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PROGRAMA DE PÓS GRADUAÇÃO EM MATEMÁTICA

Wanderley Nunes do Nascimento

Klein-Gordon models with non-effective time-dependent potential

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Resumo

Nesta tese estudamos as propriedades assintóticas para a solução do problema de Cauchy para a equação de Klein-Gordon com potencial não efetivo dependente do tempo. O principal objetivo foi definir uma energia adequada relacionada ao problema de Cauchy e derivar estimativas para tal energia. Estimativas de Strichartz e resultados de *scatering* e *scatering* modificados também foram estabelecidos. A teoria C^m e a condição de estabilização foram aplicados para tratar o caso em que o coeficiente da massa oscila muito rápido. Além disso, consideramos um modelo de onda semi-linear *scale-invariante* com massa e dissipação dependentes do tempo, nesta etapa usamos as estimativas lineares de tal modelo para provar existência global (no tempo) de solução de energia para dados iniciais suficientemente pequenos e demonstramos um resultado de *blow-up* para uma escolha adequada dos coeficientes.

Abstract

In this thesis we study the asymptotic properties for the solution of the Cauchy problem for the Klein-Gordon equation with non-effective time-dependent potential. The main goal was define a suitable energy related to the Cauchy problem and derive decay estimates for such energy. Strichartz' estimates and results of scattering and modified scattering was established. The C^m theory and the stabilization condition was applied to treat the case where the coefficient of the potential term has very fast oscillations. Moreover, we consider a semi-linear wave model scale-invariant time-dependent with mass and dissipation, in this step we used linear estimates related with the semi-linear model to prove global existence (in time) of energy solutions for small data and we show a blow-up result for a suitable choice of the coefficients.

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10th November 2015 M. Sc. Wanderley Nunes do Nascimento

To my parents.

Contents

Versão em Português

Motivação

Equações hiperbólicas são usadas na física para descrever processos evolucionários com a propriedade de que a informação se propaga com a velocidade finita. Esses processos podem ser encontrados em diversas áreas, como por exemplo na Teoria de Eletromagnetismo e Eletrodinâmica. Um dos modelos padrões é o da equação da onda livre

$$u_{tt} - c^2 \Delta u = 0,$$

que descreve uma corda vibrante para $n = 1$, membrana para $n = 2$, ou sólidos elásticos para $n = 3$. A constante c denota a velocidade de propagação e $\Delta = \sum_{i=1}^{n} \partial_i^2$ o Laplaciano com respeito a variável espacial.

Outro modelo de interesse é a equação de Klein-Gordon

$$u_{tt} - c^2 \Delta u + \Big(\frac{mc^2}{h}\Big)^2 u = 0, \tag{0.1}$$

onde h é relacionado com a constante de Planck e m é a massa constante de uma partícula. Esse modelo foi introduzido por Gordon (1926) e Klein (1927) derivando uma equação relativista para uma partícula carregada em um campo eletromagnético. Essa equação também é usada para descrever fenômenos de onda dispersiva em geral, veja [17].

Nas seções seguintes vamos discutir resultados conhecidos para esses dois modelos e também para modelos mais gerais.

Modelos clássicos de onda com e sem massa

Começaremos relembrando alguns resultados para o modelo de onda livre. Considere o seguinte problema de Cauchy para a equação da onda livre:

$$u_{tt} - \Delta u = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{0.2}$$

com $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. O problema de Cauchy (0.2) é H^s bem posto, i.e., se $u_0 \in H^s$ e $u_1 \in H^{s-1}$, então existe para todo positivo T uma solução única $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$ que depende continuamente dos dados iniciais (u_0, u_1).

Se $u \in C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$, então podemos definir a energia clássica relacionada ao problema

$$E_W(u)(t) = \frac{1}{2} \int_{\mathbb{R}^n} \big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2\big) dx, \tag{0.3}$$

e podemos provar que $E'_W(u)(t) = 0$ para todo $t \geq 0$, em outras palavras, temos conservação de energia, i.e., $E_W(u)(t) = E_W(u)(0)$ para todo $t \geq 0$.

Estimativas de Strichartz foram provadas em um primeiro momento por W. von Wahl com dado inicial $(u_0, u_1) \in C_0^\infty(\mathbb{R}^n)$. No artigo [55] ele provou, sem usar operadores integrais de Fourier, que

$$\|(u_t(t,.), \nabla_x u(t,x))\|_{L^q} \leq C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|(u_1, \nabla_x u_0)\|_{L^{p,r}}, \tag{0.4}$$

para $n \geq 2$ com p e q duais, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, onde $1 < p \leq 2$ e regularidade $r > n\left(\frac{1}{p} - \frac{1}{q}\right)$. Para esclarecer as notações usadas nesta tese veja o guia de notações no Capítulo 7. Técnicas modernas como operadores integrais de Fourier e o método da fase estacionária foram usados por Strichartz [52] e [53], Littman [39], Brenner [7] e Pecher [43] para provar a estimativa (0.4).

Outro modelo importante é o modelo clássico de Klein-Gordon introduzido em 1926,

$$u_{tt} - \Delta u + m^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{0.5}$$

com $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ e $m > 0$.

O problema de Cauchy (0.5) é H^s bem posto, i.e., se $u_0 \in H^s$ e $u_1 \in H^{s-1}$, então existe para todo T positivo uma única solução

$$u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$$

que depende continuamente dos dados iniciais (u_0, u_1).

Nesse problema a massa nos força a incluir na energia total, além das energias elástica e cinética, um terceiro componente que é a energia potencial. Podemos definir a energia total como:

$$E_{KG}(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 + m^2|u(t,x)|^2\right)dx. \tag{0.6}$$

Também é possível provar que $E'_{KG}(u)(t) = 0$ para todo $t \geq 0$, em outras palavras, temos também a propriedade de conservação de energia.

Em comparação com a equação da onda livre, a massa melhora o decaimento nas estimativas de Strichartz. W. von Wahl obteve esse resultado depois de introduzir a mundança de variável $v = v(t, x, x_{n+1})$ por

$$v(t, x, x_{n+1}) := \exp\left(-imx_{n+1}\right) u(t,x),$$

onde $x \in \mathbb{R}^n$, $x_{n+1} \in \mathbb{R}$ e $t \in \mathbb{R}_+$. Essa mudança de variável pode ser encontrada em [55]. Facilmente vemos que v é solução para a equação de onda livre (0.2) com os dados iniciais

$$v_0(x, x_{n+1}) := \exp\left(-imx_{n+1}\right) u_0(x), \;\; v_1(x, x_{n+1}) := \exp\left(-imx_{n+1}\right) u_1(x).$$

Percebemos que o novo dado inicial (v_0, v_1) não pertence a $C_0^\infty(\mathbb{R}^n)$. Contudo, a mudança de variável acima serve como motivação para conjecturar quais estimativas são esperadas. É possível provar que se o dado inicial $(u_0, u_1) \in C_0^\infty(\mathbb{R}^n)$, então as seguintes estimativas de Strichartz para o modelo clássico de Klein-Gordon

$$\|(u_t(t,.), \nabla_x u(t,x), u(t,.))\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|(u_0, u_1, \nabla_x u_0)\|_{L^{p,r}}, \tag{0.7}$$

são válidas para $n \geq 2$ com p e q duais, $p \in (1, 2]$ e regularidade $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$.

A abordagem usando operadores integrais de Fourier também foram aplicadas por Pecher [43] e Hörmander [30] para o modelo de Klein-Gordon clássico para obter as estimativas (0.7).

Para a equação de Klein-Gordon não linear relacionada com a equação clássica de Klein-Gordon

$$u_{tt} - \Delta u + m^2 u = f(u, u_t, \nabla_x u, \nabla_x^2 u), \;\; u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x), \qquad (0.8)$$

Klainerman [34] e Shatah [50] provaram existencia global de soluções para o problema de Cauchy (0.8) com dados iniciais pequenos e condições adequadas para f.

Modelos de onda com potencial dependente do tempo

Uma pergunta natural que aparece é: o que acontece quando o termo massa é dependente do tempo? O que podemos dizer sobre a definição da energia e quais estimativas podemos derivar? Nesta seção vamos descrever resultados conhecidos para o problema de Cauchy para a equação de Klein-Gordon com potencial dependente do tempo.

Considere o seguinte problema de Cauchy para a equação de Klein-Gordon:

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x), \qquad (0.9)$$

onde $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

O objetivo é definir uma energia adequada para modelos com potencial dependente do tempo e estimativas para tal energia. Um modelo importante que nos ajuda a definir esta energia é o modelo *scale-invariante* que foi estudado em [4] e [5].

Modelos *Scale-invariant*

Definir uma energia adequada não é um trabalho trivial como podemos ver no seguinte modelo que foi abordado em [5]. Vamos considerar o seguinte problema de Cauchy para a equação de Klein-Gordon

$$u_{tt} - \Delta u + \frac{\mu^2}{(1+t)^2} u = 0, \;\; u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x), \qquad (0.10)$$

com $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ e constante real $\mu \neq 0$ desenvolvendo um papel decisivo. Como $u^* = u^*(t^*, x^*) := u(t, x)$ com $1 + t = \lambda(1 + t^*)$ e $x = \lambda x^*, \lambda > 0$ arbitrário, satisfaz também o problema de Cauchy, a condição *scale-invariant* é verificada.

Uma vez satisfeita a condição *scale-invariant* podemos usar a teoria de funções especiais e introduzir a energia $E^{(\mu)}(u) = E^{(\mu)}(u)(t)$ da seguinte forma

$$E^{(\mu)}(u)(t) := \frac{1}{2}\Big(\|u_t(t, \cdot)\|_{L^2}^2 + \|\nabla_x u(t, \cdot)\|_{L^2}^2 + p_\mu(t)^2 \|u(t, \cdot)\|_{L^2}^2\Big), \qquad (0.11)$$

onde

$$p_\mu(t) = \begin{cases} (1+t)^{-\frac{1}{2}}, & \mu^2 > \frac{1}{4}, \\ (1+t)^{-\frac{1}{2}}\big(1+\ln(1+t)\big)^{-1}, & \mu^2 = \frac{1}{4}, \\ (1+t)^{-\frac{1}{2}-\frac{1}{2}\sqrt{1-4\mu^2}}, & \mu^2 \in \big(0, \frac{1}{4}\big). \end{cases} \tag{0.12}$$

Então temos a *conservação de energia generalizada*

$$p_\mu(t)^2 E^{(\mu)}(u)(0) \lesssim E^{(\mu)}(u)(t) \lesssim E^{(\mu)}(u)(0). \tag{0.13}$$

Observação 0.1. *A estimativa (0.13) exclui o blow-up da energia $E^{(\mu)}(u)(t)$ quando $t \to \infty$. E ainda temos a estimativa por baixo para o decaimento dessa energia. Vemos que a energia potencial pode ser estimada da seguinte maneira:*

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim p_\mu(t)^{-2} E^{(\mu)}(u)(0).$$

Se $\mu \to +0$, então $p_\mu(t)^{-2}$ tende para $(1+t)^2$, um comportamento assintótico que é conhecido para a energia potencial do problema de Cauchy para a equação da onda livre. Se $\mu \to \infty$, então $p_\mu(t)^{-2} = 1+t$, então a energia potencial tem um crescimento menor para $t \to \infty$.

A solução para o problema de Cauchy (0.10) com dados iniciais $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^n)$ satisfaz as estimativas de Strichartz (veja [5])

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right), \tag{0.14}$$

$$\|p_\mu(t)\, u(t,\cdot)\|_{L^q} \lesssim d_\mu(t) \left(\|u_0\|_{L^{p,r}} + \|u_1\|_{L^{p,r-1}}\right) \tag{0.15}$$

com

$$d_\mu(t) := \begin{cases} \max\left\{(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}},\ (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\right\}, & \mu^2 \geq \frac{1}{4}, \\ \max\left\{(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}-\frac{1}{2}\sqrt{1-4\mu^2}},\ (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\right\}, & \mu^2 < \frac{1}{4}, \end{cases} \tag{0.16}$$

onde $r = n\big(\frac{1}{p}-\frac{1}{q}\big)$, $\frac{1}{p}+\frac{1}{q} = 1$ com $1 < p \leq 2$. Este resultado implica que as energias cinética e elástica $\|\nabla_x u(t,\cdot)\|_{L^q}$ e $\|u_t(t,\cdot)\|_{L^q}$ medidas na norma L^q decrescem com o decaimento do tipo onda $\frac{n-1}{2}\big(\frac{1}{p}-\frac{1}{q}\big)$ como em (0.4).

O modelo anterior nos inspira a considerar dois casos diferentes para o potencial dependente do tempo. Considere o seguinte problema de Cauchy para a equação de Klein-Gordon

$$u_{tt} - \Delta u + m(t)^2 u = 0,\ \ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x), \tag{0.17}$$

com $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definição 0.1. *Dizemos que o termo potencial $m(t)^2u$ em (0.17) é efetivo se o coeficiente dependente do tempo satisfaz*

$$tm(t) \to \infty$$

quando t tende para ∞.

Definição 0.2. *Dizemos que o termo potencial $m(t)^2u$ em (0.17) é não efetivo se o coeficiente dependente do tempo satisfaz*

$$\limsup_{t\to\infty}(1+t)\int_t^\infty m(s)^2ds < \frac{1}{4},$$

e se as derivadas de $m(t)^2$ satisfazem as seguintes estimativas:

$$\Big|\frac{d^k}{dt^k}m(t)^2\Big| \lesssim (1+t)^{-(k+2)\gamma}$$

para algum $0<\gamma\le 1$, $k=1,2$ no caso $\gamma=1$ e $k=1,2,\cdots,m$ caso contrário.

Observação 0.2. *A nomenclatura anterior é motivada pelo decaimento em relação ao tempo das estimativas $L^p - L^q$. Se o decaimento é relacionado com o da equação da onda livre, então dizemos que a massa é não efetiva. Se o decaimento é relacionado com o da equação de Klein-Gordon, então dizemos que a massa é efetiva.*

Equação da onda com potencial efetivo

A tese de doutorado [4] foi direcionada ao estudo do caso efetivo, isto é, a autora estudou coeficientes decrescentes $m = m(t)$ que que satisfazem entre outras propriedades $\lim_{t\to\infty} tm(t) = \infty$. Neste caso os modelos (0.17) são chamados modelos com potencial efetivo. Em [4] foi considerado o potential efetivo com a seguinte estrutura $m(t) = \lambda(t)\nu(t) \in C^M(\mathbb{R}_+)$, $M \ge 2$, com função principal $\lambda = \lambda(t)$ e uma pequena perturbação da massa dada por uma função oscilante $\nu = \nu(t)$. Se definirmos a energia de Klein-Gordon

$$E^{(KG)} = \frac{1}{2}\int_{\mathbb{R}^n}\big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 + m(t)|u(t,x)|^2\big)dx, \tag{0.18}$$

então ocorre a conservação de energia generalizada, isto é,

$$\lambda(t)E^{(KG)}(u)(0) \lesssim E^{(KG)}(u)(t) \lesssim E^{(KG)}(u)(0). \tag{0.19}$$

Se $m(t) = \lambda(t)\nu(t) \in C^\infty(\mathbb{R}_+)$, com hipóteses adequadas para λ e ν, podemos provar para todo $t \ge 0$ a seguinte estimativa de Strichartz:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot), \lambda(t)u(t,\cdot))\|_{L^q} \quad\lesssim\quad (1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right),$$

com $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, $\frac{1}{p} + \frac{1}{q} = 1$ com $1 < p \le 2$. Este tipo de decaimento é conhecido como decaimento do tipo Klein-Gordon $\frac{n}{2}\big(\frac{1}{p} - \frac{1}{q}\big)$ no índice conjugado.

Decaimento do tipo onda

Considere o problema de Cauchy

$$u_{tt} - \Delta u + m(t)^2u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{0.20}$$

onde $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Seja $m = m(t) \in C^\infty(\mathbb{R}_+)$ satisfazendo as seguintes propriedades:

$$(B1) \quad m(t) \in L^1(\mathbb{R}_+),$$
$$(B2) \quad |d_t^k m(t)| \lesssim C_k(1+t)^{-k}, k = 0,1,2,\cdots,$$

para todo t, onde C_k são constantes positivas. Então [4] mostrou o seguinte resultado:

Teorema 0.1. *Seja* $m = m(t) \in C^\infty(\mathbb{R}_+)$ *satisfazendo* $(B1)$ *e* $(B2)$. *Então para todo tempo* t *o decaimento* $L^p - L^q$

$$\| (u_t(t,\cdot), \nabla_x u(t,\cdot)) \|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} (\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}})$$

acontece para $r = n\left(\frac{1}{p} - \frac{1}{q}\right)$, *com* $1 < p \leq 2$ *e* $\frac{1}{p} + \frac{1}{q} = 1$.

Resultado *Scattering*

A teoria *scattering* compara o comportamento da solução do problema para a onda livre com a solução do problema perturbado num tempo suficientemente grande. O principal objetivo é construir um operador que mapeia dados iniciais do problema de Cauchy para a onda livre em dados iniciais para o problema de Cauchy perturbado. Tal operador é denominado operador de onde de Moeller.

Vamos considerar que u satisfaz o problema de Cauchy para a equação de Klein-Gordon (0.9) e que v satisfaz o problema de Cauchy para a equação de onda livre (0.2).

Assuma as seguintes condições

$$m \in L^1(\mathbb{R}_+), \;\; m(t)(1+t) \leq C \;\text{ for }\; t \in [0,\infty). \tag{0.21}$$

Então o seguinte resultado pode ser encontrado em [4], Teorema 3.26:

Teorema 0.2. *Suponha que o coeficiente* $m = m(t)$ *satisfaz* (0.21). *Existe um operador scattering* $W_+ = W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *tal que os dados iniciais para os problemas de Cauchy* (0.9) *e* (0.2) *são relacionados por* $(|D|v_0, v_1)^T = W_+(D)(\langle D\rangle u_0, u_1)^T$. *Então as soluções para os problemas* (0.9) *e* (0.2) *satisfazem a equivalência assintótica*

$$\left\| (|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D \rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot) \Big) \right\|_{L^2 \times L^2} \to 0 \tag{0.22}$$

quando t *tende a infinito.*

Algumas informações sobre a tese

O que continua aberto na tese [4] é explicar as propriedades qualitativas para a solução do problema de Cauchy para a equação de Klein-Gordon com potencial dependente do tempo que não são *scattering* a equação de onda livre e que são não efetivos de acordo com a Definição 0.2. Exemplo típicos são potenciais decrescente satisfazendo $m \notin L^1(\mathbb{R}_+)$ e $\lim_{t\to\infty} tm(t) = 0$. Essa tese se concentra nesse tópico, i.e., o objetivo é definir uma energia adequada e derivar estimativas para tal energia para problemas de Cauchy para equação de Klein-Gordon com potenciais dependentes do tempo não efetivos.

Para alcançar esse objetivo vamos aplicar uma mudança de variável no problema de Cauchy para a equação de Klein-Gordon e transformá-lo em um problema de Cauchy com dissipação dependente do tempo e usar resultados conhecidos para o esse caso, veja [59]. Na próxima seção vamos descrever alguns resultados para o problema de Cauchy para a equação da onda com dissipação dependente do tempo.

Equação da onda com dissipação dependente do tempo

Um outro problema de interesse é o problema de Cauchy para a equação da onda com dissipação dependente do tempo

$$u_{tt} - \Delta u + b(t)u_t = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{0.23}$$

onde $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Se $b(t) = \mu(1+t)^{-1}$ com $\mu > 0$, isto é, estamos interessados no caso modelo *scale-invariant*, podemos encontrar em [58] as seguintes estimativas $L^p - L^q$:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \lesssim (1+t)^{\max\left\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mu}{2}, -n\left(\frac{1}{p}-\frac{1}{q}\right)-1\right\}}\big(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\big),$$

onde $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, $\frac{1}{p} + \frac{1}{q} = 1$, com $1 < p \leq 2$.
Então percebemos que o parâmetro μ influência no decaimento. Esse caso separa o caso efetivo do caso não efetivo, aqui dizemos que a dissipação é efetiva se as estimativas $L^p - L^q$ para a energia tem o decaimento no tempo relacionado com a equação de dissipação com coeficiente constante e dizemos que a dissipação é não efetiva se as estimativas $L^p - L^q$ tem um decaimento no tempo relacionado com a da equação da onda livre. Wirth provou estimativas $L^p - L^q$ para ambos os casos em [59] e [60], respectivamente. O caso importante para nós é o caso não efetivo, i.e., se o coeficiente $b = b(t)$ decai mais rapidamente que o termo do caso crítico $b(t) = \mu(1+t)^{-1}$. Temos a seguinte estimativa $L^p - L^q$ para o caso não efetivo:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\big),$$

onde $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, $\frac{1}{p} + \frac{1}{q} = 1$, com $1 < p \leq 2$. Aqui

$$\lambda(t) = \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big).$$

Além disso, se $b \in L^1$, então Wirth provou um resultado *scattering*, que as soluções se comportam assintoticamente como as soluções para equação de onda livre.

Se considerarmos v como a solução da onda livre (0.2), então Wirth provou o seguinte resultado *scattering* modificado.

Theorem 0.1. *Para qualquer dado inicial* $(u_1, u_2) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *existe um operador linear, limitado* $W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *tal que para o dado inicial de Cauchy* $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *de* (0.23) *e um dado inicial associado* $(v_0, v_2) = W_+(u_0, u_1)$ *para* (0.2) *as soluções correspondentes* $u = u(t,x)$ *e* $v = v(t,x)$ *satisfazem*

$$\|\lambda(t)(u_t(t,\cdot), \nabla_x u(t,\cdot)) - (v_t(t,\cdot), \nabla_x v(t,\cdot))\|_{L^2} \to 0 \tag{0.24}$$

quando $t \to \infty$.

Objetivo desta tese

Nesta tese estamos interessados em resultados sobre o comportamento a longo prazo para soluções do problema de Cauchy para a equação de Klein-Gordon com potencial não efetivo dependente do tempo

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{0.25}$$

onde $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Mais precisamente, focamos em resultados sobre conservação de energia generalizada, *scattering* e *scattering* modificado e estimativas $L^p - L^q$.

Também estamos interessados em estabelecer resultados para o seguinte problema semi-linear de Cauchy scale-invariante com massa e dissipação

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{\mu_2^2}{(1+t)^2} u = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{0.26}$$

onde $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. O objetivo é entender a interação entre μ_1 e μ_2 para provar existencia global no tempo de soluções de energia para dados iniciais pequenos em um espaço adequado e para valores apropriados de $p > 1$. Resultado de blow-up também será provado para uma escolha especial de μ_1 e μ_2.

Conteúdo desta tese

O conteúdo desta tese é apresentado como se segue: No Capítulo 2 estudamos o problema de Klein-Gordon com potencial dependente do tempo permitindo "oscilações muito lentas" (de acordo com a classificação de [45, 46]) focando em resultados sobre conservação de energia generalizada e resultados *scattering*. No Capítulo 3 nós estabelecemos estimativas de decaimento $L^p - L^q$ para o problema de Cauchy para a equação de Klein-Gordon com massa não efetiva dependente do tempo. Iniciamos o Capítulo 4 provando a otimalidade das estimativas obtidas no Capítulo 2 e derivamos estimativas de Strichartz para o problema de Cauchy com massa e dissipação não efetivas dependentes do tempo. No Capítulo 5 focamos na aplicação de propriedades C^m e condição de estabilização para considerar "oscilações muito rápidas" (de acordo com a classificação de [45, 46]) no coeficiente do termo da massa. Completamos esta tese considerando um problema semi-linear de Cauchy, *scale-invariant* com massa e dissipação dependentes do tempo. No Capítulo 6 usamos a teoria de funções especiais para provar estimativas lineares e consequentemente estabelecer existência global no tempo para tal problema semi-linear. Um resultado de *blow-up* completa nossas considerações.

Resultados selecionados

Resultados para modelos lineares: Vamos completar esta introdução apresentando os resultados desta tese. Por questões de simplicidade vamos assumir que o coeficiente da massa $m = m(t) \in C^m(\mathbb{R}_+)$ e satisfaz

$$|m(t)| \lesssim \frac{1}{(1+t)^\gamma}, \quad |m^{(k)}(t)| \lesssim \frac{m(t)}{(1+t)^{\gamma k}}$$

para todo $k \leq m$ e $0 < \gamma \leq 1$.

Resultado 0.1. *Se* $(1+t)m(t)^2 \in L^1$, *então para qualquer dado inicial* $(u_0, u_1) \in H^1 \times L^2$, *existe um operador linear limitado* $W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *tal que*

$$\lim_{t\to\infty} \Big\| (|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D \rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot) \Big) \Big\|_{L^2 \times L^2} = 0. \qquad (0.27)$$

Onde $u = u(t,x)$ *é a solução do problema de Cauchy para a equação de Klein-Gordon e* $v = v(t,x)$ *é a solução do problema de Cauchy para a equação de onda livre.*

Para potenciais não efetivos e não *scattering* $((1+t)m(t)^2 \notin L^1)$ temos a seguinte afirmação:

Resultado 0.2. *Suponha* $\gamma = 1$ *(oscilações muito lentas) e que*

$$m(t)^2 = \frac{\mu^2}{(1+t)^2 g(t)},$$

onde $0 < \mu < \frac{1}{4}$ *e* $g \in C^\infty(\mathbb{R}_+)$ *é uma função positiva, crescente com* $g(0) = 1$ *e*

$$|g^{(k)}(t)| \lesssim \frac{g(t)}{(1+t)^k} \quad \textit{para} \quad \textit{todo} \;\; k \in \mathbb{N}.$$

Então existe uma função positiva $\psi = \psi(t) \in C^\infty(\mathbb{R}_+)$ *tal que*

$$\limsup_{t\to\infty} 2(1+t)\frac{\psi'(t)}{\psi(t)} < 1, \qquad \Big| \frac{\psi^{(k)}(t)}{\psi(t)} \Big| \lesssim \frac{1}{(1+t)^k}$$

e

$$\int_0^\infty (1+\tau) \Big| \frac{\psi''(t)}{\psi(t)} + m(\tau)^2 \Big| d\tau \lesssim 1$$

e nós temos a estimativa $L^p - L^q$ *para as energias cinética, elástica e potencial como se segue:*

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot), p(t)u(t,\cdot))\|_{L^q} \;\; \lesssim \;\; (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}} \right)$$

para $p \in (1,2]$, p *e* q *duais,* $p(t) = (1+t)^{-1}\psi(t)$ *e regularidade* $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$.

Resultado 0.3. *Suponha* $0 < \gamma < 1$ *(oscilações muito rápidas) e*

$$m(t)^2 = \frac{\mu^2}{(1+t)^2 g(t)} + \delta(t),$$

onde $\frac{\mu^2}{(1+t)^2 g(t)}$ *é a função principal como no Resultado 1.2 e* δ *é uma função periódica limitada (função de perturbação) tal que a condição de estabilidade*

$$\Big| \int_t^\infty \delta(s) ds \Big| \leq \nu(1+t)^{\alpha-2},$$

com $\alpha \in [0,1)$, $\nu \leq \mu^2 < \frac{2-\alpha}{12}$ *e* $\gamma = \alpha + \frac{1-\alpha}{m+1}$ *é satisfeita. Então temos a seguinte estimativa para a energia da solução:*

$$\| \, (u_t(t,\cdot), \nabla_x u(t,\cdot), p(t)u(t,\cdot)) \, \|_{L^2} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2},$$

onde $p(t) = \frac{\eta(t)}{1+t}$ *e* η *são definidos em (5.5).*

Vamos considerar o seguinte problema de Cauchy para a equação de Klein-Gordon com dissipação

$$u_{tt} - \Delta u + b(t)u_t + m(t)u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{0.28}$$

onde $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $b = b(t)$ é o coeficiente da dissipação e $m = m(t)$ é o coeficiente do termo da massa satisfazendo as seguintes hipóteses:

Resultado 0.4. *Suponha que $b, m \in C^\ell(\mathbb{R}_+)$ e que para todo $k \leq \ell$ acontece*

$$\left|\frac{d^k}{dt^k}b(t)\right| \leq C_k \left(\frac{1}{1+t}\right)^{k+1} \quad \text{e} \quad \left|\frac{d^k}{dt^k}m(t)\right| \leq C_k \left(\frac{1}{1+t}\right)^{k+2}$$

e

$$\lim_{t\to\infty}(1+t)b(t) = b_0 \quad \textit{and} \quad \lim_{t\to\infty}(1+t)^2 m(t) = m_0 \tag{0.29}$$

existem e que

$$\int_1^\infty \frac{|tb(t) - b_0|^\sigma}{t}dt < \infty \quad \textit{and} \quad \int_0^\infty \frac{|t^2 m(t) - m_0|^\sigma}{t}dt < \infty,$$

seja verdadeiro com o expoente σ satisfazendo

$$\textbf{(A1)} \qquad \sigma = 1 \qquad\qquad \textit{ou} \qquad\qquad \textbf{(A2)} \qquad \sigma \in (1,2].$$

Se

$$\sigma = 1 \qquad \text{e} \qquad b_0(b_0 - 2) \leq 4m_0 \tag{0.30}$$

ou

$$\sigma \in (1,2] \qquad \text{e} \qquad b_0(b_0-2) \leq 4m_0 < (b_0-1)^2, \tag{0.31}$$

então a estimativa $L^p - L^q$

$$\big\|\big((1+t)^{-1}u(t,\cdot), u_t(t,\cdot), \nabla_x u(t,\cdot)\big)\big\|_{L^q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right)$$

acontece para $p \in (1,2]$, p e q duais com regularidade $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, onde

$$\lambda(t) = \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big).$$

Resultados para modelos semi-lineares: Considere o problema semi-linear de Cauchy com massa e dissipação *scale-invariant*

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t}u_t + \frac{\mu_2^2}{(1+t)^2}u = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{0.32}$$

com $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $p > 1$ e $\mu_1 > 0, \mu_2$ constantes reais. Defina

$$\Delta = (\mu_1 - 1)^2 - 4\mu_2^2$$

e o espaço de funções

$$\mathcal{D}_m = (H^1 \cap L^m) \times (L^2 \cap L^m),$$

com $m \in [1,2)$ e norma $\|(u,v)\|^2_{\mathcal{D}_m} = \|u\|^2_{L^m} + \|u\|^2_{H^1} + \|v\|^2_{L^m} + \|v\|^2_{H^1}$.

Resultado 0.5. *Seja $n \leq 4$, $\Delta \leq 0$ e suponha que $\mu_1 > 2$ e*

$$\begin{cases} p \geq 2 & \text{se} \quad n = 1, 2, \\ 2 \leq p \leq 3 & \text{se} \quad n = 3, \\ p = 2 = p_{GN}(4) & \text{se} \quad n = 4. \end{cases} \tag{0.33}$$

Existe uma constante positiva $\varepsilon_0 > 0$ tal que para todo $(u_0, u_1) \in \mathcal{D}_1$ com

$$\|(u_0, u_1)\|_{\mathcal{D}_1} \leq \varepsilon_0$$

existe uma única solução de energia para (0.32) em $\mathcal{C}([0, \infty), H^1) \cap \mathcal{C}^1([0, \infty), L^2)$. Além do mais, existe uma constante $C > 0$ tal que a sulução satisfaz as estimativas de decaimento

$$\begin{aligned} \|(u_t(t, \cdot), \nabla_x u(t, \cdot))\|_{L^2} &\leq C(1+t)^{-\frac{\mu_1}{2}}(1 + \ln(1+t))^{\gamma}\|(u_0, u_1)\|_{\mathcal{D}_1}, \\ \|u(t, \cdot)\|_{L^2} &\leq C(1+t)^{-\frac{\mu_1}{2}}\widetilde{q}_{\Delta}(t)\|(u_0, u_1)\|_{\mathcal{D}_1}, \end{aligned}$$

onde $\gamma = 1$ se $\Delta = 0$, $\gamma = 0$ se $\Delta < 0$ e

$$\widetilde{q}_0(t) = \begin{cases} 1 + \ln(1+t) & \text{para} \quad n > 1, \\ (\ln(1+t))^{\frac{1}{2}}\,(1 + \ln(1+t)) & \text{para} \quad n = 1, \end{cases}$$

e

$$\widetilde{q}_{\Delta}(t) = \begin{cases} 1 & \text{para} \quad n > 1, \\ (\ln(1+t))^{\frac{1}{2}} & \text{para} \quad n = 1, \end{cases}$$

para $\Delta < 0$.

Resultado 0.6. ***(Resultado Blow-up)*** *Suponha que $u \in \mathcal{C}^2\left([0, T) \times \mathbb{R}^n\right)$ seja a solução para o problema de Cauchy (0.32) com $\Delta = 1$ e dado inicial $(u_0, u_1) \in \mathcal{C}_0^2(\mathbb{R}^n) \times \mathcal{C}_0^1(\mathbb{R}^n)$ tais que $u_0, u_1 > 0$. Se*

$$p \in (1, p_{\mu_1}(n)],$$

então $T < \infty$, onde

$$p_{\mu_1}(n) = \max\left\{p_{Fuj}\left(n - 1 + \frac{\mu_1}{2}\right); p_0(n + \mu_1)\right\}. \tag{0.34}$$

Aqui $p_{Fuj}(n)$ e $p_0(n)$ denotam os expoentes de Fujita e Strauss, respectivamente (veja Capítulo 6).

1 Introduction

1.1 Motivation

Hyperbolic equations are used in physics to describe evolutionary processes with the property that information propagate with a finite speed. These processes can be found in several areas for example in the Theory of Electromagnetic Waves and Electrodynamics. One of the standard models is the free wave equation

$$u_{tt} - c^2 \Delta u = 0,$$

which describes a vibrating string for $n = 1$, membrane for $n = 2$, or elastic solid for $n = 3$. Here c denotes the speed of propagation and $\Delta = \sum_{i=1}^{n} \partial_i^2$ the Laplacian with respect to the spatial variables.

Another model of interest is the Klein-Gordon equation

$$u_{tt} - c^2 \Delta u + \Big(\frac{mc^2}{h}\Big)^2 u = 0, \tag{1.1}$$

where h is related to the Planck constant and m is a constant mass of the particle. This model was introduced by Gordon (1926) and Klein (1927) deriving a relativistic equation for a charged particle in an electromagnetic field. This equation is also used to describe dispersive wave phenomena in general, see [17].

We will discuss known properties of these two, and, of more general models, in the following sections.

1.2 Some classical wave models with and without mass

Let us at the beginning recall some results on free wave models. Consider the following Cauchy problem for the free wave equation:

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.2}$$

with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. The Cauchy problem (1.2) is H^s well-posed, i.e., if $u_0 \in H^s$ and $u_1 \in H^{s-1}$, then there exists for all positive T a unique solution $u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$ that depends continuously on the data (u_0, u_1).

If $u \in C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$, then we can define the classical energy

$$E_W(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2\big)dx, \tag{1.3}$$

and we can prove that $E_W'(u)(t) = 0$ for all $t \geq 0$, in other words, we have conservation of the energy, i.e., $E_W(u)(t) = E_W(u)(0)$ for all $t \geq 0$.

Strichartz estimates were proved in a first moment by W. von Wahl with data $(u_0, u_1) \in C_0^\infty$. In the paper [55] he proved, without using Fourier integral operators, that

$$\|(u_t(t,.), \nabla_x u(t,x))\|_{L^q} \leq C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|(u_1, \nabla_x u_0)\|_{L^{p,r}}, \tag{1.4}$$

for $n \geq 2$ with p and q from the conjugate line, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, with $1 < p \leq 2$ and regularity $r > n\left(\frac{1}{p} - \frac{1}{q}\right)$. To clarify the notations used in this thesis see the notation-guide in Chapter 7. Modern techniques like Fourier integral operators and the method of stationary phase were used by Strichartz [52] and [53], Littman [39], Brenner [7] and Pecher [43] to prove the estimate (1.4).

