties are ilustraed in the following Figure 9 For example, we consider the following map

$$\Theta(s, t) = \begin{cases} s(2t, 1), & t \in \left[0, \frac{1}{2}\right], \\ s(1, 2 - 2t) & t \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{A.2}$$

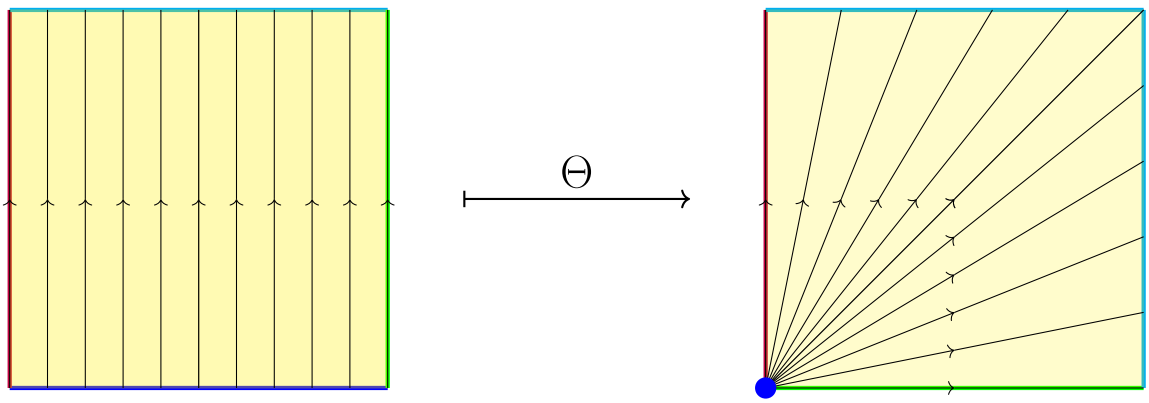

Figure 10 – How the Θ map in (A.2) works. Each vertical segment on the right is obtained at constant u and taken to the leaning ones in the write.

which is ilustrated in Figure 10 We define the homomorphism $\Phi_\Theta : C(\rho \wedge \eta) \to C(\rho) \wedge C(\eta)$ by

$$
\begin{aligned}
[(x, y)]) &\mapsto [[x], [y]] \\
[(x, b, s)]) &\mapsto [[x], [(b, s)]]) \\
[(a, y, t)] &\mapsto [[(a, t)], [b]] \\
([(a, b, t, s)]) &\mapsto [[(a, \Theta_1(t, s))], [(b, \Theta_{12}(t, s)]]
\end{aligned}
$$

where $\Theta = (\Theta_1, \Theta_2)$. One can verify that this is a pointed homeomorphism. $\square$

Remark A.2.2. We observe that, by fixing a Θ what we are really doing is fixing a natural transformation between the funtors $C' : \mathtt{Top}^2 \times \mathtt{Top}^2 \to \mathtt{Top}_*$ and $C \wedge C : \mathtt{Top}^2 \times \mathtt{Top}^2 \to \mathtt{Top}_*$ where

$$C'((\rho, \eta) \xrightarrow{(f,g),(k,kl)} (\rho', \eta')) = (C_{\rho \wedge \eta}, *) \xrightarrow{C(f,g,k,l)} (C_{\rho' \wedge \eta'}, *)$$

and

$$C((\rho, \eta) \xrightarrow{(f,g),(k,kl)} (\rho', \eta',) = (C_\rho, *) \wedge (C_\eta, *) \xrightarrow{C(f,g) \wedge C(k,l)} (C_{\rho'}, *) \wedge (C_{\eta'}, *)$$

Φ_Θ defines a natural transformation.

A.2.2 Complements of Homotopy

We collect some definitions and results which are important to this work.

Definition A.2.3 (Homotopy extension property and cofibrations)**.** A map $i : A \to X$ has the *homotopy extension property* with respect to a map $f : X \to Y$, if for every map $H : I \times A \to Y$ such that $f \circ i = H \circ i_0$, if there exists a map $\widetilde{H} : I \times X \to Y$ such that

$\widetilde{H} \circ i = H$. Schematically, we have the following commutative diagram

$$
\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & I \times A \arrow[dl, "H"] \arrow[dd, "\mathrm{id}_I \times i"] \\
& Y & \\
X \arrow[rr, "i_0"] \arrow[ur, "f"] & & I \times X \arrow[ul, dashed, "\widetilde{H}"]
\end{tikzcd}
$$

A *cofibration* is map $i : A \to X$ which has the homotopy extension property with respect to any map $f : X \to Y$.

Proposition A.2.4 (Properties of Cofibrations)**.** *A cofibration $i : A \to X$ has the following properties*

1. *i is a topological embedding;*
2. *If both A and X are Hausdorff spaces, than $i(A)$ is closed;*
3. *For any B, the map $i \times \mathrm{id}_B : A \times B \to X \times B$ is a cofibration.*
4. (stability) *Given a map $f : A \to Y$, the pushout $j : X \to X \cup_{i,j} Y$ is a cofibration.*

The reader can find proofs (or guides) for 1 and 2 in Problem 1 (p.106), for 3 (p. 103), and for 4 in Proposition 5.1.8 (p.104) all in (DIECK, 2008).

Proposition A.2.5. *If either $\mu : A \to X$ or $\nu : A \to Y$ are cofibration, then the map $\phi : M_{\mu,\nu} \to X \cup_{\mu,\nu} Y$ is a homotopy equivalence.*

For a proof of this result, the reader can consult (DIECK, 2008, Prop 5.3.2,p.112).

Proposition A.2.6. *Let $(f, g) : i \to j$ a morphism between cofibrations such that, both f and g are homotopy equivalences, then (f, g) is a homotopy equivalence*

For a proof see (MAY, 1999, p.47) or (BROWN; BROWN, 2006, 7.4.2, p.285)

A.2.3 Complements of Cohomology

Let $\Pi_2 : \mathtt{Top}_2 \to \mathtt{Top}_2$ denote the functor given by

$$\Pi_2((X, A) \xrightarrow{f} (Y, B)) = (A, \emptyset) \xrightarrow{g} (B, \emptyset)$$

Definition A.2.7 (Relative Cohomology Theory on Pairs)**.** A *relative cohomology* (h, ∂) theory over $\mathtt{Top}_2$ is a (contravariant) functor $h : \mathtt{HoTop}_2^{\mathrm{op}} \to \mathtt{GrAb}$ and a natural transformation $\partial : h^\bullet \circ \Pi \to h^{\bullet+1}$, satisfying:

Long exact sequence Associated to the cohomology theory (h, ∂) there is a a functor from the homotopy category $\mathtt{HoTop}_2$ to the category of long exact sequence defined in the following way: for each pair (X, A) we have the following long exact sequence

$$\cdots \longrightarrow h^\bullet(X, A) \xrightarrow{id_X^*} h(X) \xrightarrow{i_A^*} h^\bullet(A) \xrightarrow{\partial_{(X,A)}} h^{\bullet+1}(X, A) \longrightarrow \cdots$$

where $h(X)$ is a shorthand for $h(X, \emptyset)$ (analogously for A), $i_A : (A, \emptyset) \hookrightarrow (X, \emptyset)$ and $\mathrm{id}_X : (X, \emptyset) \to (X, A)$ are respectively the inclusion and the identity morphism. For a morphism $f : (X, A) \to (Y, B)$ we have the following morphism of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & h^\bullet(X, A) \arrow[r, "\mathrm{id}_X^*"] & h^\bullet(X) \arrow[r, "i_A^*"] & h^\bullet(A) \arrow[r, "\partial"] & h^{n+1}(X, A) \arrow[r] & \cdots \\
\cdots \arrow[r] & h^\bullet(Y, B) \arrow[r] \arrow[u, "f^*"] & h^\bullet(Y) \arrow[r] \arrow[u, "f^*"] & h^\bullet(B) \arrow[r, "\partial"] \arrow[u, "f^*"] & h^{n+1}(Y, B) \arrow[r] \arrow[u, "f^*"] & \cdots
\end{tikzcd}$$

Excision If $\overline{U} \subseteq \operatorname{int} A$, then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces isomorphism in cohomology.

Additive: Given a family of pairs $\{(X_\lambda, A_\lambda)\}_{\lambda\in\Lambda}$, let $i_\lambda : (X_\lambda, A_\lambda) \to (\bigsqcup_{\lambda\in\Lambda} X_\lambda, \bigsqcup_{\lambda\in\Lambda} A_\lambda)$ be the inclusion where $\bigsqcup_{\lambda\in\Lambda} X_\lambda$ denotes the disjoint union. The graded group $(h(\bigsqcup_{\lambda\in\Lambda} X_\lambda, \bigsqcup_{\lambda\in\Lambda} A_\lambda), (i_\lambda)^*_{\lambda\in\Lambda})$ is the direct product of the groups $h(X_\lambda, A_\lambda)$.

Before presenting the relative cohomology for pairs lets recall another version of excision axiom.

Proposition A.2.8. *The axiom of excision is equivalent to the the following statement: if $A, B \subseteq X$ are such that $\operatorname{int} A \cup \operatorname{int} B = A \cup B$, then the inclusion $i : (A, A \cap B) \to (A \cup B, B)$ induces isomorphism in cohomology.*

For a proof of this equivalence see, for example, (ROTMAN, 1988, Theorem 6.1, p. 107).

Definition A.2.9 (Reduced Cohomology Theory)**.** A *reduced cohomology theory* over $\mathtt{Top}_*$ is a (contravariant) functor $\widetilde{h} : \mathtt{HoTop}_*^{\mathrm{op}} \to \mathtt{GrAb}$ together with a natural isomorphism $s : \widetilde{h}^\bullet \circ \widetilde{\Sigma} \to \widetilde{h}^{\bullet-1}$, called *suspension*, satisfying:

Cofiber sequence: Given a pointed map $f : (X, x) \to (Y, y)$ the following sequence is exact

$$\widetilde{h}^\bullet(\widetilde{C}_f, *) \xrightarrow{i_Y^*} \widetilde{h}^\bullet(Y, y) \xrightarrow{f^*} \widetilde{h}(X, x)$$

where f^* stands for $\widetilde{h}(f)$

Aditivity: Given a family of pointed spaces $\{(X_\lambda, x_\lambda)\}_{\lambda\in\Lambda}$, let $i_\lambda : (X_\lambda, x_\lambda) \to (\bigvee_{\lambda\in\Lambda} X_\lambda, \vee_{\lambda\in\Lambda} x_\lambda)$ the inclusion maps. The group $\left(\widetilde{h}(\bigvee_{\lambda\in\Lambda} X_\lambda, \vee_{\lambda\in\Lambda} x_\lambda), (i_\lambda)^*_{\lambda\in\Lambda}\right)$ is the direct product of $\widetilde{h}(X_\lambda, x_\lambda)$.

Proposition A.2.10. *The data of a reduced cohomology theory $(\widetilde{h}, s)$ give us a functor from the arrow category of $\mathtt{HoTop}_*$ to the category of exacts sequences define in the following way: for each morphism $\rho : (A, a) \to (X, x)$ we have a long exact sequence*

$$\ldots \widetilde{h}^{\bullet-1}(A, a) \xrightarrow{\sigma} \widetilde{h}^{\bullet}(C_\rho, *) \xrightarrow{i_X^*} \widetilde{h}^{\bullet}(X, x) \xrightarrow{\rho^*} \widetilde{h}^{\bullet}(A, a) \ldots$$

and for each morphism $f : \rho \to \eta$ we have the following morphism between exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{h}^{\bullet-1}(A,a) \arrow[r, "\sigma_\rho"] & \widetilde{h}^{\bullet}(\widetilde{C}_\rho,*) \arrow[r, "i_X^*"] & \widetilde{h}^{\bullet}(X,x) \arrow[r, "\rho^*"] & \widetilde{h}^{\bullet}(A,a) \arrow[r] & \cdots \\
\cdots \arrow[r] & \widetilde{h}^{\bullet-1}(B,b) \arrow[u, "g^*"] \arrow[r, "\sigma_\eta"] & \widetilde{h}^{\bullet}(\widetilde{C}_\eta) \arrow[u, "C(f^*,g^*)"] \arrow[r, "i_Y^*"] & \widetilde{h}^{\bullet}(Y,y) \arrow[u, "f^*"] \arrow[r, "\eta^*"] & \widetilde{h}^{\bullet}(B,b) \arrow[u, "g^*"] \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
(A,a) \arrow[r, "\rho"] & (X,x) \arrow[r, "i_X"] & (\widetilde{C}_\rho,*) \arrow[r, "i_{\widetilde{C}_\rho}"] \arrow[d, "id_{\widetilde{C}_\rho}"] & (\widetilde{C}_{i_X},*) \arrow[r, "i_{\widetilde{C}_{i_X}}"] \arrow[d, "\phi_{i_\rho}"] & (\widetilde{C}_{i_{\widetilde{C}_\rho}},*) \arrow[r] \arrow[d, "\phi_{i_{\widetilde{C}_\rho}}"] \arrow[dl, Rightarrow] & \cdots \\
 & & (\widetilde{C}_\rho,*) \arrow[r, "q_\rho"] & (\widetilde{\Sigma}A,*) \arrow[r, "-\widetilde{\Sigma}\rho"] \arrow[d, "t_A"] & (\widetilde{\Sigma}X,*) \arrow[d, "t_X"] & \\
 & & & (\widetilde{\Sigma}A,*) \arrow[r, "\widetilde{\Sigma}\rho"] & (\widetilde{\Sigma}X,*) \arrow[r, "i_{\Sigma A}"] & (\widetilde{C}_{\widetilde{\Sigma}\rho},*) \arrow[r] & \cdots
\end{tikzcd}$$

which is commutative except for the square

$$\begin{tikzcd}
(\widetilde{C}_{i_X},*) \arrow[r, "i_{\widetilde{C}_{i_\rho}}"] \arrow[d, "\phi_{i_X}"] & (\widetilde{C}_{i_{\widetilde{C}_\rho}},*) \arrow[d, "\phi_{i_{\widetilde{C}_\rho}}"] \arrow[dl, Rightarrow] \\
(\widetilde{\Sigma}A,*) \arrow[r, "-\widetilde{\Sigma}\rho"] & (\widetilde{\Sigma}X,*)
\end{tikzcd}$$

which commutes only uo to homotopy (ARKOWITZ, 2011, Lema 3.5.6, p.108). Here $-\Sigma\rho := \Sigma\rho \circ t_A$ where

$$t_Z : (\widetilde{\Sigma}Z, *) \to (\widetilde{\Sigma}Z, *)$$
$$[s, z] \mapsto [\overline{s}, z]$$

and $\phi_{i_\rho} : (\widetilde{C}_{i_\rho}, *) \to (\widetilde{\Sigma}A, *)$ and $\phi_{i_{\widetilde{C}_\rho}} : (\widetilde{C}_{i_{\widetilde{C}_\rho}}, *) \to (\widetilde{\Sigma}X, *)$ are homotopy equivalences.Applying the exactness of cofiber axiom, we see that the first row is exact:

$$\widetilde{h}^{\bullet}(\widetilde{C}_{\widetilde{C}^*_\rho}) \xrightarrow{i^*_{C_{i_X}}} \widetilde{h}^{\bullet}(C_{i_X}) \xrightarrow{i^*_{\widetilde{C}_\rho}} \widetilde{h}^{\bullet}(C_\rho) \xrightarrow{i_X^*} \widetilde{h}^{\bullet}(X) \xrightarrow{\rho^*} \widetilde{h}^{\bullet}(A)$$

as we can split in the exacts sequences:

$$\widetilde{h}^\bullet(\widetilde{C}_\rho,*) \xrightarrow{i_X^*} \widetilde{h}^\bullet(X) \xrightarrow{\rho^*} \widetilde{h}^\bullet(A)$$

$$\widetilde{h}^\bullet(\widetilde{C}_{i_X},*) \xrightarrow{i_{C_\rho}^*} \widetilde{h}^\bullet(\widetilde{C}_\rho,*) \xrightarrow{i_X^*} \widetilde{h}^\bullet(X)$$

$$\widetilde{h}(\widetilde{C}_{i_{C_\rho}},*) \xrightarrow{i^*_{\widetilde{C}_{i_X}}} \widetilde{h}^\bullet(\widetilde{C}_{i_X},*) \xrightarrow{i^*_{C_\rho}} \widetilde{h}(\widetilde{C}_\rho,*)$$

The homotopy commutative diagram

$$\begin{tikzcd}
(\widetilde{C}_\rho,*) \arrow[r, "i_{\widetilde{C}_\rho}"] \arrow[d, "id_{\widetilde{C}_\rho}"] & (\widetilde{C}_{i_X},*) \arrow[r, "i_{\widetilde{C}_{i_X}}"] \arrow[d, "\phi_\rho"] \arrow[dr, phantom, "\Leftarrow"] & (\widetilde{C}_{i_{\widetilde{C}_\rho}},*) \arrow[d, "\phi_{i_X}"] \\
(\widetilde{C}_\rho,*) \arrow[r, "q_X"] & (\widetilde{\Sigma}A,*) \arrow[r, "-\Sigma\rho"] & (\widetilde{\Sigma}X,*)
\end{tikzcd}$$

give us the exactness of the exactness of

$$\widetilde{h}^\bullet(\widetilde{\Sigma}X) \xrightarrow{(-\Sigma\rho)^*} \widetilde{h}^\bullet(\widetilde{\Sigma}A) \xrightarrow{q_\rho^*} \widetilde{h}^\bullet(\widetilde{C}_\rho)$$

Now, the the commutative diagram

$$\begin{tikzcd}
(\widetilde{C}_\rho,*) \arrow[r, "q_\rho"] \arrow[d, "id_{\widetilde{C}_\rho}"] & (\widetilde{\Sigma}A,*) \arrow[r, "-\Sigma\rho"] \arrow[d, "t_A"] & (\widetilde{\Sigma}X,*) \arrow[d, "t_X"] \\
(\widetilde{C}_\rho,*) \arrow[r, "t_A\circ q_\rho"] & (\widetilde{\Sigma}A,*) \arrow[r, "\widetilde{\Sigma}\rho"] & (\widetilde{\Sigma}X,*)
\end{tikzcd}$$

give us exactness of the sequence

$$\widetilde{h}^\bullet(\widetilde{\Sigma}X,*) \xrightarrow{(\Sigma\rho)^*} \widetilde{h}^\bullet(\widetilde{\Sigma}A,*) \xrightarrow{(t_A\circ q_\rho)^*} \widetilde{h}^\bullet(\widetilde{C}_\rho,*)$$

which results in the long exact sequence

$$\cdots \longrightarrow \widetilde{h}^\bullet(\widetilde{C}_{\widetilde{\Sigma}\rho},*) \xrightarrow{i_{\widetilde{\Sigma}X}} \widetilde{h}^\bullet(\widetilde{\Sigma}X,*) \xrightarrow{\Sigma\rho^*} \widetilde{h}^\bullet(\widetilde{\Sigma}A,*) \xrightarrow{(t_A\circ q_\rho)^*} \widetilde{h}^\bullet(\widetilde{C}_\rho,*) \xrightarrow{i_X^*} \widetilde{h}^\bullet(X,x) \xrightarrow{\rho^*} \widetilde{h}^\bullet(A,a)$$

Since we have a homeormorphism $\theta_\rho : \widetilde{C}_{\widetilde{\Sigma}\rho} \to \widetilde{\Sigma}\widetilde{C}_\rho$ (ARKOWITZ, 2011, Prop. 3.2.14,p.81) such that the following diagram commutes

$$\begin{tikzcd}
 & (\widetilde{\Sigma}X,*) \arrow[dr, "\widetilde{\Sigma}i_X"] & \\
(\widetilde{C}_{\widetilde{\Sigma}\rho},*) \arrow[ur, "i_{\widetilde{\Sigma}X}"] \arrow[rr, "\theta"] & & (\widetilde{\Sigma}(\widetilde{C}_\rho),*)
\end{tikzcd}$$

we can replace the replace in the sequence by

$$\cdots \longrightarrow \widetilde{h}^\bullet(\widetilde{\Sigma}\widetilde{C}_\rho,*) \xrightarrow{\widetilde{\Sigma}i_X} \widetilde{h}^\bullet(\widetilde{\Sigma}X,*) \xrightarrow{\Sigma\rho^*} \widetilde{h}^\bullet(\widetilde{\Sigma}A,*) \xrightarrow{(t_A\circ q_\rho)^*} \widetilde{h}^\bullet(\widetilde{C}_\rho,*) \xrightarrow{i_X^*} \widetilde{h}^\bullet(X,x) \xrightarrow{\rho^*} \widetilde{h}^\bullet(A,a)$$

Now we apply the suspension isomorphism to the highlighted sequence and get the desired result

$$\cdots \longrightarrow \widetilde{h}^{\bullet-1}(\widetilde{C}_\rho, *) \xrightarrow{i_X} \widetilde{h}^{\bullet-1}(X,x) \xrightarrow{\rho^*} \widetilde{h}^{\bullet-1}(A,a) \xrightarrow{\partial} \widetilde{h}^{\bullet}(\widetilde{C}_\rho, *) \xrightarrow{i_X^*} \widetilde{h}^{\bullet}(X,x) \xrightarrow{\rho^*} \widetilde{h}^{\bullet}(A,a) \to \cdots$$

where $\partial = (t_A \circ q_\rho)^* \circ s^{-1}$. □

Let $t : (S^1, 1) \to (S^1, 1)$ denote the conjugation $t(z) = \overline{z}$

Proposition A.2.11. *The following diagram is commutative*

$$\begin{tikzcd} \widetilde{h}^{\bullet}(S^1,1) \arrow[r, "s"] \arrow[d, "t"] & \widetilde{h}^{\bullet-1}(S^1,1) \arrow[d, "-\mathrm{id}"] \\ \widetilde{h}(S^1,1) \arrow[r, "s"] & \widetilde{h}^{\bullet-1}(S^1,1) \end{tikzcd}$$

Proof. In (DIECK, 2008, Prop. 20.5.2, p.257), it is shown that $\sigma_+ = -\sigma_{-1}$. In the relative setting, the homomorphism can be identified with σ and $\sigma \circ t$. □

A.3 Complements of Differential Topology

A.3.1 Manifolds

Denote by C^n the non negative orthant of $\mathbb{R}^n$, that is

$$C^n = \{x \in \mathbb{R}^n | x_i \geq 0\}.$$

Givem some map $f : U \to W$, where $U \subseteq C^n$ and $W \subseteq C^m$ are open sets we say that f is smooth if there exists a smooth extension $F : \mathbb{R}^n \to \mathbb{R}^m$ of f.