Another important classical wave model was introduced by Klein/Gordon in 1926,

$$u_{tt} - \Delta u + m^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.5}$$

with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $m > 0$. This problem is the so-called Cauchy problem to the Klein-Gordon equation.

The Cauchy problem (1.5) is H^s well-posed, i.e., if $u_0 \in H^s$ and $u_1 \in H^{s-1}$, then there exists for all positive T a unique solution

$$u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n))$$

that depends continuously on the data (u_0, u_1).

In this problem the mass term forces us to include into the total energy besides the elastic and the kinetic energy a third component, which is the potential energy. We can define the total energy

$$E_{KG}(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 + m^2|u(t,x)|^2\big)dx. \tag{1.6}$$

Here we can also prove that $E'_{KG}(u)(t) = 0$ for all $t \geq 0$, in other words, we have the property of conservation of the energy, too.

In comparison to the free wave equation the mass term has an improving character on the decay rate in Strichartz' estimates. W. von Wahl obtained this improvement after introducing $v = v(t, x, x_{n+1})$ by

$$v(t, x, x_{n+1}) := \exp\left(-imx_{n+1}\right) u(t,x),$$

where $x \in \mathbb{R}^n$, $x_{n+1} \in \mathbb{R}$ and $t \in \mathbb{R}_+$. This change of variables can be found in [55]. Easily we see that v is the solution to the free wave equation (1.2) with non-standard Cauchy data

$$v_0(x, x_{n+1}) := \exp\left(-imx_{n+1}\right) u_0(x), \;\; v_1(x, x_{n+1}) := \exp\left(-imx_{n+1}\right) u_1(x).$$

Hence, the new data (v_0, v_1) do not belong to $C_0^\infty(\mathbb{R}^n)$. However, the above change of variable is a motivation to guess which kind of estimates do we expect. It's possible to prove that if the Cauchy data $(u_0, u_1) \in C_0^\infty(\mathbb{R}^n)$, then the Strichartz estimates for the classical Klein-Gordon model

$$\|(u_t(t,.), \nabla_x u(t,x), u(t,.))\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|(u_0, u_1, \nabla_x u_0)\|_{L^{p,r}}, \tag{1.7}$$

are valid for $n \geq 2$ with p and q from the conjugate line, $p \in (1,2]$ and regularity $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$.

The approach using Fourier integral operators was also applied by Pecher [43] and Hörmander [30] to the classical Klein-Gordon model to obtain the estimate (1.7).

For the non-linear Klein-Gordon equation related with the classical linear Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = f(u, u_t, \nabla_x u, \nabla_x^2 u), \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.8}$$

Klainerman [34] and Shatah [50] proved the global existence of solutions to the Cauchy problem (1.8) with small data and suitable conditions for f.

1.3 Wave models with time-dependent potential

The natural question that appears is: what happens when the mass term is time-dependent? What can we say about the definition and estimates for the energy? In this section we will write known results for the Klein-Gordon Cauchy problem with time-dependent potential.

Consider the following Cauchy problem for the Klein-Gordon equation

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.9}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

We are looking for a suitable energy for models with time-dependent potential and estimates for a suitable energy. An important model that helps us to define such an energy is the scale-invariant model. It was studied in [4] in 2011 and [5] in 2012.

1.3.1 Scale-invariant models

To define a suitable energy is not a trivial thing as the following model from [5] shows: Let us consider the following Cauchy problem for Klein-Gordon equation

$$u_{tt} - \Delta u + \frac{\mu^2}{(1+t)^2} u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.10}$$

with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and a real constant $\mu \neq 0$ playing a decisive role. Since $u^* = u^*(t^*, x^*) := u(t,x)$ with $1 + t = \lambda(1 + t^*)$ and $x = \lambda x^*, \lambda > 0$ arbitrarily, solves also the Cauchy problem, a scale-invariant condition is verified.

Once satisfied the scale-invariant condition we can apply the theory of special function and introduce the energy $E^{(\mu)}(u) = E^{(\mu)}(u)(t)$ in the form

$$E^{(\mu)}(u)(t) := \frac{1}{2}\Big(\|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla_x u(t,\cdot)\|_{L^2}^2 + p_\mu(t)^2 \|u(t,\cdot)\|_{L^2}^2 \Big), \tag{1.11}$$

where

$$p_\mu(t) = \begin{cases} (1+t)^{-\frac{1}{2}}, & \mu^2 > \frac{1}{4}, \\ (1+t)^{-\frac{1}{2}} \big(1 + \ln(1+t)\big)^{-1}, & \mu^2 = \frac{1}{4}, \\ (1+t)^{-\frac{1}{2} - \frac{1}{2}\sqrt{1-4\mu^2}}, & \mu^2 \in \left(0, \frac{1}{4}\right). \end{cases} \tag{1.12}$$

Then the *generalized energy conservation*

$$p_\mu(t)^2 E^{(\mu)}(u)(0) \lesssim E^{(\mu)}(u)(t) \lesssim E^{(\mu)}(u)(0) \tag{1.13}$$

holds.

Remark 1.1. *The estimate* (1.13) *excludes a blow-up behavior of the energy $E^{(\mu)}(u)(t)$ for $t \to \infty$. Moreover, it yields a lower bound of the decay behavior for this energy. We see that the potential energy can be estimated in the following way:*

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim p_\mu(t)^{-2} E^{(\mu)}(u)(0).$$

If $\mu \to +0$, then $p_\mu(t)^{-2}$ tends to $(1+t)^2$, an asymptotic profile which is known for the potential energy of solutions to the Cauchy problem for the free wave equation. If $\mu \to \infty$, then $p_\mu(t)^{-2} = 1+t$, so the potential energy has a smaller growth for $t \to \infty$.

The solutions to the Cauchy problem (1.10) with Cauchy data $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^n)$ satisfy the Strichartz estimates (see [5])

$$\begin{aligned} \|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} &\lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right), & (1.14) \\ \|p_\mu(t)\, u(t,\cdot)\|_{L^q} &\lesssim d_\mu(t) \left(\|u_0\|_{L^{p,r}} + \|u_1\|_{L^{p,r-1}}\right) & (1.15) \end{aligned}$$

with

$$d_\mu(t) := \begin{cases} \max\left\{(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}},\, (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\right\}, & \mu^2 \geq \frac{1}{4}, \\ \max\left\{(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}-\frac{1}{2}\sqrt{1-4\mu^2}},\, (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\right\}, & \mu^2 < \frac{1}{4}, \end{cases} \tag{1.16}$$

where $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$, $\frac{1}{p}+\frac{1}{q} = 1$ with $1 < p \leq 2$. This result implies that the elastic and kinetic energies $\|\nabla_x u(t,\cdot)\|_{L^q}$ and $\|u_t(t,\cdot)\|_{L^q}$ measured in the L^q norm decrease with the wave type decay rate $\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$ as in (1.4).

The previous models inspire us to consider two different cases for the time-dependent potential. Consider the following Cauchy problem for the Klein-Gordon equation

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.17}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 1.1. *We say that the potential term $m(t)^2 u$ in* (1.17) *is effective if the time-dependent coefficient satisfies*

$$tm(t) \to \infty$$

as t tends to ∞.

Definition 1.2. *We say that the potential term $m(t)^2 u$ in* (1.17) *is non-effective if the time-dependent coefficient satisfies*

$$\limsup_{t\to\infty} (1+t) \int_t^\infty m(s)^2 ds < \frac{1}{4},$$

and if the derivatives of $m(t)^2$ satisfy the following estimates:

$$\left|\frac{d^k}{dt^k} m(t)^2\right| \lesssim (1+t)^{-(k+2)\gamma}$$

for some $0 < \gamma \leq 1$, $k = 1, 2$ in the case $\gamma = 1$ and $k = 1, 2, \cdots, m$ otherwise.

Remark 1.2. *The above classification is inspired by the time decay behavior of the $L^p - L^q$ estimates. If the decays are related with the decays of the free wave equation, then we call the mass non-effective. If the decays are related with the decays of the classical Klein-Gordon equation, then we call the mass effective.*

1.3.2 Wave models with effective potential

The PhD thesis [4] are devoted to study the effective case. The author studies decreasing coefficients $m = m(t)$ which satisfy among other things $\lim_{t\to\infty} tm(t) = \infty$. In this case models (1.17) are called models with effective potential . In [4] the case was considered, where $m(t) = \lambda(t)\nu(t) \in C^M(\mathbb{R}_+),\ M \geq 2$, with the shape function $\lambda = \lambda(t)$ and a small perturbation of the mass given by the oscillating function $\nu = \nu(t)$. If we define the Klein-Gordon energy

$$E^{(KG)} = \frac{1}{2}\int_{\mathbb{R}^n} \big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 + m(t)|u(t,x)|^2\big)dx, \tag{1.18}$$

then the generalized energy conservation holds, that is,

$$\lambda(t)E^{(KG)}(u)(0) \lesssim E^{(KG)}(u)(t) \lesssim E^{(KG)}(u)(0). \tag{1.19}$$

If $m(t) = \lambda(t)\nu(t) \in C^\infty(\mathbb{R}_+)$, under suitable hypothesis for λ and ν, we can prove for all $t \geq 0$ the following Strichartz estimates:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot), \lambda(t)u(t,\cdot))\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right),$$

where $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p \leq 2$. This type of decay estimate is known as Klein-Gordon type decay estimate with the Klein-Gordon decay rate $\frac{n}{2}\big(\frac{1}{p} - \frac{1}{q}\big)$ on the conjugate line.

1.3.3 Wave type decay estimates

We consider the Klein-Gordon Cauchy problem

$$u_{tt} - \Delta u + m(t)^2 u = 0,\ \ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x), \tag{1.20}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Let $m = m(t) \in C^\infty(\mathbb{R}_+)$ satisfying the following properties:

$$\begin{aligned} &(B1) \quad m(t) \in L^1(\mathbb{R}_+), \\ &(B2) \quad |d_t^k m(t)| \lesssim C_k(1+t)^{-k}, k = 0,1,2,\cdots, \end{aligned}$$

for all t, where C_k are positive constants. Then [4] shows the following result:

Theorem 1.1. *Let $m = m(t) \in C^\infty(\mathbb{R}_+)$ satisfy $(B1)$ and $(B2)$. Then for all times t the $L^p - L^q$ decay estimate*

$$\|\,(u_t(t,\cdot), \nabla_x u(t,\cdot))\,\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right)$$

holds for $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, with $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

1.3.4 Scattering result

The scattering theory compares the behavior of the solution of the free wave problem to the solution of the perturbed problem in distant time. The main goal is construct an operator that maps initial data of Cauchy problem to initial data of an perturbed Cauchy problem. This operator is denoted as Moeller wave operator.

O principal objetivo é construir um operador que mapeia dados iniciais do problema de Cauchy para a onda livre em dados iniciais para o problema de Cauchy perturbado. Tal operador é denominado operador de onde de Moeller.

Let us consider that u solves the Cauchy problem for the Klein-Gordon equation (1.9) and that v solves the Cauchy problem for free the wave equation (1.2).
Let us assume the conditions

$$m \in L^1(\mathbb{R}_+), \ \ m(t)(1+t) \leq C \ \text{ for } \ t \in [0,\infty). \tag{1.21}$$

Then the following result can be found in [4], Theorem 3.26:

Theorem 1.2. *Let the coefficient* $m = m(t)$ *satisfy* (1.21). *There exists a scattering operator* $W_+ = W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *such that the Cauchy data to the problems* (1.9) *and* (1.2) *are related by* $(|D|v_0, v_1)^T = W_+(D)(\langle D\rangle u_0, u_1)^T$. *Then for the solutions of the problems* (1.9) *and* (1.2) *the asymptotic equivalence*

$$\Big\|(|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D\rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot)\Big)\Big\|_{L^2\times L^2} \to 0 \tag{1.22}$$

holds as t tends to infinity.

1.4 Some more information about the thesis

What remains open in the thesis [4] is to explain qualitative properties of solutions to the Klein-Gordon Cauchy problem with a time-dependent potential which does not allow on the one hand scattering to free waves and in the other hand effective mass. Typical examples are decreasing $m(t)$ satisfying $m \notin L^1(\mathbb{R}_+)$ and $\lim_{t\to\infty} tm(t) = 0$. The present thesis concerns with this topic, i.e., the goal is to define a suitable energy and derive estimates for this energy for the Klein-Gordon Cauchy problem with non-effective time-dependent potential.

To achieve this goal we will apply a change of variable in the Klein-Gordon time-dependent Cauchy problem and transform it into a damped time-dependent Cauchy problem and use known results for this case, see [59]. In the next section we will collect some results on the Cauchy problem for the wave equation with time-dependent dissipation.

1.4.1 Some more explanations about the background

A further problem of interest is the Cauchy problem for the wave equation with time-dependent dissipation

$$u_{tt} - \Delta u + b(t)u_t = 0, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), \tag{1.23}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. If $b(t) = \mu(1+t)^{-1}$ with $\mu > 0$, that is, we are interested in the scale-invariant case, we can find in [58] the following $L^p - L^q$ estimates:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \lesssim (1+t)^{\max\{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\mu}{2}, -n(\frac{1}{p}-\frac{1}{q})-1\}} \big(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\big),$$

where $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, $\frac{1}{p} + \frac{1}{q} = 1$, with $1 < p \le 2$.
Thus, it appears that the parameter μ influences the decay rate. This case separates the effective case from the non-effective case, here we say that the dissipation is effective if the time decays of the $L^p - L^q$ estimates for the energy are related with the decays of the wave equation with constant coefficients and we say that the dissipation is effective if the time decays of the $L^p - L^q$ estimates for the energy are related with decays of the free wave equation. Wirth proved $L^p - L^q$ estimates for both cases in [59] and [60], respectively. The important case for us is the non-effective case, i.e., if the coefficient $b = b(t)$ decays faster than the critical term $b(t) = \mu(1+t)^{-1}$. This implies the following $L^p - L^q$ decay estimate:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})} \big(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\big),$$

where $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$, $\frac{1}{p} + \frac{1}{q} = 1$, with $1 < p \le 2$. Here

$$\lambda(t) = \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big).$$

Moreover, if $b \in L^1$, then Wirth proved a scattering result, that the solutions behave asymptotically like the solutions of the free wave equation.

If we consider v as a solution of the free wave equation (1.2), then Wirth proved the following modified scattering result.

Theorem 1.3. *For any initial data* $(u_1, u_2) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *there exists a linear, bounded operator* $W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *such that for Cauchy data* $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *of* (1.23) *and associated data* $(v_0, v_2) = W_+(u_0, u_1)$ *to* (1.2) *the corresponding solutions* $u = u(t,x)$ *and* $v = v(t,x)$ *satisfy*

$$\|\lambda(t)(u_t(t,\cdot), \nabla_x u(t,\cdot)) - (v_t(t,\cdot), \nabla_x v(t,\cdot))\|_{L^2} \to 0 \qquad (1.24)$$

as $t \to \infty$.

1.5 Objectives of this thesis

In this thesis we are interested in statements about the long-time behaviour of the solutions to Klein-Gordon problems with time-dependent non-effective potential

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \qquad (1.25)$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. More precisely, we focus on results about generalized energy conservation, scattering and modified scattering states and $L^p - L^q$ estimates.

We are also interested to establish results for the following semi-linear scale-invariant Cauchy problem with mass and dissipation

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t}u_t + \frac{\mu_2^2}{(1+t)^2}u = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \qquad (1.26)$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. The goal is to understand the interplay between μ_1 and μ_2 to prove the global existence in time of small data energy solutions in a suitable function space and for appropriate $p > 1$. We will also prove blow-up results for a special choice of μ_1 and μ_2.

1.5.1 Content of this thesis

The content is as follows: In Chapter 2 we study a Klein-Gordon problem with time-dependent potential allowing "very slow oscillations" (according with the definitions in [45] and [46]) with focus on results about generalized energy conservation and scattering results. In Chapter 3 we will derive $L^p - L^q$ decay estimates for the Klein-Gordon Cauchy problem with non-effective time-dependent mass. In Chapter 4 we start proving the sharpness of the energy estimate obtained in Chapter 2 and we derive Strichartz estimates for the Cauchy problem with non-effective time-dependent damping and mass. In Chapter 5 we focus to apply C^m properties and stabilization conditions to consider "very fast oscillations" (according with the definitions in [45] and [46]) in the coefficient of the mass term. We complete this thesis by considering a semi-linear scale-invariant time-dependent Cauchy problem with mass and dissipation. In Chapter 6 we use the special function theory to prove linear estimates and consequently establish global existence in time for this semi-linear problem. Blow-up results complete our considerations.

1.5.2 Selected Results

Results for linear models: We will complete this introduction with selected results of this thesis. For simplicity we will assume that the mass coefficient $m = m(t) \in C^m(\mathbb{R}_+)$ and satisfies

$$|m(t)| \lesssim \frac{1}{(1+t)^\gamma}, \quad |m^{(k)}(t)| \lesssim \frac{m(t)}{(1+t)^{\gamma k}}$$

for all $k \leq m$ and $0 < \gamma \leq 1$.

Result 1.1. *If $(1+t)m(t)^2 \in L^1$, then for any initial data $(u_0, u_1) \in H^1 \times L^2$, there exists a linear, bounded operator $W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that*

$$\lim_{t\to\infty} \Big\| (|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D \rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot) \Big) \Big\|_{L^2 \times L^2} = 0. \tag{1.27}$$

Here $u = u(t,x)$ is the solution to the Klein-Gordon Cauchy problem and $v = v(t,x)$ is the solution to the free wave equation.

For non-effective and non-scattering ($(1+t)m(t)^2 \notin L^1$) potential we have the following statement:

Result 1.2. *Assume $\gamma = 1$ (very slow oscillations) and that*

$$m(t)^2 = \frac{\mu^2}{(1+t)^2 g(t)},$$

where $0 < \mu < \frac{1}{4}$ and $g \in C^\infty(\mathbb{R}_+)$ is a positive, increasing function with $g(0) = 1$ and

$$|g^{(k)}(t)| \lesssim \frac{g(t)}{(1+t)^k} \quad \text{for} \quad \text{all } k \in \mathbb{N}.$$

Then there exists a positive function $\psi = \psi(t) \in C^\infty(\mathbb{R}_+)$ such that

$$\limsup_{t\to\infty} 2(1+t)\frac{\psi'(t)}{\psi(t)} < 1, \qquad \Big|\frac{\psi^{(k)}(t)}{\psi(t)}\Big| \lesssim \frac{1}{(1+t)^k}$$

and

$$\int_0^\infty (1+\tau)\Big|\frac{\psi''(t)}{\psi(t)} + m(\tau)^2\Big|d\tau \lesssim 1$$

and we have the $L^p - L^q$ estimates for the kinetic, elastic and potential energy as follows:

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot), p(t)u(t,\cdot))\|_{L^q} \quad\lesssim\quad (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right)$$

for $p \in (1,2]$, p and q on the conjugate line, $p(t) = (1+t)^{-1}\psi(t)$, and with regularity $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$.

Result 1.3. *Assume $0 < \gamma < 1$ (very fast oscillations) and*

$$m(t)^2 = \frac{\mu^2}{(1+t)^2 g(t)} + \delta(t),$$

where $\frac{\mu^2}{(1+t)^2 g(t)}$ is the shape function as in Result 1.2 and δ is a bounded oscillating function (perturbation function) such that the stabilization condition

$$\Big|\int_t^\infty \delta(s)ds\Big| \le \nu(1+t)^{\alpha-2},$$

with $\alpha \in [0,1)$, $\nu \le \mu^2 < \frac{2-\alpha}{12}$ and $\gamma = \alpha + \frac{1-\alpha}{m+1}$ holds true. Then we have the following energy estimate for the solution:

$$\|\,(u_t(t,\cdot), \nabla_x u(t,\cdot), p(t)u(t,\cdot))\,\|_{L^2} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2},$$

where $p(t) = \frac{\eta(t)}{1+t}$ and η is defined in (5.5).

Let us consider the following Cauchy problem for damped Klein-Gordon equations

$$u_{tt} - \Delta u + b(t)u_t + m(t)u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.28}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $b = b(t)$ is the coefficient in the dissipative term and $m = m(t)$ is the coefficient in the mass term under the following assumptions:

Result 1.4. *Suppose that $b, m \in C^\ell(\mathbb{R}_+)$ and that for all $k \le \ell$ it holds*

$$\left|\frac{d^k}{dt^k}b(t)\right| \le C_k \left(\frac{1}{1+t}\right)^{k+1} \quad \textit{and} \quad \left|\frac{d^k}{dt^k}m(t)\right| \le C_k \left(\frac{1}{1+t}\right)^{k+2}$$

and

$$\lim_{t\to\infty}(1+t)b(t) = b_0 \quad \textit{and} \quad \lim_{t\to\infty}(1+t)^2 m(t) = m_0 \tag{1.29}$$

exist and that

$$\int_1^\infty \frac{|tb(t)-b_0|^\sigma}{t}dt < \infty \quad \text{and} \quad \int_0^\infty \frac{|t^2m(t)-m_0|^\sigma}{t}dt < \infty,$$

holds true with an exponent σ satisfying

(A1) $\sigma = 1$ *or* **(A2)** $\sigma \in (1,2]$.

If

$$\sigma = 1 \qquad \text{and} \qquad b_0(b_0-2) \le 4m_0 \tag{1.30}$$

or

$$\sigma \in (1,2] \qquad \text{and} \qquad b_0(b_0-2) \le 4m_0 < (b_0-1)^2, \tag{1.31}$$

then the $L^p - L^q$ estimates

$$\big\|\big((1+t)^{-1}u(t,\cdot), u_t(t,\cdot), \nabla_x u(t,\cdot)\big)\big\|_{L^q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\big)$$

hold true for $p \in (1,2]$, p and q from the conjugate line and with regularity $r = n\big(\frac{1}{p}-\frac{1}{q}\big)$, where

$$\lambda(t) = \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big).$$

Results for semi-linear models: Consider the semi-linear Cauchy problem with scale-invariant mass and dissipation

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t}u_t + \frac{\mu_2^2}{(1+t)^2}u = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.32}$$

with $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $p > 1$ and $\mu_1 > 0, \mu_2$ real constants. Define

$$\Delta = (\mu_1 - 1)^2 - 4\mu_2^2$$

and the function space

$$\mathcal{D}_m = (H^1 \cap L^m) \times (L^2 \cap L^m),$$

with $m \in [1,2)$ and the norm $\|(u,v)\|^2_{\mathcal{D}_m} = \|u\|^2_{L^m} + \|u\|^2_{H^1} + \|v\|^2_{L^m} + \|v\|^2_{H^1}$.

Result 1.5. *Let $n \le 4$, $\Delta \le 0$ and suppose that $\mu_1 > 2$ and*

$$\begin{cases} p \ge 2 & \text{if} \;\; n = 1,2, \\ 2 \le p \le 3 & \text{if} \;\; n = 3, \\ p = 2 = p_{GN}(4) & \text{if} \;\; n = 4. \end{cases} \tag{1.33}$$

There exists a constant $\varepsilon_0 > 0$ such that for all $(u_0,u_1) \in \mathcal{D}_1$ with

$$\|(u_0,u_1)\|_{\mathcal{D}_1} \le \varepsilon_0$$

there exists a unique energy solution to (1.32) in $\mathcal{C}([0,\infty),H^1) \cap \mathcal{C}^1([0,\infty),L^2)$. Moreover, there exists a constant $C > 0$ such that the solution satisfies the decay estimates

$$\begin{aligned} \|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} &\le C(1+t)^{-\frac{\mu_1}{2}}(1+\ln(1+t))^\gamma \|(u_0,u_1)\|_{\mathcal{D}_1}, \\ \|u(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(t)\|(u_0,u_1)\|_{\mathcal{D}_1}, \end{aligned}$$

where $\gamma = 1$ *if* $\Delta = 0$, $\gamma = 0$ *if* $\Delta < 0$ *and*

$$\widetilde{q}_0(t) = \begin{cases} 1 + \ln(1+t) & \text{for} \quad n > 1, \\ (\ln(1+t))^{\frac{1}{2}} (1 + \ln(1+t)) & \text{for} \quad n = 1, \end{cases}$$

and

$$\widetilde{q}_\Delta(t) = \begin{cases} 1 & \text{for} \quad n > 1, \\ (\ln(1+t))^{\frac{1}{2}} & \text{for} \quad n = 1, \end{cases}$$

for $\Delta < 0$.

Result 1.6. ***(Blow-up result)*** *Assume that* $u \in \mathcal{C}^2([0,T) \times \mathbb{R}^n)$ *is a solution to* (1.32) *with* $\Delta = 1$ *and initial data* $(u_0, u_1) \in \mathcal{C}_0^2(\mathbb{R}^n) \times \mathcal{C}_0^1(\mathbb{R}^n)$ *such that* $u_0, u_1 \geq 0$ *and* $(u_0, u_1) \neq (0,0)$. *If*

$$p \in (1, p_{\mu_1}(n)],$$

then $T < \infty$, *where*

$$p_{\mu_1}(n) = \max\left\{ p_{Fuj}\left(n - 1 + \frac{\mu_1}{2}\right); p_0(n + \mu_1) \right\}. \tag{1.34}$$

Here $p_{Fuj}(n)$ *and* $p_0(n)$ *denote the Fujita, Strauss exponent, respectively (see Chapter 6).*

2 Generalized energy conservation

2.1 Motivation

Let us consider the following Cauchy problem for Klein-Gordon models

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{2.1}$$

where $m(t)^2u$ is a time-dependent potential. Here we consider that $m(t)^2u$ is a non-effective potential, this means, $\lim_{t\to\infty} tm(t) = 0$ and $m \notin L^1(\mathbb{R}_+)$ among other things (see Definition 1.2).

Since we are interested in energy estimates we are looking for a suitable energy depending on the solution u to (2.1) such that upper and lower bounds of the energy exist for all times t. To define such an energy we can exploit our good knowledge in this matter for the scale-invariant case (1.10).

In order to get some feeling for the behavior of solutions to (2.1) we can transform the time-dependent potential to a time-dependent damping and a new potential. If we introduce the change of variables given by $u(t,x) = \psi(t)v(t,x)$, then the Cauchy problem (2.1) takes the form

$$v_{tt} - \triangle v + 2\frac{\psi'(t)}{\psi(t)}v_t + \Big(\frac{\psi''(t)}{\psi(t)} + m(t)^2\Big)v = 0, \; v(0,x) = \frac{u_0(x)}{\psi(0)}, \; v_t(0,x) = v_1(x) \tag{2.2}$$

with $v_1(x) = \frac{u_1(x) - \frac{\psi'(0)}{\psi(0)}u_0(x)}{\psi(0)}$. Therefore, if we take ψ such that $\psi''(t) + m(t)^2\psi(t) = 0$, then we can apply directly results of [58]. The main difficulty is that, in general, it is not easy to obtain an explicit representation of ψ in terms of $m(t)^2$. Fortunately, [13] gives us sufficient conditions in order to exclude contributions to the energy coming from the time-dependent potential. A sufficient condition for that is to find a function ψ such that

$$\int_0^\infty (1+\tau)\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big|d\tau \lesssim 1. \tag{2.3}$$

For the damping term that appears, we use some ideas of [59] about asymptotic properties of solutions to wave equations with time-dependent non-effective dissipation.

We will give one example for $\psi''(t) + m(t)^2\psi(t) = 0$. Here we consider the scale-invariant model of Klein-Gordon type and the goal was to find a function ψ such that the Cauchy problem becomes a scale-invariant model for the wave equation with dissipation. This example inspired us to define a suitable energy for our case.

Example 2.1. *Consider the Cauchy problem*

$$u_{tt} - \Delta u + \frac{\mu^2}{(1+t)^2}u = 0, \;\; u(0,x) = u_0(x), \;\;\; u_t(0,x) = u_1(x), \tag{2.4}$$

and consider the change of variable $u(t,x)=\psi(t)v(t,x)$, *where*

$$\psi(t)=\exp\left(\sigma\int_0^t(1+\tau)m(\tau)^2d\tau\right)=(1+t)^{\sigma\mu^2}.$$

If we consider $\mu\in\left(0,\frac{1}{4}\right)$ *and* $2\sigma_{\pm}\mu^2=1\pm\sqrt{1-4\mu^2}$, *then the Cauchy problem* (2.4) *reduces to*

$$v_{tt}-\Delta v+\frac{2\sigma_{\pm}\mu^2}{1+t}v_t=0,\ \ v(0,x)=v_0(x),\ \ v_t(0,x)=u_1(x)-\sigma u_0(x).$$

If we choose σ_-, *then we can apply Wirth's results [59]. Then the suitable* ψ *in this case is*

$$\psi(t)=(1+t)^{\frac{1-\sqrt{1-4\mu^2}}{2}}.$$

Remark 2.1. *The function* $p(t)=(1+t)^{-1}\psi(t)$, *where* $\psi(t)$ *is from the Example 2.1 coincides with the function* $p_\mu(t)$ *in the scale-invariant case* (1.10).

2.2 Both sided energy estimates

We can not expect conservation of the energy in our case, but we are able to prove lower and upper bounds for the energy for all times t. Then we state the property of generalized energy conservation.
Let us define the generalized energy conservation property.

Definition 2.1. *If we define the energy*

$$E(u)(t):=\ \frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{L^2}+\|\nabla_x u(t,\cdot)\|^2_{L^2}+\eta(t)^2\|u(t,\cdot)\|^2_{L^2}\Big),$$

where the function $\eta=\eta(t)$ *depends on the potential term* $m=m(t)$, *we say that a solution* u *to the Klein-Gordon model* (2.1) *satisfies the generalized energy conservation property if the estimates*

$$\phi(t)^2E(u)(0)\lesssim E(u)(t)\lesssim E(u)(0)$$

hold for all times $t\geq 0$, *where* $\phi=\phi(t)$ *is a positive non-increasing function depending on the function* η.

Remark 2.2. *The simplest case is when* $\phi(t)\equiv c$. *In this case we have* $E(u)(0)\approx E(u)(t)$, *for all* $t\geq 0$.

First we will suppose that there exists a function $\psi=\psi(t)$ such that (2.3) holds. Under this assumption we shall formulate and prove our main theorem. Later we will find suitable potentials for which we can find ψ explicitly.

2.2.1 The main theorem

Let us consider the Cauchy problem of Klein-Gordon type (2.1) under the following assumptions:

Hypothesis 2.1. *Let $m(t) \in C(\mathbb{R}_+)$ satisfy*

$$|m(t)| \lesssim \frac{1}{1+t}. \tag{2.5}$$

Hypothesis 2.2. *There exists a positive increasing function $\psi \in C^2(\mathbb{R}_+)$ with $\psi(0) = 1$ such that*

$$\limsup_{t\to\infty} 2(1+t)\frac{\psi'(t)}{\psi(t)} < 1, \quad \Big|\frac{\psi''(t)}{\psi(t)}\Big| \lesssim \frac{1}{(1+t)^2}. \tag{2.6}$$

Besides (2.6) *we assume the following relation between $m(t)$ and $\psi(t)$:*

$$\int_0^\infty (1+\tau)\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big| d\tau \lesssim 1. \tag{2.7}$$

Moreover, we define the energy

$$E(u)(t) = \frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{L^2} + \|\nabla u(t,\cdot)\|^2_{L^2} + p(t)^2\|u(t,\cdot)\|^2_{L^2}\Big), \text{ where } p(t) = (1+t)^{-1}\psi(t). \tag{2.8}$$

Theorem 2.1. *Under Hypotheses 2.1 and 2.2 the solution of the Cauchy problem* (2.1) *satisfies the energy estimate*

$$E(u)(t) \lesssim E(u)(0). \tag{2.9}$$

Here we, additionally, assume that the data (u_0, u_1) belong to the energy space $H^1 \times L^2$.

Proof. The proof is divided into several steps. We perform the partial Fourier transformation of (2.1) with respect to x. If we denote by $\widehat{u}(t,\xi)$ the partial Fourier transform $F_{x\to\xi}(u)(t,\xi)$ we obtain

$$\widehat{u}_{tt} + |\xi|^2\widehat{u} + m(t)^2\widehat{u} = 0, \;\; \widehat{u}(0,\xi) = \widehat{u}_0(\xi), \;\; \widehat{u}_t(0,\xi) = \widehat{u}_1(\xi). \tag{2.10}$$

We divide the extended phase space $[0,\infty) \times \mathbb{R}^n$ into the *pseudo-differential zone* $Z_{pd}(N)$ and into the *hyperbolic zone* $Z_{hyp}(N)$ which are defined by

$$\begin{aligned} Z_{pd}(N) &= \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : (1+t)|\xi| \le N\}, \\ Z_{hyp}(N) &= \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : (1+t)|\xi| \ge N\}. \end{aligned}$$

The *separating curve* is given by

$$\theta_{|\xi|} : (0,N] \to [0,\infty), \quad (1+\theta_{|\xi|})|\xi| = N.$$

We put also $\theta_0 = \infty$, and $\theta_{|\xi|} = 0$ for any $|\xi| \ge N$. The pair (t,ξ) from the extended phase space belongs to $Z_{pd}(N)$ (resp. to $Z_{hyp}(N)$) if and only if $t \le \theta_{|\xi|}$ (resp. $t \ge \theta_{|\xi|}$).

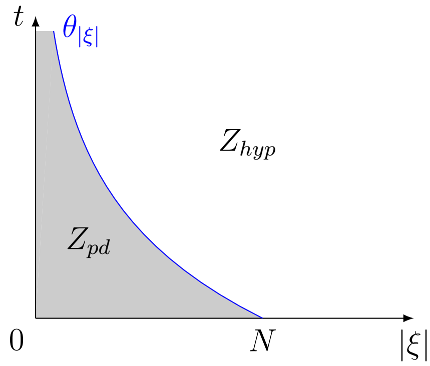

Fig. 2.1: Sketch of the zones.

We define the micro-energy

$$U(t,\xi) = \Big(h(t,\xi)\widehat{u}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \tag{2.11}$$

where

$$h(t,\xi) = \frac{1}{1+t}\phi_{pd}(t,\xi) + i|\xi|\phi_{hyp}(t,\xi).$$

Here $\phi_{pd}(t,\xi)$ is a characteristic function related to the pseudo-differential zone and $\phi_{hyp}(t,\xi)$ is a characteristic function related to the hyperbolic zone. We introduce $\phi_{hyp}(t,\xi) = \chi\left(\frac{(1+t)|\xi|}{N}\right)$ with $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(t) = 1$ for $t \leq \frac{1}{2}$, $\chi(t) = 0$ for $t \geq 2$ and $\chi'(t) \leq 0$ together with $\phi_{pd}(t,\xi) + \phi_{hyp}(\xi) = 1$. The definition of this micro-energy is related with the definition of the micro-energy from the paper [59].

Considerations in the pseudo-differential zone

In the pseudo-differential zone $Z_{pd}(N)$ the micro-energy (2.11) reduces to

$$U = \Big(\frac{\widehat{u}}{1+t}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \quad U_0(\xi) = \Big(\widehat{u}_0(\xi), \widehat{u}_1(\xi) - \frac{\psi'(0)}{\psi(0)}\widehat{u}_0(\xi)\Big)^T, \ \text{ and } \ U = \psi(t)\widetilde{U}.$$

So we have

$$\partial_t\widetilde{U}(t,\xi) = \mathcal{A}(t,\xi)\widetilde{U} := \begin{pmatrix} -\frac{1}{1+t} & \frac{1}{1+t} \\ -(1+t)\left(\frac{\psi''}{\psi} + m(t)^2 + |\xi|^2\right) & -2\frac{\psi'(t)}{\psi(t)} \end{pmatrix}\widetilde{U}. \tag{2.12}$$

We want to prove that the fundamental solution $E = E(t,s,\xi)$ to (2.12), that is, the solution to

$$\partial_t E = \mathcal{A}(t,\xi)E\,, \quad E(s,s,\xi) = I,$$

satisfies the estimate $\|E(t,0,\xi)\| \lesssim \psi(t)^{-2}$ for all $t \in [0, \theta_{|\xi|}]$. If we put $E = (E_{ij})_{i,j=1,2}$, then we can write for $j = 1,2$ the following system of coupled integral equations of Volterra type:

$$E_{1j}(t,0,\xi) \ = \ (1+t)^{-1}\Big(\delta_{1j} + \int_0^t E_{2j}(\tau,0,\xi)d\tau\Big), \tag{2.13}$$

$$E_{2j}(t,0,\xi) = \quad \psi(t)^{-2}\Big(\delta_{2j} - \int_0^t (1+\tau)\psi(\tau)^2\Big(\frac{\psi''}{\psi}(\tau) + m(\tau)^2 + |\xi|^2\Big)E_{1j}(\tau,0,\xi)d\tau\Big). \tag{2.14}$$

By replacing (2.14) into (2.13) and after integration by parts we get

$$\begin{aligned} E_{1j}(t,0,\xi) = (1+t)^{-1}\Big(\delta_{1j} + \delta_{2j}\int_0^t \psi(\tau)^{-2}d\tau\Big) - (1+t)^{-1} \\ \times \int_0^t (1+\tau)\psi(\tau)^2\Big(\frac{\psi''}{\psi}(\tau) + m(\tau)^2 + |\xi|^2\Big)E_{1j}(\tau,0,\xi)\int_\tau^t \psi(s)^{-2}ds d\tau. \end{aligned} \tag{2.15}$$

By using (2.6) (see Proposition 7 of [59]) we have

$$\int_0^t \psi(s)^{-2}ds \approx \frac{t}{\psi(t)^2}, \tag{2.16}$$

and $\frac{t}{\psi(t)^2}$ is increasing for large t. Introducing

$$h_j(t,\xi) := \|E_{1j}(t,0,\xi)\|\psi(t)^2$$

and by using $\psi(t)^2 \le 1+t$ (see (2.6)) for large t we conclude from (2.15) and (2.16) that

$$h_j(t,\xi) \le C + C\int_0^t (1+\tau)\Big(\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big| + |\xi|^2\Big)h_j(\tau,\xi)d\tau.$$

Applying Gronwall's type inequality we conclude

$$h_j(t,\xi) \le C\exp\Big(C\int_0^t (1+\tau)\Big(\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big| + |\xi|^2\Big)d\tau\Big).$$

In $Z_{pd}(N)$ we have $(1+t)|\xi| \le N$. So, from the last estimate we get

$$h_j(t,\xi) \le C\exp\Big(C\int_0^t (1+\tau)\Big(\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big|\Big)d\tau\Big).$$

Finally, by using (2.7) we get $\|E_{1j}(t,0,\xi)\| \lesssim \psi(t)^{-2}$. From the boundedness of $\|E_{1j}(t,0,\xi)\|\psi(t)^2$, using again (2.7), we can estimate $\|E_{2j}(t,0,\xi)\| \lesssim \psi(t)^{-2}$. Therefore, we proved $\|E(t,0,\xi)\| \lesssim \psi(t)^{-2}$ for all $t \in [0,\theta_{|\xi|}]$. This gives

$$\|U(t,\xi)\| \le C\psi(t)^{-1}\|U_0(\xi)\| \text{ for all } t \in (0,\theta_{|\xi|}]. \tag{2.17}$$

Considerations in the hyperbolic zone

In the hyperbolic zone $Z_{hyp}(N)$ the micro-energy (2.11) reduces to

$$U = \Big(i|\xi|\widehat{u}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \quad U_0(\xi) = \Big(i|\xi|\widehat{u}(\theta_{|\xi|},\xi), \widehat{u}_t(\theta_{|\xi|},\xi) - \frac{\psi'(\theta_{|\xi|})}{\psi(\theta_{|\xi|})}\widehat{u}(\theta_{|\xi|},\xi)\Big)^T,$$

and $U = \psi(t)\widetilde{U}$, so that

$$\partial_t\widetilde{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i|\xi|\widetilde{U} + \begin{pmatrix} 0 & 0 \\ 0 & -2\frac{\psi'(t)}{\psi(t)} \end{pmatrix}\widetilde{U} + \begin{pmatrix} 0 & 0 \\ -\frac{\psi''}{\psi}(t) - m(t)^2 & 0 \end{pmatrix}(i|\xi|)^{-1}\widetilde{U} \tag{2.18}$$

for $t \geq \theta_{|\xi|}$ with initial datum $\widetilde{U}(\theta_{|\xi|}, \xi) = \psi(\theta_{|\xi|})^{-1} U_0(\xi)$. Let P be the diagonalizer of the principal part (with respect to powers of $|\xi|$) of (2.18) given by

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

If we put $V(t,\xi) := P^{-1}\widetilde{U}(t,\xi)$, then we get

$$\partial_t V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i|\xi| V + B_0(t,\xi)\, V + B_1(t)(i|\xi|)^{-1} V, \tag{2.19}$$

where

$$B_0(t) := -\frac{\psi'(t)}{\psi(t)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B_1(t) := -\frac{1}{2} \left(\frac{\psi''}{\psi}(t) + m(t)^2 \right) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Now we define the *second diagonalizer* that depends on the anti-diagonal entries of $B_0(t)$:

$$K(t,\xi) := \begin{pmatrix} 1 & \frac{q(t)}{2i|\xi|} \\ -\frac{q(t)}{2i|\xi|} & 1 \end{pmatrix}, \qquad q(t) = \frac{\psi'(t)}{\psi(t)}. \tag{2.20}$$

Thanks to (2.6) we have

$$\frac{|q(t)|}{|\xi|} \leq \frac{C}{(1+t)|\xi|} \leq \frac{C}{N}$$

for $t \geq \theta_{|\xi|}$, hence, $|\det K| \geq 1 - C^2/(4N^2)$. Therefore, $K(t,\xi)$ and $K^{-1}(t,\xi)$ are bounded for a sufficiently large N. We replace $V(t,\xi) =: K(t,\xi) W(t,\xi)$. We get

$$\partial_t W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i|\xi| W - \frac{\psi'(t)}{\psi(t)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W + J(t,\xi) W, \tag{2.21}$$

where $J(t,\xi) = K^{-1}(t,\xi) R(t,\xi)$ with $D_0(t,\xi) = \operatorname{diag}(-i|\xi|, i|\xi|)$, $H(t,\xi) = K(t,\xi) - I$ and

$$\begin{aligned} R &= D_0 K + B_0 K - \partial_t K - K D_0 - K \operatorname{diag} B_0 + (i|\xi|)^{-1} B_1 K \\ &= B_0 + D_0 H - H D_0 - \operatorname{diag} B_0 - H \operatorname{diag} B_0 - \partial_t H + B_0 H + (i|\xi|)^{-1} B_1 K. \end{aligned}$$

By construction the sum of the first four terms of $R(t,\xi)$ vanishes. Thanks to condition (2.6) and Hypothesis 2.1 the matrix $R(t,\xi)$, and therefore also $J(t,\xi)$, satisfies the following estimate in $Z_{hyp}(N)$:

$$\|J(t,\xi)\| \leq \frac{C}{|\xi|(1+t)^2}. \tag{2.22}$$

After substituting $W(t,\xi) =: \frac{\psi(\theta_{|\xi|})}{\psi(t)} D(t,\xi) Z(t,\xi)$, where

$$D(t,\xi) = \operatorname{diag}\Big(\exp(-i|\xi|(t - \theta_{|\xi|})), \exp(i|\xi|(t - \theta_{|\xi|})) \Big),$$

we obtain the following Cauchy problem in $Z_{hyp}(N)$:

$$\begin{cases} \partial_t Z = \widetilde{J}(t,\xi)\, Z, \qquad t \geq \theta_{|\xi|}, \\ Z(\theta_{|\xi|},\xi) = K^{-1}(\theta_{|\xi|},\xi) P^{-1} \widetilde{U}(\theta_{|\xi|},\xi), \end{cases} \tag{2.23}$$

where the matrix $\widetilde{J}(t,\xi) = D^{-1}(t,\xi)J(t,\xi)D(t,\xi)$ satisfies again (2.22). For any $s,t \geq \theta_{|\xi|}$ we have

$$\int_s^t \|\widetilde{J}(\tau,\xi)\|\,d\tau \leq C\int_{\theta_{|\xi|}}^{\infty} \frac{1}{|\xi|(1+\tau)^2}\,d\tau \leq \frac{C'}{|\xi|(1+\theta_{|\xi|})} = \frac{C'}{N},$$

hence, $\|Z(t,\xi)\| \leq C\|Z(\theta_{|\xi|},\xi)\|$ and, by using Liouville's formula we may conclude, $\|Z(t,\xi)\| \geq C'\|Z(\theta_{|\xi|},\xi)\|$. Indeed, let $E = E(t,s,\xi)$ be the fundamental solution of (2.23), then $Z(t,\xi) = E(t,\theta_{|\xi|},\xi)Z(\theta_{|\xi|},\xi)$. By Liouville's formula, $\det E(t,\theta_{|\xi|},\xi) = \exp(\int_{\theta_{|\xi|}}^t \operatorname{tr}\widetilde{J}(s,\xi)ds) \approx 1$. Therefore,

$$\|Z(\theta_{|\xi|},\xi)\| = \|E^{-1}(t,\theta_{|\xi|},\xi)Z(t,\xi)\| \leq C\|Z(t,\xi)\|.$$

Summarizing we have proved in $Z_{hyp}(N)$ the both sided estimate

$$C_1\,\frac{\psi(\theta_{|\xi|})^2}{\psi(t)^2}\,\|\widetilde{U}(\theta_{|\xi|},\xi)\|^2 \leq \|\widetilde{U}(t,\xi)\|^2 \leq C_2\,\frac{\psi(\theta_{|\xi|})^2}{\psi(t)^2}\,\|\widetilde{U}(\theta_{|\xi|},\xi)\|^2. \tag{2.24}$$

Verification

We conclude the proof of (2.9) under the use of (2.8). We claim that

$$|\xi|^2|\widehat{u}(t,\xi)|^2 + |\widehat{u}_t(t,\xi)|^2 \lesssim (1+|\xi|^2)|\widehat{u}_0(\xi)|^2 + |\widehat{u}_1(\xi)|^2 \tag{2.25}$$

and

$$p(t)^2|\widehat{u}(t,\xi)|^2 \lesssim |\widehat{u}_0(\xi)|^2 + \frac{|\widehat{u}_1(\xi)|^2}{1+|\xi|^2} \tag{2.26}$$

uniformly with respect to $\xi \in \mathbb{R}^n$. By integrating these inequalities with respect to ξ and by Plancherel's Theorem we have our desired estimate (2.9).

Let us first prove (2.25). By using Cauchy-Schwarz inequality, the first estimate in (2.6) and the considerations in the pseudo-differential zone we conclude for all $t \leq \theta_{|\xi|}$ the estimates

$$\begin{aligned}
\|U(t,\xi)\|^2 &\geq \frac{1}{(1+t)^2}|\widehat{u}(t,\xi)|^2 + |\widehat{u}_t(t,\xi)|^2 + \left|\frac{\psi'(t)}{\psi(t)}\right|^2 |\widehat{u}(t,\xi)|^2 - 2|\widehat{u}_t(t,\xi)|\left|\frac{\psi'(t)}{\psi(t)}\widehat{u}(t,\xi)\right| \\
&\geq \frac{1}{(1+t)^2}|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2 - \left|\frac{\psi'(t)}{\psi(t)}\right|^2 |\widehat{u}(t,\xi)|^2 \\
&\geq \frac{3}{4(1+t)^2}|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2 \\
&\geq \frac{3}{4N^2}|\xi|^2|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2.
\end{aligned}$$

Therefore, by using (2.17) we have for all $t \leq \theta_{|\xi|}$ the estimate

$$|\xi|^2|\widehat{u}(t,\xi)|^2 + |\widehat{u}_t(t,\xi)|^2 \lesssim \|U(t,\xi)\|^2 \lesssim \frac{1}{\psi(t)^2}\|U_0(\xi)\|^2. \tag{2.27}$$

For $t \geq \theta_{|\xi|}$ we have to glue the estimate (2.24) with (2.17). By using again Cauchy-Schwarz inequality and the first estimate in (2.6) we have

$$\Big(1 - \frac{1}{4N^2}\Big)|\xi|^2|\widehat{u}(t,\xi)|^2 + \frac{1}{2}|\widehat{u}_t(t,\xi)|^2 \lesssim \|U(t,\xi)\|^2 \ \text{ for all } \ t \geq \theta_{|\xi|}.$$

By using (2.24) we get $\|U(t,\xi)\|^2 \lesssim \|U_0(\xi)\|^2$. Moreover, since $\theta_{|\xi|} = 0$ for any $|\xi| \geq N$, then

$$\|U_0(\xi)\|^2 \lesssim (1+|\xi|^2)|\widehat{u}_0(\xi)|^2 + |\widehat{u}_1(\xi)|^2 \text{ for all } |\xi| \geq N.$$

By applying (2.17) we conclude

$$\|U_0(\xi)\|^2 \lesssim \frac{1}{\psi(\theta_{|\xi|})}\big(|\widehat{u}_0(\xi)|^2 + |\widehat{u}_1(\xi)|^2\big) \text{ for all } |\xi| \leq N.$$

Therefore, (2.25) follows by taking N sufficiently large and by using that ψ is increasing.

Now let us prove (2.26). For $t \leq \theta_{|\xi|}$ we have from (2.17) the estimate

$$|\widehat{u}(t,\xi)|^2 \lesssim \frac{(1+t)^2}{\psi(t)^2}\|U_0(\xi)\|^2.$$

In order to estimate $|\widehat{u}(t,\xi)|^2$ it will be convenient to split the considerations for the hyperbolic zone into the cases $|\xi| \leq N$ and $|\xi| \geq N$. Indeed, by definition, $\theta_{|\xi|} = 0$ for all $|\xi| \geq N$, and from (2.24) we have

$$|\widehat{u}(t,\xi)|^2 \lesssim \frac{\|U(t,\xi)\|^2}{|\xi|^2} = \frac{\psi(t)^2\|\widetilde{U}(t,\xi)\|^2}{|\xi|^2} \lesssim \frac{\|\widetilde{U}(0,\xi)\|^2}{|\xi|^2} \lesssim |\widehat{u}_0(\xi)|^2 + \frac{|\widehat{u}_1(\xi)|^2}{|\xi|^2}.$$

Since $\frac{t}{\psi(t)^2}$ is increasing for large t the same is true for $\frac{t}{\psi(t)}$ and (2.26) holds. On the other hand, for $|\xi| \leq N$, from (2.24) and (2.17) we have

$$\begin{aligned}
|\widehat{u}(t,\xi)|^2 &\lesssim \frac{\|U(t,\xi)\|^2}{|\xi|^2} \lesssim \frac{\psi(\theta_{|\xi|})^2\|\widetilde{U}(\theta_{|\xi|},\xi)\|^2}{|\xi|^2} = \frac{\|U_0(\xi)\|^2}{|\xi|^2} \\
&\lesssim |\widehat{u}(\theta_{|\xi|},\xi)|^2 + \frac{1}{|\xi|^2}\Big|\widehat{u}_t(\theta_{|\xi|},\xi) - \frac{\psi'}{\psi}(\theta_{|\xi|})\widehat{u}(\theta_{|\xi|},\xi)\Big|^2 \\
&\underset{\text{by (2.17)}}{\lesssim} \frac{(1+\theta_{|\xi|})^2}{\psi(\theta_{|\xi|})^2}\|U_0(\xi)\|^2 + \frac{(1+\theta_{|\xi|})^2}{|\xi|^2(1+\theta_{|\xi|})^2\psi(\theta_{|\xi|})^2}\|U_0(\xi)\|^2 \\
&\underset{|\xi|(1+\theta_{|\xi|})=N}{\lesssim} \frac{(1+\theta_{|\xi|})^2}{\psi(\theta_{|\xi|})^2}\|U_0(\xi)\|^2.
\end{aligned}$$

Consequently, (2.26) follows again by using that $\frac{t}{\psi(t)}$ is increasing. This completes the proof of Theorem 2.1. □

Now, the purpose is to get estimates from below for solutions to our Klein-Gordon models. If we rewrite the proof that we did in the pseudo-differential zone, under the same hypothesis of Theorem 2.1 we can prove the following corollary.

Corollary 2.1. *Under the Hypothesis 2.1 and 2.2 the following estimate holds*:

$$\|\mathcal{E}(t,s,\xi)\| \lesssim \frac{\psi(s)^2}{\psi(t)^2}, \quad \textit{for all } t,s \leq \theta_{|\xi|},$$

where $\widetilde{U}(t,\xi) = \mathcal{E}(t,s,\xi)\widetilde{U}(s,\xi)$, *and* $U = \psi(t)\widetilde{U}$.

The Corollary 2.1 and the proof of Theorem 2.1 give us the following estimates from below.

Corollary 2.2. *Let u be a solution of the Cauchy problem* (2.1)*. Under the Hypothesis 2.1 and 2.2 the inequality*

$$\frac{1}{\psi(t)^2}E(u)(0) \lesssim E(u)(t)$$

is satisfied for all times $t \geq 0$.

Proof. The proof is divided into two steps.

Considerations in the pseudo-differential zone:

In the pseudo-differential zone the micro-energy (2.11) reduces to

$$U = \Big(\frac{\widehat{u}}{1+t}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \ \ U_0(\xi) = \Big(\widehat{u}_0(\xi), \widehat{u}_1(\xi) - \frac{\psi'(0)}{\psi(0)}\widehat{u}_0(\xi)\Big)^T, \ \text{ and } \ U = \psi(t)\widetilde{U}.$$

Using $\widetilde{U}(t,\xi) = \mathcal{E}(t,s,\xi)\widetilde{U}(s,\xi)$ for all $t, s \leq \theta_{|\xi|}$ it follows for $s = 0$ that

$$\|\widetilde{U}(0,\xi)\| \ = \ \|\mathcal{E}(0,t,\xi)\widetilde{U}(t,\xi)\| \lesssim \psi(t)^2\|\widetilde{U}(t,\xi)\|.$$

Then

$$\|U(0,\xi)\|^2 \lesssim \psi(t)^2\|U(t,\xi)\|^2 \ \text{ for all } \ t \leq \theta_{|\xi|}. \tag{2.28}$$

If we use (2.27) with $t = 0$ we get

$$\frac{1}{2}|\xi|^2|\widehat{u}_0(\xi)|^2 + \frac{1}{2}|\widehat{u}_1(\xi)|^2 \lesssim \|U(0,\xi)\|^2. \tag{2.29}$$

From the definition of $U(t,\xi)$ it follows that

$$\frac{1}{2}|\widehat{u}_0(\xi)|^2 \lesssim \|U(0,\xi)\|^2.$$

Then

$$\begin{aligned}\frac{1}{2}|\widehat{u}_0(\xi)|^2 + \frac{1}{2}|\xi|^2|\widehat{u}_0(\xi)|^2 + \frac{1}{2}|\widehat{u}_1(\xi)|^2 &\lesssim \|U(0,\xi)\|^2 \\ &\lesssim \psi(t)^2\|U(t,\xi)\|^2 \ \text{ for all } \ t \leq \theta_{|\xi|}.\end{aligned}$$

Considerations in the hyperbolic zone:

In the hyperbolic zone the micro-energy (2.11) reduces to

$$U = \Big(i|\xi|\widehat{u}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \quad U_0(\xi) = \Big(i|\xi|\widehat{u}(\theta_{|\xi|},\xi), \widehat{u}_t(\theta_{|\xi|},\xi) - \frac{\psi'(\theta_{|\xi|})}{\psi(\theta_{|\xi|})}\widehat{u}(\theta_{|\xi|},\xi)\Big)^T,$$

and $U = \psi(t)\widetilde{U}$. It follows from (2.24) that for all $t \geq \theta_{|\xi|}$ we have

$$\|U(t,\xi)\|^2 \approx \|U(\theta_{|\xi|},\xi)\|^2.$$

Using the considerations in the pseudo-differential zone we conclude that

$$\frac{1}{2}|\widehat{u}_0(\xi)|^2 + \frac{1}{2}|\xi|^2|\widehat{u}_0(\xi)|^2 + \frac{1}{2}|\widehat{u}_1(\xi)|^2 \lesssim \psi(\theta_{|\xi|})^2\|U(\theta_{|\xi|},\xi)\|^2 \lesssim \psi(t)^2\|U(t,\xi)\|^2.$$

Conclusion

We proved that

$$\frac{1}{2}|\widehat{u}_0(\xi)|^2 + \frac{1}{2}|\xi|^2|\widehat{u}_0(\xi)|^2 + \frac{1}{2}|\widehat{u}_1(\xi)|^2 \lesssim \psi(t)^2\|U(t,\xi)\|^2$$

for all $(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then, using Plancherel's formula we get that

$$\frac{1}{\psi(t)^2}E(u)(0) \lesssim \|U(t,\xi)\|^2 \lesssim E(u)(t).$$

□

Therefore we conclude that if u is a solution of the Cauchy problem (2.1), then u satisfies the generalized energy conservation property, i.e., we have the following theorem.

Theorem 2.2. *Under Hypotheses 2.1 and 2.2 the solution of the Cauchy problem (2.1) satisfies the generalized energy conservation property*

$$\frac{1}{\psi(t)^2}E(u)(0) \lesssim E(u)(t) \lesssim E(u)(0). \tag{2.30}$$

Here we, additionally, assume that the data (u_0, u_1) belong to the energy space $H^1 \times L^2$.

Remark 2.3. *Applying Theorem 2.1 gives the following estimate for the potential energy:*

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim (1+t)^2\psi(t)^{-2}E(u)(0).$$

Remark 2.4. *After differentiation of the Klein-Gordon energy $E_{KG}(u)(t)$ with respect to t we get*

$$E'_{KG}(u)(t) = m(t)m'(t)\int_{\mathbb{R}^n}|u(t,x)|^2dx.$$

If $m(t)$ is a positive decreasing function, then $E_{KG}(u)(t) \le E_{KG}(u)(0)$. In particular

$$\|u_t(t,\cdot)\|_{L^2}^2 + \|\nabla_x u(t,\cdot)\|_{L^2}^2 \lesssim E(u)(0).$$

What remains to prove in this case is the desired estimate to $\|u(t,\cdot)\|_{L^2}^2$ in Theorem 2.1. However, one needs to be more careful if m has some oscillations.

2.2.2 Explicit representation of ψ

At first we pose a question:
To which models can we find an explicit representation for ψ such that condition (2.7) is satisfied?

If we consider the Cauchy problem (2.1) with a coefficient $m = m(t)$ having the special structure

$$m(t) = \frac{\mu}{(1+t)g(t)} \text{ with a positive constant } \mu, \tag{2.31}$$

and a suitable function $g = g(t)$, then we are able to answer that question under some conditions for $g(t)$ as follows:

Hypothesis 2.3. *Let $g \in C^1(\mathbb{R}_+)$ be a positive, increasing function with $g(0) = 1$ satisfying*

$$g'(t) \lesssim \frac{g(t)}{1+t}. \tag{2.32}$$

Remark 2.5. *From (2.32) functions m which are given by (2.31) satisfy $|m'(t)| \lesssim \frac{m(t)}{1+t}$.*

The following remark shows how to define ψ, at least for coefficients (2.31), such that Hypothesis 2.2 holds.

Remark 2.6. *Consider $m(t)$ as in (2.31). If $(1+t)m(t)^2 \in L^1$, then we take $\psi(t) \equiv 1$. If $(1+t)m(t)^2 \notin L^1$, then we suppose that $\frac{1}{(1+t)g(t)^4} \in L^1$. If we consider*

$$\psi(t) = \exp\left(\sigma \int_0^t \frac{\mu^2}{(1+\tau)g(\tau)^2} d\tau\right),$$

then

$$\frac{\psi''(t)}{\psi(t)} = \frac{\sigma^2\mu^4}{(1+t)^2 g(t)^4} - \frac{\sigma\mu^2}{(1+t)^2 g(t)^2} - \frac{2\sigma\mu^2 g'(t)}{(1+t)g(t)^3}.$$

Therefore, if we choose $\sigma = 1$ we have

$$\frac{\psi''(t)}{\psi(t)} + m(t)^2 = \frac{\mu^4}{(1+t)^2 g(t)^4} - \frac{2\mu^2 g'(t)}{(1+t)g(t)^3}.$$

Then $(1+t)\Big(\frac{\psi''(t)}{\psi(t)} + m(t)^2\Big) \in L^1$, and we can apply Theorem 2.1.

Taking into consideration the previous remark let us suppose the following condition:

Hypothesis 2.4. *There exists an integer $N \geq 0$ such that*

$$\int_0^\infty \frac{1}{(1+\tau)g(\tau)^{2(N+1)}} d\tau \lesssim 1. \tag{2.33}$$

Remark 2.7. *From (2.33) we get $\lim_{t\to\infty} g(t) = \infty$. This implies $tm(t) \to 0$ as $t \to \infty$. So, under Hypothesis 2.4 we really consider a class of non-effectives masses $m(t)^2 u$ in (2.1).*

For coefficients (2.31) we can explicitly give the function ψ in Hypotheses 2.2. Under Hypothesis 2.4 it turns out that in the case $(1+t)m(t)^2 \in L^1$ we can take $\psi \equiv 1$. Otherwise, we choose

$$\psi(t) = \exp\Big(\sum_{k=1}^N \gamma_k \mu^{2k} \int_0^t \frac{1}{(1+\tau)g(\tau)^{2k}} d\tau\Big), \;\; \gamma_k = \sum_{\ell=1}^{k-1} \gamma_\ell \gamma_{k-\ell}, \; \gamma_1 = 1. \tag{2.34}$$

Remark 2.8. *The sequence $\{\gamma_k\}_k$ in (2.34) is well-known as Segner's recurrence formula given by Segner in 1758. It gives the solution to Euler's polygon division problem. The solution is described by the Catalan numbers which are given by the explicit formula [35]*

$$\gamma_k = \frac{(2k-2)!}{k!(k-1)!}. \tag{2.35}$$

By using (2.35) *one can explicitly compute the radius of convergence* R *of the power series* $\sum_{k=1}^{\infty}\gamma_k\mu^{2k}$ *by*

$$R=\lim_{k\to\infty}\Big|\frac{\gamma_k}{\gamma_{k+1}}\Big|=\lim_{k\to\infty}\frac{k^2}{2k(2k-1)}=\frac{1}{4},$$

and the series converges uniformly for $\mu^2<\frac{1}{4}$.

In the definition of the function ψ given by (2.34) we can take N as the smallest integer satisfying Hypothesis 2.4.

Theorem 2.3. *Under Hypotheses 2.3 and 2.4 the solution of the Cauchy problem* (2.1) *with* $m(t)$ *given by* (2.31) *satisfies the generalized energy conservation property*

$$\frac{1}{\psi(t)^2}E(u)(0)\lesssim E(u)(t)\lesssim E(u)(0), \tag{2.36}$$

where the energy $E(u)(t)$ *is defined by (2.8), with* $\psi(t)$ *given by* (2.34). *The data* (u_0,u_1) *are from the energy space* $H^1\times L^2$.

Proof. The desired statement will be a consequence of Theorem 2.2. It is clear that Hypothesis 2.3 implies that $m(t)$ from (2.31) satisfies Hypothesis 2.1. Moreover, in the case $(1+t)m(t)^2\in L^1$ it follows from Hypothesis 2.4 that Hypothesis 2.2 holds by taking $\psi\equiv 1$. Otherwise, we have $\lim_{t\to\infty}g(t)=\infty$, which implies the first condition of (2.6). It remains to prove that the function ψ which is given by (2.34) satisfies (2.7) and the second condition in (2.6). Indeed, by using the Cauchy product, i.e.,

$$\Big(\sum_{k=0}^{n}a_k\Big)\cdot\Big(\sum_{k=0}^{n}b_k\Big)=\sum_{k=0}^{2n}\sum_{i=0}^{k}a_ib_{k-i}-\sum_{k=0}^{n-1}\Big(a_k\sum_{i=n+1}^{2n-k}b_i+b_k\sum_{i=n+1}^{2n-k}a_i\Big)$$

with $a_0=b_0=0$ it follows by using the definition of the constants γ_k that

$$\begin{aligned}\frac{\psi''(t)}{\psi(t)}&=-\sum_{k=1}^{N}\frac{\gamma_k\mu^{2k}}{(1+t)^2g(t)^{2k}}-\sum_{k=1}^{N}\frac{2k\gamma_k\mu^{2k}g'(t)}{(1+t)g(t)^{2k+1}}+\Big(\sum_{k=1}^{N}\frac{\gamma_k\mu^{2k}}{(1+t)g(t)^{2k}}\Big)^2\\&=-\frac{\mu^2}{(1+t)^2g(t)^2}-\sum_{k=1}^{N}\frac{2k\gamma_k\mu^{2k}g'(t)}{(1+t)g(t)^{2k+1}}+\sum_{k=N+1}^{2N}\frac{\gamma_k\mu^{2k}}{(1+t)^2g(t)^{2k}}\\&\quad-\sum_{k=1}^{N-1}\Big(\frac{\gamma_k\mu^{2k}}{(1+t)g(t)^{2k}}\sum_{i=N+1}^{2N-k}\frac{\gamma_i\mu^{2i}}{(1+t)g(t)^{2i}}+\frac{\gamma_k\mu^{2k}}{(1+t)g(t)^{2k}}\sum_{i=N+1}^{2N-k}\frac{\gamma_i\mu^{2i}}{(1+t)g(t)^{2i}}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\psi''(t)}{\psi(t)}-m(t)^2&=-\sum_{k=1}^{N}\frac{2k\gamma_k\mu^{2k}g'(t)}{(1+t)g(t)^{2k+1}}+\sum_{k=N+1}^{2N}\frac{\gamma_k\mu^{2k}}{(1+t)^2g(t)^{2k}}\\&\quad-\sum_{k=1}^{N-1}\Big(\frac{\gamma_k\mu^{2k}}{(1+t)g(t)^{2k}}\sum_{i=N+1}^{2N-k}\frac{\gamma_i\mu^{2i}}{(1+t)g(t)^{2i}}+\frac{\gamma_k\mu^{2k}}{(1+t)g(t)^{2k}}\sum_{i=N+1}^{2N-k}\frac{\gamma_i\mu^{2i}}{(1+t)g(t)^{2i}}\Big).\end{aligned}$$

Therefore, by using Hypotheses 2.3 and 2.4 we get the second condition of (2.6) and (2.7), respectively. □

If we can not find any N satisfying Hypothesis 2.4, then we replace N by infinity in (2.34). In addition, in the case of $g_\infty = \lim_{t\to\infty} g(t) < \infty$ we introduce the following condition:

Hypothesis 2.5. *With γ_k from (2.34) we assume*

$$\sum_{k=1}^{\infty} \gamma_k \frac{\mu^{2k}}{g_\infty^{2k}} < \frac{1}{2} \text{ for all } \mu^2 < \frac{g_\infty^2}{4}. \tag{2.37}$$

Remark 2.9. *For given $t_0 > 0$ and $\mu^2 < \frac{g(t_0)^2}{4}$ the series*

$$\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t)^{2k}} \tag{2.38}$$

converges uniformly for all $t \geq t_0$. Indeed, by using that $1 \leq g(t)$ and by taking into account the benefit that $g(t)$ is an increasing function, then for all $t \geq t_0$ we have

$$\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t)^{2k}} \leq \sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t_0)^{2k}}.$$

By using that $\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t_0)^{2k}}$ converges for $\mu^2 < \frac{g(t_0)^2}{4}$ we can apply the Weierstrass M-test to conclude the uniformly convergence of the series in (2.38) for all $t \geq t_0$. Moreover, the power series $\sum_{k=1}^{\infty} \frac{2k\gamma_k \mu^{2k-1}}{g(t_0)^{2k}}$ has the same radius of convergence $\mu^2 < \frac{g(t_0)^2}{4}$, because it is the derivative on μ of the series $\sum_{k=1}^{\infty} \frac{\gamma_k \mu^{2k}}{g(t_0)^{2k}}$. This implies, together with (2.32), that for $\mu^2 < \frac{g(t_0)^2}{4}$, the series $\sum_{k=1}^{\infty} \frac{2k\gamma_k \mu^{2k} g'(t)}{g(t)^{2k+1}}$ converges uniformly for all $t \geq t_0$.

Remark 2.9 allows us to choose a function $\psi_{t_0} \in C^2[t_0, \infty)$ with a large t_0 if necessary, which is defined by

$$\psi_{t_0}(t) = \exp\Big(\sum_{k=1}^{\infty} \gamma_k \mu^{2k} \int_{t_0}^{t} \frac{1}{(1+\tau) g(\tau)^{2k}} d\tau \Big). \tag{2.39}$$

Theorem 2.4. *Under Hypotheses 2.3 and 2.5 the solution of the Cauchy problem (2.1) satisfies the generalized energy conservation property*

$$\frac{1}{\psi(t)^2} E(u)(0) \lesssim E(u)(t) \lesssim E(u)(0), \tag{2.40}$$

where the energy $E(u)(t)$ is defined by (2.8) with $\psi(t) = \psi_{t_0}(t)$ from (2.39). The data (u_0, u_1) are from the energy space $H^1 \times L^2$.

Proof. We can follow with a slightly modification the proof to Theorem 2.1. We split the pseudo-differential zone for $t \leq t_0$ and and for $t \geq t_0$ with a large t_0. For $t \leq t_0$ we are in a compact subset of this zone. Therefore, we only have to take into account the definition of ψ_{t_0} for large t. From Remark 2.9 the function ψ_{t_0} which is given by (2.39) is well-defined. The first condition of (2.6) is immediately satisfied if $g(t)$ goes to infinity as t goes to infinity and, otherwise, it follows from (2.37). By using the Cauchy product we obtain

$$\frac{\psi''(t)}{\psi(t)} + m(t)^2 = -\sum_{k=1}^{\infty} \frac{2k\gamma_k \mu^{2k} g'(t)}{(1+t)g(t)^{2k+1}}.$$

Therefore, $(1+t)\big(\frac{\psi''(t)}{\psi(t)} + m(t)^2\big) \in L^1$, and Hypothesis 2.2 is satisfied. By using Hypothesis 2.3 we get the second condition of (2.6) and the conclusion of Theorem 2.4 follows again from Theorem 2.2. □

2.2.3 Examples

We conclude this section with examples.

Example 2.2. *If $g(t)$ in (2.31) is given by $g(t)^2 = \ln(e+t)\cdots\ln^{[m]}(e^{[m]}+t)$ with $e^{[k+1]} = e^{e^{[k]}}$ and $\ln^{[k+1]}(t) = \ln(\ln^{[k]}(t))$, then we have (2.33) for $N = 1$, i.e., the conclusion of Theorem 2.3 holds with $\psi(t)$ given by (2.34). We have that $\psi(t) \approx (\ln^{[m]}(e^{[m]}+t))^{\mu^2}$.*

Example 2.3. *Let $g(t)^2 = (\ln(e+t))^{\gamma}$ for some $0 < \gamma < 1$. In order to have (2.33) one should take N such that $(N+1)\gamma > 1$. Then the conclusion of Theorem 2.3 holds with $\psi(t)$ given by (2.34).*

Example 2.4. *Let us consider the Cauchy problem (2.1) with $m(t) = \frac{\mu}{1+t}$ and $\mu \neq 0$, i.e., we consider the scale-invariant case from [5]. Here $g(t) \equiv 1$, hence, there does not exist any positive integer N such that (2.33) holds. In order to apply Theorem 2.4 one has to verify Hypothesis 2.5. Let us take the function ψ from Theorem 2.4 as*

$$\psi(t) = \exp\Big(\sum_{k=1}^{\infty}\int_0^t \frac{\gamma_k \mu^{2k}}{(1+\tau)}d\tau\Big) = (1+t)^{\sigma}$$

with $\sigma = \sum_{k=1}^{\infty}\gamma_k\mu^{2k}$. By using the infinite Cauchy product and from the definition of γ_k we get

$$\sigma^2 = \Big(\sum_{k=1}^{\infty}\gamma_k\mu^{2k}\Big)^2 = \sum_{n=2}^{\infty}\gamma_n\mu^{2n} = \sigma - \mu^2.$$

Therefore, $\sigma_{\pm} = \frac{1\pm\sqrt{1-4\mu^2}}{2}$ and $\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2 = 0$. If we take $\sigma_- = \frac{1-\sqrt{1-4\mu^2}}{2}$, then $2\sigma_- < 1$ and (2.37) holds. In this way we derived the decay estimate which is proposed by (1.12) and (1.13) for $\mu^2 \in \big(0, \frac{1}{4}\big)$.