Definition A.3.1 (Atlas and Manifolds with corners)**.** A atlas of a second countable Hausdorff space X which is locally homeomorphic to C^n is a a family of homemorphisms $\phi_i : U \to C^n$ such:

- the union of their domain is X, and
- given two maps in the Atlas $\phi : U \to C^n$ and $\phi : W \to C^n$, such that $U \cap W \neq \emptyset$, the map $\phi \circ \psi^{-1} : \mathrm{Im}(\phi) \to C^n$ is smooth in the sense of the previous paragraph.

The unique example we will deal with is the standard simplex Δ^n seem as a subset of $\mathbb{R}^n$. This definition is the same as Upmeier (2014, p.21), but as the same author, remarks there are several definitions of manifolds with corners.

A.3.2 Differential Forms

Before proceeding, we recall a classical result about extension of differential forms. Recall that a (smooth) partition of unity subordinate to an open covering $\mathfrak{U}$ is a family of smooth function $\{\rho_U\}_{U\in\mathfrak{U}}$ with $\rho_U : X \to [0,1]$ such satisfying the following conditions:

i) $\operatorname{supp}(\rho_U) \subseteq U$, where $\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}$ is the *support* of f.

ii) The family of sets $\{\operatorname{supp}\rho_U\}_{U\in\mathfrak{U}}$ is a locally finite covering of X, that is, given $x \in X$, there exists a neighourhood W of x such that $\operatorname{supp}(\rho_U) \cap W \neq \emptyset$ only for finitely many $U \in \mathfrak{U}$.

iii) $\sum\limits_{U\in\mathfrak{U}} \rho_U = 1$, where the sum is finite by condition ii).

We take for granted the existence of smooth partitions of unity of manifolds (LEE, 2013, Theorem 2.23, p.43), (RUFFINO, 2020, Seção 2.6).

Lemma A.3.2 (Extension of Smooth Sections)**.** *Let $p : E \to X$ be a smooth vector bundle and $i : A \hookrightarrow X$ a smooth closed embedding of a manifold A in X. Every smooth section $s \in \Gamma(E_A)$, admits a smooth extension $\widetilde{s} \in \Gamma(E)$.*

Proof. Since A is diffeomorphic to the submanifold $i(A) \subseteq X$, we can suppose, without loss of generality, that $A \subseteq X$. We do the extension in two steps:

1. We extend the section locally, which amounts to extend it on the trivial bundle;

2. We glue these local extensions using a smooth partition of unity.

Step 1. At each point $a \in A$, choose some (smooth) local trivialization (U, ψ) of E. We assume further that U is the domain of a slice chart (U, φ) around A where $\phi : U \to \mathbb{R}^n$ and $\phi|_{U\cap A} : U \cap A \to \mathbb{R}^l \times \{0\}^{k-l}$ is a homeomorphism. In order to do this, one can start with both a trivialization (V, ψ') and a slice chart (W, φ') around a and define $U := V \cap W$, $\psi := \psi'|_U$ and $\varphi := \varphi'|_U$,.

The smooth map $\psi|_A \circ s|_U : A \cap U \to (U \cap A) \times \mathbb{R}^n$ is of the form

$$\psi|_A \circ s|_U(a) = (a, f_U(a))$$

where $f_U : U \cap A \to \mathbb{R}^n$ is a smooth map. Using the slice chart, we can write

$$f'_U := f_U \circ \varphi^{-1} : \mathbb{R}^l \times \{0\}^{k-l} \to \mathbb{R}^n$$

Now, we wish to extend this map in a smooth way over all $\mathbb{R}^k$. We define $\widetilde{f}'_U : \mathbb{R}^k \to \mathbb{R}^n$ as

$$\widetilde{f}'_U(x) = f'_U \circ \operatorname{pr}'(x)$$

where

$$\mathrm{pr}'(x_1,\ldots,x_k)=(x_1,\ldots,x_l,0,\ldots,0).$$

Since $\widetilde{f}'_U$ is a composition of two smooth map, it is smooth. Next, we define $\widetilde{f}_U : U \to \mathbb{R}^n$ by $\widetilde{f}_U := \widetilde{f}_U \circ \varphi$. Observe that $\widetilde{f}_U|_A = f_U$. FInallu, we conclude this step by defining the local section $\widetilde{s} \in \Gamma(E_U)$ as

$$\widetilde{s}_u(x)=\psi^{-1}(x,\widetilde{f}_U(x)).$$

Since any point $a \in A$ is in some A, we get a family of local sections $\{\widetilde{s}_{U_a}\}_{a\in A}$ indexed by $a \in A$.

Step 2. Since A is closed, the set $\mathfrak{U}=\{U_a\}_{a\in A}\cup\{A^c\}$ is an open covering of X. Define $s_{A^c} := 0$, the null section over X. Consider a (smooth) partition of unity $\{\rho_U\}_{\mathfrak{U}\in\mathfrak{U}}$ subordinated to the covering $\mathfrak{U}$. Since a linear combination, with coefficients in $C^\infty(X)$, of smooth sections is a smooth section, we define

$$\widetilde{s}(x)=\sum_{U\in\mathfrak{U}}\widetilde{s}_U(x)\rho_U(x) \tag{A.3}$$

Let us verify that this sum is indeed an extension of $\widetilde{s}$. Give some point $a \in A$, we have a is in the support of ρ_{U_a} and some other $\rho_{U_{a_1}},\ldots,\rho_{U_{a_n}}$ but is not in the support of ρ_{A^c} since $\mathrm{supp}(\rho_{A^c}) \subseteq A^c$. Thus the sum in (A.3) is just

$$\widetilde{s}(a)=\widetilde{s_{U_a}}(a)\rho_a(a)+\widetilde{s}_{U_{a_1}}(a)\rho_{a_1}(a)+\cdots+\widetilde{s}_{U_{a_n}}(a)\rho_{U_{a_n}}(a)$$

and, using that $\widetilde{s}_{U_c}(b)=s(b)$ for any $b,c\in A$, it follows that

$$\widetilde{s}(a)=s(a)(\rho_a(a)+\rho_{U_{a_1}}(a)+\cdots+\rho_{U_{a_n}}(a))=s(a).$$

which concludes the result. □

As particular instance of this lemma applied to the vector bundle $\pi : \bigwedge^n T^*X \to X$ is the following corollary:

Corollary A.3.3. *Let $i : A \hookrightarrow X$ be a smooth embedding. Any differential form $\omega \in \Omega(A)$ can be extended to a differential form $\widetilde{\omega} \in \Omega(X)$ through i.*

Remark A.3.4. Sometimes we will be using this result with smooth cofibrations. Since any cofibration is a topological embedding and, whenever X is Hausdorff, it is also a closed (see Proposition A.2.4), the previous corollary can be restated as: "If $i : A \hookrightarrow X$ is a smooth cofibration, than any differential form $\omega \in \Omega(A)$ can be extended to $\widetilde{\omega} \in \Omega(X)$".

Finally, we give the promised proof of

Proposition (2.2.10)**.** *Let $p : Y \to X$ be a (n,m)-fibered manifold, $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(N)$. The following properties holds*

i) *(Homotopy Formula)*

$$\int_{\partial p} \omega = d_B \int_p \omega - \int_p d_M \omega. \tag{A.4}$$

ii) *(Projection formula)*

$$\int_f (\omega \wedge f^* \eta) = \int_f \omega \wedge \eta \tag{A.5}$$

iii) *(Functorial) Assume $q : Z \to Y$ is another fibered manifold without boundary than*

$$\int_{p \circ q} = \int_p \circ \int_q \tag{A.6}$$

iv) *(Stability) Let $p' : B \to A$ is another fibered manifold and let $(\overline{\rho}, \rho)$ be a pair of smooth maps $\rho : A \to X$ and $\overline{\rho} : B \to Y$ such that $\rho \circ p' = p \circ \overline{\rho}$ and $\overline{\rho}$ is a fiberwise diffeomorphism, that is $\overline{\rho}|_a : B_a \to Y_{\rho(a)}$ is a diffeomorphism, then*

$$\int_{p'} \overline{\rho}^* \omega = \rho^* \int_p \omega. \tag{A.7}$$

Proof. In order to proof these properties, it is enough to use the standard model of a convenient chart.

i) We prove this result using a convenient chart, which reduces the problem to show that the homotopy formula hold for integration of a form $\omega \in \Omega_v(\mathbb{R}^m \times \mathbb{H}^k)$ with respect to $\mathrm{pr}_1 : \mathbb{R}^m \times \mathbb{H}^k \to \mathbb{R}^m$. We have already remarked in the definition of the fiber integration that forms which result in non null integral can be written as a sum of differential forms of the type $f(x,y) dx_I \wedge dy_1 \wedge \cdots \wedge dy_k$. In order to $d\omega$ to be of this form, ω has only two possible forms:

- $\omega_1 = f(x,y) dx_I \wedge dy_1 \wedge \cdots \wedge dy_k$
- $\omega_2 = f(x,y) dx_I \wedge dy_J$, where $|J| = k-1$.

in each case one has

- $d\omega_1 = \dfrac{\partial f}{\partial x_j}(x,y) dx_j \wedge dx_I \wedge dy_1 \wedge \cdots \wedge dy_k$, where $j \notin I$ and,
- $d\omega_2 = (-1)^{|I|} \dfrac{\partial_j f}{\partial y_j}(x,y) dx_I \wedge dy_j \wedge dy$, where $|J| = k-1$.

There are two cases to consider:

a) Either the we are in a point $x \in \mathbb{R}^m$ where the support of f does not "touch" the boundary of the fiber, that is $y_k < 0$,

b) or we are in a point which this happens.

In case a) for for ω_1, one has

$$\left(\int_{\mathrm{pr}_1} d\omega_1\right)_x = \left(\int_{\mathbb{H}^k} \frac{\partial f}{\partial x_j}(x,y)dy_1\dots dy_k\right) dx^j \wedge dx^I = \\ d\left(\int_{\mathbb{H}^k} f(x,y)dy_1\dots dy_k\right) dx^I = d\int_{\mathrm{pr}_1}\omega$$

and for ω_2

$$\left(\int_{\mathrm{pr}_1} d\omega_2\right)_x = \left(\int_{\mathbb{H}^k} \frac{\partial f}{\partial y_j}(x,y)dy_1\dots dy_k\right) dx^I = 0 = d\int_{\mathrm{pr}_1}\omega_2.$$

where the last equality, $\int_{\mathrm{pr}_1}\omega_2 = 0$, is true for dimensional reasons.

Now, observe that under the hypothesis that $y_k < 0$ one has

$$\int_{\partial\,\mathrm{pr}_1}\omega_1 = \int_{\partial\,\mathrm{pr}_1}\omega_2 = 0$$

In the case b), we still have $\int_{\partial\,\mathrm{pr}_1}\omega_1 = 0$ (for dimensional reasons), but now

$$\int_{\partial\,\mathrm{pr}_1}\omega_2 = \left(\int_{\mathbb{R}^{k-1}} f'(x,y)dy_1\dots dy_{k-1}\right) dx^I$$

where $f'(x,y) = f(x,y_1,\dots,y_{k-1},0)$. The case of ω_1 is exactly the same, but now for ω_2 one has

$$\left(\int_{\mathrm{pr}_1} d\omega_2\right)_x = \left(\int_{\mathbb{H}^k} \frac{\partial f}{\partial y_j}(x,y)dy_1\dots dy_k\right) dx^I = \\ \left(\int_{\mathbb{R}^{k-1}} f'(x,y)dy_1\dots dy_{k-1}\right) dx^I = \int_{\partial\,\mathrm{pr}_1}\omega_2.$$

Thefore in any case we get:

$$\int d\omega = d\int\omega + \int_\partial \omega$$

ii) As before, we can deal with the problem in any chart. We choose a convenient chart and a form $\omega = f(x)dx^I$, with dx^i is the dual of the canonical basis of $\mathbb{R}^m$, which pullback to the form

$$\mathrm{pr}_1^*(\omega) = f(x)dx^I$$

on $\mathbb{R}^m \times \mathbb{H}^k$. We know that the product form $\mathrm{pr}_1^*\omega \wedge \omega'$ will have non null integral only if it of the form

$$\omega' = g(x,y)dx^J \wedge dy^1 \wedge \dots \wedge dy^k$$

in which case

$$\left(\int_{\mathrm{pr}_1}\omega'\right)_x = \left(\int_{\mathbb{H}^k} g(x,y)dy_1\dots dy_k\right) dx^J$$

and hence

$$\left(\omega \wedge \left(\int_{\mathrm{pr}_1}\omega'\right)\right)_x = \left(\int_{\mathbb{H}^k} f(x)g(x,y)dy_1\dots dy_k\right) dx^I \wedge dx^J.$$

Now we calculate $\int \pi^*\omega \wedge \omega'$. In this case, one has

$$\pi^*\omega \wedge \omega' = f(x)g(x,y)dx^I \wedge dx^J \wedge dy_1 \dots dy_k$$

which entails

$$\left(\int_{\mathrm{pr}_1} \pi^*\omega \wedge \omega'\right) = \left(\int_{\mathbb{H}^k} f(x)g(x,y)dy_1 \dots dy_k\right) dx^I \wedge dx^J$$

proving the projection form.

iii) We can always find a three charts $\psi : U \to Y \subset \mathbb{R}^{n+k}$, $\varphi : V \to \mathbb{R}^n$ and $\xi : W \to \mathbb{R}^{n-l}$ such that

$$\varphi \circ p \circ \xi^{-1} = \mathrm{pr}_{n,n-l}$$

and

$$\psi \circ q \circ \varphi^{-1} = \mathrm{pr}_{n+k,n}$$

such that

$$\psi \circ q \circ \circ \xi^{-1} = \mathrm{pr}_{n+k,n} \to \mathrm{pr}_{n,n-l} = \mathrm{pr}_{n+k,n-l}$$

where $\mathrm{pr}_{a,b} : \mathbb{R}^a \to \mathbb{R}^b$ is the projection on the first b coordinates. Next, we can carry the integration of $f : R^{n+k} \to \mathbb{R}$ with respect to $\mathrm{pr}_{n+k,n-l}$

$$\int_{\mathbb{R}^{l+k}} f\left(x_1, \dots, x_n, u_1, \dots, u_l, v_1, \dots, v_k\right) dv_1 \dots dv_k du_1 \dots du_l = \int_{\mathbb{R}^l} \left(\int_{\mathbb{R}^k} f(x_1, \dots, x_n, v_1, \dots, v_k) dv_1 \dots dv_k\right) du_1 \dots du_l$$

where we use Fubinni's theorem, an the two integral are fiber integral relative to $\mathrm{pr}_{n,n-l}$ and $\mathrm{pr}_{n+k,n}$.

iv) This follows direct from the definition since

$$\int_q \overline{\rho}^*\omega = \int_{B_a} \overline{\rho}|_a^*\omega = \rho^* \int_{X_x} \omega$$

□

A.3.3 Forms and Chains

We employ the notation of Chapter B. Given a smooth n-chain $c \in S_n^{\mathrm{sm}}(X;G)$ of the form $c = \sum_{p+q=n} g_p\sigma_q$ where $\sigma : \Delta^n \to X$ is a smooth simplex and a form $\Omega^n(X)$ we define the integration

$$\int_c \omega = \sum_{p+q=n} g_p \int_{\sigma_q} \omega_q \in G^\bullet$$

for nitty-gitty detail, the reader can consult for example (ABRAHAM; MARSDEN; RATIU, 1988, Supplement 8.A). In the relative setting the situation is completely analogous, given $c \in S_n^{\mathrm{sm}}(\rho;G)$ with $c = \sum_{p+q=n} g_p(\sigma_q, \tau_{q-1})$ and a relative form $\Omega^n(\rho)$ we define

$$\int_c^\rho (\omega, \theta) := g_p \int_{\sigma_q} \omega_q + g_p \int_{\tau_{q-1}} \theta_{q-1} \in G^\bullet \tag{A.8}$$

Recall that the map $\int : \Omega^\bullet(X) \to Z^\bullet_{\mathrm{sm}}$ given by

$$\int^X (\omega) = c \mapsto \int_c \omega$$

induces the de Rham isomorphism in cohomology[1]. The same is true in the relative setting

Proposition A.3.5 (relative de Rham Theorem)**.** *The map $\int^\rho : \Omega(\rho) \to S^\bullet_{sm}$ defined in in(A.8) induces isomorphism in cohomology.*

Proof. Consider the following commutative diagram of morphism of cochains

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{\bullet-1}(A) & \xrightarrow{-i} & \Omega^\bullet(\rho) & \xrightarrow{\pi} & \Omega^\bullet(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \int^A} & & \downarrow{\scriptstyle \int^\rho} & & \downarrow{\scriptstyle \int^X} & & \\
0 & \longrightarrow & S^{\bullet-1}_{\mathrm{sm}}(A) & \xrightarrow{-i} & S^\bullet_{\mathrm{sm}}(\rho) & \xrightarrow{\pi} & S^\bullet_{\mathrm{sm}}(X) & \longrightarrow & 0
\end{array}$$

In cohomology, it induces following commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{\bullet-1}_{\mathrm{dR}}(A) & \xrightarrow{\partial} & H^\bullet_{\mathrm{dR}}(\rho) & \xrightarrow{(\mathrm{id}_X,\varnothing_A)^*} & H^\bullet_{\mathrm{dR}}(X) & \xrightarrow{\rho^*} & H^\bullet_{\mathrm{dR}}(A) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle r_A} & & \downarrow{\scriptstyle r_\rho} & & \downarrow{\scriptstyle r_X} & & \downarrow{\scriptstyle r_A} & & \\
\cdots & \longrightarrow & H^{\bullet-1}(A) & \xrightarrow{\delta} & H^\bullet(\rho) & \xrightarrow{(\mathrm{id}_X,\varnothing_A)^*} & H^\bullet(X) & \xrightarrow{\rho^*} & H^\bullet(A) & \longrightarrow & \cdots
\end{array}$$

where the vertical arrows r_X and r_A are the de Ram isomorphisms induced by $\int^X$ and $\int^A$. Applying the five lemma (Proposition A.4.1), we conclude that r_ρ induced by $\int^\rho$ is a isomorphism too. $\square$

In the case with $G = \mathbb{Z}$ concentrated in degree 0 we shall need the following two results

Proposition A.3.6. *A relative integral form (ω, θ) is closed.*

Proof. This question can be addressed locally which implies there is no loss of generality verifying this assertion a relative form $(\omega, \theta) \in \Omega(\rho)$ where $\rho : \mathbb{R}^n \to \mathbb{R}^m$. Suppose by contradiction that (ω, θ) is not closed. Then there exists a point $p \in \mathbb{R}^m$ such that $(d\omega, \rho^*\omega - d\theta) \neq (0, 0)$. Suppose, that $d\omega \neq 0$ in p and without loss of generality suppose $\omega_p > 0$. In this case, let $D(r)$ be a disk of radius r centered aronud p such that $d\omega|_{D(r)} > 0$. Define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(r) = \int_{D(r)} d\omega$. This function is continuous and is such that $f(0) = 0$ and $f(r) > 0$ for $r > 0$. By assumption

$$0 < f(r) = \int_{\partial D(r)} \omega \in \mathbb{Z}$$

which contradicts the continuity of f thus giving us the desire contradiction by choosing the cycle $(\partial D(r), 0)$. Now, suppose $d\omega = 0$, but $\rho^*\omega - d\theta \neq 0$ By the same argument, we can form assume $\rho^*\omega - d\theta \neq>$ in some point $a \in A$. The same argument choosing the cycle $(\rho_* D(r), \partial D(r))$. shows that this cannot happen. $\square$

[1] Remarking that $H(S^\bullet_{\mathrm{sm}}) \cong H(S)$

When dealing with Cheeger-Simons differential characters we will use the following results

Proposition A.3.7. *Let $\rho : A \to X$ be a map and Y as smooth compacted oriented manifold. For a cycle $z_\rho \in Z_n^{sm}(\rho)$ and a representative of the fundamental class $Y \in Z_m^{sm}(Y)$ it holds that*

$$\int_{z_\rho \times Y} (\omega, \theta) = \int_{z_\rho} \int_{(\mathrm{pr}_Y^X, \mathrm{pr}_Y^A)} (\omega, \theta)$$

for $(\omega, \theta) \in \Omega^{n+m}(\rho \times \mathrm{id}_Y)$, where $\mathrm{pr}_Y^X : X \times Y \to Y$ and $\mathrm{pr}_Y^X : A \times Y \to Y$ are projections, and where $\times$ is defined as in the Kunneth formula (B.5).

A proof can be carried through an adaptation of theorem I in section 7.17 in (GREUB; HALPERIN; VANSTONE, 1972).

Proposition A.3.8. *Let $\rho : A \to X$ be a map and Y be a manifold. For cycles $z_\rho \in Z_n^{sm}(\rho)$ and $z_Y \in Z_m^{sm}(Y)$ it holds that*

$$\int_{z_\rho \times z_Y} ((\omega, \theta) \times \omega') = \int_{z_\rho} (\omega, \theta) \cdot \int_{z_y} \omega'$$

for $(\omega, \theta) \in \Omega^n(\rho)$, $\omega' \in \Omega^m(Y)$ and where $\times$ is defined as in the Kunneth formula (B.5).

For a proof the reader can adapt equation (GREUB; HALPERIN; VANSTONE, 1972, Equation (7.3), Section 7.12). Also, look at David E Speyer answer in <https://math.stackexchange.com/questions/29797/direct-proof-that-the-wedge-product-preserves-integral-cohomology-classes>.

A.4 Homological Algebra

A.4.1 The basic lemmas

Proposition A.4.1 (Five Lemma)**.** *Consider the following commutative diagrams of R-modules:*

(a)

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D'
\end{array}$$

If

- *the two rows are exact;*
- *β and δ are monomophisms;*
- *and α is an isomorphism,*

then γ is an monomorphism.

(b)

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "j"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "j'"] & E'
\end{tikzcd}$$

if

- *the two rows are exact;*
- *β and δ are epimorphisms;*
- *and ϵ is an isomorphism,*

then γ is an epimorphism.

(c)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "j"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "j'"] & E'
\end{tikzcd}$$

if

- *the two rows are exact;*
- *α, β, δ and ϵ are isomorphisms;*

then γ is an isomorphism.

Since the result is so standard we refer to any basic book dealing with homological algebra.

Proposition A.4.2 (Braid)**.** *Consider the following commutative diagram of R-modules:*

$$\begin{tikzcd}
& A \arrow[rr, bend left, "\alpha"] \arrow[dr, "\epsilon"] & & B \arrow[rr, bend left, "\beta"] \arrow[dr, "\theta"] & & C \arrow[rr, bend left, "\gamma"] \arrow[dr, "\lambda"] & & D \\
E \arrow[ur, "\delta"] \arrow[dr, "\nu"] & & F \arrow[ur, "\eta"] \arrow[dr, "\rho"] & & G \arrow[ur, "\kappa"] \arrow[dr, "\tau"] & & H \arrow[ur, "\mu"] & \\
& I \arrow[ur, "\pi"] \arrow[rr, bend right, "\psi"] & & J \arrow[ur, "\sigma"] \arrow[rr, bend right, "\omega"] & & K \arrow[ur, "\phi"] & &
\end{tikzcd}$$

If the sequences

1. $E \xrightarrow{\delta} A \xrightarrow{\alpha} B \xrightarrow{\theta} G \xrightarrow{\kappa} K$
2. $E \xrightarrow{\nu} I \xrightarrow{\psi} J \xrightarrow{\sigma} G \xrightarrow{\kappa} C \xrightarrow{\gamma} D$

3. $A \xrightarrow{\epsilon} F \xrightarrow{\rho} J \xrightarrow{\omega} K \xrightarrow{\phi} H \xrightarrow{\mu} D$

are exact and the composition

$$I \xrightarrow{\pi} F \xrightarrow{\eta} B$$

is the null homomorphism, then the sequence

$$I \xrightarrow{\pi} F \xrightarrow{\eta} B \xrightarrow{\beta} C \xrightarrow{\lambda} H$$

is exact.

This result, in the exact same form, can be found in (MUNKRES, 2000), but since it is often invoked but not proved, we give a proof for the sake of completeness.

Proof. Fisrt we verify that the images are subsetes of the kernels

($\mathrm{Im}(\pi) \subseteq \mathrm{Ker}(\eta)$) By hypothesis $\eta \circ \pi = 0$;

($\mathrm{Im}(\eta) \subseteq \mathrm{Ker}(\beta)$) By the commutative diagram $\beta \circ \eta = (\kappa \circ \sigma) \circ \rho = 0$;

($\mathrm{Im}(\beta) \subseteq \mathrm{Ker}(\lambda)$) By the commutative diagram $\lambda \circ \beta = \phi \circ (\tau \circ \theta) = 0$.

Now we verify that Im $\subseteq$ Ker

($\mathrm{Ker}(\eta) \subseteq \mathrm{Im}(\pi)$) Given $f \in \mathrm{Ker}(\eta)$, we have $\sigma \circ \rho(f) = \theta \circ \eta(f) = 0$, and thus we have $\rho(f) \in \ker(\sigma)$. This means there exists a $i \in I$ such that $\psi(i) = \rho(f)$. Since $\psi = \rho \circ \pi$, it follows that

$$\rho(\pi(i)) = \rho(f) \implies \rho(f - \pi(i)) = 0.$$

This implies that there exists $a \in A$ such that $\epsilon(a) = f - \pi(i)$. Composing with η to left, we get

$$\alpha(a) = \eta \circ \epsilon(a) = \eta(f) - \eta \circ \pi(i) = 0$$

which means there is $e \in E$ such that $\delta(e) = a$. Applying ϵ to the left we get $\epsilon \circ \delta(e) = \epsilon(a) = f - \pi(a)$. Since $\epsilon \circ \delta = \pi \circ \nu$ we get $\pi \circ \nu(e) = f - \pi(a)$ which implies in turn that $\pi(\nu(e) - a) = f$ which shows the desired result.

($\mathrm{Ker}(\beta) \subseteq \mathrm{Im}(\eta)$) Given $b \in \mathrm{Ker}(\beta)$, we have $\kappa \circ \theta(b) = \beta(b) = 0$. So, there exists $j \in J$ such that $\sigma(j) = \theta(b)$. Composing with τ we get

$$\omega(j) = \tau \circ \sigma(j) = \tau \circ \theta(b) = 0,$$

which implies that there exists $f \in F$ such that $\rho(f) = j$. Applying σ to the left, we get

$$\theta \circ \eta(f) = \sigma \circ \rho(f) = \sigma(j) = \theta(b),$$

which entails $\theta(\eta(f) - b) = 0$. This means that there exists $a \in A$ such that $\alpha(a) = \eta(f) - b$. Since $\alpha(a) = \eta \circ \epsilon(a)$ we get $\eta \circ \epsilon(a) = \eta(f) - b$ and finally get $\eta(\epsilon(a) - f) = b$.

$\mathrm{Ker}(\lambda) \subseteq \mathrm{Im}(\beta)$ Given $c \in \mathrm{Ker}\,\lambda$, we have $\gamma(c) = \mu \circ \lambda(c) = 0$. This means that there exists $g \in G$ such that $\kappa(g) = c$. Composing with λ and using that $\lambda \circ \kappa = \phi \circ \tau$, we conclude $\phi \circ \tau(g) = 0$. Hence, there exists $j \in J$ such that $\omega(j) = \tau(g)$. Since $\omega = \tau \circ \sigma$ we get $\tau(\sigma(j) - g) = 0$. This means that, there exists a $b \in B$ such that $\theta(b) = \sigma(j) - g$. Applying κ to the left, we conclude that

$$\beta(b) = \kappa \circ \theta(b) = \kappa \circ \sigma(j) - \kappa(g) = \kappa(g) = c,$$

which is the desired result.

□

A.4.2 Chain Complexex

Let us fix a ring commutative R (not necessarily with unity), a R-module M is called $\mathbb{Z}$-*graded* (we omit the $\mathbb{Z}$ from the graded from this point on) if there exists a direct product decomposition $M = \prod_{n\in\mathbb{Z}} M_n$. A graded $\mathbb{Z}$-module will simply be called *graded group*.

Remark A.4.3. A graded structure is usually defined as a direct sum rather than a direct product, but this would lead to some problems in our case. That is why we choose to define it as a direct product.

A non null[2] element of M_n is said to be a homogeneous element of degree n. We denote the degree of a homogeneous element c by $|c|$.

A *morphism of graded modules* of degree p is a module homomorphism $\phi : C \to D$ such that $\phi : C_n \to D_{n+p}$. Whenever we say morphism of graded modules we are tacitly assuming 0 degree, unless stated otherwise.

Given two graded modules C, D we can define their graded tensor product in a graded way as

$$C \otimes D := \prod_{p+q=k\in\mathbb{Z}} (C_p \otimes D_q)$$

A graded R-module can have a structure of a R-algebra compatible with its grading, that is, a multiplication $\cdot : C \otimes_R C \to C$ such that $|c \cdot d| = n + m$ if $|c| = n$ e $|d| = m$. A graded $\mathbb{Z}$-algebra will be called a *graded ring* and we will always suppose that it has a identity unless state otherwise.

[2] We opted to not give 0 a degree.

It is also possible that the ring R itself is a graded ring, ins this case C is a graded R-module if the scalar multiplication is graded, that is, given $c \in C$ e $r \in R$ with $|c| = n$ and $|r| = m$, one has $|c \cdot r| = n + m$.

Generally, the graded rings and graded algebras which appear in this text are not commutative, but rather anticommutative in the graded sense, which we will call commutative graded rings and algebras (implicitly understood the anticommutativity). A graded ring is (anti-)comutative if

$$c \cdot d = (-1)^{|c||d|} d \cdot c$$

for $c, d \in C$. A morphism between commutative graded algebras is a morphism of graded modules $\phi : C \to D$ of degree zero such that

$$\phi(c \cdot d) = \phi(c)\dot{\phi}(d).$$

A boundary $\partial : C \to C$ is a morphism of graded modules of degree -1 such that $\partial \circ \partial = 0$. In a dual manner, a coboundary $\delta : C \to C$ is the same as boundary but with degree 1.

A chain (C, ∂) is a graded module C together with a boundary ∂. In general, we represent it as

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

A cochain (C, δ) is a graded module with a coboundary δ which will be presented as

$$\cdots \xrightarrow{\delta} C^{n-1} \xrightarrow{\delta} C^n \xrightarrow{\delta} C^{n+1} \xrightarrow{\delta} \cdots$$

In order to emphasize the difference between chains and cochains we opted, as usual, to use the grading index as a subscript for chains and a superscript for cochains.

A *differential* $d : A \to A$ between graded algebras is a coboundary that satisfies the graded Leibnez rule:

$$d(a \cdot b) = da \cdot b + (-1)^{|a|} a \cdot b$$

A *differential graded algebra* is a graded algebra endowed with a differential.

A morphism between two chain complexes (C, ∂_D) e (D, ∂_D) is a morphism of graded algebras $faphi : C \to D$, which is compatible with boundaries, that is, $\phi \circ \partial_C = \partial_D \circ \phi$. We can represent a morphism as in the following diagram

$$\begin{array}{ccccccccc}
\xrightarrow{\partial_C} & C_{n+1} & \xrightarrow{\partial_C} & C_n & \xrightarrow{\partial_C} & C_{n-1} & \xrightarrow{\partial_C} \\
& \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & \\
\xrightarrow{\partial_D} & D_{n+1} & \xrightarrow{\partial_D} & D_n & \xrightarrow{\partial_D} & D_{n-1} & \xrightarrow{\partial_D}
\end{array}$$

A morphism of cochain is defined in a completely analogous way. In the case of differential graded algebras, the morphisms are supposed to be morphisms of graded algebras.

The *mapping cone chain complex* (C_ϕ, ∂_ϕ) of a morphism $\phi : (C, \partial_C) \to (D, \partial_D)$ is the complex $(C \oplus D, \partial)$ where $C \oplus D$ has the following gradation

$$(C \oplus D)_n = C_{n-1} \oplus D_n$$

and boundary given by

$$\partial(c, d) = (-\partial_C c, \phi(c) + \partial_D d)$$

We also have the *mapping cone complex of cochains* (C_ϕ, δ_ϕ) where $C_\phi = C \oplus D$ with gradation

$$(C \oplus D)^n = C^n \oplus D^{n-1}$$

and coboundary given by

$$\delta_\phi(c, d) = (\delta_C c, \phi(c) - \delta_D d)$$

A differential graded module (M, d) is a cochain over the graded differential algbra (A, d_A) such that

$$d_M(m \cdot a) = d_M m \cdot a + (-1)^{|m|} m \cdot d_A a$$

The mapping cone chain complex $(C(\phi), d_\phi)$ of a morphism $\phi : A \to B$ between differential graded algebra does not have the structure of a differential graded algebra, but it is a differential graded algebra over the differential graded algebra A. More precisely, given $(a, b) \in C_\phi$, we define

$$(a, b) \cdot a' := (a \cdot a', b \cdot \phi(a))$$

and one can check this product is compatible with d_ϕ.

Give a R-chain (C, ∂), we define the graded module Z by $\operatorname{Ker} d$ with grading $Z_n = \operatorname{Ker}(d : C_n \to C_{n-1})$, whose elements are called *n-cycles*, and the the grade module $B = \operatorname{Im}(d)$ with grading $B^n = \operatorname{Im}(d : C_{n+1} \to C_n)$ whose elements are called n-boundaries. We can associate a graded R-module $H(C)$ called the *Homology* given by

$$H(C) = \frac{Z}{B}$$

with its natural grading

$$H_n(C) = \frac{Z_n}{B_n}$$

Given a chain morphism $\phi : C \to D$, we can define a graded morphism

$$\phi_* : H(C) \to H(D)$$

thanks to commutativity of the boundaries. This construction is functorial: we have a functor $H : \mathtt{Ch}_R \to \mathtt{GrR-Mod}$, from the category of R-chain complex to the category of graded R-modules. The same construction holds for cochains and its associated cohomology $H : \mathtt{CoCh}_R^{\mathrm{op}} \to \mathtt{GrR-Mod}$, where $\mathtt{CoCh}_R$ is the category of cochain complexes, except that this functor is contravariant.

A.4.3 Some useful results of homological algebra

Proposition A.4.4 (Zig-Zag Lemma). *Given an exact sequence of chain complexes*

$$0 \to C \xrightarrow{\phi} C' \xrightarrow{\psi} C'' \to$$

there exists a morphism of graed modules $\Delta : H(C'') \to H(C)$ of degree one -1, called the connecting morphism, which makes the following long sequence exact:

$$\begin{tikzcd}
{} \arrow[r] & H_n(C) \arrow[r, "\phi_*"] & H_n(C') \arrow[r, "\psi_*"] & H_n(C'') \arrow[dll, "\Delta"'] & \\
& H_{n-1}(C) \arrow[r, "\phi_*"] & H_{n-1}(C') \arrow[r, "\psi_*"] & H_{n-1}(C'') \arrow[r] & {}
\end{tikzcd}$$

Moreover, this construction is functorial in the following sense: given a commutative diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\phi"] \arrow[d, "f"] & C' \arrow[r, "\psi"] \arrow[d, "f'"] & C'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & D \arrow[r, "\phi'"] & D' \arrow[r, "\psi'"] & D'' \arrow[r] & 0
\end{tikzcd}$$

where the lines are exact. The following diagram is commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(C) \arrow[r, "\phi_*"] \arrow[d, "f_*"] & H_n(C') \arrow[r, "\psi_*"] \arrow[d, "f'_*"] & H_n(C'') \arrow[r, "\Delta_C"] \arrow[d, "f''_*"] & H_{n-1}(C) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(D) \arrow[r, "\phi'_*"] & H_n(D') \arrow[r, "\psi'_*"] & H_n(D'') \arrow[r, "\Delta_D"] & H_{n-1}(D) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The proof can be found in many books on algebra, homological algebra, or algebraic topology. I particularly favor (ROTMAN, 2009, Theorem 6.10, p. 333) as it is done in detail. □

The same result holds for cochains by just reversing the arrows.

Example A.4.5. Let $i : (C, \partial_C) \hookrightarrow (D, \partial)$ a injective morphism of chain complex. Consider the short exact sequence

$$0 \longrightarrow C \overset{i}{\hookrightarrow} D \xrightarrow{q} \frac{D}{\mathrm{Im}(i)} \longrightarrow 0.$$

Applying the Zig-Zag Lemma, we obtain the long exact sequence of the quotient

$$\cdots \longrightarrow H^\bullet(C) \xrightarrow{i_*} H^\bullet(D) \xrightarrow{q_*} H^\bullet(D, C) \xrightarrow{\delta} H^{\bullet-1}(C) \longrightarrow \cdots$$

where $H(D, C) := H\left(\frac{D}{\mathrm{Im}(C)}\right)$.

Example A.4.6. Let $\phi : (C, \partial_C) \to (D, \partial_D)$ be a morphism of chain complexes and consider the following short exact sequence

$$0 \longrightarrow D_\bullet \xrightarrow{-i_D} C(\phi)_\bullet \xrightarrow{\pi_D} C_{\bullet-1} \longrightarrow 0.$$

where the negative signal at $-i_D$ is necessary for the commutaivity to hold. The Zig-Zag lemma give us the long exact sequence of the mapping cone:

$$\cdots \longrightarrow H_\bullet(D) \xrightarrow{-i_{D,*}} H_\bullet(\phi) \xrightarrow{\pi_{C,*}} H_{\bullet-1}(C) \xrightarrow{\phi_*} H_{\bullet-1}(D) \longrightarrow \cdots$$

where $H_\bullet(\phi) := H_\bullet(C(\phi))$ Notice that the connecting homomorphism Δ in this case is just ϕ_*. For a proof of this result see (ROTMAN, 2009, Lemma 10.38, p. 650).

Clearly, cohomological analogues of the previous result holds *mutatis mutandis*.

A.4.4 Method of the Acyclic Models

A distinct set $\mathcal{M}$ of objects of a category $\mathtt{C}$ will be called *models*. We say that a functor $F : \mathtt{C} \to \mathtt{Ch}_R$ from any category $\mathtt{C}$ is *free over models* $\mathcal{M}$ if

$$F(C) = \bigoplus_{M \in \mathcal{M}} \bigoplus_{\substack{f \in h_{M,C} \\ m \in M_C}} R \cdot F(f)(m)$$

where $M_C \subset F(M)$ and $h_{M,C} \subseteq \mathrm{Hom}(M, C)$. Put in another way, for each object of C in $\mathtt{C}$, the set the R-module $F(C)$ is free with basis in the image of some maps from the the modules $F(M)$ to the module $F(C)$.

We say that a functor $G : \mathtt{C} \to \mathtt{Ch}$ is acyclic on model $\mathcal{C}$ if $G(M)$ is an acylcic chain for each $M \in \mathcal{M}$., that is, the chain is exact (or equivalently, its cohomology is null).

Theorem A.4.7 (Acyclic Models Theorem). *Let* $\mathtt{C}$ *with models* $\mathcal{C}$. *Let* $F, G : \mathtt{C} \to \mathtt{Ch}_R$ *with* F *free over* $\mathcal{C}$ *and* G *acyclic and a natural transformation* $\phi : (H \circ F)_0 \to (H \circ G)_0$. *Then, there exists a natural transformation* $\psi : F \to G$ *such that* $\psi_{*,0} = \phi$.

Moreover, any two such natural transformations ψ *and* ψ' *there is a natural chain homotopy in the following sense: there exists natural transformations* $D : \mathtt{C} \to \mathtt{C}[-1]$ *such that*

$$\psi_C - \psi'_C = D_C \partial - \partial D_C.$$

There are many proves of this result: some classical ones can be find for example (DIECK, 2008), (SPANIER, 1966, Theorem 8, p.165), or (CLEMENTE, 2018) for a detailed account in Portuguese. Yet, I would like to recommend the proof in video by Roman Sauer (2021), who gives a very lively account using module categories.

A.4.5 Algebraic Kunneth Sequence

Let Tor^R be the Tor functor as described in virtually any book on homological algebra. We will not need any of this properties, except the fact that its elements for a domain R, $\mathrm{Tor}_n^R(A, B) = 0$ for any $n \geq 1$ (see (ROTMAN, 2009, Theorem 7.15, p. 414).

Lets recall a version of the algebraic kunneth formula

Proposition A.4.8 (Algebraic Kunneth Formula)**.** *Let R be a principal ideal domain. Suppose C is a chain complex consisting of free R-modules. Then there exists an exact sequence*

$$0 \longrightarrow H(C) \otimes H(D) \xrightarrow{\otimes} H(C \otimes D) \longrightarrow Tor^R(H(X), H(D)) \longrightarrow 0$$

where the map $\otimes$ is just

$$\sum_{i+j=n} [\alpha_i] \otimes_R [\beta_j] = \sum_{i+j=n} [\alpha_i \otimes_R \beta_j]$$

If also D is a free complex, then the sequence splits.

A detailed proof of this result as stated here can be found in (DIECK, 2008). Further generalizations can be found in (ROTMAN, 2009, Section 10.10, p. 678).

APPENDIX B – Differential Refinement of Singular Cohomology

B.1 Introduction

In this appendix, we present singular cohomology with coefficients in a graded abelian group as a model for ordinary relative cohomology over maps. we will also present the Cheeger-Simons differential characters as a model for a differential refinement of ordinary differential cohomology with integer coefficients.

This chapter has three purposes:

- to illustrate a non trivial yet simple model of differential cohomology as introduced in Section 3.3,
- to serve as motivation and preliminaries for 6.2, and
- some parts will be used in Hopkins-Singer model in Section D.4.2 of Appendix D.

B.2 Singular cohomology with Coefficients in a abelian graded group

In this section we briefly review the singular model of ordinary cohomology theory with coefficients in a graded abelian group.

B.2.1 Review of singular chain and cochains

We denote by

$$\Delta^n := \{(t_0, \ldots, t_n) \subseteq \mathbb{R}^n_{\geq} \mid t_0 + \ldots + t_n = 1\}$$

the *n-standard simplex* which is to be regarded as subspace of $\mathbb{R}^{n+1}$ with the subspace topology. Each n-simplex can be regarded as n-manifold with smooth corners (see section A.3). Let X be a n-manifold. A *singular n-simplex on X* is a a continuous map $\sigma : \Delta^n \to X$, that is, $\sigma \in \mathrm{Hom}_{\mathtt{Top}}(\Delta_n, X)$. In addition, if $\sigma \in \mathrm{Hom}_{\mathtt{Man}}(\Delta_n, X)$, we call it a *smooth singular n-simplex* accordingly.

The *face maps* $\delta_i^{n-1} : \Delta^{n-1} \to \Delta^n$, $i = 0, \ldots, n$ are defined in the following way

$$\delta_i^{n-1}(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, \underbrace{0}_{i}, t_i, \ldots, t_{n-1}),$$

where a zero is inserted in the i-th coordinate of the $n+1$-tuple.

Proposition B.2.1. *The face maps satisfies the following relations*

$$\delta_j^n \circ \delta_i^{n-1} = \delta_i^n \circ \delta_{j-1}^{n-1}, \tag{B.1}$$

provided $i < j$.