Remark 2.10. *This example shows us that our choice for the function ψ which is proposed in (2.34) and (2.39) is quite optimal. It shows that our choice works in a good way for the well-known scale invariant case.*

Example 2.5. *If $g(t)^2 = \ln(\ln(e^e+t))$, then we can take for $t \geq t_0$, $t_0 \gg 1$, the function*

$$\psi(t) = \exp\Big(\sum_{k=1}^{\infty}\int_{t_0}^t \frac{\gamma_k\mu^{2k}}{(1+\tau)g(\tau)^{2k}}d\tau\Big)$$

which is well defined for $\mu^2 < \frac{g(t_0)^2}{4}$. It is clear that the condition (2.37) holds and the statement of Theorem 2.4 is applicable.

2.3 New scattering results

In this section we are interested in scattering results between the solutions of the Klein-Gordon time-dependent equation

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{2.41}$$

and the free wave equation

$$v_{tt} - \Delta v = 0, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x). \tag{2.42}$$

We are interested in a scattering result under the assumption

$$(1+t)m(t)^2 \in L^1. \tag{2.43}$$

This hypothesis is weaker than the hypothesis that was assumed in the thesis [4], in other words, (1.21) implies (2.43) but not conversely.
Before stating the result we define for any $\epsilon > 0$ the following closed subset of $L^2 \times L^2$:

$$F_\epsilon := \Big\{ U_0 \in L^2 \times L^2 : \; \widehat{U}_0(\xi) = 0 \text{ for any } |\xi| \leq \epsilon \Big\} .$$

We remark that $\mathcal{L} = \cup_{\epsilon>0} F_\epsilon$ is a dense subset of $L^2 \times L^2$.

Theorem 2.5. *We assume the Hypothesis 2.1 and* $(1+t)m(t)^2 \in L^1$. *Then, for any initial data* $(u_0, u_1) \in H^1 \times L^2$*, there exists a linear, bounded operator* $W_+(D) : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *such that*

$$\lim_{t\to\infty} \Big\| (|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D \rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot) \Big) \Big\|_{L^2 \times L^2} = 0. \tag{2.44}$$

Here $u = u(t,x)$ *is the solution to the Cauchy problem* (2.41) *and* $v = v(t,x)$ *is the solution to the Cauchy problem* (2.42)*, where the initial data are related by* $(|D|v_0, v_1) = W_+(D)(\langle D \rangle_N u_0, u_1)$. *Moreover, on the dense subset* $\mathcal{L}$ *we can state the decay rate as*

$$\Big\| (|D|v(t,\cdot), v_t(t,\cdot)) - \Big(\langle D \rangle_{\frac{N}{1+t}} u(t,\cdot), u_t(t,\cdot) \Big) \Big\|_{\mathcal{L}} \lesssim \big\| (\langle D \rangle_N u_0, u_1) \big\|_{\mathcal{L}} \int_t^\infty (1+\tau) m^2(\tau) d\tau \tag{2.45}$$

as t *goes to infinity.*

Proof. With a slightly modification we can follow the proof of Theorem 3.26 of [4]. Let us define the micro-energy U by

$$U = \big(\widetilde{h}(t,\xi)\widehat{u}, D_t\widehat{u} \big)^T, \;\; \widetilde{h}(t,\xi) = \Big(|\xi|^2 + \frac{N^2}{(1+t)^2} \Big)^{1/2}.$$

Considerations in the pseudo-differential zone

Here we consider the first order system

$$D_t U(t,\xi) = \mathcal{A}(t,\xi) U := \begin{pmatrix} \frac{D_t \widetilde{h}(t,\xi)}{\widetilde{h}(t,\xi)} & \widetilde{h}(t,\xi) \\ \frac{m(t)^2 + |\xi|^2}{\widetilde{h}(t,\xi)} & 0 \end{pmatrix} U. \tag{2.46}$$

We can get an integral representation by using the fundamental solution $E = E(t, s, \xi)$ to (2.46), i.e., the solution to

$$D_t E = \mathcal{A}(t,\xi)E\,, \quad E(s,s,\xi) = I.$$

If we put $E = (E_{ij})_{i,j=1,2}$, then we can write for $j = 1, 2$ the following system of Volterra integral equations:

$$E_{1j}(t,s,\xi) = \widetilde{h}(t,\xi)\Big(\frac{\delta_{1j}}{\widetilde{h}(s,\xi)} + \int_s^t iE_{2j}(\tau,s,\xi)d\tau\Big), \tag{2.47}$$

$$E_{2j}(t,s,\xi) = \delta_{2j} + i\int_s^t \frac{m(\tau)^2+|\xi|^2}{h(\tau,\xi)}E_{1j}(\tau,s,\xi)d\tau. \tag{2.48}$$

Under the hypothesis $(1+t)m(t)^2 \in L^1$ we derive by using Gronwall's inequality the estimate $\|E(t,s,\xi)\| \leq C$. Indeed, by replacing (2.48) into (2.47) and after integration by parts we get

$$\begin{aligned} E_{1j}(t,s,\xi) &= \widetilde{h}(t,\xi)\Big(\frac{\delta_{1j}}{\widetilde{h}(s,\xi)} + i\int_s^t \Big(\delta_{2j} + i\int_s^\tau \frac{m(\sigma)^2+|\xi|^2}{h(\sigma,\xi)}E_{1j}(\sigma,s,\xi)d\sigma\Big)d\tau\Big) \\ &\underset{integration\,by\,parts}{=} \widetilde{h}(t,\xi)\Big(\frac{\delta_{1j}}{\widetilde{h}(s,\xi)} + i\int_s^t \delta_{2j}d\tau - \int_s^t \frac{m(\tau)^2+|\xi|^2}{\widetilde{h}(\tau,\xi)}E_{1j}(\tau,s,\xi)(t-\tau)d\tau\Big). \end{aligned}$$

Then, after defining $w_{1j}(t,s,\xi) := \frac{E_{1j}(t,s,\xi)}{(1+t)\widetilde{h}(t,\xi)}$ and using that $(1+t)\widetilde{h}(t,\xi) \approx 1$ whenever $(t,\xi) \in Z_{pd}$ we get

$$\begin{aligned} \|w_{1j}(t,s,\xi)\| &\lesssim \frac{1}{(1+t)\widetilde{h}(s,\xi)} + \frac{t-s}{1+t} + \int_s^t (m(\tau)^2+|\xi|^2)\|w_{1j}(\tau,s,\xi)\|(1+\tau)\frac{t-\tau}{1+t}d\tau \\ &\lesssim 1 + \int_s^t (m(\tau)^2+|\xi|^2)\|w_{1j}(\tau,s,\xi)\|(1+\tau)d\tau. \end{aligned}$$

Applying Gronwall's inequality we conclude

$$\|w_{1j}(t,s,\xi)\| \lesssim \exp\Big(C\int_s^t (1+\tau)(m^2(\tau)+|\xi|^2)d\tau\Big).$$

Since $(1+t)|\xi| \leq N$ and $(1+t)m(t)^2 \in L^1(\mathbb{R}_+)$ it follows $\|w_{1j}(t,s,\xi)\| \lesssim 1$. Therefore, $\|E_{1j}(t,s,\xi)\| \leq C$. This estimate together with (2.48) gives us that $\|E_{2j}(t,s,\xi)\|$ is also bounded in $Z_{pd}(N)$.

Considerations in the hyperbolic zone

Here we define the wave type micro-energy

$$U_W = (|\xi|\widehat{u}, D_t\widehat{u})^T.$$

This allows to derive from (2.10) the system

$$D_t U_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |\xi| U_W + \begin{pmatrix} 0 & 0 \\ m(t)^2 & 0 \end{pmatrix} (|\xi|)^{-1} U_W. \tag{2.49}$$

As in the proof of Theorem 2.1 let P be the constant diagonalizer of the principal part of (2.49). Defining $U_1 = P^{-1}U_W$, then we get the system $(D_t - D - B_1(t,\xi))U_1 = 0$, where

$$D = \begin{pmatrix} -|\xi| & 0 \\ 0 & |\xi| \end{pmatrix}, \quad B_1(t,\xi) = \frac{m(t)^2}{2|\xi|}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Let $Q_1 = Q_1(t,s,\xi)$ be the solution of $(D_t - E_1^{-1}B_1E_1)Q_1 = 0$, $Q_1(s,s,\xi) = I$, where

$$E_1(t,s,\xi) = \begin{pmatrix} e^{-i(t-s)|\xi|} & 0 \\ 0 & e^{i(t-s)|\xi|} \end{pmatrix}.$$

Then we can estimate

$$\|B_1(t,\xi)\| \le C(1+t)m(t)^2,$$

hence, after using $(1+t)m(t)^2 \in L^1$ brings $\|Q_1(t,s,\xi)\| \le C_1$ and by Liouville's formula, $\|Q_1(t,s,\xi)\| \ge C_2$. Now, let us introduce

$$H(t,\xi) := \begin{pmatrix} \frac{\widetilde{h}(t,\xi)}{|\xi|} & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that in the hyperbolic zone we have $\frac{\widetilde{h}(t,\xi)}{|\xi|} \approx C$. Then the inverse matrix H^{-1} exists and $\|H(t,\xi)\|, \|H^{-1}(t,\xi)\| \approx C$ for all $t \ge \theta_{|\xi|}$.
Since $U(t,\xi) = H(t,\xi)U_W(t,\xi) = (\widetilde{h}(t,\xi)\widehat{u}, D_t\widehat{u})^T$ we can write $U(t,\xi) = \mathcal{E}(t,0,\xi)U(0,\xi)$, where

$$\mathcal{E}(t,0,\xi) = \begin{cases} E(t,0,\xi), \ 0 \le t \le \theta_{|\xi|}, \\ H(t,\xi)PE_1(t,\theta_{|\xi|},\xi)Q_1(t,\theta_{|\xi|},\xi)P^{-1}H(\theta_{|\xi|},\xi)^{-1}E(\theta_{|\xi|},0,\xi), \ t \ge \theta_{|\xi|}. \end{cases}$$

We have proved that $\|\mathcal{E}(t,0,\xi)\| \le C$ for all t,ξ.

Scattering operator and properties

If $m \equiv 0$, then the fundamental solution of the system (2.49) can be written as $PE_1(t,s,\xi)P^{-1}$. Then, if v solves the free wave equation (2.42), by putting $V(t,\xi) = (|\xi|\widehat{v}, D_t\widehat{v})^T$, we can write $V(t,\xi) = \tilde{E}(t,s,\xi)V(s,\xi)$, where

$$\tilde{E}(t,s,\xi) = PE_1(t,s,\xi)P^{-1}.$$

Our aim is to prove that the limit

$$W_+(\xi) := \operatorname*{s-lim}_{t\to\infty} \widetilde{E}(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) \tag{2.50}$$

exists as strong limit in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. After proving this property we are able to relate the Cauchy data by

$$V(0,\xi) = W_+(\xi)\,U(0,\xi) \quad \text{for all } \xi.$$

First we prove the existence of (2.50) for $|\xi| \ge \epsilon$. Indeed, for $t \ge \theta_{|\xi|}$ we have

$$\tilde{E}^{-1}(t,0,\xi)\mathcal{E}(t,0,\xi) = P\tilde{E}_1(t,\theta_{|\xi|},\xi)Q_1(t,\theta_{|\xi|},\xi)P^{-1}H^{-1}(\theta_{|\xi|},\xi)E(\theta_{|\xi|},0,\xi)$$

with

$$\tilde{E}_1(t,\theta_{|\xi|},\xi) = E_1(0,t,\xi)P^{-1}H(t,\xi)PE_1(t,\theta_{|\xi|},\xi).$$

By using the explicit representation of $\tilde{E}_1(t,\theta_{|\xi|},\xi)$ we can prove for all $|\xi| \geq \epsilon$ that $\lim_{t\to\infty} \tilde{E}_1(t,\theta_{|\xi|},\xi) = E_1(0,\theta_{|\xi|},\xi)$, and the existence of the limit is proved if $Q_1(t,\theta_{|\xi|},\xi)$ converges for $t \to \infty$ in L^∞. For $t, s \geq \theta_{|\xi|}$ we introduce

$$C_1(t,s,\xi) := E_1(s,t,\xi)B_1(t,\xi)E_1(t,s,\xi).$$

Then the matrix $Q_1(t,\theta_{|\xi|},\xi)$ is given by

$$Q_1(t,\theta_{|\xi|},\xi) = I + \sum_{k=1}^{\infty} i^k \int_{\theta_{|\xi|}}^{t} C_1(t_1,\theta_{|\xi|},\xi) \int_{\theta_{|\xi|}}^{t_1} C_1(t_2,\theta_{|\xi|},\xi) \cdots \int_{\theta_{|\xi|}}^{t_{k-1}} C_1(t_k,\theta_{|\xi|},\xi) dt_k \cdots dt_1.$$

For $t, s \geq \theta_{|\xi|}$ we obtain the estimates

$$\begin{aligned}
&\|Q_1(t,\theta_{|\xi|},\xi) - Q_1(s,\theta_{|\xi|},\xi)\|_{L^\infty} \\
&\leq \sum_{k=1}^{\infty} \int_s^t \|C_1(t_1,\theta_{|\xi|},\xi)\| \frac{1}{(k-1)!} \Big(\int_{\theta_{|\xi|}}^{t_1} \|C_1(t_2,\theta_{|\xi|},\xi)\| dt_2 \Big)^{k-1} dt_1 \\
&\leq \int_s^t \|B_1(t_1,\xi)\| \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\int_{\theta_{|\xi|}}^{t_1} \|B_1(t_2,\xi)\| dt_2 \Big)^k dt_1 \\
&= \int_s^t \|B_1(t_1,\xi)\| \exp\Big(\int_{\theta_{|\xi|}}^{t_1} \|B_1(t_2,\xi)\| dt_2 \Big) dt_1 \\
&\lesssim \int_s^t \|B_1(t_1,\xi\| dt_1 \Big) \lesssim \int_s^t (1+t_1)m(t_1)^2 dt_1.
\end{aligned}$$

For the last inequality we used the representation of $B_1 = B_1(t,\xi)$ and the definition of the hyperbolic zone. Since $(1+t)m(t)^2 \in L^1$ it is clear that $Q_1(\infty,\theta_{|\xi|},\xi)$ exists uniformly for $|\xi| \geq \epsilon$, because $\{Q_1(t_k,\theta_{|\xi|},\xi)\}_k$ is a Cauchy sequence uniformly for $|\xi| \geq \epsilon$ in L^∞ for any sequence $\{t_k\}_k$ tending to infinity.
Then, we already proved the existence of the limit (2.50) on the dense subset $\mathcal{L}$ of $L^2 \times L^2$. By using

$$\|Q_1(\infty,\theta_{|\xi|},\xi) - Q_1(t,\theta_{|\xi|},\xi)\|_{L^\infty} \lesssim \int_t^\infty (1+\tau)m^2(\tau)d\tau,$$

where $Q_1(\infty,\theta_{|\xi|},\xi) = \lim_{t\to\infty} Q_1(t,\theta_{|\xi|},\xi)$, we conclude (2.45).
According to the estimates proved in $Z_{pd}(N)$ and $Z_{hyp}(N)$, $\mathcal{E}(t,0,\xi)$ is uniformly bounded and the same is true for $\widetilde{E}^{-1}(t,0,\xi)\mathcal{E}(t,0,\xi)$. Therefore, applying the Banach-Steinhaus Theorem 7.3 we conclude that the operator $W_+(\xi)$ is well-defined for all $\xi \in \mathbb{R}^n$.
Finally, we study the difference

$$\begin{aligned}
\|\lambda(t)U(t,\cdot) - V(t,\cdot)\|_{L^2} &= \|\lambda(t)\mathcal{E}(t,0,\cdot)U(0,\cdot) - \widetilde{\widetilde{\mathcal{E}}}_0(t,0,\cdot)V(0,\cdot)\|_{L^2} \\
&= \Big\| \Big(\lambda(t)\widetilde{\widetilde{\mathcal{E}}}_0(t,0,\cdot)^{-1}\mathcal{E}(t,0,\cdot) - W_+(\cdot) \Big) U(0,\cdot) \Big\|_{L^2},
\end{aligned}$$

under our assumption $(u_0, u_1) \in H^1 \times L^2$ and by definition of $W_+(\xi)$ we may conclude that

$$\|\lambda(t)U(t,\cdot) - V(t,\cdot)\|_{L^2} \to 0$$

as t tends to infinity. The proof is completed. □

Remark 2.11. *From a scattering result to free waves for solutions to the damped wave equation (see [58])*

$$w_{tt} - \triangle w + b(t)w_t = 0,\ w(0,x) = w_0(x),\ w_t(0,x) = w_1(x), \tag{2.51}$$

one can understand that $(1+t)m(t)^2 \in L^1$ is a reasonable condition to be assumed in Theorem 2.5. Indeed, let us assume that $b(t) = \frac{\mu}{(1+t)g(t)}$, where $g(t)$ satisfies Hypothesis 2.3. After performing the change of variables

$$w(t,x) = \exp\Big(-\frac{1}{2}\int_0^t b(s)ds\Big)u(t,x)$$

we get

$$u_{tt} - \triangle u + m(t)u = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x)$$

with

$$m(t) = -\frac{1}{4}\big(b(t)^2 + 2b'(t)\big) = \frac{1}{2}\Big(\frac{\mu}{(1+t)^2 g(t)} + \frac{\mu g'(t)}{(1+t)g(t)^2} - \frac{\mu^2}{2(1+t)^2 g(t)^2}\Big).$$

Therefore, if $g(t)$ goes to infinity for t to infinity the condition $(1+t)m(t)^2 \in L^1$ is a necessary and sufficient condition to have $b \in L^1$ which guarantees scattering behavior of solutions to (2.51) to free waves (see [58]).

Remark 2.12. *Due to the energy conservation for the free wave equation we conclude from Theorem 2.5 that*

$$E(u)(t) = \frac{1}{2}\Big(\|u_t(t,\cdot)\|^2_{L^2} + \|\langle D\rangle_{\frac{N}{1+t}} u(t,\cdot)\|^2_{L^2}\Big) \to E_w(v)(0) \ as\ t \to \infty,$$

with $E_w(v)(0) = \|\nabla v_0\|^2_{L^2} + \|v_1\|^2_{L^2}$.

Remark 2.13. *If $(1+t)m(t)^2 \in L^1$, then (2.7) holds for $\psi \equiv 1$. Consequently, $p(t) = (1+t)^{-1}$ in (2.8). This already hints to a scattering behavior to free waves. This conjecture is now proved in form of Theorem 2.5.*

Remark 2.14. *The Strichartz' estimates from Theorem 1.1 remain true for our model (2.41). This means, we can assume weaker hypothesis over $m = m(t)$, Hypothesis 2.43, and get the same $L^p - L^q$ estimates for the model (2.41). More precisely, let $m = m(t) \in C^\infty(\mathbb{R}_+)$ satisfy the following properties:*

$$\begin{aligned} &(B1) \quad (1+t)m(t)^2 \in L^1(\mathbb{R}_+), \\ &(B2) \quad |d_t^k m(t)| \lesssim C_k(1+t)^{-k}, k = 0,1,2,\cdots, \end{aligned}$$

for all t, where C_k are positives constants.

Theorem 2.6. *Let* $m = m(t) \in C^\infty(\mathbb{R}_+)$ *satisfy* $(B1)$ *and* $(B2)$. *Then for all times* t *the* $L^p - L^q$ *decay estimate*

$$\| \left(u_t(t,\cdot), \nabla_x u(t,\cdot)\right) \|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right),$$

is valid for $r = n\left(\frac{1}{p} - \frac{1}{q}\right)$, *with* $1 < p \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $u = u(t,x)$ *is the solution to the Cauchy problem* (2.41).

Example 2.6. *If* $g(t)$ *in* (2.31) *is given by* $g(t) = \ln(e+t)\cdots\ln^{[m]}(e^{[m]}+t)$, *then* $(1+t)m(t)^2 \in L^1$ *and the conclusions of Theorems 2.5 and 2.6 hold.*

Example 2.7. *Let us choose* $m(t) = \frac{\mu}{(1+t)(\ln(e+t))^{\gamma/2}}$ *in* (2.31) *for* $\gamma > 1$. *Then* $(1+t)m(t)^2 \in L^1$ *and the conclusions of Theorems 2.5 and 2.6 hold.*

3 Strichartz estimates

In this chapter we apply a diagonalization procedure to Klein-Gordon problems (2.1) with sufficiently smooth time-dependent coefficient $m = m(t)$ aiming to find a representation for the solution by Fourier multipliers and then derive $L^p - L^q$ decay estimates on the conjugate line. This procedure is well-known as WKB-analysis and was introduced by K. Yagdjian in [62], M. Reissig and K. Yagdjian in [45].

3.1 Representation of the solution

Let us consider the following Cauchy problem for Klein-Gordon models

$$u_{tt} - \Delta u + m(t)^2 u = 0, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), \tag{3.1}$$

where $u = u(t,x)$ and $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

We perform the partial Fourier transformation of (3.1) with respect to x. If we denote by $\widehat{u} = \widehat{u}(t,\xi)$ the partial Fourier transform $F_{x\to\xi}(u)(t,\xi)$ we obtain

$$\widehat{u}_{tt} + |\xi|^2 \widehat{u} + m(t)^2 \widehat{u} = 0, \ \ \widehat{u}(0,\xi) = \widehat{u}_0(\xi), \ \ \widehat{u}_t(0,\xi) = \widehat{u}_1(\xi). \tag{3.2}$$

To derive a representation of solutions we apply a diagonalization procedure to a first-order system corresponding to equation (3.1) in the hyperbolic zone.

Here we follow some ideas of Wirth [59] and Yagdjian [62]. We will consider a system with a coefficient matrix composed of a diagonal main part and a remainder part. The goal of this diagonalization is to keep the diagonal part in every step of the diagonalization. However, after every step we achieve a better normwise estimate for the remaining part in some scale of symbol classes.

We consider the Cauchy problem for the Klein-Gordon equation (3.1) under the following assumptions:

Hypothesis 3.1. *Let $m(t) \in C^\ell(\mathbb{R}_+)$ satisfy*

$$|m(t)| \lesssim \frac{1}{1+t}, \quad |m^{(k)}(t)| \lesssim \frac{m(t)}{(1+t)^k} \ \text{ for all } \ k \leq \ell. \tag{3.3}$$

Hypothesis 3.2. *There exists a positive increasing function $\psi = \psi(t) \in C^\infty(\mathbb{R}_+)$, such that*

$$\limsup_{t\to\infty} 2(1+t)\frac{\psi'(t)}{\psi(t)} < 1, \ \ \left|\frac{\psi^{(k)}(t)}{\psi(t)}\right| \lesssim \frac{1}{(1+t)^k} \ \text{ for all } \ k \in \mathbb{N}. \tag{3.4}$$

And we assume the following relation between $m(t)$ and $\psi(t)$:

$$\int_0^\infty (1+\tau)\Big|\frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2\Big| d\tau \lesssim 1. \tag{3.5}$$

In order to derive $L^p - L^q$ estimates for the solution and its derivatives we divide the extended phase space $[0,\infty)\times\mathbb{R}^n$ into three zones:

$$\begin{aligned} Z_{pd}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : (1+t)|\xi|\le N\},\\ Z^s_{hyp}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi|\le N\le(1+t)|\xi|\},\\ Z^\ell_{hyp}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi|\ge N\}, \end{aligned}$$

where N is a positive constant.

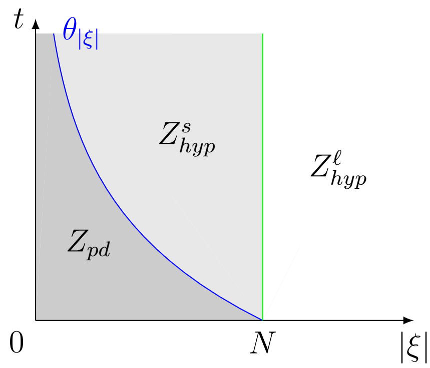

Fig. 3.1: Sketch of the zones.

Remark 3.1. *In the zone $Z^\ell_{hyp}(N)$ we consider only large frequencies and in the zones $Z_{pd}(N)$ and $Z^s_{hyp}(N)$ we consider only small frequencies. We have that the hyperbolic zone from the previous chapter is $Z_{hyp}(N) = Z^s_{hyp}(N)\cup Z^\ell_{hyp}(N)$. Furthermore, the separating curve between zones $Z_{pd}(N)$ and $Z^s_{hyp}(N)$ is given by*

$$\theta_{|\xi|} : (0,N]\to[0,\infty),\quad (1+\theta_{|\xi|})|\xi| = N.$$

We put also $\theta_0=\infty$, and $\theta_{|\xi|}=0$ for any $|\xi|\ge N$.

Remark 3.2. *For the estimates of the potential energy we need to deal with the factor $|\xi|^{-1}$. For this reason we shall divide the phase space into three zones because we shall proceed in a different way in the hyperbolic zone for small and large frequencies.*

In order to separate the extended phase space into three parts we introduce the function $\chi\in C^\infty(\mathbb{R}_+)$ such that $\chi(t)=1$ for $t\le\frac{1}{2}$, $\chi(t)=0$ for $t\ge 2$ and $\chi'(t)\le 0$.

We can define the characteristic functions φ_{pd}, φ^ℓ_{hyp} and φ^s_{hyp} of the zones $Z_{pd}(N)$, $Z^\ell_{hyp}(N)$ and $Z^s_{hyp}(N)$, respectively, by

$$\begin{aligned} \varphi_{pd}(t,\xi) &= \chi\left(|\xi|N^{-1}\right)\chi\left((1+t)|\xi|N^{-1}\right),\\ \varphi^s_{hyp}(t,\xi) &= \chi\left(|\xi|N^{-1}\right)\left(1-\chi\left((1+t)|\xi|N^{-1}\right)\right),\\ \varphi^\ell_{hyp}(\xi) &= 1-\chi\left(|\xi|N^{-1}\right) \end{aligned}$$

such that $\varphi_{pd}(t,\xi)+\varphi^\ell_{hyp}(\xi)+\varphi^s_{hyp}(t,\xi)=1$. Let us consider the same micro-energy that we defined in the Chapter 2, that is,

$$U(t,\xi)=\left(h(t,\xi)\hat{u},\hat{u}_t-\frac{\psi'(t)}{\psi(t)}\hat{u}\right)^T, \tag{3.6}$$

where

$$h(t,\xi)=\frac{1}{1+t}\varphi_{pd}(t,\xi)+i|\xi|\left(\varphi^\ell_{hyp}(\xi)+\varphi^s_{hyp}(t,\xi)\right).$$

3.1.1 Considerations in the pseudo-differential zone

In the pseudo-differential zone $Z_{pd}(N)$ the micro-energy (3.6) reduces to

$$U = \Big(\frac{\widehat{u}}{1+t}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \;\; U_0(\xi) = \Big(\widehat{u}_0(\xi), \widehat{u}_1(\xi) - \frac{\psi'(0)}{\psi(0)}\widehat{u}_0(\xi)\Big)^T, \;\; \text{and} \;\; U = \psi(t)\widetilde{U}.$$

So we have

$$\partial_t \widetilde{U}(t,\xi) = \mathcal{A}(t,\xi)\widetilde{U} := \begin{pmatrix} -\frac{1}{1+t} & \frac{1}{1+t} \\ -(1+t)\Big(\frac{\psi''}{\psi} + m(t)^2 + |\xi|^2\Big) & -2\frac{\psi'(t)}{\psi(t)} \end{pmatrix} \widetilde{U}. \qquad (3.7)$$

If we consider the fundamental solution $\mathcal{E}(t,s,\xi)$ of the system (3.7), i.e., $\widetilde{U}(t,\xi) = \mathcal{E}(t,s,\xi)\widetilde{U}(s,\xi)$ and $\mathcal{E}(s,s,\xi) = I$, then we proved in Corollary 2.1 that in the pseudo-differential zone we have the following proposition:

Proposition 3.1. *Assume the Hypothesis 3.1 and 3.2. Then the fundamental solution $\mathcal{E}(t,0,\xi)$ satisfies the estimate*

$$\|\mathcal{E}(t,0,\xi)\| \lesssim \frac{1}{\psi(t)^2},$$

for all $(t,\xi) \in Z_{pd}(N)$.

Consider $\mathcal{H}(t,s,\xi)$ such that $U(t,\xi) = \mathcal{H}(t,s,\xi)U(s,\xi)$ and $\mathcal{H}(s,s,\xi) = I$, that is, $\mathcal{H}(t,s,\xi) = \frac{\psi(t)}{\psi(s)}\mathcal{E}(t,s,\xi)$. Then it follows from the Proposition 3.1 that

$$\|\mathcal{H}(t,0,\xi)\| \lesssim \frac{1}{\psi(t)} \;\text{ for all }\; t \le \theta_{|\xi|}. \qquad (3.8)$$

The properties of the matrix $A = A(t,\xi)$ imply the following symbol-like estimate.

Lemma 3.1. *Assume the Hypothesis 3.1 and 3.2. Then for $|\xi| \le N$ the symbol-like estimates*

$$\big\|D_\xi^\alpha\big(\psi^2(\theta_{|\xi|})\mathcal{E}(\theta_{|\xi|},0,\xi)\big)\big\| \le C_\alpha |\xi|^{-|\alpha|}$$

are valid for all $|\alpha| \le \ell + 1$.

Proof. Observe that the matrix $A(t,\xi)$ has the same properties of the matrix of the Lemma 3.10 of [58], what is sufficient to show the lemma. □

From Lemma 3.1 it follows that

$$\big\|D_\xi^\alpha\big(\psi(\theta_{|\xi|})\mathcal{H}(\theta_{|\xi|},0,\xi)\big)\big\| \le C_\alpha |\xi|^{-|\alpha|}, \qquad (3.9)$$

for all $|\alpha| \le \ell + 1$.

3.1.2 Considerations in the hyperbolic zone: $Z^s_{hyp}(N) \cup Z^\ell_{hyp}(N)$

First of all let us introduce the symbol class $S_N^{\ell_1,\ell_2}\{m_1,m_2\}$ in the hyperbolic zone.

Definition 3.1. *The time-dependent amplitude function $a = a(t,\xi)$ belongs to the symbol class $S_N^{\ell_1,\ell_2}\{m_1,m_2\}$ with restricted smoothness ℓ_1,ℓ_2 if it satisfies the symbol-like estimates*

$$\left|D_t^k D_\xi^\alpha a(t,\xi)\right| \le C_{K,\alpha}|\xi|^{m_1-|\alpha|}\left(\frac{1}{1+t}\right)^{m_2+k} \tag{3.10}$$

for all $(t,\xi) \in Z_{hyp}(N)$, all non-negative integers $k \le \ell_1$ and all multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| \le \ell_2$.

We will denote by $S_N\{m_1,m_2\}$ the symbol class when $\ell_1 = \ell_2 = \infty$, that is, $S_N\{m_1,m_2\} = S_N^{\infty,\infty}\{m_1,m_2\}$.

The rules of the symbolic calculus are collected in the following proposition.

Proposition 3.2. (1) $S_N^{\ell_1,\ell_2}\{m_1,m_2\}$ *is a vector space, for all non-negative integers ℓ_1 and ℓ_2.*
(2) $S_N^{\ell_1',\ell_2'}\{m_1-k,m_2+\ell\} \subset S_N^{\ell_1,\ell_2}\{m_1,m_2\}$, *for all $\ell \ge k \ge 0, \ell_1' \ge \ell_1$ and $\ell_2' \ge \ell_2$.*
(3) $S_N^{\ell_1,\ell_2}\{m_1,m_2\} \cdot S_N^{\ell_1',\ell_2'}\{m_1',m_2'\} \subset S_N^{\widetilde{\ell}_1,\widetilde{\ell}_2}\{m_1+m_1',m_2+m_2'\}$, *for all non-negative integers ℓ_j and ℓ_j' with $\widetilde{\ell}_j = \min\{\ell_j,\ell_j'\}$ for $j=1,2$.*
(4) $D_t^k D_\xi^\alpha S_N^{\ell_1,\ell_2}\{m_1,m_2\} \subset S_N^{\ell_1-k,\ell_2-|\alpha|}\{m_1-|\alpha|,m_2+k\}$, *for all non-negative integers ℓ_1 and ℓ_2 with $k \le \ell_1$ and $|\alpha| \le \ell_2$.*
(5) $S_N^{\ell_1,\ell_2}\{-1,2\} \subset L_\xi^\infty L_t^1(Z_{hyp})$, *for all non-negative integers ℓ_1 and ℓ_2.*

In the hyperbolic zone $Z_{hyp}(N) = Z^s_{hyp}(N) \cup Z^\ell_{hyp}(N)$ the micro-energy (3.6) reduces to

$$U = \Big(i|\xi|\widehat{u}, \widehat{u}_t - \frac{\psi'(t)}{\psi(t)}\widehat{u}\Big)^T, \quad U_0(\xi) = \Big(i|\xi|\widehat{u}(\theta_{|\xi|},\xi), \widehat{u}_t(\theta_{|\xi|},\xi) - \frac{\psi'(\theta_{|\xi|})}{\psi(\theta_{|\xi|})}\widehat{u}(\theta_{|\xi|},\xi)\Big)^T,$$

and $U = \psi(t)\widetilde{U}$, so that

$$\partial_t\widetilde{U} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} i|\xi|\widetilde{U} + \begin{pmatrix}0 & 0\\ 0 & -2\frac{\psi'(t)}{\psi(t)}\end{pmatrix}\widetilde{U} + \begin{pmatrix}0 & 0\\ -\frac{\psi''}{\psi}(t) - m(t)^2 & 0\end{pmatrix}(i|\xi|)^{-1}\widetilde{U} \tag{3.11}$$

for $t \ge \theta_{|\xi|}$ with initial datum $\widetilde{U}(\theta_{|\xi|},\xi) = \psi(\theta_{|\xi|})^{-1}U_0(\xi)$.

Let M be the diagonalizer of the principal part (with respect to powers of $|\xi|$) of (3.11) given by

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}, \qquad M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

If we put $V(t,\xi) := M^{-1}\widetilde{U}(t,\xi)$, then we get

$$D_tV = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}|\xi|V - (B(t) + C(t,\xi))V, \tag{3.12}$$

where

$$B(t) = -i\frac{\psi'(t)}{\psi(t)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad C(t,\xi) = -\frac{1}{2|\xi|}\left(\frac{\psi''}{\psi}(t) + m(t)^2\right)\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Note that $B \in S_N\{0,1\}$ and $C(t,\xi) \in S_N^{\ell,\infty}\{-1,2\} \subset S_N^{\ell,\infty}\{0,1\}$.

Now we want to carry out further steps of the diagonalization procedure. The aim is to transform the previous system such that the new matrix has diagonal structure and the new remainder belongs to a special hyperbolic symbol class.

We construct recursively the diagonalizer $N_k(t,\xi)$ of order k. The construction of the diagonalizer matrix was developed by Yagdjian [62]. Let

$$N_k(t,\xi) = \sum_{j=0}^{k} N^{(j)}(t,\xi), \quad F_k(t,\xi) = \sum_{j=0}^{k} F^{(j)}(t,\xi),$$

where $N^{(0)} = I$, $B^{(0)} = B + C$ and $F^{(0)} = \text{diag}B^{(0)} = F_0(t,\xi)$. Following Yagdjian [62] and Wirth [58] the construction goes along the next scheme. We define

$$F^{(j)} := \text{diag}B^{(j)}, \tag{3.13}$$

$$N^{(j+1)} := \begin{pmatrix} 0 & \frac{-B_{12}^{(j)}}{2|\xi|} \\ \frac{B_{21}^{(j)}}{2|\xi|} & 0 \end{pmatrix}, \tag{3.14}$$

$$B^{(j+1)} := \big(D_t - D + B + C\big)N_{j+1} - N_{j+1}\big(D_t - D + F_j\big). \tag{3.15}$$

Proposition 3.3. *Assume the Hypothesis 3.1 and 3.2. Then $N^{(j)} \in S_N^{\ell-j+1,\infty}\{-j,j\}$ and $B^{(j+1)} \in S_N^{\ell-j-1,\infty}\{-j-1,j+2\}$ for all $j \geq 0$. Moreover, the matrix N_k is invertible in $Z_{hyp}(N)$ for all $k \in \mathbb{N}$.*

Proof. We will prove the statements by induction on j.

1) The Hypothesis 3.1 and 3.2 imply that

$$F^{(0)}(t,\xi) = F_0 \in S_N^{\ell,\infty}\{0,1\}, \quad N^{(1)}(t,\xi) \in S_N^{\ell,\infty}\{-1,1\}.$$

So we have,

$$\begin{aligned} B^{(1)} &= (D_t - D + B + C)\,N_1 - N_1\,(D_t - D + F_0) \\ &= (D_t - D + B + C)\left(N^{(0)} + N^{(1)}\right) - \left(N^{(0)} + N^{(1)}\right)(D_t - D + F_0) \\ &= D_tN^{(1)} + \left(B^{(0)} + [N^{(1)}, D] - F_0\right) - N^{(1)}F_0 + (B + C)N^{(1)}. \end{aligned}$$

Now $B^{(0)} + [N^{(1)}, D] - F_0 = 0$ imply that

$$B^{(1)} = D_tN^{(1)} - N^{(1)}F_0 + (B + C)N^{(1)}.$$

Taking into consideration the rules of the symbolic calculus we have

$$\begin{aligned} \text{if } N^{(1)} \in S_N^{\ell,\infty}\{-1,1\}, \text{then } D_tN^{(1)} \in S_N^{\ell-1,\infty}\{-1,2\}, \\ \text{if } F_0 \in S_N^{\ell,\infty}\{0,1\}, N^{(1)} \in S_N^{\ell,\infty}\{-1,1\}, \text{then } N^{(1)}F_0 \in S_N^{\ell,\infty}\{-1,2\}, \\ \text{if } B + C \in S_N^{\ell,\infty}\{0,1\}, N^{(1)} \in S_N^{\ell,\infty}\{-1,1\}, \text{then } (B + C)N^{(1)} \in S_N^{\ell,\infty}\{-1,2\}, \end{aligned}$$

summarizing $B^{(1)} \in S_N^{\ell-1,\infty}\{-1,2\}$.

2) For $j \geq 2$ we suppose that $B^{(m)} \in S_N^{\ell-m,\infty}\{-m, m+1\}$ for all $1 \leq m \leq j$. Then, by definition of $N^{(m+1)}$ we have from $|\xi|^{-1} \in S_N\{-1,0\}$, that $N^{(m+1)} \in S_N^{\ell-m,\infty}\{-m-1, m+1\}$ and $F^{(m)} \in S_N^{\ell-m,\infty}\{-m, m+1\}$ for all $1 \leq m \leq j$.

3) For $B^{(j+1)}$ we have

$$\begin{aligned}
B^{(j+1)} &= (D_t - D + B + C)\, N_{j+1} - N_{j+1}\,(D_t - D + F_j) \\
&= (D_t - D + B + C) \sum_{v=0}^{j+1} N^{(v)} - \sum_{v=0}^{j+1} N^{(v)} \Big(D_t - D + \sum_{v=0}^{j} F^{(v)} \Big) \\
&= (D_t - D + B + C) \sum_{v=0}^{j+1} N^{(v)} - \sum_{v=0}^{j+1} N^{(v)} \Big(D_t - D + \sum_{v=0}^{j-1} F^{(v)} \Big) - \sum_{v=0}^{j+1} N^{(v)} F^{(j)} \\
&= B^{(j)} + (D_t - D + B + C)\, N^{(j+1)} - N^{(j+1)} \Big(D_t - D + \sum_{v=0}^{j-1} F^{(v)} \Big) \\
&\quad - \sum_{v=1}^{j+1} N^{(v)} F^{(j)} - F^{(j)} \\
&= B^{(j)} + \big[N^{(j+1)}, D \big] - F^{(j)} + D_t N^{(j+1)} - N^{(j+1)} \sum_{v=0}^{j-1} F^{(v)} \\
&\quad + (B + C) N^{(j+1)} - \sum_{v=1}^{j+1} N^{(v)} F^{(j)}.
\end{aligned}$$

We have that

$$B^{(j)} + \big[N^{(j+1)}, D \big] - F^{(j)} = 0$$

therefore,

$$B^{(j+1)} = D_t N^{(j+1)} - N^{(j+1)} \sum_{v=0}^{j-1} F^{(v)} + (B + C) N^{(j+1)} - \sum_{v=1}^{j+1} N^{(v)} F^{(j)}.$$

Then $B^{(j+1)} \in S_N^{\ell-j-1,\infty}\{-j-1, j+2\}$.

Now we are able to prove that N_k is invertible on $Z_{hyp}(N)$. This follows from $N_k - I \in S_N^{\ell-k+1,\infty}\{-1,1\}$ and by the choice of a sufficiently large zone constant N. In fact,

$$\begin{aligned}
\|N_k - I\| &\lesssim \frac{1}{|\xi|(1+t)} \lesssim \frac{1}{|\xi|(1+\theta_{|\xi|})} = \frac{1}{N} \\
&\therefore \ \|N_k - I\| \to 0, N \to \infty.
\end{aligned}$$

The proof is complete. □

If we denote $R_k(t,\xi) = -N_k(t,\xi)^{-1} B^{(k)}(t,\xi)$, the previous results give us the following lemma:

Lemma 3.2. *Assume the Hypothesis 3.1 and 3.2. For each $1 \leq k \leq \ell$ there exists a zone constant N and matrix-valued symbols such that*

1. $N_k(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$ *is invertible for* $(t,\xi) \in Z_{hyp}(N)$ *with* $N_k(t,\xi)^{-1} \in S_N^{\ell-k+1,\infty}\{0,0\}$;

2. $F_{k-1}(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,1\}$ *is diagonal with* $F_{k-1}(t,\xi) - F^{(0)} \in S_N^{\ell-k+1,\infty}\{-1,2\}$;

3. $R_k(t,\xi) \in S_N^{\ell-k,\infty}\{-k,k+1\}$.

Moreover, the identity

$$\big(D_t - D(\xi) + B(t) + C(t,\xi)\big)N_k(t,\xi) = N_k(t,\xi)\big(D_t - D(\xi) + F_{k-1}(t,\xi) - R_k(t,\xi)\big) \quad (3.16)$$

holds for all $(t,\xi) \in Z_{hyp}(N)$.

The next proposition shows us that the multiplication by $e^{\pm it|\xi|}$ is not a well-defined operation on the symbol classes $S_N^{\ell_1,\ell_2}\{m_1,m_2\}$.

In order to simplify the calculations the following remark is important.

Remark 3.3. *Let* g *be a sufficiently smooth function such that* $g = g(|\xi|)$ *satisfies*

$$\big|D_{|\xi|}^{|\alpha|} g(|\xi|)\big| \lesssim |\xi|^{-|\alpha|}$$

for all $\xi \in \mathbb{R}^n$. *Then we have for all multi-indices* α *the estimates*

$$|D_\xi^\alpha g(|\xi|)| \lesssim |\xi|^{-\alpha}.$$

Indeed, applying Faà de Bruno's formula for a multivariate version we get that

$$\begin{aligned}
|D_\xi^\alpha g(|\xi|)| &= \left|\sum_{j=1}^{|\alpha|} \sum_{\beta_1+\cdots+\beta_j=\alpha} C_{\beta_1,\cdots,\beta_j} g^{(j)}(|\xi|) \prod_{i=1}^{j} D_\xi^{\beta_i}|\xi|\right| \\
&\le \sum_{j=1}^{|\alpha|} \sum_{\beta_1+\cdots+\beta_j=\alpha} |C_{\beta_1,\cdots,\beta_j}||g^{(j)}(|\xi|)| \left|\prod_{i=1}^{j} D_\xi^{\beta_i}|\xi|\right| \\
&\le \sum_{j=1}^{|\alpha|} \sum_{\beta_1+\cdots+\beta_j=\alpha} |C_{\beta_1,\cdots,\beta_j}||\xi|^{-j}|\xi|^{-(|\beta_1|+\cdots+|\beta_j|)+j} \\
&\le C_\alpha |\xi|^{-|\alpha|}.
\end{aligned}$$

Proposition 3.4. *We have that*

1. $\exp(\pm i\theta_{|\xi|}|\xi|)S_N^{\ell_1,\ell_2}\{m_1,m_2\} \subset S_N^{\ell_1,\ell_2}\{m_1,m_2\}$,

2. $\exp(\pm it|\xi|)S_N^{\ell_1,\ell_2}\{m_1,m_2\} \subset S_N^{\ell_1,\ell_2}\{m_1+\eta,m_2-\eta\}$ *with* $\eta = \ell_1+\ell_2$.

Proof. In fact, let $a(t,\xi) \in S_N^{\ell_1,\ell_2}\{m_1,m_2\}$.

First let us prove 1. If $a(t,\xi) \in S_N^{\ell_1,\ell_2}\{m_1,m_2\}$, then

$$
\begin{aligned}
&\left|D_t^k D_\xi^\alpha \left(\exp\left(\pm i\theta_{|\xi|}|\xi|\right)a(t,\xi)\right)\right| = \left|\sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} D_\xi^{\alpha_1}\left(e^{\pm i\theta_{|\xi|}|\xi|}\right) D_t^k D_\xi^{\alpha_2} a(t,\xi)\right| \\
\leq &\sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2}\left(C'_{\alpha_1}\left|e^{\pm i\theta_{|\xi|}|\xi|}\sum_{j=1}^{|\alpha_1|}\sum_{\ell_1+\cdots\ell_j=|\alpha_1|}\prod_{i=1}^{j}\partial_{|\xi|}^{\ell_i}\left(\theta_{|\xi|}|\xi|\right)\right|\right)\left|D_t^k D_\xi^{\alpha_2} a(t,\xi)\right| \\
\leq &\sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2}\left(C'_{\alpha_1}\sum_{j=1}^{|\alpha|}\sum_{\ell_1+\cdots\ell_j=|\alpha_1|}\prod_{i=1}^{j}|\xi|^{-\ell_i}\right)\left|D_t^k D_\xi^{\alpha_2} a(t,\xi)\right| \\
\leq &\sum_{\alpha_1+\alpha_2=\alpha} \widetilde{C}_{k,\alpha_1,\alpha_2}|\xi|^{m_1-(|\alpha_1|+|\alpha_2|)}\left(\frac{1}{1+t}\right)^{m_2+k} \\
= \ &C_{k,\alpha}|\xi|^{m_1-|\alpha|}\left(\frac{1}{1+t}\right)^{m_2+k}.
\end{aligned}
$$

To prove 2 let $a(t,\xi) \in S_N^{\ell_1,\ell_2}\{m_1,m_2\}$. Then

$$
\begin{aligned}
&\left|D_t^k D_\xi^\alpha \left(\exp\left(\pm it|\xi|\right)a(t,\xi)\right)\right| \\
&= \left|\sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C_{k_1,k_2,\alpha_1,\alpha_2} D_t^{k_1} D_\xi^{\alpha_1} e^{\pm it|\xi|} D_t^{k_2} D_{|\xi|}^{\alpha_2} a(t,\xi)\right| \\
&\leq \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C' t^{|\alpha_1|}\left|D_t^{k_1} e^{\pm it|\xi|}\right|\left|D_t^{k_2} D_{|\xi|}^{\alpha_2} a(t,\xi)\right| \\
&\leq \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C'\left(\frac{1}{1+t}\right)^{m_2+k_2-|\alpha_1|}|\xi|^{m_1-|\alpha_2|+k_1} \\
&\leq C_{k,\alpha}|\xi|^{m_1+\eta-|\alpha|}\left(\frac{1}{1+t}\right)^{m_2-\eta+k}.
\end{aligned}
$$

□

3.1.3 Fundamental solution of the diagonalized system

We are interested to find a representation for the fundamental solution of the diagonalized system

$$
\begin{cases} \left(D_t - D(\xi) + F_{k-1}(t,\xi) - R_k(t,\xi)\right)\mathcal{E}_k(t,s,\xi) = 0, \\ \mathcal{E}_k(s,s,\xi) = I. \end{cases} \tag{3.17}
$$

To find such a representation we are going to consider the fundamental solution $\mathcal{E}_0 = \mathcal{E}_0(t,s,\xi)$ of the free wave system, i.e., the fundamental solution to $D_t - D(\xi)$. Then we study the influence of diag B and, finally, we check the influence of $F_{k-1} - F^{(0)}$, diag C and R_k on the construction of the fundamental solution.

Step 1: We have the fundamental solution $\mathcal{E}_0(t,s,\xi)$ for

$$
\left(D_t - D(\xi)\right)\mathcal{E}_0(t,s,\xi) = 0, \ \ \mathcal{E}_0(s,s,\xi) = I,
$$

as

$$\mathcal{E}_0(t,s,\xi) = \exp\left(i(t-s)D(\xi)\right) = \begin{pmatrix} e^{i(t-s)|\xi|} & 0 \\ 0 & e^{-i(t-s)|\xi|} \end{pmatrix}. \tag{3.18}$$

Step 2: To analyze the influence of diag B let us define $\widetilde{\mathcal{E}_0} = \widetilde{\mathcal{E}_0}(t,s,\xi) = \frac{\psi(s)}{\psi(t)}\mathcal{E}_0(t,s,\xi)$. Then

$$\begin{aligned} D_t\widetilde{\mathcal{E}_0}(t,s,\xi) &= D_t\left(\frac{\psi(s)}{\psi(t)}\right)\mathcal{E}_0(t,s,\xi) + \frac{\psi(s)}{\psi(t)}D_t\mathcal{E}_0(t,s,\xi) \\ &= \left(i\frac{\psi(s)\psi'(t)}{\psi(t)^2}I + \frac{\psi(s)}{\psi(t)}D(\xi)\right)\mathcal{E}_0(t,s,\xi) \\ &= \left(i\frac{\psi'(t)}{\psi(t)}I + D(\xi)\right)\frac{\psi(s)}{\psi(t)}\mathcal{E}_0(t,s,\xi) \\ &= \left(i\frac{\psi'(t)}{\psi(t)}I + D(\xi)\right)\widetilde{\mathcal{E}_0}(t,s,\xi) \\ &= \left(D(\xi) - \text{diag } B(t)\right)\widetilde{\mathcal{E}_0}(t,s,\xi). \end{aligned}$$

Therefore $\widetilde{\mathcal{E}_0}(t,s,\xi)$ satisfies

$$D_t\widetilde{\mathcal{E}_0}(t,s,\xi) = \big(D(\xi) - \text{diag } B\big)\widetilde{\mathcal{E}_0}(t,s,\xi). \tag{3.19}$$

Step 3: Now we will study the influence of $F_{k-1} - F^{(0)}$, diag C and R_k. Therefore, we define

$$\Phi_k(t,s,\xi) \;=\; \widetilde{\mathcal{E}_0}(s,t,\xi)\left(-F_{k-1}(t,\xi) + R_k(t,\xi) + F^{(0)}(t,\xi) - \text{diag } C\right)\widetilde{\mathcal{E}_0}(t,s,\xi).$$

But $F_{k-1}(t,\xi)$, $F^{(0)}(t,\xi)$, and diag C are diagonal. Hence,

$$\Phi_k(t,s,\xi) = -(F_{k-1} - F^{(0)}(t,\xi)) + \mathcal{E}_0(s,t,\xi)R_k(t,\xi)\mathcal{E}_0(t,s,\xi) - \text{diag } C. \tag{3.20}$$

Let us consider the following system:

$$\begin{cases} D_t\mathcal{Q}_k(t,s,\xi) = \Phi_k(t,s,\xi)\mathcal{Q}_k(t,s,\xi), \\ \mathcal{Q}_k(s,s,\xi) = I. \end{cases} \tag{3.21}$$

The fundamental solution $\mathcal{E}_k = \mathcal{E}_k(t,s,\xi)$ of the diagonalized system can be represented as

$$\mathcal{E}_k(t,s,\xi) = \widetilde{\mathcal{E}_0}(t,s,\xi)\mathcal{Q}_k(t,s,\xi) = \frac{\psi(s)}{\psi(t)}\mathcal{E}_0(t,s,\xi)\mathcal{Q}_k(t,s,\xi). \tag{3.22}$$

In fact,

$$\begin{aligned} D_t\mathcal{E}_k &= (D_t\widetilde{\mathcal{E}_0})\mathcal{Q}_k + \widetilde{\mathcal{E}_0}\left(D_t\mathcal{Q}_k\right) \\ &= \left(D - \text{diag } B\right)\widetilde{\mathcal{E}_0}\mathcal{Q}_k + \widetilde{\mathcal{E}_0}\left(\Phi_k\mathcal{Q}_k\right) \\ &= \left(D - \text{diag } B\right)\widetilde{\mathcal{E}_0}\mathcal{Q}_k + \left(-F_{k-1} + R_k + F^{(0)} - \text{diag } C\right)\widetilde{\mathcal{E}_0}\mathcal{Q}_k \\ &= \left(D - \text{diag } B - \text{diag } C - F_{k-1} + R_k + F^{(0)}\right)\mathcal{E}_k \\ &= \left(D - F^{(0)} - F_{k-1} + R_k + F^{(0)}\right)\mathcal{E}_k \\ &= \left(D - F_{k-1} + R_k\right)\mathcal{E}_k. \end{aligned}$$

We have that $R_k(t,\xi) \in S_N^{\ell-k,\infty}\{-k,k+1\}$, and if $\ell - k \geq 0$ we can prove the following lemma:

Lemma 3.3. *The matrix-valued function $\Phi_k(t,s,\xi)$ belongs to the symbol class*

$$\Phi_k(t,s,\xi) \in S_N^{0,k-1}\{-1,2\} \tag{3.23}$$

for all $s \geq \theta_{|\xi|}$.

Proof. Let us consider the case where $s = \theta_{|\xi|}$, the case $s > \theta_{|\xi|}$ is analogous. The representation of $\Phi_k(t,s,\xi)$ in (3.20) implies that

$$\Phi_k(t,\theta_{|\xi|},\xi) = -(F_{k-1}(t,\xi) - F^{(0)}(t,\xi)) + \mathcal{E}_0(\theta_{|\xi|},t,\xi)R_k(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi) - \text{diag } C.$$

We have that $\mathrm{diag}\ C \in S_N^{\ell,\infty}\{-1,2\} \subset S_N^{0,k-1}\{-1,2\}$ and the Lemma 3.2 implies that $F_{k-1}(t,\xi) - F^{(0)}(t,\xi) \in S_N^{\ell-k+1,\infty}\{-1,2\} \subset S_N^{0,k-1}\{-1,2\}$. The only remainder term is

$$\widetilde{\Phi}_k(t,\theta_{|\xi|},\xi) = \mathcal{E}_0(\theta_{|\xi|},t,\xi)R_k(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi) = \begin{pmatrix} a_{11}^k & a_{12}^k e^{2i(\theta_{|\xi|}-t)|\xi|} \\ a_{21}^k e^{2i(t-\theta_{|\xi|})|\xi|} & a_{22}^k \end{pmatrix},$$

where by Lemma 3.2 the entries $a_{ij}^k \in S_N^{\ell-k,\infty}\{-k,k+1\} \subset S_N^{0,k-1}\{-1,2\}$. Applying Proposition 3.4 we deduce that $\widetilde{\Phi}_k(t,\theta_{|\xi|},\xi) \in S_N^{0,k-1}\{-k+\eta,k+1-\eta\} = S_N^{0,k-1}\{-1,2\}$, where $\eta = k-1$. □

Remark 3.4. *Moreover, we may conclude that $\Phi_k(t,s,\xi) \in S_N^{\ell_1,\ell_2}\{-1,2\}$ if $\eta = \ell_1 + \ell_2 < k-1$.*

Taking account of $S_N^{0,k-1}\{-1,2\} \subset L_\xi^\infty L_t^1(Z_{hyp})$ it is allowed to apply the Peano-Baker formula (7.2) and to conclude that the fundamental solution of the system (3.21) is given by

$$\mathcal{Q}_k(t,s,\xi) = I + \sum_{\ell=1}^{\infty} i^\ell \int_s^t \Phi_k(t_1,s,\xi) \int_s^{t_1} \Phi_k(t_2,s,\xi) \cdots \int_s^{t_{\ell-1}} \Phi_k(t_\ell,s,\xi)dt_\ell \cdots dt_1. \tag{3.24}$$

This series representation for solutions to (3.21) and Proposition 7.2 imply that

$$\|\mathcal{Q}_k(t,s,\xi)\| \lesssim 1.$$

In fact, first of all look that $F_{k-1} - F_0 \in S_N^{\ell-k,\infty}\{-1,2\}$, $R_k(t,\xi) \in S_N^{\ell-k,\infty}\{-1,2\}$, diag $C \in S_N^{\ell-k,\infty}\{-1,2\}$ and $\|\mathcal{E}_0(t,s,\xi)\| = 1$. Then

$$\|\Phi_k(t,s,\xi)\| \leq \|F_{k-1} - F_0\| + \|R_k(t,\xi)\| + \|\text{diag } C\| \lesssim \frac{1}{(1+t)^2|\xi|}.$$

Then,

$$
\begin{aligned}
\Big\| \mathcal{Q}_k(t,s,\xi) \Big\| &= \Big\| I + \sum_{\ell=1}^{\infty} i^\ell \int_s^t \Phi_k(t_1,s,\xi) \int_s^{t_1} \Phi_k(t_2,s,\xi) \cdots \int_s^{t_{\ell-1}} \Phi_k(t_\ell,s,\xi) dt_\ell \cdots dt_1 \Big\| \\
&\le 1 + \sum_{\ell=1}^{\infty} \int_s^t \|\Phi_k(t_1,s,\xi)\| \int_s^{t_1} \|\Phi_k(t_2,s,\xi)\| \cdots \int_s^{t_{\ell-1}} \|\Phi_k(t_\ell,s,\xi)\| dt_\ell \cdots dt_1 \\
&\lesssim 1 + \sum_{\ell=1}^{\infty} \int_s^t \frac{1}{|\xi|(1+t)^2} \int_s^{t_1} \frac{1}{|\xi|(1+t_1)^2} \cdots \int_s^{t_{\ell-1}} \frac{1}{|\xi|(1+t_\ell)^2} dt_\ell \cdots dt_1 \\
&\le 1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left(\int_s^t \frac{d\tau}{|\xi|(1+\tau)^2} \right)^\ell \\
&= \exp\left(\int_s^t \frac{d\tau}{|\xi|(1+\tau)^2} \right) \\
&\le \exp\left(\int_{\theta_{|\xi|}}^{\infty} \frac{d\tau}{|\xi|(1+\tau)^2} \right) = \exp\left(\frac{1}{N}\right) \lesssim 1.
\end{aligned}
$$

This leads to the following estimate for the fundamental solution of the diagonalized system in the hyperbolic zone:

$$\|\mathcal{E}_k(t,s,\xi)\| \lesssim \frac{\psi(s)}{\psi(t)}.$$

The goal is now to estimate the ξ-derivatives of $\mathcal{Q}_k(t,s,\xi)$.

Lemma 3.4. *Assume that* $a(t,s,\xi) \in S_N^{0,k-1}\{-1,2\}$. *Then*

$$b(t,s,\xi) = 1 + \sum_{j=1}^{\infty} \int_s^t a(t_1,s,\xi) \int_s^{t_1} a(t_2,s,\xi) \cdots \int_s^{t_{j-1}} a(t_j,s,\xi) dt_j \cdots dt_1 \quad (3.25)$$

defines a symbol from $S_N^{0,k-1}\{0,0\}$ *uniformly in* $s \ge \theta_{|\xi|}$.

Proof. First of all let us analyze the α derivatives with respect to ξ in the representation (3.25) for $b(t,s,\xi)$. We have that

$$D_\xi^\alpha b(t,s,\xi) = \sum_{\ell=1}^{\infty} D_\xi^\alpha \left(\int_s^t a(t_1,s,\xi) \int_s^{t_1} a(t_2,s,\xi) \cdots \int_s^{t_{\ell-1}} a(t_\ell,s,\xi) dt_\ell \cdots dt_1 \right).$$

Let us consider the terms of the form

$$\int_s^t D_\xi^{\alpha_1} a(t_1,s,\xi) \int_s^{t_1} D_\xi^{\alpha_2} a(t_2,s,\xi) \cdots \int_s^{t_{\ell-1}} D_\xi^{\alpha_j} a(t_\ell,s,\xi) dt_\ell \cdots dt_1$$

with $\sum_{i=1}^{j} \alpha_i = \alpha$.

If $s > \theta_{|\xi|}$, then the Lemma 3.3 implies that the norm of these term can be estimated by

$$
\begin{aligned}
& C'_{\alpha,k,N}\int_s^t |\xi|^{-1-|\alpha_1|}\left(\frac{1}{1+t_1}\right)^2 \int_s^{t_2}|\xi|^{-1-|\alpha_2|}\left(\frac{1}{1+t_2}\right)^2\\
& \times\cdots\times\int_s^{t_{\ell-1}}|\xi|^{-1-|\alpha_j|}\left(\frac{1}{1+t_j}\right)^2 dt_\ell\cdots dt_1\\
& \leq C_{\alpha,k,N}|\xi|^{-|\alpha|}.
\end{aligned}
$$

If $s=\theta_{|\xi|}$, then we also have to care for derivatives of the lower integral bound $\theta_{|\xi|}$. Then there arise terms as

$$D_\xi^{\alpha-\beta}\left(a(\theta_{|\xi|},\theta_{|\xi|},\xi)D_\xi^\beta\theta_{|\xi|}\right).$$

For $|\beta|=1$ we can estimate as follows:

$$
\begin{aligned}
\left\|D_\xi^{\alpha-\beta}\left(a(\theta_{|\xi|},\theta_{|\xi|},\xi)\partial_{|\xi|}\theta_{|\xi|}\right)\right\| &= \left\|\sum_{|\alpha_1|+|\alpha_2|=|\alpha|-1} C_{\alpha_1,\alpha_2}D_\xi^{\alpha_1}a(\theta_{|\xi|},\theta_{|\xi|},\xi)D_\xi^{\alpha_2+\beta}\theta_{|\xi|}\right\|\\
&\leq \widetilde{C}_\alpha\sum_{|\alpha_1|+|\alpha_2|=|\alpha|-1}|\xi|^{-1-|\alpha_1|}\left(\frac{1}{1+\theta_{|\xi|}}\right)^2|\xi|^{-2-|\alpha_2|}\\
&\leq C_{\alpha,N}|\xi|^{-|\alpha|}.
\end{aligned}
$$

We use that the terms $D_\xi^{\alpha_1}a(\theta_{|\xi|},\theta_{|\xi|},\xi)$ can be estimate in the following way:

$$\|D_\xi^{\alpha_1}a(\theta_{|\xi|},\theta_{|\xi|},\xi)\|\lesssim|\xi|^{-1-|\alpha|}\left(\frac{1}{1+\theta_{|\xi|}}\right)^2.$$

Indeed, following the Remark 3.4 we know that if $\ell_1+\ell_2\leq k-1$, then $\Phi_k(t,s,\xi)\in S_N^{\ell_1,\ell_2}$. If we denote by $a(\theta_{|\xi|},\theta_{|\xi|},\xi)=G(\theta_{|\xi|},\xi)$ and apply the generalized version of Faa di Bruno's formula, see Lemma 7.6, we obtain for the case $|\alpha|=n$

$$D_\xi^\alpha G(\theta_{|\xi|},\xi)=\sum_0\sum_1\cdots\sum_n C(n,k_i,q_{ij})\frac{\partial^\kappa G}{\partial_{\theta_{|\xi|}}\partial_\xi^{\alpha_2}}(\theta_{|\xi|},\xi)\prod_{i=1,|\alpha_i|=i}\left(D_\xi^{\alpha_i}\theta_{|\xi|}\right)^{q_{i1}}\left(D_\xi^{\alpha_i}\xi\right)^{q_{i2}},$$

where the respective sums are taken over all non-negative integer solution of the Diophantine equations as follows:

$$
\begin{aligned}
\sum_0 &\rightarrow k_1+2k_2+\cdots+nk_n=n\\
\sum_1 &\rightarrow q_{11}+q_{12}+\cdots+q_{1r}=k_1\\
&\vdots\\
\sum_n &\rightarrow q_{n1}+q_{n2}+\cdots+q_{nr}=k_n,
\end{aligned}
$$

and

$$
\begin{aligned}
p_1 &= \sum_{i=1}^n q_{ij},\ |\alpha_2|=\sum_{i=1}^n q_{i2}\\
|\kappa| &= k_1+k_2+\cdots+k_n=p_1+|\alpha_2|.
\end{aligned}
$$

By virtue of $\partial_{\xi_k}\xi_L = \delta_{kl}$ we may conclude that $q_{i2} = 0,$ for all $i \geq 2$ and $|\alpha_2| = q_{12}$. This yields the estimate

$$
\begin{aligned}
\|D_\xi^\alpha G(\theta_{|\xi|},\xi)\| &\lesssim \sum_0 \sum_1 \cdots \sum_n |\xi|^{-1-|\alpha_2|}\left(\frac{1}{1+\theta_{|\xi|}}\right)^{2+p_1} \prod_{i=1}^{n} |\xi|^{-(|\alpha_i|+1)q_{i1}} \\
&= \sum_0 \sum_1 \cdots \sum_n |\xi|^{-1-q_{12}}\left(\frac{1}{1+\theta_{|\xi|}}\right)^{2+p_1} |\xi|^{-p_1-n-q_{12}} \\
&\lesssim |\xi|^{-1-n}\left(\frac{1}{1+\theta_{|\xi|}}\right)^{2}.
\end{aligned}
$$

Therefore $|D_\xi^\alpha b(t,s,\xi)|$ can be estimate by $C_{\alpha,N}|\xi|^{-|\alpha|}$. This complete the proof.

□

After all these discussions we arrive at the following representation of the fundamental solution to system (3.17).

Theorem 3.1. *Assume that the Hypothesis 3.1 and 3.2 are satisfied. Then the fundamental solution $\mathcal{E}_k(t,s,\xi)$ of the diagonalized system* (3.17) *can be represented in the hyperbolic zone as*

$$
\mathcal{E}_k(t,s,\xi) = \frac{\psi(s)}{\psi(t)}\mathcal{E}_0(t,s,\xi)\mathcal{Q}_k(t,s,\xi) \text{ for all } t,s \geq \theta_{|\xi|} \tag{3.26}
$$

with a symbol $\mathcal{Q}_k(t,s,\xi)$ subjected to the symbol like estimates

$$
\left\|D_\xi^\alpha \mathcal{Q}_k(t,s,\xi)\right\| \leq C_\alpha |\xi|^{-|\alpha|} \text{ for all } t,s \geq \theta_{|\xi|} \tag{3.27}
$$

and for all multi-indices $|\alpha| \leq k-1$.

Remark 3.5. *One use of this representation is to derive later $L^p - L^q$ estimates for the Cauchy problem for Klein-Gordon models* (3.1). *To derive these estimates we will apply the Marcinkiewicz's multiplier Theorem 7.1. One basic assumption to apply this theorem is that an amplitude $b = b(\xi) \in C^m(\mathbb{R}^n - \{0\})$ is subjected to the symbol-like estimates*

$$
|D_\xi^\alpha b(\xi)| \lesssim |\xi|^{-|\alpha|} \text{ for all } |\xi| \leq m,
$$

where $m = \left\lceil \frac{n}{2} \right\rceil + 1.$

The previous remark shows us how many steps of diagonalization are necessary, at least, for applying the Marcinkiewicz's theorem. In other words, if k is the number of steps of diagonalization, then $k-1 \geq \left\lceil \frac{n}{2} \right\rceil + 1$. Here the regularity of the mass $m(t)$ should be at the least equal to $2(k-1)$.

Transforming back to the original problem:

After constructing the fundamental solution $\mathcal{E}_k(t,s,\xi)$ we want to transform back to the original problem and get in the hyperbolic zone the representation that we are looking for. We know that,

$$
N_k(t,\xi)^{-1}\big(D_t - D(\xi) + B + C\big)N_k(t,\xi) = \big(D_t - D(\xi) + F_{k-1}(t,\xi) - R_k(t,\xi)\big).
$$

If $\mathcal{E}_k(t,s,\xi)$ is the fundamental solution to $\big(D_t - D(\xi) + F_{k-1}(t,\xi) - R_k(t,\xi)\big)$, then

$$\big(D_t - D(\xi) + B + C\big)N_k(t,\xi)\mathcal{E}_k(t,s,\xi) = 0.$$

In this way we have that $\mathcal{E}^{(0)}(t,s,\xi)N_k(s,\xi)$ and $N_k(t,\xi)\mathcal{E}_k(t,s,\xi)$ satisfy the same initial value problem, where $\mathcal{E}^{(0)}(t,s,\xi)$ is the fundamental solution to $D_t - D(\xi) + B + C$. Therefore,

$$\mathcal{E}^{(0)}(t,s,\xi)N_k(s,\xi) = N_k(t,\xi)\mathcal{E}_k(t,s,\xi),\;\; \mathcal{E}^{(0)}(t,s,\xi) = N_k(t,\xi)\mathcal{E}_k(t,s,\xi)N_k(s,\xi)^{-1},$$

respectively. Moreover, $\mathcal{E}^{(0)}(t,s,\xi)$ and $M^{-1}\mathcal{E}(t,s,\xi)M$ satisfy the same initial value problem. So, the representation of the fundamental solution in the hyperbolic zone is

$$\begin{aligned}\mathcal{E}(t,s,\xi) &= MN_k(t,\xi)\mathcal{E}_k(t,s,\xi)N_k(s,\xi)^{-1}M^{-1}\\ &= \frac{\psi(s)}{\psi(t)}MN_k(t,\xi)\mathcal{E}_0(t,s,\xi)\mathcal{Q}_k(t,s,\xi)N_k(s,\xi)^{-1}M^{-1},\end{aligned}$$

with uniformly bounded coefficient matrices $N_k, N_k^{-1} \in S_N^{\ell-k+1,\infty}\{0,0\}$.

The above representation will be used for large frequencies. We will consider a different representation for small frequencies, because in this case we shall use the "gluing procedure". For small frequencies we should remember that for $0 \le s \le \theta_{|\xi|} \le t$ we have

$$\mathcal{E}(t,s,\xi) \;=\; \mathcal{E}(t,\theta_{|\xi|},\xi)\mathcal{E}(\theta_{|\xi|},s,\xi).$$

Taking this into account for $\mathcal{E}(t,\theta_{|\xi|},\xi)$ we get

$$\mathcal{E}(t,s,\xi) = \frac{1}{\psi(t)}MN_k(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi)\mathcal{Q}_k(t,\theta_{|\xi|},\xi)N_k(\theta_{|\xi|},\xi)^{-1}M^{-1}\psi(\theta_{|\xi|})\mathcal{E}(\theta_{|\xi|},s,\xi)$$

for $0 \le s \le \theta_{|\xi|} \le t$.

If $U(t,\xi) = \mathcal{H}(t,s,\xi)U(s,\xi)$ and $\mathcal{H}(s,s,\xi) = I$, then $\mathcal{H}(t,s,\xi) = \dfrac{\psi(t)}{\psi(s)}\mathcal{E}(t,s,\xi)$.