Proof. By one side, we have

$$\begin{aligned}
\delta_j^n \circ \delta_i^{n-1}(t_0, \dots, t_{n-1}) =& \delta_j^n(t_0, \dots, t_{i-1}, \underbrace{0}_{i}, t_i, \dots, t_{n-1}) \\
=& (t_0, \dots t_{i-1}, \underbrace{0}_{i}, t_i, \dots t_{j-2}, \underbrace{0}_{j}, t_{j-1} \dots, t_{n-1}),
\end{aligned}$$

and by the other

$$\begin{aligned}
\delta_i^n \circ \delta_{j-1}^{n-1}(t_0, \dots, t_{n-1}) =& \delta_i^n(t_0, \dots, t_{j-2}, \underbrace{0}_{j-1}, t_{j-1}, \dots, t_{n-1}) \\
=& (t_0, \dots t_{i-1}, \underbrace{0}_{i}, t_i, \dots t_{j-2}, \underbrace{0}_{j}, t_{j-1} \dots, t_{n-1}).
\end{aligned}$$

□

Definition B.2.2 (Singular Chain Complex)**.** The *singular chain complex* $(S(X;G), \partial)$ associated to a topological space X with coefficients in a $\mathbb{Z}$-graded abelian group $G = \bigoplus_{m\in\mathbb{Z}} G^m$ is defined as

$$S^k(X;G) = \bigoplus_{n+m=k} G^m \otimes_{\mathbb{Z}} \mathbb{Z}\,\mathrm{Hom}_{\mathtt{Top}}(\Delta^n, X), \tag{B.2}$$

where $\mathbb{Z}\,\mathrm{Hom}_{\mathtt{Top}}(\Delta^n, X)$ is the free abelian group generated over the n-singular simplex. The boundary $\partial : S^k(X;G) \to S^{k-1}(X;K)$ is the degree -1 homomorphism

$$\partial^k(g_m\sigma_n) = g_m \sum_{i=0}^{n} (-1)^i \sigma_n \circ \delta_i^{n-1},$$

where $g_m\sigma_n$ is a shorthand for $g_m \otimes \sigma_n$, and extended to all $S^k(X;G)$ by linearity. The *smooth singular chain complex* $S^{\mathrm{sm}}(X;G)$, associated to smooth manifold with corners X, is defined in the same way just replacing in (B.2) the singular simpleces $\mathrm{Hom}_{\mathtt{Top}}$ by the smooth singular simpleces $\mathrm{Hom}_{\mathtt{Man}}$.

Let's verify that $\partial \circ \partial = 0$. By one side one has

$$\begin{aligned}
\partial^{k-1} \circ \partial^k(g_m\sigma_n) =& g_m \sum_{i=0}^{n-1} (-1)^i \left(\sum_{j=0}^{n} (-1)^j \sigma \circ \delta_j^{n-1} \right) \circ \delta_i^{n-2} \\
=& g_m \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} \sigma \circ \delta_j^{n-1} \circ \delta_i^{n-2} \\
=& g_m \left(\sum_{i,j=0}^{n-1} (-1)^{i+j} \sigma \circ \delta_j^{n-1} \circ \delta_i^{n-2} + \sum_{i=0}^{n-1} (-1)^{n+i} \sigma \circ \delta_n^{n-1} \circ \delta_i^{n-2} \right),
\end{aligned} \tag{B.3}$$

where we split the sum. Observe that

$$\begin{aligned}
\sum_{i,j=0}^{n-1}(-1)^{i+j}\sigma\circ\delta_j^{n-1}\circ\delta_i^{n-2} &= \sum_{\substack{i,j=0\\ i<j}}^{n-1}(-1)^{i+j}\sigma\circ\delta_j^{n-1}\circ\delta_i^{n-2}+\sum_{\substack{i,j=0\\ i\geq j}}^{n-1}(-1)^{i+j}\sigma\circ\delta_j^{n-1}\circ\delta_i^{n-2}\\
&\overset{(B.1)}{=}\sum_{\substack{i,j=0\\ i<j}}^{n-1}(-1)^{i+j}\sigma\circ\delta_i^{n-1}\circ\delta_{j-1}^{n-2}+\sum_{\substack{i,j=0\\ i\geq j}}^{n-1}(-1)^{i+j}\sigma\circ\delta_j^{n-1}\circ\delta_i^{n-2}
\end{aligned}$$

At last, one gets

$$\begin{aligned}
&\sum_{j=i+1}^{n-1}\sum_{i=0}^{n-1}(-1)^{i+j}\sigma\circ\delta_i^{n-1}\circ\delta_{j-1}^{n-2}+\sum_{i=j}^{n-1}\sum_{j=0}^{n-1}(-1)^{i+j}\sigma\circ\delta_j^{n-1}\circ\delta_i^{n-2}=\\
&\qquad\sum_{k=i}^{n-2}\sum_{i=0}^{n-1}(-1)^{i+k-1}\sigma\circ\delta_i^{n-1}\circ\delta_k^{n-2}+\sum_{k=i}^{n-1}\sum_{i=0}^{n-1}(-1)^{k+i}\sigma\circ\delta_i^{n-1}\circ\delta_k^{n-2}
\end{aligned}$$

where we exchanged indices $k=j-1$ in the first sum and renamed the indices in the second sum in order to get a clear picture of the canceling terms. Every term of the first sum of the second line cancels with the second sum, lefting only the terms $k=n-1$ of the second sum, from which we get

$$\sum_{i,j=0}^{n-1}(-1)^{i+j}\sigma\circ\delta_j^{n-1}\circ\delta_i^{n-2}=\sum_{i=0}^{n-1}(-1)^{n+i-1}\sigma\circ\delta_i^{n-1}\circ\delta_{n-1}^{n-2}$$

With respect to the last term of (B.3), note that

$$\sum_{i=0}^{n-1}(-1)^{n+i}\sigma\circ\delta_n^{n-1}\circ\delta_i^{n-2}\overset{(B.1)}{=}\sum_{i=0}^{n-1}(-1)^{n+i}\sigma\circ\delta_i^{n-1}\circ\delta_{n-1}^{n-2}$$

It follows that

$$\begin{aligned}
\partial^{k-1}\circ\partial^k(g_m\sigma_n) =& g_m\left(\sum_{i,j=0}^{n-1}(-1)^{i+j}\sigma\circ\delta_j^{n-1}\circ\delta_i^{n-2}+\sum_{i=0}^{n-1}(-1)^{n+i}\sigma\circ\delta_n^{n-1}\circ\delta_i^{n-2}\right)\\
=& g_m\left(\sum_{i=0}^{n-1}(-1)^{n+i-1}\sigma\circ\delta_i^{n-1}\circ\delta_{n-1}^{n-2}+\sum_{i=0}^{n-1}(-1)^{n+i}\sigma\circ\delta_i^{n-1}\circ\delta_{n-1}^{n-2}\right)\\
=& g_m0=0
\end{aligned}$$

Given some map $f:X\to Y$, we define a pushfoward morphism $f_\#:S(X)\to S(Y)$ by putting $f_\#(\sigma)=f\circ\sigma$ for each singular simplex and extended by linearity. In other words, the the singular chains defines a functor $S:\texttt{Top}\to\texttt{Ch}$ where $\texttt{Ch}$ is the category of chain complexes and their morphisms given by

$$S(X\xrightarrow{f}Y;G)=S(X;G)\xrightarrow{f_\#}S(Y;G)$$

The boundaries of $S(\cdot;G)$ will be denote by $B(\cdot;G) := \mathrm{Im}(\partial)$ and the cycles by $Z(\cdot;G) := \ker(\partial)$. We denote by $H(\cdot,G) = \frac{Z(\cdot;G)}{(\cdot;G)}$ the *singular homology functor* , and use the notation $f_* = H(f_\#;G)$ for the morphisms. If $G = \mathbb{Z}$ concentrated at degree 0 we write H for $H(;\mathbb{Z})$.

We quote, without proof, the classical result of homotopy invariance of this functor

Theorem B.2.3. *If $f : X \to Y$ and $g : X \to Y$ are homotopic, then $f_* = g_*$.*

Observe that, although we are working with an ordinary cohomology with coefficients in a graded abelian group G, the usual proof works in the same way (see (HATCHER, 2002) for the usual proof).

Definition B.2.4 (Singular Cochain Complex)**.** The *singular cochain complex* associated to a topological space X $(S^\bullet(X;G),\delta)$ is defined as

$$S^n(X;G) := \bigoplus_{p+q=n} \mathrm{Hom}_{\mathbb{Z}}(S_p(X), G^q)$$

with the coboundary morphism $\delta : S^\bullet(X;G) \to S^{\bullet+1}(X;G)$ given by $\delta s = s\circ\partial$. The *smooth singular chain complex* associated to a smooth manifold with corners $(S^\bullet_{\mathrm{sm}}(X),\delta)$ is obtained by replacing $S_p(X)$ by $S_p^{\mathrm{sm}}(X)$ in (B.2.4).

If $f : X \to Y$ is a (smooth) continuous map, we define $f^\# : S^\bullet(Y) \to S^\bullet(X)$ ($f^\# : S^\bullet_{\mathrm{sm}}(Y) \to S^\bullet_{\mathrm{sm}}(X)$) as $f^\#(s) = s\circ f_\#$. We denote by $Z^\bullet(\cdot,G) = \ker\delta$ and $B^\bullet(\cdot,G) = \mathrm{Im}\,\delta$ the cocyles and the coboundaries of $S^\bullet(\cdot,G)$ respectively. We also denote by $H^\bullet(;G) = \frac{Z^\bullet}{B^\bullet}$ the *singular cohomology functor*. This functor also respects the homotopy invariance in the sense that, if f is homotopic to g, then $f^* = g^*$.

B.2.2 Relative singular chains chains

There are two flavours of relative singular chain/cochains:

R1) Given a topological pair (X,A), we define the relative chain complexe of X with respect to A with coefficients in G, denoted by $S(X,A;G)$, as the the cokernel of the homomorphism induce by inlcusion $i_A : A \hookrightarrow X$, that is,

$$S_{\mathrm{par}}(X,A;G) = \mathrm{coker}\,(i_\#) := \frac{S(X;G)}{i_\#(S(A;G))} \tag{B.4}$$

with the boundary homomorphism the same as $S(X;G)$.

R2) Given a continuous map $\rho : A \to X$, we define the relative chain complex of ρ denote by $S(\rho;G)$, as the mapping cone complex of the morphism $\rho_\# : S(A;G) \to S(X;G)$, more precisely

$$S_\bullet(\rho;G) = S_\bullet(X) \oplus S_{\bullet-1}(A)$$

where $\partial(s,t) = (\partial s + \rho_\# t, -\partial t)$

both sections give rise to a long exact sequences in homology through the use of the Zig-Zag Lemma (Proposition A.4.4) applied to each long exact sequence

1. The quotient sequence (Example A.4.5)

$$0 \longrightarrow S(A) \xrightarrow{i_\#} S(X) \xrightarrow{q} S_{\mathrm{par}}(X,A) \longrightarrow 0$$

2. The mapping cone sequence (Example A.4.6)

$$0 \longrightarrow S(A) \xrightarrow{-i_2} S(\rho) \xrightarrow{\mathrm{pr}_1} S(X) \longrightarrow 0$$

The homology theory which we emerges from the quotient sequence is the usual one on maps of pairs. Here we are interested in the second construction.

We define the homology $H_\bullet(\rho;G)$ as the homology of the complex $S(\rho)$. The long exact sequence associated

$$\cdots \longrightarrow H_\bullet(A;G) \xrightarrow{\rho_*} H_\bullet(X;G) \xrightarrow{(\mathrm{id}_X,\varnothing_A)_*} H_\bullet(\rho;G) \xrightarrow{\delta} H_{\bullet-1}(A;G) \longrightarrow \cdot$$

arises from the zig-zag lemma (Proposition A.4.4) to the long exact sequence of the cone. Moreover, applying the five lemma (Proposition A.4.1) to the long exact sequences, we can verify that the theory is a homotopy invariant. So, in order to conclude that $(H_\bullet(\rho;G),\partial)$ is a homology theory, it is enough to prove excision. We could prove this directly, through an argument of barycentric subdivision, but we have opted to prove that there exists a natural isomorphism

$$H(\rho) \cong H(S_{\mathrm{par}}(C_\rho,*))$$

Assuming that we know that the relative homology on pairs satisfies excision, this will show that the same holds in the relative homology on maps.

First, observe that

Proposition B.2.5. *The cone $C(\Delta^{n-1})$ can be canonically identified with the standard simplex Δ^n through the following pointed homeomorphism*

$$\begin{aligned} h : (C(\Delta^n),*) &\to (\Delta^{n+1}, e_0) \\ [t,(s_0,\dots,s_n)] &\mapsto (t,(1-t)s_0,\dots,(1-t)s_n) \end{aligned}$$

Proof. This pointed map is well defined as $[1,s]=[1,s']$. It is also continuous since, denoting by $q_n : I\times\Delta^n \to C(\Delta^n)$ the quotient application and $\widetilde{h} = h\circ q$, where

$$\begin{aligned} \widetilde{h} : I\times\Delta^n &\to \Delta^{n+1} \\ (t,(s_0,\dots,s_n)) &\mapsto (t,(1-t)s_0,\dots,(1-t)s_n) \end{aligned}$$

is continuous, the universal property of the quotient topology ensures the continuity of h. Also, notice that the h is injective with inverse given by

$$\begin{aligned} h^{-1} : \Delta^{n+1} \to & C(\Delta^n) \\ (t_0, \dots, t_{n+1}) \mapsto & \left[t_0, \left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0} \right) \right]. \end{aligned}$$

Since the cone $C(\Delta^n)$ is compact and Hausdorff and h is continuous and bijective, from which follows n is an homeomorphism. □

Definition B.2.6 (Cone operator)**.** Given a singular simplex on A $\sigma \in \mathrm{Hom}_{\mathtt{Top}}(\Delta^n, A)$, let $C(\sigma) \in \mathrm{Hom}_{\mathtt{Top}}(C(\Delta^{n+1}), C(A))$ be the map given by

$$C(\sigma)(t_0, \dots, t_{n+1}) = \left[t_0, \sigma \left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0} \right) \right].$$

Given some singular chain $a = \sum_{n+m=k} g_m \sigma_m$ in $S_k(A; G)$, we define the *cone operator* $C^k : S_k(A, a; G) \to S_{k+1}(C(A), *; G)$ as

$$C^k(a) := \sum_{n+m=k} g_m C^n(\sigma_n).$$

Proposition B.2.7. *The homomorphism*

$$\begin{aligned} \varphi_\rho : S(\rho) \to & S(C_\rho, *) \\ (x, a) \mapsto & (i_X)_\# x + (i_{C(A)})_\# C(a) \end{aligned}$$

is a natural chain quasi-isomorphism.

The map which establishes the isomorphism can be better appreciated in Figure 11 The proof of this result can be found in (SHAHBAZI, 2004, Theorem 2.5.1, p.17).

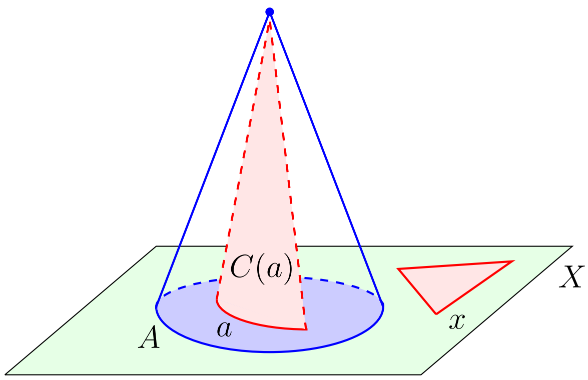

Figure 11 – Example of the map φ in the particular example of the mapping cyclinder of a inclusion.

Source: Figura 3.6.4 of (CLEMENTE, 2018) (with adaptation for notation)

B.2.3 Eilenberg-Zilber Morphism, Eilenberg-Maclane and Alexander Whitney Maps

Fix a ring R. Adapting definitions from Dieck (2008)

Definition B.2.8. A pair of Einleberg-Zilber maps is any pair of natural chain morphisms $P^{X,Y} : S(X;R) \otimes_R S(Y;R) \to S(X \times Y;R)$ and $Q^{X,Y} : S(X \times Y;G) \to S(X;R) \otimes_R S(Y;R)$ which are a homotopy equivalences between the chain complexes $S(X;R) \otimes S(Y;R)$ and $S(X \times Y;R)$.

Proof (Sketch). We apply the acyclic models theorem A.4.7 thoerem for $S(;\mathbb{Z})$.

Considering models (Δ^p, Δ^q) one can verify that both $S(\cdot) \otimes S(\cdot)$ and $S(\cdot \times \cdot)$ are free and acyclic on models.

Consider the morphisms $\phi^{X,Y} : (S(X) \otimes S(X))_0 \to S_0(X \times Y;G)$ given by $\phi^{X,Y}(x \otimes y) = (x,y)$ and $\psi^{X,Y} : S(X \times Y) \to S(X) \otimes S(Y)$ given by $\psi^{X,Y}(x,y) = x \otimes y$. The acyclic models asserts the existence of natural chain maps $P^{X,Y} : S(X) \otimes S(Y) \to S(X \times Y)$ and $Q^{X,Y} : S(X \times Y) \to S(X) \otimes S(Y)$, unique up to algebraic homotopy, with $P_0^{X,Y} = \phi^{X,Y}$ and $Q_0^{X,Y} = \psi^{X,Y}$.

Moreover, since $P_0 \circ Q_0$ (reciprocally $Q_0 \circ P_0$) is the identity, the acyclic models implies that $P \circ Q$ ($Q \circ P$) is homotopic to the identity which proves that P and Q are a pair of Eilenberg-Zilber maps.

To finish the proof, we "tensorize" with R. $\square$

One can show that a pair of Eilenberg-Zilber maps are associative and commutative up to homotopy. Nevertheless, there exists a special pair of Eilenberg-Zilber maps which are associative and commutative at chain level.

Th Alexander-Whitney A and Eilenberg-Maclane E (also called Eilenberg-Zilber or shuffle) maps are a pair of Eilenberg-Zilber maps

$$\begin{aligned} A^{X,Y} &: S_\bullet(X \times Y;R) \to (S(X) \otimes_R S(Y;R))_\bullet \\ E^{X,Y} &: (S(X;R) \otimes_R S(Y;R))_\bullet \to S_\bullet(X \times Y;R) \end{aligned}$$

such that $A \circ E = \mathrm{id}$ and $E \circ A$ is chain homotopic to id. Whereas a generic Eilenberg-Zilber morphism is only associative and commutative in homology, these maps are associative at chain level.

We can extend these maps to the relative setting. Observe that, by one side

$$\begin{aligned} S(\rho;R) \otimes_R S(Y;R) &= (S(X;R) \oplus S(A;R)) \otimes_R S(Y;R) \\ &= (S(X;R) \otimes_R S(Y;R)) \oplus (S(A;R) \otimes_R S(Y;R)) \end{aligned}$$

where the operations are understood in the graded sense. By other side, note that

$$S(\rho \times \mathrm{id}_Y; R) = S(X \times Y; R) \oplus S(A \times Y; R)$$

Considering the pair of maps

$$S(X \times Y; R) \oplus S(A \times Y; R) \underset{A^{\rho,y}}{\overset{E^{\rho,Y}}{\rightleftarrows}} S(X; R) \otimes_R S(Y; R) \oplus (S(A; R)) \otimes_R S(Y; R))$$

where $A_\bullet^{\rho,Y} := A_\bullet^{X,Y} \oplus A_{\bullet-1}^{A,Y}$ and $E_\bullet^{\rho,Y} := E_\bullet^{X,Y} \oplus E_{\bullet-1}^{A,Y}$. we obtain a chain homotopy between $S(\rho) \otimes S(Y)$ and $S(\rho \times \mathrm{id}_Y)$. Supposing R is a principal ideal domain, we apply the algebraic Kunneth formula (Proposition A.4.8) and get the following split short exact sequence

$$0 \longrightarrow H(\rho; R) \otimes H(Y; R) \xrightarrow{\times} H(\rho \times \mathrm{id}_Y; R) \longrightarrow \mathrm{Tor}^R(H(\rho; R), H(Y; R)) \longrightarrow 0 \tag{B.5}$$

where $\alpha \times \beta = E(\alpha \otimes \beta)$

B.2.4 Topological S^1 Integration

Fix a ring R. The topological S^1 integration in $H(; R)$ can be obtained from the homological *slant product* $/ : H^p(\mathrm{id}_Y \times \rho; R) \times H_p(Y) \to H^{p-1}(\rho; R)$. In fact, we define

$$\int_{S^1} : H^p(\mathrm{id}_{S^1} \times \rho; R) \to H^{p-1}(\rho; R)$$
$$\alpha \mapsto \alpha/\iota$$

where $\iota \in H^1(S^1; R) \cong R$ is a generator. The slant product α/β at chain level can be written as

$$/ : S_{p-q}(\rho; R) \xrightarrow{\mathrm{id}_{S_{p-q}}(\cdot)\otimes\beta} S_{p-q}(\rho; R) \otimes S_q(Y; R) \xrightarrow{EZ} S_p(\rho \times \mathrm{id}_Y; R) \xrightarrow{\alpha} R$$

in other words, $(\alpha/\beta)(c) = \alpha(c \times \beta))$. More generally, we can define a S^1 integration at chain level

$$\int_{S^1} : S^\bullet(\rho \times \mathrm{id}_{S^1}) \to S^{\bullet-1}(\rho)$$
$$c \mapsto c/d$$

where we choose $d : \Delta^1 \to S^1$ as $d(t_0, t_1) = e^{2\pi t_0}$ which is a generator of $H^1(S^1; R)$.

Finishing these observations about singular cohomology, it would be instructive to present the chain level product $S(\rho) \times S(\eta) \to S(\rho \wedge \eta)$, but we will not use it anywhere. In fact, we will just use both the version $S(\rho) \times S(Y) \to S(\rho \times \mathrm{id}_Y)$, which will be used in the next section, and the version $S(X, A) \times S(Y, B) \to S(X \times Y, A \times Y \cup X \times B)$ for excisive pairs $(X \times Y, A \times Y), (X \times Y, X \times B)$, used in D. Since these products are classical, we refer the reader to the usual references such as (HATCHER, 2002).

B.2.5 The Chern Character

The Chern-Dold character in ordinary cohomology with coefficients G is obtained by tensorization with $\mathbb{R}$.

$$\begin{aligned} ch : H(\rho, G) &\to H(\rho, G \otimes \mathbb{R}) \\ \alpha &\mapsto \alpha \otimes_{\mathbb{Z}} 1_{\mathbb{R}} \end{aligned}$$

B.3 Ordinary Differential Cohomology

In this section, we introduce ordinary differential cohomology with coefficients in $\mathbb{Z}$ through the Cheeger-Simmons differential characters. The main reference for this section is the book by Bär and Becker (2014).

Given a sngular chain $(\sigma, \tau) \in S_*(\rho)$ and a relative form $(\omega, \theta) \in \Omega^\bullet(\rho)$ we define[1]

$$\int_{(\sigma,\tau)} (\omega, \theta) := \int_\sigma \omega + \int_\tau \theta$$

Definition B.3.1 (Differential Characters)**.** A *relative differential character* of degree n over a smooth map $\rho : A \to X$ is a pair $(\chi, (\omega, \theta))$ where $\chi : Z^{\mathrm{sm}}_{n-1}(\rho) \to \frac{\mathbb{R}}{\mathbb{Z}}$ is a homomorphism and $(\omega, \theta) \in \Omega^n(\rho)$ is a relative form such that

$$\chi(\partial(\sigma, \tau)) = \int_{(\sigma,\tau)} (\omega, \theta) \mod \mathbb{Z}. \tag{B.6}$$

Remark B.3.2. Consider a differential character $(\chi, (\omega, \theta))$ of degree n:

- The relative form (ω, θ) has integer periods, since, for a cycle $z \in Z^{\mathrm{sm}}_n(\rho)$, one has

$$0 = \chi(\partial z) = \int_z (\omega, \theta) \mod \mathbb{Z}$$

  which implies $\int_z (\omega, \theta) \in \mathbb{Z}$. By Proposition A.3.6, the form (ω, θ) is also closed.

- According to Bär and Becker (2014, p. 116), the relative form (ω, θ) is uniquely determined by the character, that is, if $(\chi, (\omega, \theta))$ and $(\chi, (\omega', \theta'))$ are two differential characters, than $(\omega, \theta) = (\omega', \theta')$. Because of this we will generally write χ instead of $(\chi, (\omega, \theta))$.

We denote by $\widehat{H}(\rho)$ the abelian group of differential characters over $\rho : A \to X$ under pointwise addition. In fact, $\widehat{H} : \mathtt{Man}^{2,\mathrm{op}} \to \mathtt{GrAb}$ is a contravariant functor acting on morphisms $(f, g) : \rho \to \eta$ as

$$\begin{aligned} (f, g)^* := \widehat{H}(f, g) : \widehat{H}(\eta) &\to \widehat{H}(\rho) \\ (\chi, (\omega, \theta)) &\mapsto (\chi \circ (f, g)_\#, (f, g)^*(\omega, \theta)) \,. \end{aligned}$$

[1] See section A.3.3 for more information.

Given $(\chi, (\omega, \theta)) \in \widehat{H}(\rho)$, we define its curvature $R(\chi)$ by $R(\chi) := (\omega, \theta)$. The curvature $R : \widehat{H}(\rho) \to \Omega_{\mathrm{cl}}(\rho)$ is a group homomorphism.

Since the abelian group $Z^{\mathrm{sm}}_{n-1}(\rho)$ is free (it is the direct sum of two free groups) we can extend a morphism $\chi : Z^{\mathrm{sm}}_{n-1}(\rho) \to \frac{\mathbb{R}}{\mathbb{Z}}$ to a homomorphism $\widetilde{\chi} : Z_{n-1}(\rho) \to \mathbb{R}$. Lets define the homomorphism $I(\widetilde{\chi}) : S^{\mathrm{sm}}_n(\rho) \to \mathbb{R}$

$$I(\widetilde{\chi})(c) = \int_c R(\chi) - \widetilde{\chi}(\partial c)$$

one can verify that:

- $I(\widetilde{\chi})$ assumes values in $\mathbb{Z}$, which give us a cochain $I(\widetilde{\chi}) \in C(\rho)$
- This cochain is actually a cocyle, that is, $\delta I(\widetilde{\chi}) = I(\widetilde{\chi}) \circ \partial = 0$.
- The cohomology class $[I(\widetilde{\chi})] \in H(\rho)$ is the same for every lift $\widetilde{\chi}$ of χ. We denote this cohomology class by $I(\chi)$.
- One can verify that that $I : \widehat{H}(\rho) \to H(\rho)$ is group homomorphism.

Given some relative form $(\omega, \theta) \in \Omega^{\bullet-1}(\rho)$ we associate the differential character $(a(\omega, \theta), d(\omega, \theta))$ where

$$a(\omega, \theta)(z) := \int_z (\omega, \theta) \mod \mathbb{Z}$$

Observe that, if $(\omega', \theta') = (\omega, \theta') + d(\mu, \nu)$ then $a(\omega, \theta) = a(\omega', \theta')$. Thus we get a homomorphism $a : \frac{\Omega(\rho)}{\mathrm{Im}(d)} \to \widehat{H}(\rho)$

Proposition B.3.3. *The data $(\widehat{H}, R, I, a)$ is a differential refinement of (H, ∂).*

Proof. We start by verifying axiom A1 (3.8). Clearly,

$$R \circ a(\omega, \theta) = d(\omega, \theta).$$

For the commutative square, notice that, since $I(\chi) = [c \mapsto \int_c R(\chi) + \widetilde{\chi}(\partial c)]_{\mathbb{Z}}$. Observe that, since $\mathbb{R}$ is a divisible group[2], we can extend $\widetilde{\chi} : Z_{n-1}(\rho) \to \mathbb{R}$ to a $\overline{\chi} : S_{n-1}(\rho) \to \mathbb{R}$. From this follows that $ch \circ I(\chi) = [\int_c R(\chi)]_{\mathbb{R}}$, since $ch([\overline{\chi}(\partial c)]_{\mathbb{Z}}) = [\delta\overline{\chi}(c)]_{\mathbb{R}} = 0$ as a real chain. Thus we conclude

$$ch \circ I(\chi) = r \circ q_{\mathrm{dR}}(R(\chi)).$$

Now we verify that the following sequence is exact

$$H^{\bullet-1}(\rho) \xrightarrow{ch} \frac{\Omega^{\bullet-1}(\rho)}{\mathrm{Im}(d)} \xrightarrow{a} \widehat{H}^{\bullet}(\rho) \xrightarrow{I} H^{\bullet}(\rho) \longrightarrow 0$$

[2] Hence a injective group.

- (*exactness at $H^\bullet(\rho)$*). Given a class $[\alpha] \in H$ represented by a cochain level cocycle $\alpha : S(\rho) \to \mathbb{Z}$. The de Rham isomorphism tell us that there exists a closed differential form (ω, θ) such that
$$\int_c (\omega, \theta) - \alpha(c) = \partial\tilde{\alpha} \tag{B.7}$$
for some real cochain $\tilde{\alpha} : S^{\mathrm{sm}}_{\bullet-1}(\rho) \to \mathbb{R}$. Define $\chi = \tilde{\alpha}|_{Z_{n-1}}(\rho) \mod \mathbb{Z}$. It follows that χ is a differential character with $I(\chi) = \alpha$.

- (*exactness at $\widehat{H}^\bullet(\rho)$*) It is clear that $I \circ a = 0$, since $c \mapsto \int_c (\omega, \theta)$ is a real lift of $a(\omega, \eta)$. On the other hand, let $I(\chi) = 0$. In this case $\int_c R(\chi) - \overline{\chi}(\partial c) = \partial h$, where $\overline{\chi} : S_{\bullet-1}(\rho) \to \mathbb{R}$ is an extension of $\tilde{\chi}$ and h is some some integer-valued cochain. Notice that $\delta(\overline{\chi} + h)(c) = \int_c (\omega, \theta)$ which, implies by the Rham isomorphism, that (ω, θ) is exact. Let $(\mu, \nu) \in \Omega^{n-1}(\rho)$ such that $d(\mu, \nu) = (\omega, \theta)$. We have
$$\overline{\chi}(z) + h(z) = \int_z (\mu, \nu)$$
which implies $\chi(z) = \int_z (\mu, \nu) \mod \mathbb{Z}$.

- (*exactness at $\frac{\Omega(\rho)}{\mathrm{Im}(d)}$*) The forms $\Omega_{ch}(\rho)$ which lie in the image of ch are precisely the differential forms with integral periods. By definition of a, the exactness is clear.

Next, we verify Axiom A2 (3.9). Given $(\chi, (\omega, \theta)) \in \widehat{H}(\rho)$ its covariance is $cov(\chi) = \theta$, therefore
$$a(\theta)(c) = \int_c \theta \mod \mathbb{Z}$$
On the other hand, $(\mathrm{id}_X, \varnothing_A)^*(\chi) = \chi((z, 0))$ with $z \in Z_{\mathrm{sm}}(X)$. By pulling back,
$$\chi((\rho_\sharp s, 0)) = \chi(\partial(0, s)) = \int_s \theta \mod \mathbb{Z}.$$

$\square$

B.3.1 S^1-integration

We define the differential S^1 integration map by
$$\int_{S^1} \chi(z) = (-1)^z \chi(z \times \iota)$$
where $\iota \in Z_1^{\mathrm{sm}}(S^1)$ is a cycle representing the fundamental class S^1. This is indeed a Cheeger-Simmons character since
$$\begin{aligned}\int_{S^1} \chi(\partial c) &= (-1)^{|z|} \chi(\partial(\iota \times c)) \\ &= (-1)^{|z|} \int_{\iota \times c} R(\chi) \\ &= \int_c \int_{S^1} R(\chi)\end{aligned}$$

In particular, this proves the compatibility with R as well. The other conditions can be verified in similar ways. For example, for the trivialization a we have

$$\begin{aligned}\int_{S^1} a(\omega,\theta)(z) &= a(\omega,\theta)(z\times\iota)\\ &= \int_z\int_{S^1}(\omega,\theta)\\ &= a\left(\int_{S^1}(\omega,\theta)\right)(z).\end{aligned}$$

B.3.2 Multiplicative structures

Before presenting the absolute-relative product as described in Section 3.5.1, we will need some special results. We say that a cycle $t \in Z_\bullet(\rho)$ is a torsion cycle if it induces a torsion class in cohomology, that is, $nt = \partial c$ for some chain $c \in S_{\bullet+1}(\rho)$ and $n \in \mathbb{N}$.

Lemma B.3.4. *Given a differential character χ and a torsion cocyle $t \in Z_{\bullet-1}(\rho)$ such that $nt = \partial c$ for some $c \in S_\bullet(\rho)$ and $n \in \mathbb{Z}$, we have*

$$\chi(t) = \frac{1}{n}\left(\int_c R(\chi) - \widetilde{I}(\chi)(c)\right) \mod \mathbb{Z} \tag{B.8}$$

where $\widetilde{I}(\chi)$ is a cocyle representing $I(\chi)$.

Proof. Choose a lift $\widetilde{\chi} : S(\rho) \to \mathbb{R}$ lifting and extending χ and recall that

$$I(\widetilde{\chi})(c) = \int_c(\omega,\theta) - \widetilde{\chi}(\partial c) \tag{B.9}$$

Since z is a torsion cycle, we have $nz = \partial c$ for some n. As $\widetilde{\chi}$ is a lift

$$\chi(z) = \frac{1}{n}\widetilde{\chi}(\partial c) \mod \mathbb{Z}$$

Using (B.9) we get

$$\chi(z) = \frac{1}{n}\left(\int_c(\omega,\theta) - I(\widetilde{\chi})(c)\right) \mod \mathbb{Z}$$

In order to see that this expression does not depend on the lift $I(\widetilde{\chi})$ observe that, if $\widetilde{I}(\chi)$ and $\widetilde{I}(\chi)'$ are two cocyles representing $I(\chi)$, than there exists a cochain $\beta : S_{\bullet-1}(\rho) \to \mathbb{Z}$ such that

$$\widetilde{I}(\chi)'(c) - \widetilde{I}(\chi)(c) = \beta(\partial c)$$

Notice that

$$\frac{1}{n}\left(\widetilde{I}(\chi)'(c) - \widetilde{I}(\chi)(c)\right) = \frac{1}{n}\beta(\partial c) = \beta(z) \in \mathbb{Z}$$

which in turn implies

$$\frac{1}{n}\widetilde{I}(\chi)'(c) = \frac{1}{n}\widetilde{I}(\chi)(c) \mod \mathbb{Z}.$$

∎

Consider the following two short exact sequences sequences:

$$\begin{tikzcd} 0 \arrow[r] & Z_\bullet(\rho) \arrow[r, hook, "i_\rho"] & S_\bullet(\rho) \arrow[l, bend left, "s_\rho"] \arrow[r, two heads, "\partial"] & B_\bullet(\rho) \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & Z_\bullet(Y) \arrow[r, hook, "i_Y"] & S_\bullet(Y) \arrow[l, bend left, "s_Y"] \arrow[r, two heads, "\partial"] & B_{\bullet-1}(Y) \arrow[r] & 0 \end{tikzcd}$$

Both these sequence splits since $B_{\bullet-1}(\rho)$ and $B_{\bullet-1}(Y)$ are free as they are subgroups of free groups. We denote by $s_\rho : S_\bullet(\rho) \to Z_\bullet(\rho)$ and $s_X : S_\bullet(X) \to Z_\bullet(X)$ the split maps as depicted in the diagrams. Let $E : S(\rho) \otimes S(Y) \to S(\rho \times \mathrm{id}_Y)$ and $A : S(\rho \times \mathrm{id}_Y) \to S(\rho) \otimes S(X)$ be the Eilenberg-Zilber and the Alexander-Whitney chain morphisms respectively as defined in section B.2.3. We define, as usual, the map $\times : Z_p(\rho) \otimes Z_q(Y) \to S(\rho \times \mathrm{id}_Y)$ as the composition

$$\times : (Z_\bullet(\rho) \otimes Z_\bullet(Y))_{p+q} \xrightarrow{i_\rho \otimes i_Y} (S(\rho) \otimes S(Y))_{p+q} \xrightarrow{E} S_{p+q}(\rho \times \mathrm{id}_Y)$$

These facts are expressed in the following commutative diagram

$$\begin{tikzcd} (Z(\rho) \otimes Z(Y))_{p+q} \arrow[r, shift left, "i_\rho \otimes i_Y"] \arrow[drr, shift left, "\times"] & (S(\rho) \otimes S(Y))_{p+q} \arrow[l, shift left, "s_\rho \otimes s_Y"] \arrow[d, shift left, "E"] \\ & & S_{p+q}(\rho \times \mathrm{id}_Y) \arrow[ull, shift left, "S"] \arrow[u, shift left, "A"] \end{tikzcd}$$

where $S(z) = (s_\rho \otimes s_X) \circ A(z)$. Observe also that $S \circ \times = \mathrm{id}$.

Lemma B.3.5. *Any cycle $z \in Z^n(\rho \times \mathrm{id}_Y)$ can be written as*

$$z = \sum_{p+q=n} x_p \times y'_q + t$$

where with $x_p \in Z_p(\rho)$ and $y_q \in Z_q(Y)$ and $t \in Z_n(\rho \times \mathrm{id}_Y)$ is a torsion cycle.

Proof. Recall the relative Kunneth sequence in homology (B.5):

$$\begin{tikzcd} 0 \arrow[r] & (H(\rho) \otimes H(Y))_{p+q} \arrow[r, "\times"] & H_{p+q}(\rho \times \mathrm{id}_Y) \arrow[r] & Tor(H(\rho), H(Y))_{p+q} \arrow[l, bend left, "\phi"] \arrow[r] & 0. \end{tikzcd}$$

The map $\times$ is just the homology version of the chain map $\times$ defined at chain level in Section B.2.3. Since this sequence is splits (not naturally though) we have for $z \in Z_n(\rho \times \mathrm{id}_Y)$

$$z = \sum_{q+p=n} x_p \times y_q + t$$

where $t = z - \sum_{q+p=n} x_p \times y_q$. Since $\sum_{q+p=n} x_p \times y_q$ is in the image of $\times$, we have t in the image of ϕ. Since $Tor(H(\rho), H(Y))_{p+q}$ is a torsion group the cycle t is a torsion cycle. □

Definition B.3.6 (Absolute Relative Product)**.** Let $z \in H(\rho \times \mathrm{id}_Y)$ and write it as $z = x \times y + t$ as in the previous lemma. We define the product $\times : \widehat{H}^p(\rho) \times \widehat{H}^q(Y) \to \widehat{H}^{p+q}(\rho \times \mathrm{id}_Y)$ as

$$(\chi \times \chi')(z) = (\chi \times \chi')(x \times x') + (\chi \times \chi')(t)$$

where, for products,

$$(\chi \times \chi')(x \times y) = \chi(x) \cdot \int_y R(\chi')) + \int_x (R(\chi)) \cdot \chi'(y) \mod \mathbb{Z}$$

with the conventions

- $\chi((\sigma, \tau)) = 0$ if $(\sigma, \tau) \notin Z_{p-1}(\rho)$ and $\chi(\sigma') = 0$ if $\sigma' \notin Z_{q-1}(Y)$, and
- $\int_{(\sigma,\tau)}(R(\chi)) = 0$ if $(\sigma, \tau) \notin Z_p(\rho)$ and $\int_{\sigma'}(R(\chi)) = 0$ if $\sigma' \notin Z_q(Y)$,

and, for cycles,

$$(\chi \times \chi')(t) = \frac{1}{n} \int_c R(\chi) \times R(\chi') + \left(\widetilde{I(\chi) \times I}(\chi') \right)(c)$$

where $nt = \partial c$.

Lets show that $\chi \times \chi'$ is indeed a differential character.

Proposition B.3.7. *The map $(\chi \times \chi') : Z_{n-1}(\rho \times \mathrm{id}_Y) \to \frac{\mathbb{R}}{\mathbb{Z}}$ defined above is a differential character*

Proof. If $z = \partial c$ with $c \in S_n(\rho \times \mathrm{id}_Y)$ one has

$$\begin{aligned}
S(\partial c) &= (s_\rho \otimes s_Y)(E(\partial c)) \\
&= (s_\rho \otimes s_Y)(\partial E(c)) \\
&= (s_\rho \otimes s_Y)(\sum_{p+q=n} \partial a_p \otimes b_q + (-1)^q a_p \otimes \partial b_q) \\
&= \sum_{p+q=n} s_\rho(\partial a_p) \otimes s_Y(b_q) + (-1)^q s_\rho(a_q) \otimes s_Y(\partial b_p) \\
&= \sum_{p+q=n} \partial a_q \otimes s_Y(b_q) + s_\rho(a_q) \otimes \partial(b_q)
\end{aligned}$$

where we used $s_\rho(\partial a_p) = \partial a_p$ and analogously to $s_Y(\partial b_q) = \partial b_q$. In other words, we can write

$$\partial c = \sum_{p+q=n} \partial a_p \times x_q + (-1)^q y_p \times \partial b_q + t$$

where x and y are cycles.

$$
\begin{aligned}
(\chi\times\chi')(\partial c) &= \sum_{p+q=n} \chi(\partial a_p)\cdot \int_{y_p} (R(\chi')) + (-1)^q \int_{x_q} (R(\chi)\cdot \chi'(\partial b_q)) \\
&= \sum_{p+q=n} \int_{a_p} R(\chi)\cdot \int_{y_q} R(\chi') + (-1)^q \int_{x_p} R(\chi)\cdot \int_{b_q} R(\chi') \\
&\overset{A.3.8}{=} \sum_{p+q=n} \int_{a_p\times b_{p-1}} R(\chi)\times R(\chi') + (-1)^q \int_{a_{p-1}\times b_p} R(\chi)\times R(\chi') \\
&= \int_{\sum_{p+q=n} a_p\times b_{p-1}+a_{p-1}\times b_p} R(\chi)\times R(\chi') \\
&= \int_{\partial c} R(\chi)\times R(\chi')
\end{aligned}
$$

The torsion cycle is just a boundary $t=\partial d$ for some $d\in S_n(\rho\times \mathrm{id}_Y)$. Using (B.8) with $n=1$

$$
(\chi\times\chi)(\partial d) = \int_d R(\chi)\times R(\chi') + (I(\chi)\,\widetilde{\times}\, I(\chi'))(d) = \int_d R(\chi)\times R(\chi') \mod \mathbb{Z}
$$

since $I(\xi)\,\widetilde{\times}\, I(\xi)$ is integer-valued. □

Proposition B.3.8. *The product defined above is a relative absolute product in the sense of Definition 3.5.1*

The proof of the compatibility with R has already been done in the previous proposition. The proof of the compatibility with a is easy, in fact, the product was defined with it in view. Compatibility with I is the main challenge concerning the compatibilities. The proof of the associativity and naturality are harder and uses explicitly the associativity at chain level of Einlenberg-Maclane morphisms. The reader is referred to (BäR; BECKER, 2014, Theorem 26, p.147).

B.3.3 Parallel Differential Characters

Definition B.3.9 (Parallel Characters)**.** A parallel differential character over a smooth pair (X,A) is a pair (χ,ω) where $\chi: Z^{\mathrm{sm}}_{\mathrm{par},n-1}(X,A)\to \frac{\mathbb{R}}{\mathbb{Z}}$ and a parallel differential form $\omega\in \Omega^n_{\mathrm{par}}(X,A)$ such that

$$
\chi(\partial c) = \int_c \omega.
$$

In this definition we regard $S^{\mathrm{sm}}_{\mathrm{par}}(X,A)$ as the quotient $\frac{S^{\mathrm{sm}}(X)}{i_\# S^{\mathrm{sm}}(A)}$.

We denote the set of parallel differential characters by $\widehat{H}'_{\mathrm{par}}(\rho)$. As in the previous section, we define the R_{par}, I_{par} and a_{par}.

Proposition B.3.10. *The data $(\widehat{H}'_{par}, R_{par}, I_{par}, a_{par})$ give us parallel differential cohomology as in Definition 3.3.11.*

The proof of this result can be carried in a completely analogous as in Proposition B.3.3.

Proposition B.3.11. *Let (X, A) be a manifold pair and denote by $i_A : A \hookrightarrow X$ the inclusion. The following map*

$$\begin{aligned} i : \widehat{H}'_{par}(X, A) &\to \widehat{H}_{par}(X, A) \\ (\chi, \omega) &\mapsto ((\chi \circ i), (\omega', 0)) \,, \end{aligned}$$

is an isomorphism. Here, $i : S^{sm}(X, A) \to S^{sm}_{par}(X, A)$ is defined by

$$i(\sigma, \tau) = \sigma \mod \operatorname{Im}((i_A)_\#)$$

The proof can be found in (BäR; BECKER, 2014, Theorem 17, p. 132).

APPENDIX C – K-Theory

C.1 Introduction

The goal of this appendix is to introduce the relative differential model of K-theory with the same aim as the previous appendix. In order to do so we need a relative version of K-theory on maps. Because of this, we start developing a K-theory on maps and after this give an account of differential K-theory. Here, we will just scratch the main points omitting the majority of the proofs, The interested reader may consult the thesis of (NUñEZ, 2021, Chapter 4), in which this appendix is based, for a full account.

C.2 Relative K-Theory

In this section we briefly review the topological complex K-theory (KU-theory) on maps. The construction which we are going to carry is similar to the L functor of Atiyah and Anderson (2018, p.87a) (see also (KAROUBI, 1978, 2.13, p.61), (HUSEMöLLER, 1994, Chapter 10, Section 4, p.129)).