Summarizing we arrive at

$$\mathcal{H}(t,s,\xi) = MN_k(t,\xi)\mathcal{E}_0(t,s,\xi)\mathcal{Q}_k(t,s,\xi)N_k(s,\xi)^{-1}M^{-1},\;\; t,s \ge \theta_{|\xi|}, \tag{3.28}$$

and

$$\mathcal{H}(t,s,\xi) = MN_k(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi)\mathcal{Q}_k(t,\theta_{|\xi|},\xi)N_k(\theta_{|\xi|},\xi)^{-1}M^{-1}\mathcal{H}(\theta_{|\xi|},s,\xi) \tag{3.29}$$

for $0 \le s \le \theta_{|\xi|} \le t$.

3.2 $L^p - L^q$ decay estimates on the conjugate line

In Chapter 2 we proved results about generalized energy conservation. In this way we find out $L^2 - L^2$ estimates for energy solutions to Cauchy problems for Klein-Gordon models. In this section we will use the representations for the solution, we have discussed in the previous section, to extend our $L^2 - L^2$ estimates to $L^p - L^q$ decay estimates on the conjugate line for solutions to Cauchy problems for non-effective Klein-Gordon models.

Basically the $L^p - L^q$ decay estimates are given by the decay estimates of the fundamental solution operator $\mathcal{E}_0(t,s,D)$ of the free wave equation. This result is well-known and we will present it in the next theorem and give a proof.

Theorem 3.2. *The fundamental solution operator of the free wave equation satisfies*

$$\|\mathcal{E}_0(t,0,D)\|_{L^{p,r}\to L^q} \le C_{p,q}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \tag{3.30}$$

for $p \in (1,2]$, p and q on the conjugate line and with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Proof. Let us divide the extended phase space into three zones:

$$\begin{aligned} Z_1 &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : (1+t)|\xi| \le N\},\\ Z_2 &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi| \le N \le (1+t)|\xi|\},\\ Z_3 &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi| \ge N\}. \end{aligned}$$

If we consider the function $\chi \in C^\infty(\mathbb{R}_+)$ such that $\chi(s)=1$ for $s \le \frac{1}{2}$, $\chi(s)=0$ for $s \ge 2$ and $\chi'(s) \le 0$. Then the functions $\chi_1 = \chi_1(t,\xi)$, $\chi_2 = \chi_2(t,\xi)$ and $\chi_3 = \chi_3(\xi)$ are defined by

$$\begin{aligned} \chi_1(t,\xi) &= \chi(|\xi|N^{-1})\chi((t+1)|\xi|N^{-1})\\ \chi_2(t,\xi) &= \chi(|\xi|N^{-1})\left(1-\chi((t+1)|\xi|N^{-1})\right)\\ \chi_3(\xi) &= 1-\chi(|\xi|N^{-1}), \end{aligned}$$

such that $\chi_1+\chi_2+\chi_3=1$. These functions are the characteristic functions for the zones Z_1, Z_2 and Z_3, respectively.
The diagonal matrix $\mathcal{E}_0(t,0,\xi)$ has entries $e^{\pm it|\xi|}$. Therefore, we shall investigate the Fourier multipliers

$$F^{-1}(e^{\pm it|\xi|}F(v)) \text{ for } v \in \mathcal{S}.$$

Considerations in Z_1

In Z_1 we have that $\|\mathcal{E}_0(t,0,\xi)\chi_1(t,\xi)\| \lesssim 1$. How $q \ge 2$ if we suppost that $v = v(x) \in L^p(\mathbb{R}^n)$ follows the estimate

$$\begin{aligned} \left\|F^{-1}\left(\mathcal{E}_0(t,0,\xi)\chi_1(t,\xi)\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^q} &\le \|\mathcal{E}_0(t,0,\cdot)\chi_1(t,\cdot)\widehat{v}\|_{L^{p'}}\\ &\le \|\mathcal{E}_0(t,0,\cdot)\|_{L^\infty}\|\chi_1(t,\cdot)\|_{L^{\frac{pq}{q-p}}}\|\widehat{v}\|_{L^q}\\ &\lesssim (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^p}, \end{aligned}$$

which is a better decay estimate in comparison with the statement of the theorem.

Considerations in Z_3

In this zone we will consider large frequencies $|\xi| \ge N$. We have to analyze

$$\|F^{-1}(\chi_3(\xi)e^{\pm it|\xi|}\widehat{v}(\xi))(t,\cdot)\|_{L^q}$$

with $2 \le q < \infty$ and $v \in \mathcal{S}$.

Take $\phi = \phi(s) \in C_0^\infty(\mathbb{R}_+)$ with $\operatorname{supp}\phi \subset \left[\frac{1}{2},2\right]$ such that $\sum_{j=-\infty}^{\infty}\phi(2^{-j}s)=1$.

Let us define a dyadic decomposition $\{\phi_j\}_{j\in\mathbb{Z}}$ by $\phi_j(\xi) = \phi\left(2^{-j}|\xi|N^{-1}\right)$.

Since $\chi_3(\xi)\phi_j(\xi) = 0$ for $j < 0$ we have

$$\chi_3(\xi) \leq \sum_{j=0}^{\infty} \phi_j(\xi).$$

This enables us to investigate every sum stated in the right-hand side for $j \in \mathbb{N}$ separately. The goal here is apply the Riesz-Thorin interpolation Theorem 7.2.

For every $j \in \mathbb{N}$ let us examine the oscillatory integral

$$F^{-1}(\phi_j(\xi)|\xi|^{-r}e^{\pm it|\xi|}).$$

$L^1 - L^\infty$ **estimates:** Let us substitute $\xi = 2^j N\eta$. Therefore,

$$\begin{aligned}
\left\|F^{-1}\left(\phi_j(\xi)|\xi|^{-r}e^{\pm it|\xi|}\right)(t,\cdot)\right\|_{L^\infty} &= 2^{j(n-r)}\left\|F^{-1}\left(\phi(|\eta|)e^{\pm i2^j tN|\eta|}|\eta|^{-r}\right)(t,\cdot)\right\|_{L^\infty} \\
&\leq K2^{j(n-r)}\left(1+2^jNt\right)^{-\frac{n-1}{2}}\sum_{|\alpha|\leq M}\|D^\alpha\phi(|\eta|)|\eta|^{-r}\|_{L^\infty} \\
&\leq C2^{j(n-r)}\left(1+2^jNt\right)^{-\frac{n-1}{2}}\sup_{\frac{1}{2}\leq|\eta|\leq 2}|\eta|^{-r} \\
&\leq \widetilde{C}2^{j(n-r)}(1+t)^{-\frac{n-1}{2}}.
\end{aligned}$$

Here we use Lemma 7.1 in the first inequality with a suitably positive constant M. Moreover, we have that $(1+t) \lesssim (1+2^jNt)$ for $j > 0$ and N sufficiently large.

Therefore (1) in Lemma 7.2 implies that

$$\left\|F^{-1}\left(\phi_j(\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^\infty} \lesssim 2^{j(n-r)}(1+t)^{-\frac{n-1}{2}}\|v\|_{L^1}.$$

$L^2 - L^2$ **estimates:** We have that

$$\|\phi_j(\xi)|\xi|^{-r}e^{\pm it|\xi|}\|_{L^\infty} \lesssim \sup_{\frac{1}{2}\leq|\eta|\leq 2}\phi(|\eta|)\left(2^jN|\eta|\right)^{-r} \lesssim 2^{-jr}.$$

The application of (2) in Lemma 7.2 yields the estimate

$$\left\|F^{-1}\left(\phi_j(\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^2} \lesssim 2^{-jr}\|v\|_{L^2}.$$

$L^p - L^q$ **estimates:** Applying Riesz-Thorin's Theorem 7.2 we have

$$\left\|F^{-1}\left(\phi_j(\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^q} \lesssim 2^{j\left(n\left(\frac{1}{p}-\frac{1}{q}\right)-r\right)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^p}$$

for $1 < p \leq 2$ and p, q from the conjugate line.

The hypothesis for the regularity r allows us to estimate the right-hand side uniformly for all $j \geq 0$. Thus, due the Lemma 7.3 we conclude

$$\left\|F^{-1}\left(\chi_3(\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^p}.$$

Considerations in Z_2

In this zone we consider small frequencies $|\xi| \leq N$ and $(1+t)|\xi| \geq N$. We have to analyze

$$\|F^{-1}(\chi_2(t,\xi)e^{\pm it|\xi|}\widehat{v}(\xi))(t,\cdot)\|_{L^q}$$

with $2 \leq q < \infty$ and $v \in \mathcal{S}$.

We introduce a dyadic decomposition $\{\phi_j\}_{j\in\mathbb{Z}}$ in this part of the extended phase space by defining $\phi_j(t,\xi) = \phi(2^{-j}(1+t)|\xi|N^{-1})$, $j \in \mathbb{Z}$, where $\phi = \phi(s) \in C_0^\infty(\mathbb{R}_+)$ with $\operatorname{supp}\phi \subset \left[\frac{1}{2}, 2\right]$ such that $\sum\limits_{j=-\infty}^{\infty} \phi(2^{-j}s) = 1$, $t > 0$. Then the product $\chi_2(t,\xi)\phi_j(t,\xi)$ vanishes for all $j < 0$. This implies

$$\chi_2(t,\xi) \;\; \leq \;\; \sum_{j=0}^{\infty} \phi_j(t,\xi),$$

so we have to investigate the Fourier multipliers for every $j \geq 0$ separately.

For every $j \in \mathbb{N}$ let us examine the oscillatory integrals

$$F^{-1}(\phi_j(t,\xi)|\xi|^{-r}e^{\pm it|\xi|}).$$

$L^1 - L^\infty$ **estimates:** Let us substitute $(1+t)\xi = 2^j N\eta$. Therefore,

$$\begin{aligned}
&\left\|F^{-1}\left(\phi_j(t,\xi)|\xi|^{-r}e^{\pm it|\xi|}\right)(t,\cdot)\right\|_{L^\infty} \\
\lesssim\;& 2^{j(n-r)}(1+t)^{-n+r}\left\|F^{-1}\left(\phi(|\eta|)e^{\pm i2^j\frac{t}{t+1}N|\eta|}|\eta|^{-r}\right)(t,\cdot)\right\|_{L^\infty} \\
\lesssim\;& 2^{j(n-r)}(1+t)^{-n+r}(1+2^jN)^{-\frac{n-1}{2}}\sum_{|\alpha|\leq M}\|D^\alpha\phi(|\eta|)|\eta|^{-r}\|_{L^\infty} \\
\lesssim\;& 2^{j\left(\frac{n+1}{2}-r\right)}(1+t)^{-n+r}\sup_{\frac{1}{2}<|\xi|\leq 2}|\eta|^{-r} \\
\lesssim\;& 2^{j\left(\frac{n+1}{2}-r\right)}(1+t)^{-n+r},
\end{aligned}$$

where M is a positive constant and $t \geq 1$. For $t \leq 1$ it suffices to observe that the set $\{(t,\xi); t \leq 1 \text{ and } |\xi| \leq N\}$ is compact. Summarizing,

$$\left\|F^{-1}\left(\phi_j(t,\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^\infty} \lesssim 2^{j\left(\frac{n+1}{2}-r\right)}(t+1)^{-n+r}\|v\|_{L^1}.$$

$L^2 - L^2$ **estimates:** We have that

$$\|\phi_j(t,\xi)|\xi|^{-r}e^{\pm it|\xi|}\|_{L^\infty} \lesssim 2^{-jr}(1+t)^r.$$

Therefore, it follows from (2) in Lemma 7.2 that

$$\left\|F^{-1}\left(\phi_j(t,\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^2} \lesssim 2^{-jr}(1+t)^r\|v\|_{L^2}.$$

$L^p - L^q$ **estimates:** Applying Riesz-Thorin's Theorem 7.2 gives

$$\left\|F^{-1}\left(\phi_j(t,\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^q} \lesssim 2^{j\left(\frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-r\right)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)+r}\|v\|_{L^p}.$$

If we take $r = \frac{n+1}{2}\big(\frac{1}{p} - \frac{1}{q}\big)$, then we can estimate the right-hand side uniformly for all $j \geq 0$. Thus, due the Lemma 7.3 we conclude

$$\left\|F^{-1}\left(\chi_2(t,\xi)|\xi|^{-r}e^{\pm it|\xi|}\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^p}.$$

All this considerations imply that the corresponding operators possess the mapping property

$$e^{\pm it|D|} : \dot{B}^r_{p,2} \to L^q$$

with the regularity $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$. Finally, the embedding relation $L^{p,r} \subset \dot{B}^r_{p,2} \cap L^p$ for $r > 0$ and $p \in (1,\infty)$ yields the desired result. □

Remark 3.6. *The decay in the previous theorem comes from the hyperbolic zone. The regularity comes from the large frequencies in the hyperbolic zone.*

Using Theorem 3.2 we can deduce $L^p - L^q$ estimates for the fundamental solution operator of the Cauchy problem for Klein-Gordon equations (3.1).

Theorem 3.3. *Assume that the Hypothesis 3.1 and 3.2 are satisfied. Then the fundamental solution operator of the Cauchy problem for Klein-Gordon equation satisfies*

$$\begin{aligned}\|\mathcal{H}(t,0,D)\varphi_{pd}(t,D)\|_{L^{p,r}\to L^q} &\leq C_{p,q}\frac{1}{\psi(t)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)},\\ \left\|\left(\mathcal{H}(t,0,D)\varphi^{\ell}_{hyp}(D), \mathcal{H}(t,0,D)\varphi^{s}_{hyp}(t,D)\right)\right\|_{L^{p,r}\to L^q} &\leq C_{p,q}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\end{aligned}$$

for $p \in (1,2]$, p and q on the conjugate line and with regularity $r = n\big(\frac{1}{p} - \frac{1}{q}\big)$.

Proof. The proof is divided into two steps.

Considerations in the pseudo-differential zone

In the pseudo-differential zone we have the estimate

$$\|\mathcal{H}(t,0,\xi)\varphi_{pd}(t,\xi)\| \lesssim \frac{1}{\psi(t)}.$$

If $v = v(x) \in L^p(\mathbb{R}^n)$, then

$$\begin{aligned}\left\|F^{-1}\left(\mathcal{H}(t,0,\xi)\varphi_{pd}(t,\xi)\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^q} &\leq \|\mathcal{H}(t,0,\cdot)\varphi_{pd}(t,\cdot)\widehat{v}\|_{L^p}\\ &\leq \|\mathcal{H}(t,0,\cdot)\|_{L^\infty}\|\varphi_{pd}(t,\cdot)\|_{L^{\frac{pq}{q-p}}}\|\widehat{v}\|_{L^q}\\ &\lesssim \frac{1}{\psi(t)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^p},\end{aligned}$$

which is a better decay estimate in comparison to the statement of the theorem.

Considerations in the hyperbolic zone

For large frequencies we have from (3.28) that

$$\mathcal{H}(t,0,D)\varphi^{\ell}_{hyp}(D) = \underbrace{MN_k(t,D)}_{L^q \to L^q}\underbrace{\mathcal{E}_0(t,0,D)}_{L^{p,r}\to L^q}\underbrace{\mathcal{Q}_k(t,0,D)}_{L^{p,r}\to L^{p,r}}\underbrace{N_k(0,D)^{-1}M^{-1}}_{L^{p,r}\to L^{p,r}}\varphi^{\ell}_{hyp}(D).$$

Indeed, the mapping properties of all operators appearing in this representation can be explained as follows:

- We know that $MN_k(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$, then $MN_k(t,\xi) \in \dot{S}^0_{\ell-k+1}$, Marcinkiewicz's Theorem 7.1 implies that $MN_k(t,\xi) \in M_q^q$ uniformly in t. Here it is essential that $\ell - k + 1 \geq \lceil \frac{n}{2} \rceil + 1$.

- Theorem 3.2 implies $\mathcal{E}_0(t,0,D) : L^{p,r} \to L^q$ with a decay rate $(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$.

- Theorem 3.1 implies that $\mathcal{Q}_k(t,0,\xi) \in \dot{S}^0_{k-1}$. Then Marcinkiewicz's Theorem 7.1 implies that $\mathcal{Q}_k(t,0,\xi) \in M^{p,r}_{p,r}$ uniformly in t. It is essential that, $k-1 \geq \lceil \frac{n}{2} \rceil + 1$.

Let us take $\ell = 2(k-1)$ and $k-1 \geq \lceil \frac{n}{2} \rceil + 1$. Then we can apply the Marcinkiewicz's Theorem 7.1. Therefore, if $v = v(x) \in L^{p,r}(\mathbb{R}^n)$, then

$$\|F^{-1}\left(\mathcal{H}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{v}(\xi)\right)(t,\cdot)\|_{L^q} \quad \lesssim \quad (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^{p,r}}.$$

For small frequencies we have from (3.29)

$$\mathcal{H}(t,0,D)\varphi^{s}_{hyp}(t,D) = \underbrace{MN_k(t,D)}_{L^q\to L^q}\underbrace{\mathcal{E}_0(t,\theta_{|D|},D)}_{L^{p,r}\to L^q}\underbrace{\mathcal{Q}_k(t,\theta_{|D|},D)}_{L^{p,r}\to L^{p,r}}\underbrace{N_k(\theta_{|D|},D)^{-1}M^{-1}}_{L^{p,r}\to L^{p,r}}$$
$$\times\underbrace{\mathcal{H}(\theta_{|D|},0,D)}_{L^{p,r}\to L^{p,r}}\varphi^{s}_{hyp}(t,D).$$

In fact, the mapping properties of all operators appearing in this representation can be explained as follows:

- We know that $MN_k(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$, then $MN_k(t,\xi) \in \dot{S}^0_{\ell-k+1}$, Marcinkiewicz's Theorem 7.1 implies that $MN_k(t,\xi) \in M_q^q$ uniformly in t.

- Follows from part 1 of Proposition 3.4 that $\mathcal{E}_0(0,\theta_{|\xi|},\xi) \in \dot{S}^0_\infty$. In fact, we have that the entries of the matrix $\mathcal{E}_0(0,\theta_{|\xi|},\xi)$ are $e^{\pm i\theta_{|\xi|}|\xi|}$. How the constant function $1 \in S_N\{0,0\}$ then $e^{\pm i\theta_{|\xi|}|\xi|}\cdot 1 \in S_N\{0,0\}$. Therefore Theorem 3.2 and the propagator property $\mathcal{E}_0(t,\theta_{|\xi|},\xi) = \mathcal{E}_0(t,0,\xi)\mathcal{E}_0(0,\theta_{|\xi|},\xi)$ implies that $\mathcal{E}_0(t,\theta_{|D|},D) : L^{p,r} \to L^q$ with decay rate $(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$.

- Theorem 3.1 implies that $\mathcal{Q}_k(t,\theta_{|\xi|},\xi) \in \dot{S}^0_{k-1}$. Then Marcinkiewicz's Theorem 7.1 implies that $\mathcal{Q}_k(t,\theta_{|\xi|},\xi) \in M^{p,r}_{p,r}$ uniformly in t. It is essential that $k-1 \geq \lceil \frac{n}{2} \rceil + 1$.

- We have that $\mathcal{H}(\theta_{|\xi|},0,\xi) = \frac{1}{\psi(\theta_{|\xi|})}\psi(\theta_{|\xi|})\mathcal{H}(\theta_{|\xi|},0,\xi)$. Lemma 3.1 guarantee that $\psi(\theta_{|\xi|})\mathcal{H}(\theta_{|\xi|},0,\xi) \in \dot{S}^0_{\ell+1}$ and Hypothesis 3.2 ensure $\frac{1}{\psi(\theta_{|\xi|})} \in \dot{S}^0_\infty$. Therefore, $\mathcal{H}(\theta_{|\xi|},0,\xi) \in M^{p,r}_{p,r}$.

If $v = v(x) \in L^{p,r}(\mathbb{R}^n)$, then

$$\|F^{-1}\left(\mathcal{H}(t,0,\xi)\varphi^s_{hyp}(t,\xi)\widehat{v}(\xi)\right)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^{p,r}}$$

for small frequencies. This completes the proof. □

The following lemma is important to derive Strichartz estimates for the potential energy in the hyperbolic zone for small frequencies.

Lemma 3.5. *Assume Hypothesis 3.2. Then* $\frac{p(t)}{p(\theta_{|\xi|})} \in \dot{S}^0_\infty$, *where* $p(t) = (1+t)^{-1}\psi(t)$, $t \geq \theta_{|\xi|}$, *and* t *is sufficiently large.*

Proof. We know that $p(t)$ is decreasing for large t, i.e., $\frac{p(t)}{p(\theta_{|\xi|})} \lesssim 1$.

Applying Faà de Bruno's formula (Lemma 7.4) we get

$$d^n_{|\xi|}\left(\frac{p(t)}{p(\theta_{|\xi|})}\right) = \sum \frac{n!}{m_1!1!^{m_1}m_2!2!^{m_2}\cdots m_n!n!^{m_n}}p(t)\left(\frac{1}{p}\right)^{(m_1+m_2+\cdots+m_n)}(\theta_{|\xi|})\prod_{j=1}^n\left(d^j_{|\xi|}\theta_{|\xi|}\right)^{m_j}, \tag{3.31}$$

where the sum is taken over all $n-$tuples of non-negative integers $(m_1, m_2, \cdots, m_n)$ satisfying

$$1m_1 + 2m_2 + \cdots + nm_n = n.$$

We have that $\left|d^j_{|\xi|}\theta_{|\xi|}\right| \lesssim |\xi|^{-1-j}$ for every $j = 1, 2, \cdots, n$. The Hypothesis 3.2 together with $\frac{p(t)}{p(\theta_{|\xi|})} \leq 1$ implies that

$$p(t)\Big(\frac{1}{p}\Big)^{(n)}(\theta_{|\xi|}) \lesssim |\xi|^n \quad \text{for} \quad \text{all} \quad n \geq 0.$$

Indeed, first of all we have

$$p^{(n)}(t) = \sum_{i=0}^n c_i \frac{\psi^{(i)}(t)}{(1+t)^{n+1-i}}.$$

Then Hypothesis 3.2 implies

$$\Big|\frac{p^{(n)}(t)}{p(t)}\Big| = \Big|\sum_{i=0}^n c_i \frac{\psi^{(i)}(t)}{\psi(t)}\frac{1}{(1+t)^{n-i}}\Big| \lesssim \sum_{i=0}^n |c_i|\frac{1}{(1+t)^n} \approx \frac{1}{(1+t)^n}. \tag{3.32}$$

Applying the Faà de Bruno Formula (Lemma 7.4) for the function $\frac{1}{p(t)}$ we arrive in

$$\Big(\frac{1}{p}\Big)^{(n)}(t) = \frac{1}{p(t)}\sum \frac{n!}{m_1!1!^{m_1}m_2!2!^{m_2}\cdots m_n!n!^{m_n}}\frac{1}{p(t)^{m_1+\cdots+m_n}}\prod_{j=1}^n\left(p^{(j)}(t)\right)^{m_j}$$

where the sum is taken over all $n-$tuples of non-negative integers $(m_1, m_2, \cdots, m_n)$ satisfying

$$1m_1 + 2m_2 + \cdots + nm_n = n.$$

Therefore we conclude from (3.32)

$$\Big|\left(\frac{1}{p}\right)^{(n)}(t)\Big| \lesssim \frac{1}{p(t)}\sum \frac{n!}{m_1!1!^{m_1}m_2!2!^{m_2}\cdots m_n!n!^{m_n}}\frac{1}{(1+t)^{1m_1+2m_2+\cdots+nm_n}}$$
$$\lesssim \frac{1}{p(t)}\frac{1}{(1+t)^n}.$$

Summarizing,

$$\Big|\left(\frac{1}{p}\right)^{(n)}(\theta_{|\xi|})\Big| \lesssim \frac{1}{p(\theta_{|\xi|})}\frac{1}{(1+\theta_{|\xi|})^n}$$
$$\Rightarrow\ p(t)\left(\frac{1}{p}\right)^{(n)}(\theta_{|\xi|}) \lesssim \frac{p(t)}{p(\theta_{|\xi|})}\frac{1}{(1+\theta_{|\xi|})^n} \lesssim |\xi|^n.$$

Therefore (3.31) imply

$$\left|d_{|\xi|}^n\left(\frac{p(t)}{p(\theta_{|\xi|})}\right)\right| \lesssim |\xi|^{-n},$$

what we wanted to prove. □

After these considerations we can formulate the following corollary.

Corollary 3.1. *Assume Hypotheses 3.1 and 3.2. If the Cauchy data $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n)$, then we have the $L^p - L^q$ estimates for the the kinetic, elastic and potential energy as follows:*

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot), p(t)u(t,\cdot))\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right)$$

for $p \in (1,2]$, p and q on the conjugate line, $p(t) = (1+t)^{-1}\psi(t)$, and with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Proof. The proof is divided into two steps.

Step 1: First let us prove estimates for the derivatives of the solution.

Considerations in the pseudo-differential zone

In the pseudo-differential zone we have the following relation:

$$\begin{pmatrix} 1 & 0 \\ -(1+t)\frac{\psi'(t)}{\psi(t)} & 1 \end{pmatrix}\begin{pmatrix} \frac{\widehat{u}(t,\xi)}{1+t} \\ \widehat{u}_t(t,\xi) \end{pmatrix}\varphi_{pd}(t,\xi) = \mathcal{H}(t,0,\xi)\varphi_{pd}(t,\xi)\begin{pmatrix} 1 & 0 \\ -\psi'(0) & 1 \end{pmatrix}\begin{pmatrix} \widehat{u}_0 \\ \widehat{u}_1 \end{pmatrix}.$$

If we denote by $\mathcal{H}_{ij}$, $i,j = 1,2$, the entries of the matrix $\mathcal{H} = \mathcal{H}(t,0,\xi)$, we have

$$\frac{\widehat{u}(t,\xi)}{1+t}\varphi_{pd}(t,\xi) = \sum_{j=1}^{2}\mathcal{H}_{1j}(t,0,\xi)\varphi_{pd}(t,\xi)\delta_j\widehat{u}_0(\xi) + \mathcal{H}_{12}(t,0,\xi)\varphi_{pd}(t,\xi)\delta_j\widehat{u}_1(\xi),$$

$$\widehat{u}_t(t,\xi)\varphi_{pd}(t,\xi) = \sum_{j=1}^{2}\Big(\mathcal{H}_{1j}(t,0,\xi)\varphi_{pd}(t,\xi)\eta_j(t) + \mathcal{H}_{2j}(t,0,\xi)\varphi_{pd}(t,\xi)\delta_j\Big)\widehat{u}_0(\xi)$$
$$+\sum_{j=1}^{2}\mathcal{H}_{1j}(t,0,\xi)\varphi_{pd}(t,\xi)\widetilde{\delta}_j(t)\widehat{u}_1(\xi),$$

where

$$\delta_j := \begin{cases} 1, & j=1 \\ -\psi'(0), & j=2 \end{cases}, \quad \widetilde{\delta}_j(t) := \begin{cases} (1+t)\frac{\psi'}{\psi}(t), & j=1 \\ 1, & j=2 \end{cases},$$

and

$$\eta_j(t) := \begin{cases} (1+t)\frac{\psi'}{\psi}(t), & j=1 \\ -(1+t)\frac{\psi'}{\psi}(t), & j=2 \end{cases}.$$

The Hypothesis 3.2 implies that $(1+t)\frac{\psi'(t)}{\psi(t)}$ is bounded. Therefore the Theorem 3.3 implies

$$\begin{aligned} \left\| F^{-1}\left(|\xi|\widehat{u}(t,\xi)\varphi_{pd}(t,\xi)\right)(t,\cdot)\right\|_{L^q} &\lesssim \frac{1}{\psi(t)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^p}+\|u_1\|_{L^p}\right), \\ \left\| F^{-1}\left(\widehat{u}_t(t,\xi)\varphi_{pd}(t,\xi)\right)(t,\cdot)\right\|_{L^q} &\lesssim \frac{1}{\psi(t)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^p}+\|u_1\|_{L^p}\right). \end{aligned}$$

In this part of the extended phase space we do not need to assume higher regularity for the data. Moreover, the decay behavior is better than those from the theorem.

Considerations in the hyperbolic zone

In the hyperbolic zone we divide our considerations for large and small frequencies. For large frequencies it holds

$$U(t,\xi) = \mathcal{H}(t,\theta_{|\xi|},\xi)U(\theta_{|\xi|},\xi) = \mathcal{H}(t,0,\xi)U(0,\xi).$$

Therefore,

$$\begin{pmatrix} i & 0 \\ -\frac{1}{|\xi|}\frac{\psi'(t)}{\psi(t)} & 1 \end{pmatrix}\begin{pmatrix} |\xi|\widehat{u}(t,\xi) \\ \widehat{u}_t(t,\xi) \end{pmatrix}\varphi^{\ell}_{hyp}(\xi) = \mathcal{H}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\begin{pmatrix} \frac{i|\xi|}{\langle\xi\rangle} & 0 \\ -\frac{\psi'(0)}{\langle\xi\rangle} & 1 \end{pmatrix}\begin{pmatrix} \langle\xi\rangle\widehat{u}_0 \\ \widehat{u}_1 \end{pmatrix}.$$

Then,

$$\begin{aligned} |\xi|\widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi) &= \frac{|\xi|}{\langle\xi\rangle}\mathcal{H}_{11}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\langle\xi\rangle\widehat{u}_0(\xi) - i\psi'(0)\mathcal{H}_{12}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{u}_0(\xi) \\ &\quad -i\mathcal{H}_{12}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{u}_1(\xi), \\ \widehat{u}_t(t,\xi)\varphi^{\ell}_{hyp}(\xi) &= \sum_{i=1}^{2}\mathcal{H}_{i1}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\zeta_j(t,\xi)\langle\xi\rangle\widehat{u}_0(\xi) \\ &\quad -\sum_{i=1}^{2}\mathcal{H}_{i2}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widetilde{\zeta}_j(t,\xi)\widehat{u}_0(\xi) \\ &\quad +\sum_{i=1}^{2}\mathcal{H}_{i2}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\varepsilon(t,\xi)\widehat{u}_1(\xi), \end{aligned}$$

where

$$\zeta_i(t,\xi) := \begin{cases} -\frac{1}{\langle\xi\rangle}\frac{\psi'(t)}{\psi(t)}, & i=1 \\ i\frac{|\xi|}{\langle\xi\rangle}, & i=2 \end{cases}, \quad \widetilde{\zeta}_i(t,\xi) := \begin{cases} \frac{i}{|\xi|}\frac{\psi'(t)}{\psi(t)}, & i=1 \\ \psi'(0), & i=2 \end{cases},$$

and

$$\varepsilon_i(t) := \begin{cases} \frac{i}{|\xi|}\frac{\psi'(t)}{\psi(t)}, & i=1 \\ 1, & i=2 \end{cases}.$$

Note that $\frac{1}{|\xi|}\frac{\psi'(t)}{\psi(t)}, \frac{1}{\langle\xi\rangle}\frac{\psi'(t)}{\psi(t)}, \frac{|\xi|}{\langle\xi\rangle} \in S_N\{0,0\}$. Then we can apply the Marcinkiewicz's theorem for these multipliers. Applying Theorem 3.3 the following estimates can be concluded:

$$\begin{aligned} \left\|F^{-1}\left(|\xi|\widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi)\right)(t,\cdot)\right\|_{L^q} &\lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}}+\|u_1\|_{L^{p,r}}\right), \\ \left\|F^{-1}\left(\widehat{u}_t(t,\xi)\varphi^{\ell}_{hyp}(\xi)\right)(t,\cdot)\right\|_{L^q} &\lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}}+\|u_1\|_{L^{p,r}}\right). \end{aligned}$$

For small frequencies we have

$$\begin{aligned} &\begin{pmatrix} i & 0 \\ -\frac{1}{|\xi|}\frac{\psi'(t)}{\psi(t)} & 1 \end{pmatrix}\begin{pmatrix} |\xi|\widehat{u}(t,\xi) \\ \widehat{u}_t(t,\xi) \end{pmatrix}\varphi^{s}_{hyp}(t,\xi) \\ &= \mathcal{H}(t,\theta_{|\xi|},\xi)\varphi^{s}_{hyp}(t,\xi)\begin{pmatrix} i|\xi|\widehat{u}(\theta_{|\xi|}) \\ -\frac{\psi'(\theta_{|\xi|})}{\psi(\theta_{|\xi|})}\widehat{u}(\theta_{|\xi|})+\widehat{u}_t(\theta_{|\xi|}) \end{pmatrix} \\ &= \mathcal{H}(t,\theta_{|\xi|},\xi)\varphi^{s}_{hyp}(t,\xi)\mathcal{H}(\theta_{|\xi|},0,\xi)U(0,\xi) \\ &= \mathcal{H}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\begin{pmatrix} i|\xi| & 0 \\ -\psi'(0) & 1 \end{pmatrix}\begin{pmatrix} \widehat{u}_0 \\ \widehat{u}_1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\xi|\widehat{u}(t,\xi)\varphi^{s}_{hyp}(t,\xi) &= \frac{|\xi|}{\langle\xi\rangle}\mathcal{H}_{11}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\langle\xi\rangle\widehat{u}_0(\xi) - i\psi'(0)\mathcal{H}_{12}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_0(\xi) \\ &\quad -i\mathcal{H}_{12}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_1(\xi), \\ \widehat{u}_t(t,\xi)\varphi^{s}_{hyp}(t,\xi) &= \sum_{i=1}^{2}\frac{|\xi|}{\langle\xi\rangle}\mathcal{H}_{i1}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\zeta_j(t,\xi)\langle\xi\rangle\widehat{u}_0(\xi) \\ &\quad -\sum_{i=1}^{2}\mathcal{H}_{i2}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widetilde{\zeta}_j(t,\xi)\widehat{u}_0(\xi) \\ &\quad +\sum_{i=1}^{2}\mathcal{H}_{i2}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\varepsilon(t,\xi)\widehat{u}_1(\xi), \end{aligned}$$

where

$$\zeta_i(t,\xi) := \begin{cases} i\frac{1}{|\xi|}\frac{\psi'(t)}{\psi(t)}, & i=1 \\ i, & i=2 \end{cases}, \quad \widetilde{\zeta}_i(t,\xi) := \begin{cases} \frac{i\psi'(0)}{|\xi|}\frac{\psi'(t)}{\psi(t)}, & i=1 \\ -\psi'(0), & i=2 \end{cases},$$

and

$$\varepsilon_i(t) := \begin{cases} \frac{i}{|\xi|}\frac{\psi'(t)}{\psi(t)}, & i=1 \\ 1, & i=2 \end{cases}.$$

Analogous as above we may conclude

$$\begin{aligned} \left\|F^{-1}\left(|\xi|\widehat{u}(t,\xi)\varphi^{s}_{hyp}(t,\xi)\right)(t,\cdot)\right\|_{L^q} &\lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}}+\|u_1\|_{L^{p,r}}\right), \\ \left\|F^{-1}\left(\widehat{u}_t(t,\xi)\varphi^{s}_{hyp}(t,\xi)\right)(t,\cdot)\right\|_{L^q} &\lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}}+\|u_1\|_{L^{p,r}}\right). \end{aligned}$$

Step 2: Now let us devote to estimates for the potential energy.