In this appendix, **all vector bundles are assumed to be complex with compact base**.

C.2.1 Relative bundles

Definition C.2.1. Let $\rho : A \to X$ be a continuous map. We say that a triple $\mathbf{E} = (E_1, E_2, \psi)$ is a *vector bundle triple* over ρ when

- $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are vector bundles, and
- $\psi : \rho^* E_1 \to \rho^* E_2$ is a vector bundle isomorphism over A.

An important example of a vector bundle triple, henceforth only called triple, is the *elementary triple* which is any vector bundle triple of the form (G, G, γ) with γ homotopic to the identity id_G.

Definition C.2.2. A *morphisms between vector bundles triples* (E_1, E_2, ψ) and (E_1', E_2', ψ') is a pair of vector bundle morphisms $f : E_1 \to E_1'$ and $g : E_2 \to E_2'$ over X such that the following diagram commutes:

$$\begin{tikzcd} \rho^* E_1 \arrow[r, "\psi"] \arrow[d, "\rho^* f"] & \rho^* E_2 \arrow[d, "\rho^* g"] \\ \rho^* E_1' \arrow[r, "\varphi"] & \rho^* E_2', \end{tikzcd}$$

A morphism $(f, g) : (E_1, E_2, \psi) \to (E_1', E_2', \psi')$ is an *isomorphism* if both f and g are isomorphisms of vector bundles.

We denote by $\mathrm{Vec}(\rho)$ the set of equivalence classes of triples over ρ under the relation given by isomorphisms and denote a generic element of it by $\langle E, E', \psi\rangle$.

Definition C.2.3 (Pullback Triple)**.** Let $(f, g) : \eta \to \rho$ be a morphism between two continuous maps and let (E_1, E_2, ψ) be a triple over $\rho : A \to X$. The *pullback triple* $(f, g)^*(E_1, E_2, \psi)$ is the vector bundle over triple $\eta : B \to Y$ defined by

$$(f, g)^*(E_1, E_2, \psi) := (f^*E_1, f^*E_2, g^*\psi)$$

The pullback of a triple is compatible with isomorphisms. This tell us that $\mathrm{Vec} : \mathtt{Top}^{2,\mathrm{op}} \to \mathtt{Set}$ defines a contravariant functor from $\mathtt{Top}^2$ to the category $\mathtt{Set}$ of sets and functions.

We further endow $\mathrm{Vec}(\rho)$ with the structure of a commutative semigroup[1] by defining

$$\langle E_1, E_2, \psi\rangle + \langle F_1, F_2, \phi\rangle = \langle E_1 \oplus F_1, E_2 \oplus F_2, \psi \oplus \phi\rangle,$$

which can be verified to be well defined since it is compatible with isomorphisms.

C.2.2 Relative K-theory

We say that two classes of triples of vector bundles $\langle E_1, E_2, \psi\rangle$ and $\langle F_1, F_2, \phi\rangle$ are *equivalent under stabilization* if there exist two elementary triples (G, G, γ) and (H, H, κ) such that

$$\langle E_1, E_2, \psi\rangle + \langle G, G, \gamma\rangle = \langle F_1, F_2, \psi'\rangle + \langle H, H, \kappa\rangle.$$

This relation is compatible with pullbacks.

Definition C.2.4 (Relative K Groups)**.** We define the relative K group over ρ, $K(\rho)$ as the quotient of $\mathrm{Vec}(\rho)$ under the stabilization relation.

We denote the class of $\langle E_1, E_2, \psi\rangle$ in $K(\rho)$ by $[E_1, E_2, \psi]$. One can verify that $K(\rho)$ is indeed an abelian group with the class of elementary triples as the identity and the opposite element $-[E_1, E_2, \psi]$ of $[E_1, E_2, \psi]$ as $[E_2, E_1, \psi^{-1}]$. One can also show that the stabilization equivalence is compatible with pullbacks. Moreover, given $(f, g) : \rho \to \eta$,

$$(f, g)^* : K(\eta) \to K(\rho)$$

is a homomorphism of groups. Summing up, the K groups define a contravariant functor $K : \mathtt{Top}^2 \to \mathtt{GrAb}$.

[1] It is in fact a monoid, since $\langle 0, 0, \mathrm{id}_0\rangle$ is a identity element, where 0 represents the 0 dimensional bundle over X, but we will not use this fact.

Now, we will define the lower order K groups. We denote by T^n be the n-dimensional torus, that is, $T^n := \underbrace{S^1 \times \cdots \times S^1}_{n}$ and consider the embeddings

$$(i_j \times \mathrm{id}_X, i_j \times \mathrm{id}_A) : \mathrm{id}_{T^{n-1}} \times \rho \hookrightarrow \mathrm{id}_{T^n} \times \rho$$

where $i_j : S^{n-1} \to S^n$ is the map

$$i_j(s_1, \ldots, s_n) = (s_1, \ldots, s_{j-1}, 1, s_j, \ldots, s_n).$$

We write $I_j := (i_j \times \mathrm{id}_X, i_j \times \mathrm{id}_A)$ for short.

Definition C.2.5. The relative K-theory groups in negative degrees are given by

$$K^{-n}(\rho) := \bigcap_{j \in 1^n} \mathrm{Ker}(I_j^*)$$

where $n \in \mathbb{N}$.

We would like to exhibit a connecting morphism $\partial : K^{-n}(A) \to K^{-n+1}(\rho)$ and prove that this is indeed a cohomology theory. Rather than do this will show that

$$K^{-n}(\rho) \simeq K^{-n}(M_\rho, A)$$

in a natural way, where the K to right is just the usual relative K group of topological K-theory as presented (ATIYAH; ANDERSON, 2018),(HUSEMöLLER, 1994) or (KAROUBI, 1978). We take for granted that this is a cohomology theory.

The proof of following proposition can be found in detail in (NUñEZ, 2021), Nevertheless, we sketch a proof.

Proposition C.2.6. *Let $\rho : A \to X$ be a continuous map between compact topological spaces and $\iota_1 : j_{A\times I}(A \times \{1\}) \hookrightarrow M_\rho$ the inclusion of the top of the mapping cylinder M_ρ of ρ. There exists a natural isomorphism $c : K(\rho) \to K(\iota_1)$.*

Proof. (sketch). The collapse map $c : M_\rho \to X$ defined in (A.1) makes the following diagram commutative:

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, "i_0"] & X \arrow[d, "j_X"] \arrow[ddr, bend left, "\mathrm{id}_X"] & \\
A \times I \arrow[r, "j_{A\times I}"] \arrow[drr, bend right, "\rho\circ\mathrm{pr}_A"'] & M_\rho \arrow[dr, "c"] & \\
& & X
\end{tikzcd}$$

Consider the tautological isomorphisms $\theta : j_{A\times I}(A \times \{1\}) \xrightarrow{\sim} A \times \{1\} \xrightarrow{\sim} A$ and $\kappa : A \xrightarrow{\sim} A \times \{0\} \xrightarrow{\sim} A \times \{1\} \xrightarrow{\sim} j_{A\times I}(A \times \{1\})$. The map

$$\begin{aligned}
\Psi : K(\iota_1) &\to K(\rho)\\
[E_1, E_2, \psi] &\mapsto [j_X^* E_1, j_X^* E_2, \theta^* \psi] .
\end{aligned}$$

is a well-defined isomorphism with inverse given by

$$\begin{aligned}\Phi : K(\rho) &\to K(\iota_1)\\ [F_1, F_2, \phi] &\mapsto [c^*F_1, c^*F_2, \kappa^*\phi]\end{aligned}$$

. □

C.2.3 Multiplicative Structure

We will only consider the relative-absolute topological product in this text

$$\times : K^{-p}(\rho) \times K^{-q}(Y) \to K^{-p-q}(\rho \times \mathrm{id}_Y)$$

since we will not use the complete product. Consider the homeomorphism

$$h^{A,Y}_{pq} : T^{p+q} \times A \times Y \to T^p \times A \times T^q \times Y \tag{C.1}$$

$$(s, s', a, y) \mapsto (s, a, s', y) \tag{C.2}$$

where $(s, s') \in T^{p+q}$ with $s \in T^p$ and $s' \in T^q$. We define the morphism $\phi_{pq} : \mathrm{id}_{T^{p+q}} \times\rho \times \mathrm{id}_Y \to \mathrm{id}_{T^p} \times\rho \times \mathrm{id}_{T^q} \times \mathrm{id}_Y$ as the map $\varphi_{pq} := (h^{X,Y}_{pq}, h^{A,Y}_{pq})$ as depict in the following diagram

$$\begin{tikzcd}
T^{p+q} \times A \times T^q \times Y \arrow[r, "h^{A,Y}_{pq}"] \arrow[d, "\mathrm{id}_{T^{p+q}} \times\rho\times\mathrm{id}_Y"'] & T^p \times A \times T^p \times Y \arrow[d, "\mathrm{id}_{T^p} \times\rho\times\mathrm{id}_{T^q} \times \mathrm{id}_Y"] \\
T^{p+q} \times A \times T^q \times Y \arrow[r, "h^{X,Y}_{pq}"] & T^p \times X \times T^p \times Y
\end{tikzcd} \tag{C.3}$$

With this notation, we define the product between $\mathbf{E} = [E_1, E_2, \psi] \in K^{-p}(\rho)$ and $\mathbf{F} = [F_1, F_2, \varnothing] \in K^{-q}(Y)$ as

$$\mathbf{E} \times \mathbf{F} = \varphi^*_{p,q} \left([E_1 \boxtimes F_1, E_2 \boxtimes F_1, \phi \boxtimes \mathrm{id}_{F_1}] - [E_1 \boxtimes F_2, E_2 \boxtimes F_2, \phi \boxtimes \mathrm{id}_{F_2}]\right)$$

where $E \boxtimes F$ is the external tensor product of bundles $p : E \to X$ and $q : F \to Y$, which can be defined as the bundle $\mathrm{pr}^*_X E \otimes \mathrm{pr}^*_Y F$ where $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ are the projections.

C.2.4 Bott-Periodity and the Extension for positive degrees

Let us consider a generator $\eta - 1 \in \widetilde{K}(T^2) \cong \mathbb{Z}$. The Bott-periodicity theorem in K-theory states that for any compact space X, the following map

$$\begin{aligned}B : \widetilde{K}^{-n-2}(X) &\to \widetilde{K}^{-n}(X)\\ \alpha &\mapsto \alpha \times [\eta - 1]\end{aligned}$$

is isomorphism. Applying this result to C_ρ and noticing that $K(\rho) \cong K(M_\rho, A) \cong \widetilde{K}(C_\rho)$ we get the following version of the Bott periodicity

Theorem C.2.7 (Bott-Periodicty Theorem for maps)**.** *For any compact space $\rho : A \to X$, the following map*

$$B : K^{-n-2}(\rho) \to K^{-n}(\rho)$$
$$\alpha \mapsto \alpha \times [\eta - 1]$$

is isomorphism.

With aid of this result, we extend K theory for positive degrees, thus finishing its description.

As a final remark, observe that the coefficient group of topological K-theory is $\mathfrak{K} := \mathbb{Z}[t, t^{-1}]$ as graded ring, where t has degree -2. This implies that

$$\Omega^n \mathfrak{K}_{\mathbb{R}} = \begin{cases} \Omega^{\text{ev}} := \bigoplus_p \Omega^{2p}, & \text{for } n \text{ even,} \\ \Omega^{\text{odd}} := \bigoplus_p \Omega^p, & \text{for } n \text{ odd} \end{cases}$$

C.2.5 Chern Character

In order to present the Chern-Dold character and the relative differential model, we need to review the relative Chern-Weil theory. In this section we assume that all bundles are smooth.

Recall that the *Chern form* associated to a connection ∇ on a smooth vector bundle $p : E \to X$ is the even form

$$ch(\nabla) = \exp\left(\frac{i}{2\pi} tr(R^{\nabla})\right)$$

where R^{∇} denotes the curvature form associated to ∇, tr its trace, and $\exp : \Omega^2(X) \to \Omega^{\text{ev}}(X)$ is the series

$$\exp(\omega) = \sum_{i=0}^{n} \frac{1}{n} \underbrace{\omega \wedge \cdots \wedge \omega}_{n}.$$

The Chern form has the following properties

(i) $ch(\nabla) \in \Omega^{\text{ev}}_{cl}(X)$;

(ii) $ch(\nabla \oplus \nabla') = ch(\nabla) + ch(\nabla')$;

(iii) $ch(\nabla \otimes \nabla) = ch(\nabla) \wedge ch(\nabla')$, where $\nabla \oplus \nabla = \nabla \otimes \text{id} + \text{id} \otimes \nabla$;

(iv) $ch(f^*\nabla) = f^* ch(\nabla)$ for any smooth map $f : Y \to X$.

Its cohomology class does not depend on the connection. Indeed, given two connections ∇_0 and ∇_1 over $p : E \to X$, we can define its standard interpolation

$$\widetilde{\nabla}_{0,1} := (1-t)\nabla_0 + t\nabla_1 + \partial_s \tag{C.4}$$

which is a connection on $p : \mathrm{pr}_X^* E \to I \times X$ such that $i_0^* \widetilde{\nabla}_{0,1} = \nabla_0$ and $i_1^* \widetilde{\nabla}_{0,1} = \nabla_1$. We define the *Chern-Simons form*

$$cs(\nabla_0, \nabla_1) = \int_{\mathrm{pr}_I} ch(\widetilde{\nabla}_{0,1})$$

which satisfies the following property

$$ch(\nabla_1) - ch(\nabla_0) = d \int_I cs(\nabla_0, \nabla_1).$$

Remark C.2.8. As a matter of fact, there is nothing special with the connection $\widetilde{\nabla}_{0,1}$. We could as well replaced it by any connection $\widetilde{\nabla}$ over $\mathrm{pr}_X^* E \to I \times X$ such that

$$i_0^* \widetilde{\nabla} = \nabla_0 \text{ and } i_1^* \widetilde{\nabla} = \nabla_1 \tag{C.5}$$

obtaining the same result.

Associated to the Chern-Simons form we have the *Chern-Simons class* $CS(\nabla, \nabla')$, which is just

$$cs(\nabla, \nabla') \mod \mathrm{Im}_d .$$

The Chern-Simons form and Chern-Simons class have the following properties

1. $\rho^* cs(\nabla_0, \nabla_1) = cs(\rho^* \nabla_0, \rho^* \nabla_1)$
2. $CS(\nabla_0, \nabla_1) = -CS(\nabla_1, \nabla_0)$
3. $CS(\nabla_0, \nabla_1) + CS(\nabla_1, \nabla_2) = CS(\nabla_0, \nabla_2)$
4. $CS(\nabla_0 \oplus \nabla_0', \nabla_0 \oplus \nabla_1) = CS(\nabla_0, \nabla_0') + CS(\nabla_1, \nabla_1')$

As we noted (Remark C.2.8), we could choose any connection satisfying the conditions (C.5). In this section, this datum will be of relevance:

Definition C.2.9 (Interpolation Connection)**.** Given two connections ∇ and ∇' over a vector bundle $E \to X$, we say that a connection $\widetilde{\nabla}$ on $\mathrm{pr}_X E \to I \times X$ is *interpolation connection*, or path connection, if it satisfies the

The connection $\nabla_{0,1}$ defined in (C.4) will perform a special role ahead. We call it the *standard interpolation connection*

Definition C.2.10 (Relative Connection)**.** A *relative connection* $(\nabla_1, \nabla_2, \widetilde{\nabla})$ over a relative vector bundle $\mathbf{E} = (E_1, E_2, \psi)$ on ρ is given by the following data

- A connection ∇_1 on E_1 and a connection ∇_2 in E_2;
- A *interpolating connection* $\widetilde{\nabla}$ between $\rho^* \nabla_1$ and $\rho^* \nabla_2 \circ \psi$.

As with regular connections, we can define

- the pullback, $(f,g)^*(\nabla,\nabla',\widetilde{\nabla}) := (f^*\nabla, f^*\nabla, (\mathrm{id}_I \times g)^*\widetilde{\nabla})$
- the direct sum,

$$(\nabla_1,\nabla_2,\widetilde{\nabla}) \oplus (\nabla_1',\nabla_2',\widetilde{\nabla}') := (\nabla_1 \oplus \nabla_1', \nabla_2 \oplus \nabla_2', \widetilde{\nabla} \oplus \widetilde{\nabla}')$$

- and the tensor product of between a relative connection and an absolute connection

$$(\nabla_1,\nabla_2,\widetilde{\nabla}) \otimes \nabla' := (\nabla_1 \otimes \nabla', \nabla_2 \otimes \nabla', \widetilde{\nabla} \otimes \nabla')$$

We will often write $\nabla^{\mathbf{E}} = (\nabla_1,\nabla_2,\widetilde{\nabla})$ for a relative connection on the relative bundle $\mathbf{E} = (E_1, E_2, \psi)$. We remark that any relative bundle can be endowed with a relative connection. This is true since any bundle can be endowed with a connection and we can always choose the standard interpolation connection.

We can extend the Chern form to the relative setting, thus obtaining the relative Chern form.

Definition C.2.11 (Relative Chern)**.** Given some relative connection $\nabla^{\mathbf{E}} = (\nabla_1,\nabla_2,\widetilde{\nabla})$ on a relative bundle on $\mathbf{E} = (E, F, \psi)$ over map ρ, we define its *relative Chern form*, $ch(\nabla^E, \nabla^F, \nabla^E) \in \Omega^{\mathrm{ev}}_{\mathrm{cl}}(\rho)$, by

$$ch(\nabla_1,\nabla_2,\widetilde{\nabla}) := \left(ch(\nabla_1) - ch(\nabla_2), \int_I ch(\widetilde{\nabla})\right)$$

This relative form shares the same characteristics as the usual Chern form, namely

(i) $ch(\nabla^{\mathbf{E}}) \in \Omega^{\mathrm{ev}}_{cl}(\rho)$;

(ii) $ch(\nabla^{\mathbf{E}} \oplus \nabla^{\mathbf{F}}) = ch(\nabla^{\mathbf{E}}) + ch(\nabla^{\mathbf{F}})$;

(iii) $ch(\nabla^{\mathbf{E}} \otimes \nabla^{F}) = ch(\nabla^{\mathbf{E}}) \times ch(\nabla^{F})$, where ∇^F is connection on a bundle $p : E \to Y$;

(iv) $ch((f,g)^*\nabla) = (f,g)^* ch(\nabla)$, for any smooth morphism of maps $(f,g) : \eta \to \rho$.

Let see for example (i):

$$\begin{aligned} d\left(ch(\nabla^E) - ch(\nabla^F), \int_I ch(\widetilde{\nabla})\right) &= \left(d(ch(\nabla^E) - ch(\nabla^F)), \rho^*(ch(\nabla^E) - ch(\nabla^F)) - d\int_I ch(\widetilde{\nabla})\right) \\ &= \left(0, \rho^*(ch(\nabla^E) - ch(\nabla^F)) - i_1^* ch(\widetilde{\nabla}) - i_0^* ch(\widetilde{\nabla})\right) \\ &= (0,0) \end{aligned}$$

where we used (2.3.7), and properties of the absolute Chern form.

Similar to the absolute Chern-form, the de Rham cohomology class associated to $ch(\nabla^{\mathbf{E}})$ does not depend on the relative connection. The proof is analogous to the absolute case. We introduce the *relative Chern-Simons form* between two relative connection ∇ and ∇'

$$cs(\nabla, \nabla') = \int_{I,I} ch(\widetilde{\nabla}^1_{0,1}, \widetilde{\nabla}_{0,1}, \widetilde{\widetilde{\nabla}}_{0,1}))$$

where the $(\widetilde{\nabla}^1_{0,1}, \widetilde{\nabla}_{0,1}, \widetilde{\widetilde{\nabla}}_{0,1})$ is a relative analogue of the standard connection: $\widetilde{\nabla}^i_{0,1}$, $i \in \{1, 2\}$ is the standard interpolation between ∇_i and ∇'_i and $\widetilde{\widetilde{\nabla}}$ is the standard interpolation between $\widetilde{\nabla}$ and $\widetilde{\nabla}'$. In a similar manner we define the *Chern-Simons relative class* $[CS]$ as CS modulo exact forms, *i.e.*, $cs \mod \operatorname{Im}(d)$.

Since every relative smooth bundle admits a connection and the de Rham class of its associated Chern form is independent of the connection, it makes sense to define the $ch(\mathbf{E}) = r \circ [ch(\nabla^{\mathbf{E}})] \in H^{\mathrm{ev}}(\rho)$ for a relative bundle $\mathbf{E}$ over ρ. The properties of the Chern-form tell us that this de Rham relative class is compatible with the isomorphism and stability, which lead to define a morphism $ch : K \to H^{\mathrm{ev}}$. We define the (real) chern character $ch : K \to H\mathfrak{K}$ as

$$ch^{-n}(\mathbf{E}) = r \circ \left[\int_{T^n} ch(\mathbf{E})\right]$$

which is the Chern-Dold character of K-theory for smooth bundle[2].

C.3 Differential K-Theory

We present a relative differential refinement of complex topological K-theory based on the Freed-Lott-Klonoff model (FREED; LOTT, 2010; KLONOFF, 2008).

C.3.1 Differential Vector Bundles

Definition C.3.1 (Differential Vector Bundle)**.** A *differential vector bundle* $\widehat{\mathbf{E}} := (\mathbf{E}, \nabla^{\mathbf{E}}, (\omega, \theta))$ over a smooth function $\rho : A \to X$ is triple where

- $\mathbf{E} = (E_1, E_2, \psi)$ is a relative complex vector smooth bundle over ρ;
- $\nabla^{\mathbf{E}} = (\nabla_1, \nabla_2, \widetilde{\nabla})$ is a relative connection;
- $(\omega, \eta) \in \Omega^{odd}(\rho)$

We say that two differential vector bundles over ρ, $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{F}}$, are equivalent, if there exists a diffeomorphism of complex vector smooth bundle $(f, g) : \mathbf{E} \to \mathbf{F}$ such that

$$(\omega, \eta) - (\omega', \eta') \in CS(\nabla^{\mathbf{E}}, \nabla^{\mathbf{F}} \circ (f, g))$$

where $\nabla^{\mathbf{F}} \circ (f, g) = (\nabla_1 \circ f, \nabla_2 \circ f, \widetilde{\nabla} \circ \mathrm{pr}_A^*(g))$. We denote the set of equivalence classes of differential vector bundle by $\mathrm{DiffVec}(\rho)$. We can endow these vector bundle with a

[2] Clearly, we can obtain the general **rational** Chern character using Chern-classes and the splitting principle without resorting to smooth structures. But we will have no use for it here.

additive structure which turns it into a monoid. Given two differential vector bundles $\widehat{\mathbf{E}} = (\mathbf{E}, \nabla, (\omega, \theta))$ and $\widehat{\mathbf{E}}' = (\mathbf{F}, \nabla', (\omega', \theta'))$ we put

$$\widehat{\mathbf{E}} + \widehat{\mathbf{F}} = (\mathbf{E} + \mathbf{F}, \nabla^{\mathbf{E}} + \nabla^{\mathbf{F}}, (\omega, \theta) + (\omega', \theta'))$$

One can verify that this sum is compatible with the equivalence above. We call a differential vector bundle of the form $((E, E, id), (\nabla, \nabla, \widetilde{\nabla}_{0,1})), (0, 0)$ an *elementary differential bundle*. Observe that the sum of elementary bundle is elementary.

C.3.2 Differential K-Groups

We define the following stabilization relation over $VecDiff(\rho)$: give two relative vector bundle $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{F}}$ we say that they are equivalent under satbilization, if there exists two elementary bundles $\widehat{\mathbf{G}}_1$ and $\widehat{\mathbf{G}}_2$ such that

$$\widehat{\mathbf{E}} + \widehat{\mathbf{G}}_1 \cong \widehat{\mathbf{F}} + \widehat{\mathbf{G}}_2$$

This relation is compatible with the sum as well.

Definition C.3.2 (Relative $\widehat{K}^0$-Groups)**.** We define $(\widehat{K}^0(\rho), +)$ as the monoid $(VecDiff(\rho), +)$ under the stabilization relation.

We denote by $[\mathbf{E}, \nabla^{\mathbf{E}}, (\omega, \theta)]$ the class associated to the relative differential bundle $(\mathbf{E}, \nabla^{\mathbf{E}}, (\omega, \theta))$. It turns out that this is an abelian group. The inverse element being

$$-[\mathbf{E}, \nabla_1, \nabla_2, \widetilde{\nabla}, (\omega, \eta)] = [-\mathbf{E}, \nabla_2, \nabla_1, \overline{\widetilde{\nabla}}, -(\omega, \eta)]$$

where $\overline{\widetilde{\nabla}}$ denotes the path connection parameterized in the reverse way. We define the following maps

- The curvature $R^0([\mathbf{E}, \nabla^{\mathbf{E}}, (\omega, \theta)]) = ch(\nabla^{\mathbf{E}}) - d(\omega, \theta)$
- The forgetful map $I^0([\mathbf{E}, \nabla^{\mathbf{E}}, (\omega, \theta)]) = [\mathbf{E}]$
- The trivialization $a^0(\omega, \theta) = (0, 0, -(\omega, \theta))$

Next, we define K^{-n} with $n > 0$. as the subgroup $\widehat{K}^{-n} \subset \bigcap_j \ker(I_j^*)$, with

$$R^0(\mathbf{E}, \nabla^{\mathbf{E}}, (\omega, \theta)) = (\omega', \theta') \times dt_1 \times \cdots dt_n$$

where $dt \in \Omega^1(S^1)$. Given $\widehat{\mathbf{E}} = [\mathbf{E}, \nabla^{\mathbf{E}}, (\omega, \eta)] \in K^{-n}(\rho)$, we define

- The curvature $R^{-n}(\widehat{\mathbf{E}}) = \int_{T^n} R(\widehat{\alpha})$, where $\int_{T^n}$ stands for $\int_{\mathrm{pr}_X, \mathrm{pr}_A}$.
- The forgetful map $I^{-n}(\widehat{\mathbf{E}}) = [\mathbf{E}]$

- The trivialization $a^{-n}(\omega,\eta) = (-1)^{n+1}a^0(dt_1 \times \cdots dt_n \times (\omega,\theta))$.

Proposition C.3.3. *The relative $\widehat{K}$-group together with the natural transformations R, I, a are a cohomology theory*

The proof of this result is quite simple except for the exactness in Axiom A1 (3.8) at $\frac{\Omega^{\bullet-1}\mathfrak{K}}{\mathrm{Im}(d)}$.

Remark C.3.4. We can express an absolute class $\widehat{K}(X)$ either in this language or as a class $[F,\nabla,\omega]$ where F is a vector bundle, ∇ is a connection on F and $\omega \in \Omega^{\mathrm{odd}}(X)$. The isomorphism between the two models is given by

$$[E_1,E_2,\varnothing,\nabla_1,\nabla_2,(\omega,0)] \mapsto [E_1,\nabla_1,\omega] - [E_2,\nabla_2,0].$$

C.3.3 Multiplicative Structure

Definition C.3.5 (Relative-Absolute Product)**.** Given a relative differential class $\widehat{\alpha} := [\mathbf{E},\nabla^{\mathbf{E}},(\omega,\theta)] \in \widehat{K}^{-p}(\rho)$ and a absolute class $\widehat{\beta} := [F,\nabla^F,\omega] \in \widehat{K}^{-q}(Y)$ we define the product $\widehat{\alpha}\times\widehat{\beta} \in \widehat{K}^{-p-q}(Y)$ in the following way:

$$\widehat{\alpha}\times_0\widehat{\beta} := [\mathbf{E}\times F',\nabla^{\mathbf{E}}\times\nabla^F,(\omega,\theta)\times R(\omega') + R(\omega,\theta)\times\omega'] \in K^0(\mathrm{id}_{T^p}\times\rho\,\mathrm{id}_{T^q}\times\mathrm{id}_Y)$$

where $\mathbf{E}\times F$ was defined in and $\nabla^{\mathbf{E}}\times\nabla^{\mathbf{F}}$ is a shorthand for

$$\nabla^E\boxtimes\nabla^F,\nabla^{E'}\boxtimes\nabla^F,\widetilde{\nabla}\boxtimes\nabla^F.$$

and we put

$$\widehat{\alpha}\times\widehat{\beta} = \phi^*_{p,q}\widehat{\alpha}\times_0\widehat{\beta}$$

where $\phi_{p,q}$ is the same as defined in (C.3).

Proposition C.3.6. *The product defined above is a relative parallel product as defined in 3.5.1 but* ***without unit****.*

In order to extend the differential K-theory for positive degree we prove that the Bott Periodicty holds in the differential setting. Recall that the topological Bott isomorphism (see Theorem C.2.7) $B^{-n}: K^{-n-2}(\rho) \to K^n(\rho)$ is given by

$$B^{-n}(\alpha) = (\eta-1)\times\alpha$$

where $\kappa - 1 \in K(S^1\times S^1) \cong \mathbb{Z}$ is a generator. We define the differential Bott map $\widehat{B}^{-n}: \widehat{K}^{-n-2}(\rho) \to \widehat{K}^{-n}(\rho)$ is defined as a map

$$\widehat{B}^{-n}(\widehat{\alpha}) = (\widehat{k}-1)\times\widehat{\alpha}$$

where $\eta \in \widehat{K}(T^2)$ is such that $I(\widehat{\eta}-1) = \eta - 1$, $R(\widehat{\eta}-1) = dt_1\times dt_2$ and $i_1^*(\widehat{\eta}-1) = i_2^*(\widehat{\eta}-1) = 0$, for $i_1, i_2 : T^1 \to T^2$. Hence $\widehat{\kappa}-1 \in \mathfrak{K}^{-2}(pt)$ has curvature 1. It turns out that this homomorphism is also a isomorphism.

Proposition C.3.7 (Differential Bott Periodicity)**.** *The map $\widehat{B}^{-n}$ is a isomorphism.*

With this, we can finish the description of differential K theory defining K^n for positive degree using the periodicity.

C.3.4 Parallel model for differential K-theory

In the main corpus of this text (Chapter 6 to be more specific) we will need a special model for the parallel classes as was done in the de Rham model.

Definition C.3.8 (Parallel relative connections)**.** A parallel relative connection on a relative bundle (E_1, F_1, ψ) over ρ is a pair (∇_1, ∇_2) where

- ∇_1 is a connection over E_1;
- ∇_2 is a connection over E_2;
- $\psi : (\rho^* E_1, \rho^* \nabla_1) \to (\rho^* E_2, \rho^* \nabla_2)$ is a geometric isomorphism in the sense that $\rho^* \nabla_2 \circ \psi = \rho^* \nabla_1$.

The *parallel Chern form* associated to a parallel connection is simply $ch(\nabla_1, \nabla_2) = ch(\nabla_1) - ch(\nabla_2) \in \Omega^{\text{ev}}_{\text{par}}(\rho)$ and the parallel Chern-Simons form and the Chern-Simons class as

$$cs((\nabla_1, \nabla_2), \nabla_1', \nabla_2) = \int_I ch(\nabla^1_{0,1}, \nabla^2_{0,1}) \text{ and } CS = cs \mod Im(d)$$

where $\nabla^i_{0,1}$ is the standard interpolation between ∇_i and ∇_i' for $i \in \{1, 2\}$.

Definition C.3.9 (Parallel Differential Bundles)**.** A parallel differential bundle $[\mathbf{E}, \nabla^{\mathbf{E}}, \omega]$ is given by the following data

- A vector bundle triple $\mathbf{E} = (E_1, E_2, \psi)$;
- A relative parallel connection $\nabla^{\mathbf{E}} = (\nabla_1, \nabla_2)$;
- A parallel form $\omega \in \Omega_{\text{par}}(\rho)$

Two parallel differential vector bundles will be said isomorphic, if there exist a morphism of relative bundles $(f, g) : \mathbf{E} \to \mathbf{F}$ over ρ such that

$$\omega - \omega' \in CS((\nabla_1, \nabla_1')(\nabla_2, \nabla_2')).$$

As before we form the stabilization set of these classes.

Definition C.3.10. The special parallel differential group $\widehat{K}_{\text{par},I}$ is stabilization of $\text{DiffVec}_{\text{par}}(\rho)$

Consider the following map

Proposition C.3.11. *If $\rho : A \to X$ is a cofibration, the map*

$$\begin{aligned} i : \widehat{K}_{par,I}(\rho) &\to \widehat{K}_{par}(\rho) \\ [\mathbf{E}, (\nabla_1, \nabla_2), \omega] &\mapsto [\mathbf{E}, (\nabla_1, \nabla_2, \widetilde{\nabla_{0,1}}, (\omega, 0)] \end{aligned}$$

is a isomorphism.

APPENDIX D – Spectral Cohomology

D.1 Introduction

This appendix is intended to be a review of the Hopkins-Singer of differential cohomology in the relative setting. This model was developed by Hopkins and Singer (2005) and provided a refinement to any cohomology theory using spectra.

The relative model of Hopkins Singer which we present here is due to Ruffino (2015) which is based on the presentation of Upmeier (2014) who also developed the multiplicative structure. Our presentation follows Ruffino and Barriga (2021).

Before proceeding to the model, we briefly review some notions of spectra.

D.2 Spectrum

We recall some basic facts about (classical sequential) spectrum referring the reader to either (ADAMS, 1969),(SWITZER, 2002), or (RUDYAK, 1998). Since we are dealing with the category of spaces[1], we can focus on Ω-spectrum.

We define the *loop space* $\Omega(X, x)$ of a pointed space (X, x) as the space

$$\Omega(X, x) = \{x \in \mathrm{Hom}_{\mathtt{Top}}(I, X) : x(0) = x(1) = x\}$$

where $\mathrm{Hom}_{\mathtt{Top}}(I, X)$ is the space of continuous maps from the interval I to X. We can see Ω as functor $\Omega : \mathtt{Top}_* \to \mathtt{Top}_*$ which is right adjoint to the reduced suspension functor $\widetilde{\Sigma}$. The adjoint $f^\perp : (X, x) \to \Omega(X, x)$ of a map $f : \widetilde{\Sigma}(X, x) \to (X, x)$ is the map

$$f^\perp(x)(t) := f[t, x]$$

Definition D.2.1 (Ω-spectrum)**.** A Ω-spectrum is a sequence of pointed spaces CW-complexes $(E_n, *)$ and pointed maps $\epsilon_n : \widetilde{\Sigma} E_n \to E_{n+1}$ such that $\epsilon_n^\perp : E_n \to \Omega E_{n+1}$ is a homeomorphism.

Remark D.2.2. In the definition of Ω spectrum we required that the maps $\epsilon_n^\perp$ are homeomorphisms. In the literature it is common to find the same definition but requiring only homotopy equivalence. According to (UPMEIER, 2014, p. 32) this can always be arranged.

In order to describe both the multiplication as well as the Chern-Dold character, we need to use the smash product between spectrum. We will treat this topic as a "black box", using only its properties. The construction in the classical sequential way can be found, for example, in (SWITZER, 2002, Chapter 13).

[1] And not in general spectra.

D.3 Spectral cohomology

We denote by $\widetilde{E}^n(X) = [X, E_n]$ the set of morphisms in $\mathtt{HoTop}_*$. The set $\widetilde{E}^n(X)$ has a group structure, since it is equivalent to $[X, \Omega E_{n+1}]$ and we can define the product by composition of maps. Moreover, a Ekcmann-Hilton like argument can be used to verify that is abelian.

Observe that we have a map $s : \widetilde{E}^n(\widetilde{\Sigma}X) = [\widetilde{\Sigma}X, E_n] \cong [X, \Omega E_n] = [X, E_{n-1}] \cong \widetilde{E}^{n-1}(\widetilde{\Sigma}X)$ where the first equivalence is through adjunction and the other via composition with a homeorphism. Now, we can view $\widetilde{E}$ as functor from $\mathtt{Top}_* \to \mathtt{GrAb}$. It is possible to verify that $(\widetilde{E}, s)$ is a reduced cohomology theory and that every reduced cohomology theory arrives in this way.

Before proceeding, we observe that we can define a relative cohomology theory on pairs as we have done in the section as $E(\rho) = \widetilde{E}(C_\rho)$.

Since we can identify the set of classes of homotopies $[(M_\rho, A), (C_\rho, *)]$ with $[(C_\rho, *), (E_n, e_n)]$, we will prefer to think of $E(\rho)$ as the set $[(M_\rho, A), (C_\rho, *)]$. This will be our model of relative cohomology over maps.

D.3.1 Chern-Dold character and fundamental cocyles

We follow Upmeier (2014) in this section. The (real) *Cher-Dold* character[2] associated to a spectrum is a multiplicative ring spectrum morphisms $ch : E \to H\mathfrak{E}_{\mathbb{R}}$, where $\mathfrak{E}_{\mathbb{R}}$ is the coefficient group of the cohomology generated by the spectrum. Recall that we can define the rationalization of a spectrum (under certain circumstances) as the smash product with the rational Moore spectrum $M\mathbb{Q}$. The case here is completely analogous: we define $E_{\mathbb{R}} := E \wedge M\mathbb{R}$. The nice fact in this case is that $E_{\mathbb{R}}$ is equivalent to the graded Eilenberg-MacLane spectrum $H\mathfrak{h}_{\mathbb{R}}$. The (real) Chern-Dold character is the map composition

$$ch : E \xrightarrow{i} E_{\mathbb{R}} \xrightarrow{\sim} H\mathfrak{h}_{\mathbb{R}}.$$

We will not prove here that there exists the equivalence $E_{\mathbb{R}} \xrightarrow{\sim} H\mathfrak{E}_{\mathbb{R}}$ (this proof can be found in (RUDYAK, 1998) for the rational case and (UPMEIER, 2014) for the our case), neither we prove that this equivalence is multiplicative. Clearly, this map naturally induces a multiplicative natural transformation $ch : E^\bullet \to H\mathfrak{E}^\bullet_{\mathbb{R}}$ in cohomology (reduced and therefore relative) which will also call this map **Chern-Dold character**

Since the Chern-Dold character give us a map in cohomology $ch : \widetilde{E}^\bullet(E) \to \widetilde{H\mathfrak{h}_{\mathbb{R}}}^\bullet$, we can find a singular cocyle $\iota \in S^n(E_n, *; \mathfrak{E}_{\mathbb{R}})$ (see Appendix B for the notation) which implements the Chern-Dold character in the sense that $\iota_n = ch(\mathrm{id}_{E_n})$. We call this cocycle a *fundamental cocyle*. In particular, observe that given $[f] \in E(\rho)$, we can write

[2] Also knows as the generalized Chern character

$ch([f]) = ch([f \circ \mathrm{id}_{E_n}]) = ch(f^*[\mathrm{id}_{E_n}]) = f^* ch([\mathrm{id}_{E_n}]) = f^*\iota_n$. But we are interest in some *special* fundamental cocycles which are compatible with integration

Proposition D.3.1 (Special Fundamental Cocyle)**.** *There exists fundamental cocycles ι_n which satisfies the following property*

$$\iota_n = \int_{S^1} \epsilon_n^* \iota_{n+1}$$

Here $\int_{S^1} : S^n(\widetilde{\Sigma} E_n, e_n) \to S^{n+1}(E_n, e_n)$ is the S^1 integration at chain level as defined in section B.2.4 in the reduced case. The proof of this result can be found in (UPMEIER, 2014, p. 33-34) or in (HOPKINS; SINGER, 2005, Definition 4.32, p. 378).

D.3.2 Rationally even Cohomology Theories

Some results on differential cohomology such as uniqueness of the refinement require some hypothesis on the underlying cohomology theory. A very common requirement is that the underlying cohomology theory is *rationally even*

Definition D.3.2. We say that a cohomology theory (h, ∂) is rationally even, if $\mathfrak{h}^{2n+1}_{\mathbb{Q}} = 0$ for $n \in \mathbb{Z}$.

With respect to this definition, we will only use the following proposition

Proposition D.3.3. *Consider a spectrum (E_n, ϵ_n) and suppose its cohomology is rationally even. It holds that*

$$H\mathfrak{E}^{\bullet}_{\mathbb{Q}}(E_n, *) = 0 \ \ (n \ \textit{even})$$
$$H\mathfrak{E}^{\bullet}_{\mathbb{Q}}(E_n \times E_m, * \times *) = 0 \ \ (n, m \ \textit{even})$$

where $\mathfrak{E}_{\mathbb{Q}}$ is the ratioanlized coefficient group of E.

A discussion about this result can be found in (UPMEIER, 2014, Lema 8.2, p.115).

D.4 Hopkin-Singer model

Here we briefly review the definition of Hopkins-Singer model in the relative case. This model was introduced by Hopkins and Singer (2005) and settle the problem of the existence of a differential cohomology of any cohomology theory represented by a spectrum. The existence a multiplicative structure was done by Upmeier (2014). The model developed by this author is the one we are employing but in the relative setting. In fact, the author gives a relative version of the model which coincides with our parallel version. The model we present here was first discussed in (RUFFINO, 2014) and latter developed in (RUFFINO; BARRIGA, 2021).

D.4.1 Differential functions

We have defined the set $E^n(\rho)$ of a map $\rho : A \to X$ as $[(M_\rho, A), (E_n, e_n)]$ which turns out to be a nuisance in the differential case. That is because the space M_ρ is not a manifold in general. In order to solve this we give a smooth structure to M_ρ in a *ad hoc* way.

Given a smooth function $\rho : A \to X$, we consider the corresponding mapping cylinder M_ρ and the natural map

$$\begin{aligned}\iota_\rho : M_\rho &\to X \times I \\ p &\mapsto \begin{cases} (x, 0), & \text{if } p = [x] \\ (\rho(a), t) & \text{if } p = [a, t] \end{cases}\end{aligned}$$

If Y is a manifold, a continuous map $f : Y \to M_\rho$ is said to be *smooth* if and only if $\iota_\rho \circ f : Y \to X \times I$ is a smooth map.

For another smooth map $\eta : B \to Y$, we say that a continuous map $f : M_\eta \to M_\rho$ is smooth if and only if, for any manifold Z and any smooth map $\xi : Z \to M_\eta$, the composition $f \circ \xi$ is smooth in the sense of the previous paragraph.

With these preliminaries, we can define smooth singular chains $S^{\text{sm}}_\bullet(M_\rho; G)$ and cochains $S^\bullet_{\text{sm}}(M_\rho; G)$ on M_ρ with coefficienrs in a graded group G as usual.

Fix a graded real vector space V and let $c_\rho : M_\rho \to X$ be the collapse map defined in (A.1). Any form $\omega \in \Omega V^\bullet(X)$ naturally induces smooth singular chain in $S^\bullet_{\text{sm}}(M_\rho; V)$ through the following map:

$$\begin{aligned}\chi_\rho : \Omega V^\bullet(X) &\to S^\bullet_{\text{sm}}(M_\rho; V) \\ \omega &\mapsto c_\rho^* \chi_X(\omega).\end{aligned} \tag{D.1}$$

where $\chi_X : \Omega V^\bullet \to S^\bullet_{\text{sm}}(\cdot, V)$ is the de Rham map at chain level.

Let $\rho : A \to X$ be a smooth function, Y a topological space and $\kappa_n \in S^n(Y, V)$ a singular cocycle.

Definition D.4.1 (Differential Function)**.** A *differential function* $(f, h, \omega) : \rho \to (Y, \kappa_n)$ from ρ to (Y, κ_n) is a triple (f, h, ω) such that:

- $f : M_\rho \to Y$ is a continuous function;
- $h \in S^{n-1}_{\text{sm}}(M_\rho; V)$ is a smooth singular cochain with values in V;
- $\omega \in \Omega_{\text{cl}} V^n(X)$ is a n differential form with values[3] in V;

[3] We assumed that V is a real graded vector space, hence $V \otimes \mathbb{R} \cong V$.

satisfying the following condition:

$$\delta h = \chi_\rho(\omega) - f^* \kappa_n,$$

with χ_ρ defined in (D.1).

Definition D.4.2 (Homotopy between differential functions)**.** A *homotopy* between two differential functions (f_0, h_0, ω_0) and (f_1, h_1, ω_1) is a differential function $(F, H, \mathrm{pr}_X^* \omega)$: $\rho \times \mathrm{id}_I \to (Y, \kappa_n)$, where $\mathrm{pr}_X : X \times I \to X$ is the projection, such that:

- $\omega_0 = \omega_1$;
- F is a homotopy between f_0 and f_1;
- $H|_{(M_\rho \times \{i\}, A \times \{i\})} = h_i$ for $i = 0, 1$.