Considerations in the pseudo-differential zone

In the pseudo-differential zone we have

$$\begin{pmatrix} 1 & 0 \\ -(1+t)\frac{\psi'(t)}{\psi(t)} & 1 \end{pmatrix} \begin{pmatrix} \frac{\widehat{u}(t,\xi)}{1+t} \\ \widehat{u}_t(t,\xi) \end{pmatrix} \varphi_{pd}(t,\xi) = \mathcal{H}(t,0,\xi)\varphi_{pd}(t,\xi) \begin{pmatrix} 1 & 0 \\ -\psi'(0) & 1 \end{pmatrix} \begin{pmatrix} \widehat{u}_0 \\ \widehat{u}_1 \end{pmatrix}.$$

This leads to the representation

$$\begin{aligned} \frac{\widehat{u}(t,\xi)}{1+t}\varphi_{pd}(t,\cdot) &= \Big(\mathcal{H}_{11}(t,0,\xi)\varphi_{pd}(t,\xi) - \psi'(0)\mathcal{H}_{12}(t,0,\xi)\varphi_{pd}(t,\xi)\Big)\widehat{u}_0(\xi) \\ &\quad + \mathcal{H}_{12}(t,0,\xi)\varphi_{pd}(t,\cdot)\widehat{u}_1(\xi). \end{aligned}$$

The application of Theorem 3.3 implies

$$\begin{aligned} \Big\| F^{-1}\Big(\frac{\widehat{u}(t,\xi)}{1+t}\varphi_{pd}(t,\xi)\Big)(t,\cdot)\Big\|_{L^q} &\lesssim \frac{1}{\psi(t)}\,(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^p} + \|u_1\|_{L^p}\right), \\ \Rightarrow \left\| F^{-1}\left(p(t)\widehat{u}(t,\xi)\varphi_{pd}(t,\xi)\right)(t,\cdot)\right\|_{L^q} &\lesssim (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^p} + \|u_1\|_{p}\right). \end{aligned}$$

In this part of the extended phase space we have a better decay estimate than the expected one from the theorem.

Considerations in the hyperbolic zone

In the hyperbolic zone we have for large frequencies

$$\begin{pmatrix} \widehat{u}(t,\xi) \\ \widehat{u}_t(t,\xi) \end{pmatrix} \varphi^{\ell}_{hyp}(\xi) = \begin{pmatrix} \frac{1}{i|\xi|} & 0 \\ \frac{1}{i|\xi|}\frac{\psi'(t)}{\psi(t)} & 1 \end{pmatrix} \mathcal{H}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi) \begin{pmatrix} \frac{i|\xi|}{\langle\xi\rangle} & 0 \\ -\frac{\psi'(0)}{\langle\xi\rangle} & 1 \end{pmatrix} \begin{pmatrix} \langle\xi\rangle\widehat{u}_0 \\ \widehat{u}_1 \end{pmatrix}.$$

Hence, it follows

$$\begin{aligned} \widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi) &= \left(\mathcal{H}_{11}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi) - \frac{\psi'(0)}{|\xi|}\mathcal{H}_{12}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\right)\widehat{u}_0(\xi) \\ &\quad + \frac{1}{i|\xi|}\mathcal{H}_{12}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{u}_1(\xi). \end{aligned}$$

For large frequencies we have that $|\xi|^{-1}$ is bounded, so $|\xi|^{-1} \in S_N\{0,0\}$. We can conclude from Theorem 3.3 that

$$\left\| F^{-1}\left(\widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi)\right)(t,\cdot)\right\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}} + \|u_1\|_{L^{p,r}}\right),$$

which is a better decay estimate in comparison with that one from the theorem. In the hyperbolic zone we have for small frequencies

$$\begin{pmatrix} p(t)\widehat{u}(t,\xi) \\ \widehat{u}_t(t,\xi) \end{pmatrix} \varphi^{s}_{hyp}(t,\xi) = \begin{pmatrix} \frac{p(t)}{i|\xi|} & 0 \\ \frac{1}{i|\xi|}\frac{\psi'(t)}{\psi(t)} & 1 \end{pmatrix} \mathcal{H}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi) \begin{pmatrix} i|\xi| & 0 \\ -\psi'(0) & 1 \end{pmatrix} \begin{pmatrix} \widehat{u}_0 \\ \widehat{u}_1 \end{pmatrix}.$$

Hence, we get the representation

$$\begin{aligned} p(t)\widehat{u}(t,\xi)\varphi^{s}_{hyp}(t,\xi) &= p(t)\mathcal{H}_{11}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_0(\xi) \\ &\quad - p(t)\frac{\psi'(0)}{|\xi|}\mathcal{H}_{12}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_0(\xi) + \frac{p(t)}{i|\xi|}\mathcal{H}_{12}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_1(\xi). \end{aligned}$$

As before the Fourier multiplier related to the first term is estimated by $p(t)(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_0\|_{L^{p,r+1}}$. Due to the decreasing behavior of p we get a better estimate as we wanted to have in all extended phase space.

We need to deal with the terms $\frac{p(t)}{i|\xi|}\mathcal{H}_{12}(t,0,\xi)\varphi^s_{hyp}(t,\xi)F(v)$. The representation of the solution for small frequencies (3.29) gives

$$\frac{p(t)}{|D|}\mathcal{H}(t,0,D)\varphi^s_{hyp}(t,D) = \underbrace{\frac{p(t)}{Np(\theta_{|D|})}}_{L^q\to L^q}\underbrace{MN_k(t,D)}_{L^q\to L^q}\underbrace{\mathcal{E}_0(t,\theta_{|D|},D)}_{L^{p,r}\to L^q}$$
$$\times\underbrace{\mathcal{Q}_k(t,\theta_{|D|},D)}_{L^{p,r}\to L^{p,r}}\underbrace{N_k(\theta_{|D|},D)^{-1}M^{-1}}_{L^{p,r}\to L^{p,r}}\underbrace{\psi(\theta_{|D|})\mathcal{H}(\theta_{|D|},0,D)}_{L^{p,r}\to L^{p,r}}\varphi^s_{hyp}(t,D).$$

Here we use (3.9) together with Lemma 3.5. Summarizing

$$\left\|F^{-1}\left(p(t)\widehat{u}(t,\xi)\varphi^s_{hyp}(t,\xi)\right)(t,\cdot)\right\|_{L^q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|u_0\|_{L^{p,r+1}}+\|u_1\|_{L^{p,r}}\right).$$

We obtained all desired L^p-L^q decay estimates on the conjugate line. This completes the proof. □

Remark 3.7. *The decay behavior for the potential energy comes from the small frequencies in the hyperbolic zone. The regularity of the data comes from the hyperbolic zone.*

Remark 3.8. *The $L^p - L^q$ decay estimates for the kinetic and elastic energy from Corollary 3.1 coincide with the corresponding estimates from Theorem 2.6.*

3.2.1 Examples

We conclude this section with examples. In Chapter 2 we explained the function ψ for models with masses having the following structure:

$$m(t) = \frac{\mu}{(1+t)g(t)} \tag{3.33}$$

with a positive constant μ and with a function g satisfying Hypotheses 2.3 and 2.4.

If we assume estimates for further derivatives of g, i.e.,

$$|g^{(k)}(t)| \lesssim \frac{g(t)}{(1+t)^k}, \quad \text{for} \quad \text{all} \quad k\in\mathbb{N}, \tag{3.34}$$

then we can find explicitly $\psi=\psi(t)$. The mass term m given by (3.33) satisfies Hypothesis 3.1.

Example 3.1. *If $g(t)$ in* (3.33) *is given by $g(t)^2=\ln(e+t)\cdots\ln^{[m]}(e^{[m]}+t)$ with $e^{[k+1]}=e^{e^{[k]}}$ and $\ln^{[k+1]}(t)=\ln(\ln^{[k]}(t))$, then we have* (2.33) *for $N=1$, i.e., the conclusion of Corollary 3.1 holds with $\psi(t)$ given by* (2.34)*. We have that $\psi(t)\sim(\ln^{[m]}(e^{[m]}+t))^{\mu^2}$. Observe that ψ satisfies Hypothesis 3.2.*

Example 3.2. *Let $g(t)^2=(\ln(e+t))^\gamma$ for some $0<\gamma<1$. In order to have* (2.33) *one should take N such that $(N+1)\gamma>1$. Then the conclusion of Corollary 3.1 holds with $\psi(t)$ given by* (2.34).

If Hypothesis 2.3 is not satisfied, then we can find functions ψ if Hypothesis 2.4 is satisfied for models with coefficient (3.33). In the following example we have that situation.

Example 3.3. *Let us consider the Cauchy problem* (3.1) *with* $m(t) = \frac{\mu}{1+t}$ *and* $\mu \neq 0$*, i.e., we consider the scale-invariant case from [5]. Let us take the function* ψ *as*

$$\psi(t) = \exp\Big(\sum_{k=1}^{\infty}\int_0^t \frac{\gamma_k \mu^{2k}}{(1+\tau)}d\tau\Big) = (1+t)^{\sigma}$$

with $\sigma = \sum_{k=1}^{\infty}\gamma_k\mu^{2k}$*. By using the infinite Cauchy product and from the definition of* γ_k *we get*

$$\sigma^2 = \Big(\sum_{k=1}^{\infty}\gamma_k\mu^{2k}\Big)^2 = \sum_{n=2}^{\infty}\gamma_n\mu^{2n} = \sigma - \mu^2.$$

If we take $\sigma_- = \frac{1-\sqrt{1-4\mu^2}}{2}$*, then the Corollary 3.1 holds.*

The last example shows us that if we have the scale-invariant case, for $\mu^2 \in \big(0, \frac{1}{4}\big)$, we derived the same $L^p - L^q$ estimates on the conjugate line for the kinetic and elastic energy, which is proposed by (1.14) and (1.15). Although, for the potential energy we have the same estimates only for $p = q = 2$.

Example 3.4. *If* $g(t)^2 = \ln(\ln(e^e + t))$*, then we can take for* $t \geq t_0$*,* $t_0 \gg 1$*, the function*

$$\psi(t) = \exp\Big(\sum_{k=1}^{\infty}\int_{t_0}^t \frac{\gamma_k \mu^{2k}}{(1+\tau)g(\tau)^{2k}}d\tau\Big)$$

which is well-defined for $\mu^2 < \frac{g(t_0)^2}{4}$*. It is clear that the condition* (2.37) *holds, i.e., Hypothesis 3.1 is satisfied and the statement of Corollary 3.1 is applicable.*

4 Wave models with mass and dissipation

The main goal of this chapter is to prove the sharpness of the energy estimates obtained in Chapter 2 and to derive Strichartz estimates for solutions to the Cauchy problem for damped Klein-Gordon equations. For the first reason we will prove in Section 4.1 a modified scattering result to solutions for the Cauchy problems for wave equations with scattering time-dependent mass term and non-effective time-dependent dissipation. Later we will investigate in Section 4.2 damped Klein-Gordon equations with variable in time mass and dissipation analyzing the interplay between both coefficients and asymptotic properties of solution as time tends to infinity. In former papers authors introduced precise classifications of effective or non-effective mass, see [5, 6, 19], or dissipation, see [57, 59, 60], terms. There exists a "grey zone" around the scale-invariant models where difficulties appear in a systematic study. If models are scale-invariant, then theory of special functions allows to derive precise results. But already "small perturbations" make the treatment difficult. The goal of Section 4.2 is to study models from this "grey zone".

4.1 Scattering producing time-dependent mass versus non-effective time-dependent dissipation

Let us consider the following Cauchy problem for wave equations with time-dependent mass and dissipation

$$u_{tt} - \Delta u + b(t)u_t + m(t)u = 0,\;\; u(0,x) = u_0(x),\;\; u_t(0,x) = u_1(x), \qquad (4.1)$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $b = b(t) \geq 0$ the dissipative term and $m = m(t) \geq 0$ the mass term under the following assumptions:

Hypothesis 4.1. *Suppose that $b(t)$ and $m(t)$ satisfy*

$$\left|\frac{d^k}{dt^k} b(t)\right| \leq C_k \left(\frac{1}{1+t}\right)^{k+1} \;\; \textit{and} \;\; m(t) \leq C \left(\frac{1}{1+t}\right)^2,$$

for $k = 0, 1$.

Hypothesis 4.2. *Suppose that $b(t)$ and $m(t)$ satisfy*

$$\limsup_{t\to\infty} t b(t) < 1 \;\; \textit{and} \;\; (1+t)m(t) \in L^1.$$

Remark 4.1. *The Hypothesis 4.1 and 4.2 coincide with the hypothesis from Section 2.2. To be more precise, if we consider the Cauchy problem* (2.1) *and choose a function ψ like in Hypothesis 2.2 performing the change of variable $u(t,x) = \psi(t)v(t,x)$, then we have the same model of the Cauchy problem* (4.1) *under the above hypothesis.*

4.1.1 Representation of the solution

Applying the partial Fourier transformation in (4.1) we obtain

$$\widehat{u}_{tt}+|\xi|^2\widehat{u}+b(t)\widehat{u}_t+m(t)\widehat{u}=0,\ \ \widehat{u}(0,\xi)=\widehat{u}_0(\xi),\ \ \widehat{u}_t(0,\xi)=\widehat{u}_1(\xi). \tag{4.2}$$

We divide the extended phase space $[0,\infty)\times\mathbb{R}^n$ into the *pseudo-differential zone* $Z_{pd}(N)$ and into the *hyperbolic zone* $Z_{hyp}(N)$ which are defined by

$$\begin{aligned} Z_{pd}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n:(1+t)|\xi|\le N\},\\ Z_{hyp}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n:(1+t)|\xi|\ge N\},\end{aligned}$$

with N determined later on. The *separating curve* is given by

$$\theta_{|\xi|}:(0,N]\to[0,\infty),\quad (1+\theta_{|\xi|})|\xi|=N.$$

We put also $\theta_0=\infty$, and $\theta_{|\xi|}=0$ for any $|\xi|\ge N$. The pair (t,ξ) from the extended phase space belongs to $Z_{pd}(N)$ (resp. to $Z_{hyp}(N)$) if and only if $t\le\theta_{|\xi|}$ (resp. $t\ge\theta_{|\xi|}$). We define the micro-energy

$$U(t,\xi)=\big(h(t,\xi)\widehat{u},D_t\widehat{u}\big)^T, \tag{4.3}$$

where

$$h(t,\xi)=\frac{1}{1+t}\phi_{pd}(t,\xi)+|\xi|\phi_{hyp}(t,\xi).$$

Here $\phi_{pd}(t,\xi)$ is a characteristic function related to the pseudo-differential zone and $\phi_{hyp}(t,\xi)$ is a characteristic function related to the hyperbolic zone. We introduce the function $\phi_{hyp}(t,\xi)=\chi\big(\frac{(1+t)|\xi|}{N}\big)$ with $\chi\in C^\infty(\mathbb{R}^n)$, $\chi(t)=1$ for $t\le\frac{1}{2}$, $\chi(t)=0$ for $t\ge 2$ and $\chi'(t)\le 0$ together with $\phi_{pd}(t,\xi)+\phi_{hyp}(t,\xi)=1$.

Considerations in the pseudo-differential zone

In the pseudo-differential zone $Z_{pd}(N)$ the micro-energy (4.3) reduces to

$$U=\Big(\frac{\widehat{u}}{1+t},D_t\widehat{u}\Big)^T.$$

So we have

$$\partial_tU(t,\xi)=\mathcal{A}(t,\xi)U:=\begin{pmatrix} & \frac{i}{1+t} & \frac{1}{1+t}\\ & (1+t)\,(m(t)+|\xi|^2) & ib(t)\end{pmatrix}U. \tag{4.4}$$

We will prove estimates for the fundamental solution $\mathcal{E}=\mathcal{E}(t,s,\xi)$ to (4.4), that is, the solution to

$$\partial_t\mathcal{E}=\mathcal{A}(t,\xi)\mathcal{E}\,,\quad \mathcal{E}(s,s,\xi)=I.$$

Let us define the function

$$\lambda(t)=\exp\left(\frac{1}{2}\int_0^t b(\tau)d\tau\right). \tag{4.5}$$

that describes the influence of the dissipative term.

Lemma 4.1. *Assume Hypothesis 4.1 and Hypothesis 4.2. Then the fundamental solution $\mathcal{E}(t,0,\xi)$ to (4.4) satisfies the estimate*

$$\|\mathcal{E}(t,0,\xi)\| \lesssim \frac{1}{\lambda(t)^2} \tag{4.6}$$

uniformly in t and for all $(t,\xi) \in Z_{pd}(N)$.

Proof. If we put $\mathcal{E} = (\mathcal{E}_{ij})_{i,j=1,2}$, then we can write for $j = 1,2$ the following system of coupled integral equations of Volterra type:

$$\begin{aligned} \mathcal{E}_{1j}(t,0,\xi) &= (1+t)^{-1}\Big(\delta_{1j} + i\int_0^t \mathcal{E}_{2j}(\tau,0,\xi)d\tau\Big), &(4.7)\\ \mathcal{E}_{2j}(t,0,\xi) &= \lambda^{-2}(t)\Big(\delta_{2j} - i\int_0^t (1+\tau)\lambda(\tau)^2\big(m(\tau)+|\xi|^2\big)\mathcal{E}_{1j}(\tau,0,\xi)d\tau\Big). &(4.8) \end{aligned}$$

After replacing (4.8) into (4.7) and integration by parts we get

$$\begin{aligned} \mathcal{E}_{1j}(t,0,\xi) = (1+t)^{-1}\Big(\delta_{1j} + i\delta_{2j}\int_0^t \lambda(\tau)^{-2}d\tau\Big) + (1+t)^{-1} \\ \times \int_0^t (1+\tau)\lambda(\tau)^2\big(m(\tau)+|\xi|^2\big)\mathcal{E}_{1j}(\tau,0,\xi)\int_\tau^t \lambda(s)^{-2}dsd\tau. \end{aligned} \tag{4.9}$$

By using Hypothesis 4.2 (see Proposition 7 of [59]) we have

$$\int_0^t \lambda(s)^{-2}ds \approx \frac{t}{\lambda(t)^2}, \tag{4.10}$$

and $\frac{t}{\lambda(t)^2}$ is increasing for large t. Introducing

$$h_j(t,\xi) := \|\mathcal{E}_{1j}(t,0,\xi)\|\lambda(t)^2$$

and by using $\lambda(t)^2 \le 1+t$ (see Hypothesis 4.2) for large t we conclude from (4.9) and (4.10) that

$$h_j(t,\xi) \le C + C\int_0^t (1+\tau)\big(m(\tau)+|\xi|^2\big)h_j(\tau,\xi)d\tau.$$

Applying Gronwall's type inequality we conclude

$$h_j(t,\xi) \le C\exp\Big(C\int_0^t (1+\tau)\big(m(\tau)+|\xi|^2\big)d\tau\Big).$$

In $Z_{pd}(N)$ we have $(1+t)|\xi| \le C$. So, from the last inequality we get

$$h_j(t,\xi) \le C\exp\Big(C\int_0^t (1+\tau)m(\tau)d\tau\Big).$$

Finally, by using Hypothesis 4.2 we get $\|\mathcal{E}_{1j}(t,0,\xi)\| \lesssim \lambda(t)^{-2}$. From the boundedness of $\|\mathcal{E}_{1j}(t,0,\xi)\|\lambda(t)^2$, using again Hypothesis 4.2, we can estimate $\|\mathcal{E}_{2j}(t,0,\xi)\| \lesssim \lambda(t)^{-2}$. Therefore, we proved $\|\mathcal{E}(t,0,\xi)\| \lesssim \lambda(t)^{-2}$ for all $t \in [0,\theta_{|\xi|}]$. ☐

Considerations in the hyperbolic zone

In the hyperbolic zone we will carry out two steps of diagonalization aiming to derive decay estimates for the energy and then to derive a modified scattering result. The ansatz is the same as in the paper [61].

In the hyperbolic zone the micro-energy (4.3) becomes

$$U(t,\xi) = (|\xi|\widehat{u}, D_t\widehat{u})^T\,,$$

then

$$D_tU = \left[\begin{pmatrix} 0 & |\xi| \\ |\xi| & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{m(t)}{|\xi|} & ib(t) \end{pmatrix}\right] U(t,\xi). \tag{4.11}$$

The goal is to transform the principal part in a diagonal matrix such that the remainder has a suitable normwise estimate. Take the matrices

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \ \ and\ \ M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$M^{-1}\begin{pmatrix} 0 & |\xi| \\ |\xi| + \frac{m(t)}{|\xi|} & ib(t) \end{pmatrix} M = D(\xi) + A(t) + B(t,\xi),$$

where

$$D(\xi) = \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix}, \quad A(t) = i\frac{b(t)}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ \ and\ \ B(t,\xi) = \frac{m(t)}{2|\xi|}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

In the second step we want to transform the second matrix on the right side without changing the structure of the first and the third one. For this we set

$$\begin{aligned} N^{(1)}(t,\xi) &= \frac{i}{2}\begin{pmatrix} 0 & \frac{b(t)}{2|\xi|} \\ -\frac{b(t)}{2|\xi|} & 0 \end{pmatrix}, \\ B^{(1)}(t,\xi) &= D_tN^{(1)}(t,\xi) - i\frac{b(t)}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N^{(1)}(t,\xi) - \frac{m(t)}{2|\xi|}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

Then we have by construction

$$\begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix} N^{(1)}(t,\xi) - N^{(1)}(t,\xi)\begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix} = i\frac{b(t)}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.12}$$

such that with $N_1(t,\xi) = I - N^{(1)}(t,\xi)$ the following operator identity holds:

$$\begin{aligned} &\left(D_t - \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix} - i\frac{b(t)}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{m(t)}{2|\xi|}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right) N_1(t,\xi) \\ = \;& D_t + N_1(t,\xi)\begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix} - i\frac{b(t)}{2}N_1(t,\xi) - B^{(1)}(t,\xi). \end{aligned}$$

Choosing a sufficiently large constant, we have that

$$\det N_1(t,\xi) = 1 - \frac{b^2(t)}{16|\xi|^2} \geq 1 - \frac{b^2(t)}{16(1+t)^2|\xi|^2} \geq 1 - \frac{1}{16N^2} > 0.$$

Then $N_1(t,\xi)$ is uniformly bounded away from zero on $Z_{hyp}(N)$. Therefore $N_1(t,\xi)^{-1}$ exist and $N_1(t,\xi)$, $N_1(t,\xi)^{-1}$ are both uniformly bounded on $Z_{hyp}(N)$.

Setting $R_1(t,\xi) = N_1(t,\xi)^{-1}B^{(1)}(t,\xi)$ we obtain

$$\begin{aligned} & N_1(t,\xi)^{-1}\left(D_t - \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix} - i\frac{b(t)}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{m(t)}{2|\xi|}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right) N_1(t,\xi) \\ = \; & D_t - \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix} - i\frac{b(t)}{2}N_1(t,\xi) - R_1(t,\xi) \end{aligned}$$

with the remainder term $R_1(t,\xi)$ subjected to the pointwise estimate

$$\|R_1(t,\xi)\| \lesssim \frac{1}{|\xi|(1+t)^2}. \tag{4.13}$$

After this considerations we are able to derive the main result of this section.

Lemma 4.2. *Assume Hypothesis 4.1. Then the fundamental solution of* (4.11) *can be represented by*

$$\mathcal{E}(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)} M^{-1} N_1(t,\xi)^{-1} \mathcal{E}_0(t,s,\xi) \mathcal{Q}_1(t,s,\xi) N_1(s,\xi) M \tag{4.14}$$

for $t \geq s$ and $(s,\xi) \in Z_{hyp}(N)$, where

1. *the function $\lambda(t) = \exp\left(\frac{1}{2}\int_0^t b(\tau)d\tau\right)$ describes the main influence of the coefficient $b = b(t)$ in the dissipation;*
2. *the matrices $N_1(t,\xi), N_1(t,\xi)^{-1}$ are uniformly bounded on $Z_{hyp}(N)$ tending on $\{\xi : |\xi| \geq \varepsilon\}$ uniformly to the identity matrix I for all $\varepsilon > 0$;*
3. *the matrix $\mathcal{E}_0(t,s,\xi)$ is given by*

$$\mathcal{E}_0(t,s,\xi) = \begin{pmatrix} e^{i(t-s)|\xi|} & 0 \\ 0 & e^{-i(t-s)|\xi|} \end{pmatrix} \tag{4.15}$$

   and is the fundamental solution of the free wave equation;

4. *$\mathcal{Q}_1(t,s,\xi)$ is uniformly bounded on $Z_{hyp}(N)$ tending uniformly on $\{\xi : |\xi| \geq \varepsilon\}$ to the invertible matrix $\mathcal{Q}_1(\infty,s,\xi)$ for all $\varepsilon > 0$.*

Proof. The construction of the representation of solution will be done in two steps.

Step 1: If

$$\widetilde{\mathcal{E}}_0(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\mathcal{E}_0(t,s,\xi), \tag{4.16}$$

then

$$D_t\widetilde{\mathcal{E}}_0(t,s,\xi) = \left(i\frac{b(t)}{2}I + \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix}\right)\widetilde{\mathcal{E}}_0(t,s,\xi), \quad \widetilde{\mathcal{E}}_0(s,s,\xi) = I. \tag{4.17}$$

Step 2: If we define

$$\Phi(t,s,\xi) := \mathcal{E}_0(s,t,\xi)R_1(t,\xi)\mathcal{E}_0(t,s,\xi) \tag{4.18}$$

and consider the Cauchy problem

$$D_t\mathcal{Q}_1(t,s,\xi) = \Phi(t,s,\xi)\mathcal{Q}_1(t,s,\xi) \quad \mathcal{Q}_1(s,s,\xi) = I, \tag{4.19}$$

then the fundamental solution of the transformed operator

$$D_t - \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix} - i\frac{b(t)}{2}I + R_1(t,\xi)$$

can be represented by $\widetilde{\mathcal{E}}_0(t,s,\xi)\mathcal{Q}_1(t,s,\xi)$. After transforming back to the starting problem it follows (4.14).

Using that $\mathcal{E}_0(t,s,\xi)$ is unitary we see that $\Phi(t,s,\xi)$ satisfies the same estimates as $R_1(t,\xi)$. This allows us to estimate in a second step the solution $\mathcal{Q}_1(t,s,\xi)$ directly from the representation by Peano-Backer formula (7.2)

$$\mathcal{Q}_1(t,s,\xi) = I + \sum_{i=1}^{\infty}\int_s^t \Phi(t_1,s,\xi)\int_s^{t_1}\Phi(t_2,s,\xi)\cdots\int_s^{t_{k-1}}\Phi(t_k,s,\xi)dt_k\cdots dt_1. \tag{4.20}$$

Therefore,

$$\begin{aligned}\|\mathcal{Q}_1(t,s,\xi) - I\| &\le \int_s^t \|R_1(\tau,\xi)\|d\tau \exp\left(\int_s^t \|R_1(\tau,\xi)\|d\tau\right) \\ &\le \frac{C}{|\xi|}\int_s^t \frac{d\tau}{(1+\tau)^2}\exp\left(\frac{C}{|\xi|}\int_s^t \frac{d\tau}{(1+\tau)^2}\right) \le \frac{C}{N}\exp\left(\frac{C}{N}\right)\end{aligned}$$

uniformly for $t \ge s$ and $(s,\xi) \in Z_{hyp}(N)$. A large N implies that Q_1 is uniformly invertible on $Z_{hyp}(N)$.

If $|\xi| \ge \varepsilon$, then

$$\|N_1(t,\xi) - I\| = \|N^{(1)}(t,\xi)\| \lesssim \frac{1}{|\xi|(1+t)} \le \frac{1}{\varepsilon(1+t)} \to 0, \tag{4.21}$$

when $t \to \infty$. Analogously we can show that $N_1(t,\xi)^{-1}$ converges to I on $\{\xi : |\xi| \ge \varepsilon\}$ for all $\varepsilon > 0$. Finally, assuming $|\xi| \ge \varepsilon$ and using the representation (4.20) for $\mathcal{Q}_1(t,s,\xi)$ we get

$$\begin{aligned}\|\mathcal{Q}_1(\infty,s,\xi) - \mathcal{Q}_1(t,s,\xi)\| &\lesssim \int_t^\infty \|R_1(\tau,\xi)\|d\tau \exp\left(\int_s^t \|R_1(\tau,\xi)\|d\tau\right) \\ &\le \frac{C}{\varepsilon(1+t)} \to 0,\end{aligned}$$

when $t \to \infty$. Thus $\mathcal{Q}_1(t,s,\xi)$ converges uniformly on $\{\xi : |\xi| \ge \varepsilon\}$ to $Q_1(\infty,s,\xi)$ for all $\varepsilon > 0$. The lemma is proved. □

4.1.2 Modified scattering result

First let us find an energy estimate for the solutions to Cauchy problem (4.1). This result is a directly consequence of the Lemma 4.1, Lemma 4.2 and the definition of the micro-energy (4.3). The next theorem is a particular case of the paper [13].

Theorem 4.1. *Let u be the solution to* (4.1) *for data* $(u_0, u_1) \in H^1 \times L^2$. *Assume Hypothesis 4.1 and Hypothesis 4.2. Then the estimate*

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \tag{4.22}$$

holds true, where $\lambda = \lambda(t)$ is defined by (4.5).

In this section we are interested in modified scattering results between the solutions of

$$u_{tt} - \Delta u + b(t)u_t + m(t)u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{4.23}$$

and

$$v_{tt} - \Delta v = 0, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x), \tag{4.24}$$

where we assume Hypothesis 4.1 and Hypothesis 4.2 for the coefficients b and m.

The goal is construct an operator that maps initial data of Cauchy problem (4.23) to initial data of Cauchy problem (4.24) such that after multiplication by $\lambda(t)$ the asymptotic behavior of the energy of solution to (4.23) coincides with the asymptotic behaviour of the energy of solutions to the related Cauchy problem (4.24) for large times. The operator relating (u_0, u_1) to (v_0, v_1) is denoted as Moeller wave operator which was mentioned in Lax-Phillips approach [36].

Theorem 4.2. *Assume Hypothesis 4.1 and Hypothesis 4.2 . Then there exists a bounded operator*

$$W_+ : (u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to (v_0, v_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$$

such that for Cauchy data (u_0, u_1) of (4.23) *and associated data* $(v_0, v_1) = W_+(u_0, u_1)$ *to* (4.24) *the corresponding solutions* $u = u(t,x)$ *and* $v = v(t,x)$ *satisfy*

$$\|\lambda(t)(u_t(t,\cdot), \nabla_x u(t,\cdot)) - (v_t(t,\cdot), \nabla_x v(t,\cdot))\|_{L^2} \to 0$$

as $t \to \infty$.

Proof. Let us define for any $\varepsilon > 0$ the following closed subset of $L^2 \times L^2$:

$$F_\epsilon := \Big\{U_0 \in L^2 \times L^2 : \; \widehat{U}_0(\xi) = 0 \text{ for any } |\xi| \le \epsilon\Big\}.$$

We remember that $\mathcal{L} = \cup_{\epsilon>0} F_\epsilon$ is a dense subset of $L^2 \times L^2$. If we introduce $\mathcal{E}_0 = \mathcal{E}_0(t,s,\xi)$ as in (4.15) and if v solves the free wave equation (4.24), after defining $V(t,\xi) = (|\xi|\widehat{v}, D_t\widehat{v})^T$ we can write $V(t,\xi) = \widetilde{\widetilde{\mathcal{E}}}_0(t,s,\xi)V(s,\xi)$, where

$$\widetilde{\widetilde{\mathcal{E}}}_0(t,s,\xi) = M^{-1}\mathcal{E}_0(t,s,\xi)M.$$

The proof is based on an explicit representation of the modified wave operator W_+. Our goal is to prove that the limit

$$W_+(\xi) = \lim_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) \tag{4.25}$$

exists uniformly in $|\xi| > \varepsilon$ for all $\varepsilon > 0$. After proving this property we are able to relate the Cauchy data by

$$W_+(\xi)U(0,\xi) = V(0,\xi).$$

From Lemma 4.2 we know that the limit

$$\mathcal{Q}_1(\infty, \theta_{|\xi|}, \xi) = \lim_{t\to\infty} \mathcal{Q}_1(t, \theta_{|\xi|}, \xi)$$

exists uniformly when $|\xi| \geq \epsilon$ for any $\epsilon > 0$. Hence, if we consider $\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi)$ on $\mathcal{L}$ we obtain in hyperbolic zone

$$\begin{aligned}
& \lim_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) \\
= \; & \lim_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,\theta_{|\xi|},\xi)\mathcal{E}(\theta_{|\xi|},0,\xi) \\
= \; & \lim_{t\to\infty} \lambda(\theta_{|\xi|})\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}M^{-1}N_1(t,\xi)^{-1}\mathcal{E}_0(t,\theta_{|\xi|},\xi)\mathcal{Q}_1(t,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)M\mathcal{E}(\theta_{|\xi|},0,\xi) \\
= \; & \lim_{t\to\infty} \lambda(\theta_{|\xi|})M^{-1}\mathcal{E}_0(0,t,\xi)N_1(t,\xi)^{-1}\mathcal{E}_0(t,\theta_{|\xi|},\xi)\mathcal{Q}_1(t,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)M\mathcal{E}(\theta_{|\xi|},0,\xi) \\
= \; & \lambda(\theta_{|\xi|})M^{-1}\mathcal{E}_0(0,\theta_{|\xi|},\xi)\mathcal{Q}_1(\infty,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)M\mathcal{E}(\theta_{|\xi|},0,\xi),
\end{aligned}$$

using the fact that

$$\mathcal{E}_0(0,t,\xi)N_1(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi) = \mathcal{E}_0(0,\theta_{|\xi|},\xi) + \mathcal{E}_0(0,t,\xi)(N_1(t,\xi) - I)\mathcal{E}_0(t,\theta_{|\xi|},\xi)$$

and $N_1(t,\xi) \to I$ uniformly for $|\xi| \geq \epsilon$. In the pseudo-differential zone the boundness of the fundamental solution of the free wave equation and the estimative in Lemma 4.1 guarantee that the limit 4.25 goes to zero.

According to the estimates for the energy from Theorem 4.1 we conclude that

$$\lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi)$$

is uniformly bounded on $\mathcal{L}$. Therefore applying Banach-Steinhaus Theorem 7.3 we conclude that

$$W_+(D) = \operatorname*{s-lim}_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,D)^{-1}\mathcal{E}(t,0,D)$$

exists as strong limit in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Finally, we study the difference

$$\begin{aligned}
\|\lambda(t)U(t,\cdot) - V(t,\cdot)\|_{L^2} &= \|\lambda(t)\mathcal{E}(t,0,\cdot)U(0,\cdot) - \tilde{\tilde{\mathcal{E}}}_0(t,0,\cdot)V(0,\cdot)\|_{L^2} \\
&= \left\| \Big(\lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\cdot)^{-1}\mathcal{E}(t,0,\cdot) - W_+(\cdot)\Big)U(0,\cdot)\right\|_{L^2},
\end{aligned}$$

under our assumption $(u_0,u_1) \in H^1 \times L^2$ and by definition of $W_+(\xi)$ we may conclude that

$$\|\lambda(t)U(t,\cdot) - V(t,\cdot)\|_{L^2} \to 0$$

as t tends to infinity. The proof is completed. □

Remark 4.2. *The statement from Theorem 4.2 gives us the sharpness of the energy estimates derived in Chapter 2. In fact, suppose that $w = w(t,x)$ satisfies the Cauchy problem (2.1) and that $v = v(t,x)$ satisfies the Cauchy problem for the free wave equation (4.24). Choose a function ψ as in Hypothesis 2.2 and perform the change of variable $w(t,x) = \psi(t)u(t,x)$. Then the Cauchy problem (2.1) takes the form of the Cauchy problem (4.23) with $\lambda(t) = \psi(t)$. The Hypotheses 2.1 and 2.2 allow us to apply Theorem 4.2. Then there exists a bounded operator that maps initial data of the Cauchy problem (4.23) to initial data of the Cauchy problem (4.24) such that the corresponding solutions satisfy*

$$\|\psi(t)(u_t(t,\cdot),\nabla_x u(t,\cdot)) - (v_t(t,\cdot),\nabla_x v(t,\cdot))\|_{L^2} \to 0,$$

or, equivalently,

$$\left\|\left(w_t(t,\cdot) - \frac{\psi'(t)}{\psi(t)}w(t,\cdot),\nabla_x w(t,\cdot)\right) - (v_t(t,\cdot),\nabla_x v(t,\cdot))\right\|_{L^2} \to 0.$$

Taking into consideration the energy conservation for the free wave equation it follows

$$\begin{aligned}
0 &< c_0 \le \left\|\left(w_t(t,\cdot) - \frac{\psi'(t)}{\psi(t)}w(t,\cdot),\nabla_x w(t,\cdot)\right)\right\|_{L^2}^2 \\
&= \left\|w_t(t,\cdot) - \frac{\psi'(t)}{\psi(t)}w(t,\cdot)\right\|_{L^2}^2 + \|\nabla_x w(t,\cdot)\|_{L^2}^2 \\
&\le \|w_t(t,\cdot)\|_{L^2}^2 + 2\|w_t(t,\cdot)\|_{L^2}\frac{\psi'(t)}{\psi(t)}\|w(t,\cdot)\|_{L^2} + \left(\frac{\psi'(t)}{\psi(t)}\right)^2\|w(t,\cdot)\|_{L^2}^2 + \|\nabla_x w(t,\cdot)\|_{L^2}^2 \\
&\le 2\,\|w_t(t,\cdot)\|_{L^2}^2 + 2p(t)^2\|w(t,\cdot)\|_{L^2}^2 + 2\,\|\nabla_x w(t,\cdot)\|_{L^2}^2 \\
&\sim E(w)(t),
\end{aligned}$$

where $E(w)(t)$ is defined in (2.8).