We write $(F, H, \mathrm{pr}_X^* \omega) : (f_0, h_0, \omega_0) \simeq (f_1, h_1, \omega_1)$ to indicate that the functions (f_0, h_0, ω_0) and (f_1, h_1, ω_1) are homotopic though the homotopy $(F, H, \mathrm{pr}_X^* \omega)$.

Remark D.4.3. In the previous definition it is possible that $A = \emptyset$. In this case we get a (absolute) differential function from the manifold X to (Y, κ_n).

Definition D.4.4. Let X be a manifold. Given:

- a differential function $(f, h, \omega) : X \to (Y, \kappa_n)$;
- a marked point $* \in Y$ and the constant function $k_* : X \to Y$;
- a $(n-1)$-form $\theta \in \Omega^{n-1}(X, V^\bullet)$;

We define a *strong topological trivialization* of (f, h, ω) induced by θ as a homotopy

$$(F, H, \mathrm{pr}_X^* \omega) : (f, h, \omega) \simeq (k_*, \chi_X(\omega), d\theta)$$

between (f, h, ω) and $(k_*, \chi_X(\theta), d\theta)$, where $\chi_X : \Omega V(X) \to S^\bullet_{\mathrm{sm}}(X; V)$ is the de Rham isomorphism at chain level.

Remark D.4.5. From the definition above, it follows that, if $(F, H, \mathrm{pr}_X^* \omega)$ is strongly trivialization of (f, h, ω) induced by θ, than $\omega = d\theta$.

D.4.2 Relative Hopkins-Singer model

Given a smooth function $\rho : A \to X$ between manifolds, we denote by:

- $\iota_{M(A)} : M(A) \to M_\rho$ the natural map defined by $(a, t) \mapsto [(a, t)]$;
- $\mathrm{pr}_A : M(A) \to A$ the projection;

- $A \times \{1\}$ the upper base of $M(A)$ and M_ρ.

For a differential function $(f, h, \omega) : \rho \to (Y, *)$, we define the pullbacks:

$$\rho^*(f, h, \omega) := (f \circ \rho, \rho^*(h|_X), \rho^*\omega) : A \to (Y, \kappa_n),$$

$$\iota^*_{M(A)}(f, h, \omega) := (f \circ \iota_{M(A)}, \iota^*_{M(A)}h, \mathrm{pr}^*_A \rho^*\omega) : M(A) \to (Y, \kappa_n).$$

Now, we fix a spectrum $((E_n, e_n), \epsilon_n)$ and denote its coefficient group by $\mathfrak{E}$. We set $\mathfrak{E}_\mathbb{R} := \mathfrak{E} \otimes \mathbb{R}$ and fix special fundamental cocycles $\iota_n \in S^n(E_n, e_n; \mathfrak{E}_\mathbb{R})$ as in (D.3.1)

Definition D.4.6. We denote by $\widehat{E}^n(\rho)$ the set of equivalence classes $[(f, h, \omega, \theta)]$, where:

- $(f, h, \omega) : \rho \to (E_n, \iota_n)$ is a differential function such that
$$\iota^*_{M(A)}(f, h, \omega) : \rho^*(f, h, \omega) \simeq (k_{e_n}, \chi(\omega), d\theta).$$
- Two representatives $(f_0, h_0, \omega, \theta)$ and $(f_1, h_1, \omega, \theta)$ are equivalent if there exists a homotopy $(F, H, \mathrm{pr}^*\,\omega) : (f_0, h_0, \omega) \simeq (f_1, h_1, \omega)$ that is constant on the upper base of the cylinder, i.e. such that $(F, H, \mathrm{pr}^*\,\omega)|_{A\times\{1\}\times I} = (c_{e_n}, \chi_{A\times I}(\mathrm{pr}^*_A\,\theta), \mathrm{pr}^*_A\,d\theta)$.

This definition implies that $\rho^*\omega = d\theta$, which, together with the fact that ω is closed, implies that $(\omega, \theta) \in \Omega^n_{\mathrm{cl}}(\rho; \mathfrak{E}_\mathbb{R})$.

Given two maps $\rho : A \to X$ and $\eta : B \to Y$ and a morphism $(f, g) : \eta \to \rho$, there is a natural induced map

$$\begin{aligned}(f, g)_\# : M_\eta &\to M_\rho\\ p &\mapsto \begin{cases} [f(y)] & , \text{ if } p = [(y)] \\ [(g(b), t)] & , \text{ if } p = [(b, t)] \end{cases}\end{aligned}$$

which induces the pullback

$$\begin{aligned}(f, g)^* : \widehat{E}(\rho) &\to \widehat{E}(\eta)\\ [(f, h, \omega, \theta)] &\mapsto [(f \circ (f, g)_\#, (f, g)^*_\# h, f^*\omega, g^*\theta)]\end{aligned}$$

.

Now, we show how one can endow $\widehat{E}^n(\rho)$ with an abelian group structure under the additional hypothesis that $E^\bullet$ is rationally even. We, fix the following data:

- A sequence of maps $\alpha_n : E_n \times E_n \to E_n$ representing the addition in cohomology, i.e. such that, for any topological space X and continuous functions $f, g : X \to E_n$ inducing $(f, g) : X \to E_n \times E_n$, one has $[f] + [g] = [\alpha_n \circ (f, g)]$. We require that, calling $\phi_n : \Sigma(E_n \times E_n) \to E_{n+1} \times E_{n+1}$ the structure maps of the spectrum $E_n \times E_n$ (defined via the factorization $\Sigma(E_n \times E_n) \to \Sigma E_n \times \Sigma E_n \to E_{n+1} \times E_{n+1}$), one has $\varepsilon_{n-1} \circ \Sigma\alpha_{n-1} = \alpha_n \circ \phi_{n-1}$.

- We call $\mathrm{pr}_{1,n}, \mathrm{pr}_{2,n} : E_n \times E_n \to E_n$ the two projections: their homotopy classes correspond to two elements of $E^n(E_n \times E_n)$, whose sum is represented by $\alpha_n \circ (\mathrm{pr}_{1,n}, \mathrm{pr}_{2,n}) = \alpha_n$, since $(\mathrm{pr}_{1,n}, \mathrm{pr}_{2,n}) = \mathrm{id}_{En \times En}$. From this follows that

$$\mathrm{pr}_{1,n}^*[\iota_n] + \mathrm{pr}_{2,n}^*[\iota_n] = \mathrm{ch}([\mathrm{pr}_{1,n}]) + \mathrm{ch}([\mathrm{pr}_{2,n}]) = \mathrm{ch}(\mathrm{pr}_{1,n} + \mathrm{pr}_{1,n}) = ch[\alpha_n] = \alpha_n^*[\iota_n]), \tag{D.2}$$

hence there exists $A_{n-1} \in S^{n-1}(E_n \times E_n, e_n \times e_n, \mathfrak{E}_{\mathbb{R}})$ such that:

$$\mathrm{pr}_{1,n}^*(\iota_n) + \mathrm{pr}_{2,n}^*(\iota_n) - \alpha_n^*(\iota_n) = \delta^{n-1} A_{n-1} \tag{D.3}$$

Since we are assuming that $E^\bullet$ is rationally even, it follows from Proposition D.3.3 that A_{n-1} is unique up to a coboundary for n even. We set $A_{n-2} := -\int_{S^1} \phi_{n-1}^* A_{n-1}$ where ϕ_{n-1} is the structure map of the spectrum $E_n \times E_n$ defined above. In this way, A_{n-1} is uniquely defined up to a coboundary for every n.

We define the sum in $\widehat{E}_n(\rho)$ as follows:

$$[(f_0, h_0, \omega_0, \theta_0)] + [(f_1, h_1, \omega_1, \theta_1)] := \\ [(\alpha_n \circ (f_0, f_1), h_0 + h_1 + (f_0, f_1)^* A_{n-1}, \omega_0 + \omega_1, \theta_0 + \theta_1)].$$

As above, we denoted by $(f_0, f_1) : X \to E_n \times E_n$ the map induced by f_0 and f_1.

Let us show that we get a differential extension of $E^\bullet$ by constructing the corresponding natural transformations. Define

$$R : \widehat{E}^\bullet(\rho) \to \Omega^\bullet_{\mathrm{cl}} \mathfrak{E}_{\mathbb{R}}(\rho), \quad [(f, h, \omega, \theta)] \mapsto (\omega, \theta);$$
$$I : \widehat{E}^\bullet(\rho) \to E^\bullet(\rho), \quad [(f, h, \omega, \theta)] \mapsto [f];$$
$$a : \frac{\Omega^{\bullet-1} \mathfrak{E}_{\mathbb{R}}(\rho)}{\mathrm{Im}} \to \widehat{E}^\bullet(\rho), \quad (\omega, \theta) \mapsto [(k_{e_n}, \chi_\rho(\omega, \theta), d(\omega, \theta)]$$

where the smooth singular cochain[4] $\chi^n_\rho(\omega, \theta) \in S^n_{\mathrm{sm}}(M_\rho; \mathfrak{E}_{\mathbb{R}})$ is defined as follows:

We fix a real number $\epsilon \in (0, 1)$ and we take a smooth non-decreasing function $\lambda : I \to I$ such that $\lambda(t) = 0$ for $t \leq \epsilon$ and $\lambda(1) = 1$. We fix the open cover $\{U, W\}$ of M_ρ defined by $U := A \times (\frac{\epsilon}{3}, 1]$ and $W := A \times [0, \frac{\epsilon}{2}) \cup_\rho X$. For each smooth chain $\sigma : \Delta_n \to M_\rho$, we take the iterated barycentric subdivision, so that the image of each sub-chain is contained in U or in W; then, for each small chain σ', we set

$$\chi^n_\rho(\omega, \theta)(\sigma') = \begin{cases} \chi^n_{A \times I}(\mathrm{pr}_A^* \rho^* \omega - d(\lambda\, \mathrm{pr}_A^* \rho^* \theta) \text{ if } \sigma' \subseteq U \\ \chi^n_X(\omega)(\mathrm{pr}_X \circ \sigma') \text{ if } \sigma' \subseteq U, \end{cases} \tag{D.4}$$

where $\mathrm{pr}_X : W \to X$ is the natural projection defined by $[a, t] \mapsto \rho(a)$ and $[x] \mapsto x$. Note that the morphism is well defined for $\sigma' \subseteq U \cap W$, since $\lambda(t) = 0$ for $t \leq \epsilon$. The cochain $\chi^n_\rho(\omega, \theta)$ depends on the choice of the function λ up to coboundaries.

4 One should confuse this with the other notation ξ_ρ introduced for a form $\omega \in \Omega\mathfrak{E}_{\mathbb{R}}(X)$.

D.4.3 Differential S^1-integration

Given a class $\widehat{\alpha} \in \widehat{E}^{n+1}(\rho \times \mathrm{id}_{S^1})$, we define $\int_{S^1} \widehat{\alpha} \in \widehat{E}^n(\rho)$ as follows. We consider the canonical split

$$E^{n+1}(\rho \times \mathrm{id}_{S^1}) \cong E^{n+1}(\rho) \oplus E^n(\rho) \tag{D.5}$$

that, using the representation through the cylinder, can be described as follows: we observe that $M_{\rho \times \mathrm{id}_{S^1}} \cong M_\rho \times S^1$, hence $E^{n+1}(\rho \times \mathrm{id}_{S^1}) \cong E^{n+1}(M_\rho \times S^1, A \times S^1)$, where we denote by A the upper base $A \times \{1\}$. We consider the projection $\mathrm{pr}_1 : (M_\rho \times S^1, A \times S^1) \to (M_\rho, A)$ and, fixing a marked point on S^1, the corresponding embedding $i_1 : (M_\rho, A) \hookrightarrow (M_\rho \times S^1, A \times S^1)$. The term $E^{n+1}(\rho)$ in (D.5), as a subgroup of $E^{n+1}(\rho \times \mathrm{id}_{S^1})$, is the image of pr_1^*, and the term $E^n(\rho)$ is the kernel of i_1^*.

Given $\widehat{\alpha} \in \widehat{E}^{n+1}(\rho \times \mathrm{id}_{S^1})$, we set $\widehat{\alpha}' := \widehat{\alpha} - \mathrm{pr}_1^*(\widehat{\alpha})$ and we represent it as $\widehat{\alpha}' = [(f, h, \omega, \theta)]$. Since $i_1^*(\alpha') = 0$ by construction, we have $i_1^* I(\widehat{\alpha}') = 0$. Therefore the function $f : (M_\rho \times S^1, A \times S^1) \to (E_{n+1}, e_{n+1})$ can be chosen in such a way that $f(i_1(M_\rho, A)) = e_n$. Hence, we have and adjoint $\widetilde{f} : (M_\rho, A) \to (\Omega E_{n+1}, c_{e_{n+1}})$, where $c_{e_{n+1}}$ is the constant loop. Composing with the inverse of the adjoint of the structure map, we get $\int_{S^1} f := \widetilde{\varepsilon}_n \circ \widetilde{f} : (M_\rho, A) \to (E_n, e_n)$, hence we set $\int_{S^1}[(f, h, \omega, \theta)] := [(\int_{S^1} f, \int_{S^1} h, \int_{S^1}(\omega, \theta))]$.

D.4.4 Product

In order to define the exterior product between absolute and relative classes, we call $\mu_{n,m} : E_n \wedge E_m \to E_{n+m}$ the maps making E a ring spectrum and we fix the following data:

- Given two topological spaces with marked point (X, x_0) and (Y, y_0) and two maps $f : (X, x_0) \to (E_n, e_n)$ and $g : (Y, y_0) \to (E_m, e_m)$, representing the reduced cohomology classes $[f] \in \widetilde{h}^n(X)$ and $[g] \in \widetilde{h}^m(Y)$, the cohomology class $[f] \times [g] \in \widehat{E}^{n+m}(X \wedge Y)$ is represented by the following composition:

  $$X \wedge Y \xrightarrow{f \wedge g} E_n \wedge E_m \xrightarrow{\mu_{n,m}} E_{n+m}.$$

  Therefore, one has $\mathrm{ch}[f] \times \mathrm{ch}[g] = \mathrm{ch}([f] \times [g]) = \mathrm{ch}([\mu_{n,m} \circ (f \wedge g)])$. Choosing $f = \mathrm{id}_{E_n}$ and $g = \mathrm{id}_{E_m}$, we get $\mathrm{ch}[id_{E_n}] \times \mathrm{ch}[id_{E_m}] = \mathrm{ch}[\mu_{n,m}]$. Hence, there exists $M_{n,m} \in S^{n+m-1}(E_n \wedge E_m, e_n \wedge e_m, \mathfrak{E}_{\mathbb{R}})$ such that:

  $$\delta_{n+m-1} M_{n,m} = \iota_n \times \iota_m - \mu_{n,m}^* \iota_{n+m}. \tag{D.6}$$

  Since we are assuming that $\mathfrak{E}_{\mathbb{R}}^{\mathrm{odd}}$, it follows that $M_{n,m}$ is unique up to a coboundary for n and m even (Proposition D.3.3)

- We fix a chain homotopy between the wedge product of differential forms and the cup product of the associated singular cochains. In particular, given two manifolds

X and Y, we consider the two maps

$$P, Q : \Omega^n(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \otimes \Omega^m(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) \to C^{n+m}(X \times Y; \mathfrak{h}^\bullet_{\mathbb{R}})$$

defined by $P(\omega_0 \otimes \omega_1) := \chi_{X\times Y}(\omega_0 \wedge \omega_1)$ and $Q(\omega_0 \otimes \omega_1) := \chi_X(\omega_0) \cup \chi_Y(\omega_1)$, where $\chi_* : \Omega^\bullet(*; \mathfrak{h}^\bullet_{\mathbb{R}}) \to C^\bullet(*; \mathfrak{h}^\bullet_{\mathbb{R}})$ is the natural homomorphism. The coboundary of $\Omega^n(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \otimes \Omega^m(Y; \mathfrak{h}^\bullet_{\mathbb{R}})$ is defined as $d(\omega_0 \otimes \omega_1) := d\omega_0 \otimes \omega_1 + (-1)^n \omega_0 \otimes d\omega_1$. There exists a chain homotopy

$$B : \Omega^n(X; \mathfrak{h}^\bullet_{\mathbb{R}}) \otimes \Omega^m(Y; \mathfrak{h}^\bullet_{\mathbb{R}}) \to C^{n+m-1}(X \times Y; \mathfrak{h}^\bullet_{\mathbb{R}})$$

between P and Q, which by definition satisfies

$$\chi_{X\times Y}(\omega_0 \wedge \omega_1) - \chi_X(\omega_0) \cup \chi_Y(\omega_1) = \delta B(\omega_0 \otimes \omega_1) + Bd(\omega_0 \otimes \omega_1).$$

Given $\widehat{\alpha} = [(f, h, \omega, \theta)] \in \widehat{E}^n(\rho)$, where $\rho : A \to X$ is a smooth map, and $\widehat{\beta} = [(f', h', \omega')] \in \widehat{E}^m(Y)$, with n and m even, the class $\widehat{\alpha} \times \widehat{\beta} \in \widehat{E}_{n+m}(\rho \times \mathrm{id}_Y)$ is defined by

$$\begin{aligned}[(f, h, \omega, \theta)] \times [(f', h', \omega')] := [(&\mu_{n,m} \circ (f \times f'),\\ &h \times \chi_Y(\omega') + \chi_\rho(\omega) \times h' + B(\omega \otimes \omega') - h \times \delta h' + (f, f')^* M_{n,m}, \omega \times \omega', \theta \times \omega')].\end{aligned}$$

In the first entry, $\mu_{n,m} \circ (f_0 \times f_1)$, we actually considered the following composition:

$$M(\rho \times id_Y) \approx M_\rho \times Y \xrightarrow{f_0 \times f_1} E_n \times E_m \longrightarrow E_n \wedge E_m \xrightarrow{\mu_{n,m}} E_{n+m}. \tag{D.7}$$

For any $\widehat{\alpha} \in \widehat{E}^n(\rho)$ (without restrictions on n), there exists a unique class $\widehat{\alpha}' \in \widehat{E}^{n+1}(\rho \times \mathrm{id}_{S^1})$ such that $\int_{S^1} \widehat{\alpha}' = \widehat{\alpha}$ and $R(\widehat{\alpha}') = dt \wedge \mathrm{pr}^*_{1,\rho}(\widehat{\alpha})$, where $\mathrm{pr}_{1,\rho} : \rho \times \mathrm{id}_{S^1} \to \rho$ is the projection. The same statement holds replacing $\rho \times \mathrm{id}_{S^1}$ by $Y \times S^1$. Hence, still supposing that n and m are even, we define:

- for $\widehat{\alpha} \in \widehat{E}^{n-1}(\rho)$ and $\widehat{\beta} \in \widehat{E}^m(Y)$, $\widehat{\alpha} \times \widehat{\beta} := \int_{S^1} \mathrm{pr}^*_{1,\rho} \widehat{\alpha} \times \widehat{\beta}'$;
- for $\widehat{\alpha} \in \widehat{E}^n(\rho)$ and $\widehat{\beta} \in \widehat{E}^{m-1}(Y)$, $\widehat{\alpha} \times \widehat{\beta} := \int_{S^1} \widehat{\alpha}' \times \mathrm{pr}^*_{1,Y} \widehat{\beta}$;
- for $\widehat{\alpha} \in \widehat{E}^{n-1}(\rho)$ and $\widehat{\beta} \in E^{m-1}(Y)$, $\widehat{\alpha} \times \widehat{\beta} := -\int_{S^1} \int_{S^1} \mathrm{pr}^*_{1,\rho} \widehat{\alpha'} \times \mathrm{pr}^*_{1,Y} \widehat{\beta'}$.

D.4.5 Parallel classes

When $\rho : A \hookrightarrow X$ is a closed embedding, we can simplify the model described above, without considering the cylinder, but simply requiring that a differential function from X to E_n restricts on A to $(c_{e_n}, \chi_A(\theta), d\theta)$. We are interested in this model in particular for parallel classes, as in the previous sections.

Definition D.4.7. We denote by $\widehat{E}^n_{I,\mathrm{par}}(X, A)$ the set of equivalence classes $[(f, h, \omega)]$, where:

- (f, h, ω) is a differential function from X to (E_n, ι_n) such that $(f, h, \omega)|_A = (c_{e_n}, 0, 0)$.
- Two representatives (f_0, h_0, ω) and (f_1, h_1, ω) are equivalent if there exists a homotopy $(F, H, \mathrm{pr}^* \omega)$ between (f_0, h_0, ω) and (f_1, h_1, ω) that is constant on A, i.e. such that $(F, H, \mathrm{pr}^* \omega)|_{A\times I} = (c_{e_n}, 0, 0)$.

We could equivalently require that $f : (X, A) \to (E_n, e_n)$, $h \in S^{n-1}_{\mathrm{sm}}(X, A)$ and $\omega \in \Omega^n(X, A; \mathfrak{E}_{\mathbb{R}})$, two representatives being equivalent if there exits a homotopy $(F, H, \mathrm{pr}_X^* \omega)$ of this form on $(X \times I, A \times I)$. The sum is defined similarly to (D.4.2), i.e.:

$$[(f_0, h_0, \omega_0)] + [(f_1, h_1, \omega_1)] := [(\alpha_n \circ (f_0, f_1), h_0 + h_1 + (f_0, f_1)^* A_{n-1}, \omega_0 + \omega_1)] \quad \text{(D.8)}$$

where $(f_0, f_1) : (X, A) \to (E_n \times E_n, e_n \times e_n)$. The natural transformations that characterize the parallel theory are defined as $I[(f, h, \omega)] := [f]$, $R[(f, h, \omega)] := \omega$ and $a(\omega) := [(0, 0, \omega)]$. We define S^1-integration like in section D.4.3, replacing the pair (M_ρ, A) by (X, A).

Given a closed embedding $\rho : A \hookrightarrow X$, we denote by $\widehat{E}^n_{\mathrm{par}}(\rho)$ the parallel subgroup of the model defined above. We have the natural morphism $\widehat{E}^n_{I,\mathrm{par}}(X, A) \to \widehat{E}^n_{\mathrm{par}}(\rho)$, $[(f, h, \omega)] \mapsto [(f \circ c_\rho, c_\rho^* h, \omega, 0)]$, where $c_\rho : M_\rho \to X$ is the collapse map (see (A.1)). In (RUFFINO; BARRIGA, 2021, Theorem 3.2) it is proven that it is an isomorphism.

Index