This guarantees the sharpness of our energy estimate.

4.1.3 Examples

We will conclude this section with examples.

Example 4.1. *Let $\mu,\sigma > 0$ and $s \in \mathbb{N}$ with $s \ge 1$. Then we consider*

$$b(t) = \frac{\mu}{(e^{[s]}+t)\log(e^{[s]}+t)\cdots\log^{[s]}(e^{[s]}+t)} \quad \text{and} \quad m(t) = \frac{\sigma}{(1+t)^\gamma},$$

with $\gamma > 2$. Then the hypotheses of Theorem 4.1 and Theorem 4.2 are satisfied and we obtain

$$\lambda(t) = \left(\log^{[s]}\left(e^{[s]}+t\right)\right)^{\frac{\mu}{2}}.$$

The decay estimate for the energy is

$$\|(u_t(t,\cdot),\nabla_x u(t,\cdot))\|_{L^2} \approx \frac{1}{\lambda(t)}.$$

Example 4.2. *If we consider*

$$b(t) = \frac{2 + \cos(\alpha \log(e+t))}{2(e+t)} \quad \textit{and} \quad m(t) = \frac{\log(e+t)}{(1+t)^\gamma}$$

with $\gamma > 3$, then the hypotheses of Theorem 4.1 and Theorem 4.2 are satisfied and we obtain

$$\lambda(t) = \exp\left(\frac{1}{2}\log(e+t) + \frac{1}{4\alpha}\sin(\alpha\log(e+t))\right).$$

The decay estimate for the energy is

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^2} \approx \frac{1}{\lambda(t)}.$$

4.2 Non-effective dissipation versus non-effective potential

In this section we will prove $L^p - L^q$ estimates for $p \in (1,2]$ on the conjugate line and modified scattering results for Cauchy problems as in (4.1), but now we will consider in the models non-effective time-dependent mass and dissipation. The results of this section was published in the paper [41].

4.2.1 Representation of the solution

Let us consider the following Cauchy problem for damped Klein-Gordon equations

$$u_{tt} - \Delta u + b(t)u_t + m(t)u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{4.26}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $b = b(t)$ is the coefficient in the dissipative term and $m = m(t)$ is the coefficient in the mass term under the following assumptions:

Hypothesis 4.3. *Suppose that $b, m \in C^\ell(\mathbb{R}_+)$ and that for all $k \leq \ell$ it holds*

$$\left|\frac{d^k}{dt^k}b(t)\right| \leq C_k\left(\frac{1}{1+t}\right)^{k+1} \quad \textit{and} \quad \left|\frac{d^k}{dt^k}m(t)\right| \leq C_k\left(\frac{1}{1+t}\right)^{k+2},$$

the number ℓ will be specified later on. Some statements need a higher regularity.

Hypothesis 4.4. *Suppose that the following limits*

$$\lim_{t\to\infty}(1+t)b(t) = b_0 \quad \textit{and} \quad \lim_{t\to\infty}(1+t)^2 m(t) = m_0 \tag{4.27}$$

exist and that

$$\int_1^\infty \frac{|tb(t) - b_0|^\sigma}{t}dt < \infty \quad \textit{and} \quad \int_0^\infty \frac{|t^2 m(t) - m_0|^\sigma}{t}dt < \infty,$$

holds true with exponent σ satisfying

(A1) $\sigma = 1$ *or* **(A2)** $\sigma \in (1,2]$.

Results will depend on relations between the constants b_0 and m_0. It will not be necessary to restrict considerations to $b_0 \geq 0$ and $m_0 \geq 0$, results will however depend on the constraint $4m_0 > b_0(b_0-2)$ or additional conditions imposed on initial data.

We further define the auxiliary function

$$\lambda(t) = \exp\left(\frac{1}{2}\int_0^t b(\tau)\mathrm{d}\tau\right) \tag{4.28}$$

related to the dissipative term $b(t)u_t$. It will play an important role in the resulting estimates. Under part (A1) of Hypothesis 4.4 it follows that

$$\lambda(t) \approx (1+t)^{\frac{b_0}{2}} \text{ for } t \to \infty. \tag{4.29}$$

When assuming (A2) a further sub-polynomial correction term appears.

Zones and general strategy

Applying the partial Fourier transformation in (4.26) we obtain

$$\widehat{u}_{tt} + |\xi|^2\widehat{u} + b(t)\widehat{u}_t + m(t)\widehat{u} = 0, \ \ \widehat{u}(0,\xi) = \widehat{u}_0(\xi), \ \ \widehat{u}_t(0,\xi) = \widehat{u}_1(\xi). \tag{4.30}$$

In order to derive $L^p - L^q$ estimates for the solution and its derivatives we divide the extended phase space $[0,\infty)\times\mathbb{R}^n$ into three zones:

$$\begin{aligned} Z_{pd}(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : (1+t)|\xi| \leq N\}, \\ Z_{hyp}^s(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi| \leq N \leq (1+t)|\xi|\}, \\ Z_{hyp}^\ell(N) &= \{(t,\xi)\in[0,\infty)\times\mathbb{R}^n : |\xi| \geq N\}, \end{aligned}$$

where N is a positive constant that will be specified later on.

Remark 4.3. *In the zone $Z_{hyp}^\ell(N)$ we consider only large frequencies and in the zones $Z_{pd}(N)$ and $Z_{hyp}^s(N)$ we consider small frequencies. Furthermore, the separating curve between both zones $Z_{pd}(N)$ and $Z_{hyp}^s(N)$ is given by*

$$\theta_{|\xi|} : (0,N] \to [0,\infty), \quad (1+\theta_{|\xi|})|\xi| = N.$$

We put also $\theta_0 = \infty$, and $\theta_{|\xi|} = 0$ for any $|\xi| \geq N$.

In order to divide the extended phase space into three parts we introduce the function $\chi \in C^\infty(\mathbb{R}_+)$ such that $\chi(t) = 1$ for $t \leq \frac{1}{2}$, $\chi(t) = 0$ for $t \geq 2$ and $\chi'(t) \leq 0$. We define the characteristic functions φ_{pd}, φ_{hyp}^s and φ_{hyp}^ℓ related to the zones $Z_{pd}(N)$, $Z_{hyp}^s(N)$ and $Z_{hyp}^\ell(N)$, respectively, by

$$\begin{aligned} \varphi_{pd}(t,\xi) &= \chi\left(|\xi|N^{-1}\right)\chi\left((1+t)|\xi|N^{-1}\right), \\ \varphi_{hyp}^s(t,\xi) &= \chi\left(|\xi|N^{-1}\right)\left(1-\chi\left((1+t)|\xi|N^{-1}\right)\right), \\ \varphi_{hyp}^\ell(\xi) &= 1-\chi\left(|\xi|N^{-1}\right), \end{aligned}$$

such that $\varphi_{pd}(t,\xi) + \varphi_{hyp}^s(t,\xi) + \varphi_{hyp}^\ell(\xi) = 1$. Let us consider the same micro-energy that we defined in the Section 4.1, i.e.,

$$U(t,\xi) = \left(h(t,\xi)\widehat{u}, D_t\widehat{u}\right)^T, \tag{4.31}$$

where

$$h(t,\xi)=\frac{1}{1+t}\varphi_{pd}(t,\xi)+|\xi|\big(\varphi^s_{hyp}(t,\xi)+\varphi^\ell_{hyp}(\xi)\big).$$

In the hyperbolic zone we apply a diagonalization procedure to a first-order system corresponding to equation (4.30) in order to derive a representation for the fundamental solution. We follow some ideas of Wirth [59] and Yagdjian [62]. We will consider a system with a coefficient matrix composed of a diagonal main part and a remainder part. The goal of this diagonalization is to keep the diagonal part in every step of the diagonalization and to improve the remainder terms. The strategy is the same one as in Chapter 3.

To derive the asymptotic behavior of the fundamental solution to (4.30) in the pseudo-differential zone we will perform, for L^1 condition (A1), one step of diagonalization and apply the Levinson Theorem 7.5 and, for L^σ condition (A2), we will apply the Hartman–Wintner Theorem 7.6. For the L^σ condition we need one more step of diagonalization (see proof of Theorem 7.6).

Considerations in the pseudo-differential zone

In the pseudo-differential zone the micro-energy (4.31) becomes

$$U(t,\xi)=\left(\frac{1}{1+t}\widehat{u},D_t\widehat{u}\right).$$

Therefore we shall consider the system

$$D_tU(t,\xi)=\widetilde{A}(t,\xi)U(t,\xi):=\begin{pmatrix}\frac{i}{1+t} & \frac{1}{1+t}\\ (1+t)\big(|\xi|^2+m(t)\big) & ib(t)\end{pmatrix}U(t,\xi). \tag{4.32}$$

Let us consider the fundamental solution $\mathcal{E}(t,s,\xi)$ of the system (4.32). The strategy is to apply Levinson's Theorem 7.5 obtaining the asymptotic behavior of the solution in the pseudo-differential zone. For this reason we shall apply steps of diagonalization on the matrix $\widetilde{A}=\widetilde{A}(t,\xi)$.
Note that the Hypothesis 4.4 proposes us to rewrite (4.32) in the following way:

$$\begin{aligned}(1+t)\partial_tU(t,\xi)&=\begin{pmatrix}-1 & i\\ im_0 & -b_0\end{pmatrix}U(t,\xi)\\ &+\begin{pmatrix}0 & 0\\ i(1+t)^2|\xi|^2+i\big((1+t)^2m(t)-m_0\big) & -(1+t)b(t)+b_0\end{pmatrix}U(t,\xi).\end{aligned}$$

Let us denote by $R=R(t,\xi)$ the matrices

$$A=\begin{pmatrix}-1 & i\\ im_0 & -b_0\end{pmatrix},\quad R=\begin{pmatrix}0 & 0\\ i(1+t)^2|\xi|^2+i\big((1+t)^2m(t)-m_0\big) & -(1+t)b(t)+b_0\end{pmatrix}.$$

Therefore, we study the system

$$(1+t)\partial_tU(t,\xi)=\big(A+R(t,\xi)\big)U(t,\xi). \tag{4.33}$$

By Hypothesis 4.4 in the form (A1) and the definition of the zone we know that

$$\sup_{|\xi|<N}\int_1^{\theta_{|\xi|}}\|R(t,\xi)\|\frac{\mathrm{d}t}{t}<\infty \tag{4.34}$$

and $R(t,\xi)$ is a remainder term in the sense of Theorem 7.5. Furthermore, as $\operatorname{tr} A = -1 - b_0$ and $\det A = b_0 + m_0$ the eigenvalues of A are given as

$$\mu_\pm = -\frac{b_0+1}{2} \pm \sqrt{\frac{(b_0-1)^2}{4} - m_0}. \tag{4.35}$$

In particular we see that

$$4m_0 \neq (b_0-1)^2 \tag{4.36}$$

implies that the eigenvalues are distinct.

Theorem 4.3. *Assume Hypothesis 4.4 with $\sigma = 1$ together with* (4.36). *Then the matrix-valued fundamental solution of the system* (4.33) *satisfies*

$$\|\mathcal{E}(t,s,\xi)\| \lesssim \left(\frac{1+t}{1+s}\right)^{\operatorname{Re}\mu_+} \tag{4.37}$$

uniformly in $0 \le s \le t$ and $(t,\xi) \in Z_{pd}(N)$.

Proof. This follows from Theorem 7.5 applied to (4.33) with $R(t,\xi)$ extended by zero outside $Z_{pd}(N)$. Let P be the diagonalizer of A given by

$$P = \begin{pmatrix} 1 & 1 \\ \frac{im_0+1+\mu_+}{i+b_0+\mu_+} & \frac{im_0+1+\mu_-}{i+b_0+\mu_-} \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{\det P}\begin{pmatrix} \frac{im_0+1+\mu_-}{i+b_0+\mu_-} & -1 \\ -\frac{im_0+1+\mu_+}{i+b_0+\mu_+} & 1 \end{pmatrix}, \tag{4.38}$$

with $\det P = \frac{im_0+1+\mu_-}{i+b_0+\mu_-} - \frac{im_0+1+\mu_+}{i+b_0+\mu_+}$. Then, if we define $U^{(0)} = P^{-1}U$ we will get

$$(1+t)\partial_t U^{(0)}(t,\xi) = \left[\begin{pmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{pmatrix} + P^{-1}R(t,\xi)P\right] U^{(0)}(t,\xi). \tag{4.39}$$

We have that

$$\begin{aligned} P^{-1}R(t,\xi)P &= \frac{i(1+t)^2|\xi|^2}{\det P}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \frac{(1+t)b(t)-b_0}{\det P}\begin{pmatrix} c_+ & c_- \\ -c_+ & -c_- \end{pmatrix} \\ &+ \frac{i\left((1+t)^2 m(t) - m_0\right)}{\det P}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \end{aligned}$$

with $c_\pm = \frac{im_0+1+\mu_\pm}{i+b_0+\mu_\pm}$. Note that $c_\pm$ is an imaginary number. In fact,

$$\begin{aligned} c_\pm &= \frac{\mu_\pm^2 + (b_0+1)\mu_\pm + (m_0+b_0)}{1+(b_0+\mu_\pm)^2} - i\frac{1+(1-m_0)\mu_\pm - m_0 b_0}{1+(b_0+\mu_\pm)^2} \\ &= -i\frac{1+(1-m_0)\mu_\pm - m_0 b_0}{1+(b_0+\mu_\pm)^2}. \end{aligned}$$

From $\mu_+ \neq \mu_-$ and Hypothesis 4.4 together with the zone definition implies that $P^{-1}R(t,\xi)P \in L^1\left([0,\infty), \frac{dt}{t}\right)$ uniformly in $\nu = |\xi|$. Then from Levinson's Theorem 7.5 we can conclude that there exist two linearly independent solutions to (4.33) of the form

$$U_\pm(t,\xi) = (e_\pm + o(1))(1+t)^{\mu_\pm} \text{ for } t \to \infty \tag{4.40}$$

within $Z_{pd}(N)$ and uniformly in ξ, where $e_\pm$ are the two normalized eigenvectors corresponding to $\mu_\pm$, i.e.,

$$e_\pm = \left(1, \frac{im_0 + 1 + \mu_\pm}{i + b_0 + \mu_\pm}\right). \tag{4.41}$$

Constructing the fundamental solution as in Remark 7.2, we see that

$$\mathcal{E}(t,0,\xi) = \big(U_-(t,\xi)|U_+(t,\xi)\big)\big(U_-(0,\xi)|U_+(0,\xi)\big)^{-1}, \tag{4.42}$$

and, hence, we obtain

$$\|\mathcal{E}(t,0,\xi)\| \lesssim (1+t)^{\operatorname{Re}\mu_+} \tag{4.43}$$

for any $(t,\xi) \in Z_{pd}(N)$. Using the scaling from Remark 7.3 (taking into account the shift in time) we obtain (4.37) uniformly in $0 \le s \le t \le \theta_{|\xi|}$. □

In order to treat the form (A2) of Hypothesis 2 by the Hartmann–Wintner Theorem 7.6, we need to ensure that $\operatorname{Re}\mu_+ \ne \operatorname{Re}\mu_-$. This happens if both are real and distinct. The latter is equivalent to

$$4m_0 < (b_0 - 1)^2. \tag{4.44}$$

Theorem 4.4. *Assume Hypothesis 4.4 with $\sigma \in (1,2]$ together with* (4.44). *Let further σ' be the conjugate line to σ. Then the fundamental solution of the system* (4.33) *satisfies*

$$\|\mathcal{E}(t,s,\xi)\| \lesssim \left(\frac{1+t}{1+s}\right)^{\mu_+} \exp\left(C\left(\ln\frac{1+t}{1+s}\right)^{\frac{1}{\sigma'}}\right) \tag{4.45}$$

uniformly in $0 \le s \le t$ and $(t,\xi) \in Z_{pd}(N)$.

Proof. As in the previous case we extend $R(t,\xi)$ by zero outside $Z_{pd}(N)$ and denote by $e_\pm$ normalised eigenvectors of A corresponding to $\mu_\pm$. Forming the unitary matrix $P = (e_+|e_-)$ with these eigenvectors as columns and defining $\widetilde{R}(t,\xi) = P^{-1}R(t,\xi)P$ given by (4.38) allows to rewrite (4.33) in the new unknown vector $\widetilde{U}(t,\xi) = PU(t,\xi)$ as

$$(1+t)\partial_t\widetilde{U}(t,\xi) = \big(\operatorname{diag}(\mu_+,\mu_-) + \widetilde{R}(t,\xi)\big)\widetilde{U}(t,\xi) \tag{4.46}$$

We apply Theorem 7.6 to this system. As $\mu_\pm$ are real and distinct, they clearly satisfy (7.36). Furthermore, the matrix $\widetilde{R}(t,\xi)$ contains combinations of $(1+t)b(t) - b_0$ and $(1+t)^2m(t) - m_0$ controlled by (A2) and terms of the form $(1+t)^2|\xi|^2$ which are uniformly bounded and integrable with respect to $\mathrm{d}t/t$ by the definition of the zone. Hence, Hypothesis 4.4 in the form (A2) implies (7.35) with $\sigma \in (1,2]$. Therefore, Theorem 7.6 applies and gives a matrix $N(t,\xi) \in L^\sigma(\mathbb{R}_+, \mathrm{d}t/t)$ transforming (4.33) for $t \ge t_0$ into Levinson form

$$(1+t)\partial_t V(t,\xi) = \big(\operatorname{diag}(\mu_+ + \widetilde{r}_{++}, \mu_- + \widetilde{r}_{--}) + \widetilde{R}_1(t,\xi)\big)V(t,\xi) \tag{4.47}$$

in the new unknown vector $V(t,\xi) = (I + N(t,\xi))^{-1}\widetilde{U}(t,\xi)$ and with the new remainder $\widetilde{R}_1 \in L^1([t_0,\infty), \mathrm{d}t/t)$. By $\widetilde{r}_{++}(t,\xi)$ and $\widetilde{r}_{--}(t,\xi)$ we denote the diagonal entries of

$\widetilde{R}(t,\xi)$. The new diagonal part satisfies the dichotomy condition (7.11), the additional diagonal entries satisfy by Hölder's inequality

$$\int_s^t |\widetilde{r}_{++}(\tau,\xi)| \frac{\mathrm{d}\tau}{1+\tau} \le C\left(\ln\frac{1+t}{1+s}\right)^{\frac{1}{\sigma'}} \tag{4.48}$$

with σ' the dual index and are thus small compared to

$$\int_s^t (\mu_+ - \mu_-)\frac{\mathrm{d}\tau}{1+\tau} = (\mu_+ - \mu_-)\left(\ln\frac{1+t}{1+s}\right). \tag{4.49}$$

Hence, Levinson's Theorem 7.5 yields a fundamental system of solutions together with the estimate

$$\|\mathcal{E}(t,t_0,\xi)\| \le (1+t)^{\mu_+} \exp\left(C\left(\ln(1+t)\right)^{\frac{1}{\sigma'}}\right), \quad t \ge t_0, \tag{4.50}$$

for the matrix-valued fundamental solution to the transformed system. The scaling argument from Remark 7.3 extends this estimate to variable starting times $t_0 \le s \le t \le \theta_{|\xi|}$ as

$$\|\mathcal{E}(t,s,\xi)\| \lesssim \left(\frac{1+t}{1+s}\right)^{\mu_+} \exp\left(C\left(\ln\frac{1+t}{1+s}\right)^{\frac{1}{\sigma'}}\right). \tag{4.51}$$

Transforming back to the original system combined with compactness of the remaining bit of $Z_{pd}(N)$ where the transform was not defined yields the desired statement. The theorem is proved. □

Remark 4.4. *If $2\operatorname{Re}\mu_+ < -b_0$, i.e., if*

$$b_0(b_0-2) < 4m_0, \tag{4.52}$$

then Theorems 4.3 and 4.4 imply

$$\|\mathcal{E}(t,s,\xi)\| \lesssim \frac{\lambda(s)}{\lambda(t)}, \tag{4.53}$$

for all $0 \le s \le t$ and $(t,\xi) \in Z_{pd}(N)$. In the first case this is obvious, while in the second case we observe that for all $\varepsilon > 0$ there exists a constant c_ε such that

$$\exp\left(C\left(\ln\frac{1+t}{1+s}\right)^{\frac{1}{\sigma'}}\right) \le c_\varepsilon\left(\frac{1+t}{1+s}\right)^{\varepsilon}. \tag{4.54}$$

Therefore

$$\left(\frac{1+t}{1+s}\right)^{\mu_+} \exp\left(C\left(\ln\frac{1+t}{1+s}\right)^{\frac{1}{\sigma'}}\right) \lesssim \left(\frac{1+t}{1+s}\right)^{\mu_+ + \varepsilon}$$
$$\lesssim \left(\frac{1+t}{1+s}\right)^{-\frac{b_0}{2}-\varepsilon} \lesssim \frac{\lambda(s)}{\lambda(t)} \tag{4.55}$$

uniformly in $0 \le s \le t$.

In order to combine the estimates from the pseudo-differential zone with the treatment in the hyperbolic zone, we need one further estimate. It is conditional in the sense that it is entirely based on the final estimate from the pseudo-differential zone and not on the precise assumptions used to prove it. It is also the first statement using Hypothesis 4.3.

Lemma 4.3. *Assume Hypothesis 4.3 and Hypothesis 4.4 in combination with* (4.52). *Then for* $|\xi| \le N$ *the symbol-like estimates*

$$\left\| D_\xi^\alpha \mathcal{E}(\theta_{|\xi|}, 0, \xi) \right\| \le C_\alpha \frac{1}{\lambda(\theta_{|\xi|})} |\xi|^{-|\alpha|} \tag{4.56}$$

are valid for all $|\alpha| \le \ell$.

Proof. Observe that the properties of the matrix $\widetilde{A}(t,\xi)$ allow to apply Lemma 3.10 of [58]. This lemma gives the desired statement. □

Remark 4.5. *The result of Lemma 3.1 can be reformulated in the following form. The symbol* $\lambda(\theta_{|\xi|})\mathcal{E}(\theta_{|\xi|}, 0, \xi)$ *is an element of the homogeneous symbol class*

$$\dot{S}^0_\ell = \{ m \in C^\infty(\mathbb{R}^n \setminus \{0\}) \;:\; |D_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \text{ for all } \; |\alpha| \le \ell \} \tag{4.57}$$

of order zero and restricted smoothness ℓ.

Considerations in the hyperbolic zone

First of all, let us introduce symbol classes $S_N^{\ell_1,\ell_2}\{m_1, m_2\}$ in the hyperbolic zone.

Definition 4.1. *The time-dependent amplitude function* $a = a(t,\xi)$ *belongs to the symbol class* $S_N^{\ell_1,\ell_2}\{m_1, m_2\}$ *with restricted smoothness* ℓ_1, ℓ_2 *if it satisfies the symbol-like estimates*

$$\left| D_t^k D_\xi^\alpha a(t,\xi) \right| \le C_{k,\alpha} |\xi|^{m_1 - |\alpha|} \left(\frac{1}{1+t} \right)^{m_2 + k} \tag{4.58}$$

for all $(t,\xi) \in Z_{hyp}(N)$*, all non-negative integers* $k \le \ell_1$ *and all multi-indices* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \le \ell_2$.

If $\ell_1 = \ell_2 = \infty$, then we introduce the notation $S_N\{m_1, m_2\}$.

Remark 4.6. *This symbol class coincides with the symbol class from Definition 3.1. So, Proposition 3.2 gives us the rules of the symbolic calculus in this set of symbol classes.*

In the hyperbolic zone the micro-energy (4.31) becomes

$$U(t,\xi) = (|\xi|\widehat{u}, D_t\widehat{u}) \,.$$

Then we consider the system

$$D_t U = A(t,\xi) U := \begin{pmatrix} 0 & |\xi| \\ |\xi| + \frac{m(t)}{|\xi|} & ib(t) \end{pmatrix} U. \tag{4.59}$$

We denote by $\mathcal{E} = \mathcal{E}(t,s,\xi)$ the fundamental solution of (4.59), i.e., the solution to

$$D_t\mathcal{E}(t,s,\xi) = A(t,\xi)\mathcal{E}(t,s,\xi),\ \mathcal{E}(s,s,\xi) = I. \tag{4.60}$$

The goal is to transform the "principal part" to a diagonal structure such that the remainder allows a suitable normwise estimate. We shall apply several steps of diagonalization to system (4.60). In the first step let us choose the matrix

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\ M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then we obtain

$$M^{-1}A(t,\xi)M = D(\xi) + B(t) + C(t,\xi)$$

with $D = D(\xi) = \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi| \end{pmatrix}$, $B = B(t) = \frac{ib(t)}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $C = C(t,\xi) = \frac{m(t)}{2|\xi|}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Note that $B \in S_N^{\ell,\infty}\{0,1\}$ and $C \in S_N^{\ell,\infty}\{-1,2\}$.

Now we carry out further steps of the diagonalization procedure. Like in Chapter 3 the aim is to transform the previous system to a diagonalized system with a remainder belonging in some sense to a better "symbol class". We construct recursively the diagonalizer $N_k = N_k(t,\xi)$ of order k. Let

$$N_k(t,\xi) = \sum_{j=0}^{k} N^{(j)}(t,\xi),\quad F_k(t,\xi) = \sum_{j=0}^{k} F^{(j)}(t,\xi),$$

where $N^{(0)}(t,\xi) = I$, $B^{(0)}(t,\xi) = B(t) + C(t,\xi)$ and $F^{(0)}(t,\xi) = \text{diag}B^{(0)}(t,\xi) = F_0(t,\xi)$. Following the construction of Chapter 3 we define

$$F^{(j)} := \text{diag}B^{(j)}, \tag{4.61}$$

$$N^{(j+1)} := \begin{pmatrix} 0 & \frac{-B_{12}^{(j)}}{2|\xi|} \\ \frac{B_{21}^{(j)}}{2|\xi|} & 0 \end{pmatrix}, \tag{4.62}$$

$$B^{(j+1)} := (D_t - D - B - C)\,N_{j+1} - N_{j+1}\,(D_t - D - F_j)\,. \tag{4.63}$$

Analogous to Proposition 3.3 the following result can be proved.

Proposition 4.1. *Assume the Hypothesis 4.3 with derivatives up to order ℓ. Then $N^{(j)} \in S_N^{\ell-j+1,\infty}\{-j,j\}$ and $B^{(j)} \in S_N^{\ell-j,\infty}\{-j,j+1\}$ for all $j = 1,2,\cdots,\ell$. Moreover, for any k we find a zone constant N such that N_k is invertible in $Z_{hyp}(N)$.*

If we denote $R_k(t,\xi) := -N_k(t,\xi)^{-1}B^{(k)}(t,\xi)$ the previous results yield the following statement:

Lemma 4.4. *Assume the Hypothesis 4.3. For each $1 \le k \le \ell$ there exists a zone constant N and matrix-valued symbols such that*

1. $N_k(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$ *is invertible for* $(t,\xi) \in Z_{hyp}(N)$ *with* $N_k(t,\xi)^{-1} \in S_N^{\ell-k+1,\infty}\{0,0\}$;

2. $F_{k-1}(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,1\}$ *is diagonal with* $F_{k-1}(t,\xi) - F^{(0)} \in S_N^{\ell-k+1,\infty}\{-1,2\}$*;*
3. $R_k(t,\xi) \in S_N^{\ell-k,\infty}\{-k,k+1\}$.

Moreover, the identity

$$\big(D_t - D(\xi) - B(t) - C(t,\xi)\big)N_k(t,\xi) = N_k(t,\xi)\big(D_t - D(\xi) - F_{k-1}(t,\xi) - R_k(t,\xi)\big) \quad (4.64)$$

holds for all $(t,\xi) \in Z_{hyp}(N)$.

We are now in a position to derive the main result of this subsection.

Proposition 4.2. *Assume Hypothesis 4.3. Then the fundamental solution* $\mathcal{E}_k(t,s,\xi)$ *of the diagonalized operator* $D_t - D - F_{k-1} - R_k$ *with remainder* R_k *can be represented as*

$$\mathcal{E}_k(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\mathcal{E}_0(t,s,\xi)\mathcal{Q}_k(t,s,\xi), \quad (4.65)$$

for $t \geq s$ *and* $(s,\xi) \in Z_{hyp}(N)$*, where*

1. *the function* $\lambda(t) = \exp\big(\frac{1}{2}\int_0^t b(\tau)d\tau\big)$ *describes the main influence of the dissipation* $b(t)u_t$*;*
2. *the matrices* $N_k(t,\xi), N_k(t,\xi)^{-1} \in S_N^{\ell-k+1,\infty}\{0,0\}$ *tending on* $\{\xi : |\xi| \geq \varepsilon\}$ *uniformly to the identity matrix* I*;*
3. *the matrix* $\mathcal{E}_0(t,s,\xi)$ *given by*

$$\mathcal{E}_0(t,s,\xi) = \begin{pmatrix} e^{i(t-s)|\xi|} & 0 \\ 0 & e^{-i(t-s)|\xi|} \end{pmatrix} \quad (4.66)$$

   is the fundamental solution of the free wave equation;

4. *the function* $Q_k(t,s,\xi)$ *is the fundamental solution to the operator*

$$D_t - \Phi_k(t,s,\xi), \quad (4.67)$$

   where $\Phi_k(t,s,\xi) = F_{k-1}(t,\xi) - F^{(0)}(t,\xi) + \mathcal{E}_0(s,t,\xi)R_k(t,\xi)\mathcal{E}_0(s,t,\xi) + \operatorname{diag} C$*;*

5. *the matrix* $\mathcal{Q}_k(t,s,\xi)$ *satisfies for all multi-indices* $|\alpha| \leq \min\{k-1, \ell-k-1\}$ *the symbol-like estimates*

$$\big\|D_\xi^\alpha \mathcal{Q}_k(t,s,\xi)\big\| \leq C_\alpha |\xi|^{-|\alpha|} \text{ for all } t,s \geq \theta_{|\xi|} \quad (4.68)$$

   uniformly in $\theta_{|\xi|} \leq s \leq t$.

Proof. In fact,

$$\begin{aligned} D_t\mathcal{E}_k &= \big(D + F_{k-1} + R_k + \operatorname{diag}(B) + \operatorname{diag}(C) - F^{(0)}\big)\mathcal{E}_k \\ &= \big(D + F_{k-1} + R_k\big)\mathcal{E}_k. \end{aligned}$$

From the Proposition 4.1 we have that $N_k - I \in S_N^{\ell-k+1,\infty}\{-1,1\}$. Therefore, $|\xi| \geq \varepsilon$ implies

$$\|N_1(t,\xi) - I\| \lesssim \frac{1}{|\xi|(1+t)} \leq \frac{C}{\varepsilon(1+t)} \to 0 \text{ when } t \to \infty.$$

Using that $\mathcal{E}_0(t,s,\xi)$ is unitary and Lemma 3.3 we see that $\Phi_k(t,s,\xi) \in S_N^{0,k-1}\{-1,2\}$ for all $s \geq \theta_{|\xi|}$. Taking account of $S_N^{0,k-1}\{-1,2\} \subset L_\xi^\infty L_t^1(Z_{hyp})$ it is allowed to apply the Peano-Baker formula (7.2) and to conclude that the fundamental solution to the operator (4.67) is given by

$$\mathcal{Q}_k(t,s,\xi) = I + \sum_{\ell=1}^{\infty} i^\ell \int_s^t \Phi_k(t_1,s,\xi) \int_s^{t_1} \Phi_k(t_2,s,\xi) \cdots \int_s^{t_{\ell-1}} \Phi_k(t_\ell,s,\xi) dt_\ell \cdots dt_1. \tag{4.69}$$

Therefore, Lemma 3.4 implies the estimate (4.68). □

The next result is important for deriving a modified scattering result for the Cauchy problem (4.26).

Lemma 4.5. *Assume Hypothesis 4.3. Then $\mathcal{Q}_k(t,s,\xi)$ is invertible on $Z_{hyp}^\ell(N) \cup Z_{hyp}^s(N)$ tending uniformly to the invertible matrix $\mathcal{Q}_k(\infty,s,\xi)$ for $\{\xi : |\xi| \geq \varepsilon\}$.*

Proof. It follows from (4.69) that

$$\mathcal{Q}_k(t,s,\xi) = I + \sum_{\ell=1}^{\infty} i^\ell \int_s^t \Phi_k(t_1,s,\xi) \int_s^{t_1} \Phi_k(t_2,s,\xi) \cdots \int_s^{t_{\ell-1}} \Phi_k(t_\ell,s,\xi) dt_\ell \cdots dt_1, \tag{4.70}$$

where $\Phi_k(t,s,\xi) \in S_N^{0,k-1}\{-1,2\}$. Therefore, the desired result follows by the Cauchy criterion applied to the series (4.70) or by the estimate

$$\begin{aligned} \|\mathcal{Q}_k(\infty,s,\xi) - \mathcal{Q}_k(t,s,\xi)\| &\leq \int_t^\infty \|\Phi_k(\tau,s,\xi)\| d\tau \exp\left(\int_s^t \|\Phi_k(\tau,s,\xi)\| d\tau\right) \\ &\lesssim \frac{1}{|\xi|} \int_t^\infty \frac{1}{(1+\tau)^2} d\tau \leq \frac{C}{\varepsilon(1+t)}. \end{aligned}$$

The invertibility of $\mathcal{Q}_k(t,s,\xi)$ follows from

$$\begin{aligned} \|\mathcal{Q}_k(t,s,\xi) - I\| &\leq \int_s^t \|R_k(\tau,\xi)\| d\tau \exp\left(\int_s^t \|R_k(\tau,\xi)\| d\tau\right) \\ &\leq \frac{C}{|\xi|} \int_s^t \frac{d\tau}{(1+\tau)^2} \exp\left(\frac{C}{|\xi|} \int_s^t \frac{d\tau}{(1+\tau)^2}\right) \leq \frac{C}{N} \exp\left(\frac{C}{N}\right) \end{aligned}$$

uniformly for $t \geq s$ and $(s,\xi) \in Z_{hyp}^\ell(N) \cup Z_{hyp}^s(N)$. A large N implies that Q_k is uniformly invertible on $Z_{hyp}^\ell(N) \cup Z_{hyp}^s(N)$. This completes the proof. □

Transforming back to the original problem:

After constructing the fundamental solution $\mathcal{E}_k(t,s,\xi)$ we transform back to the original problem and get in the hyperbolic zone the representation of fundamental solution

$$\mathcal{E}(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)} M^{-1} N_k(t,\xi) \mathcal{E}_0(t,s,\xi) \mathcal{Q}_k(t,s,\xi) N_k(s,\xi)^{-1} M \tag{4.71}$$

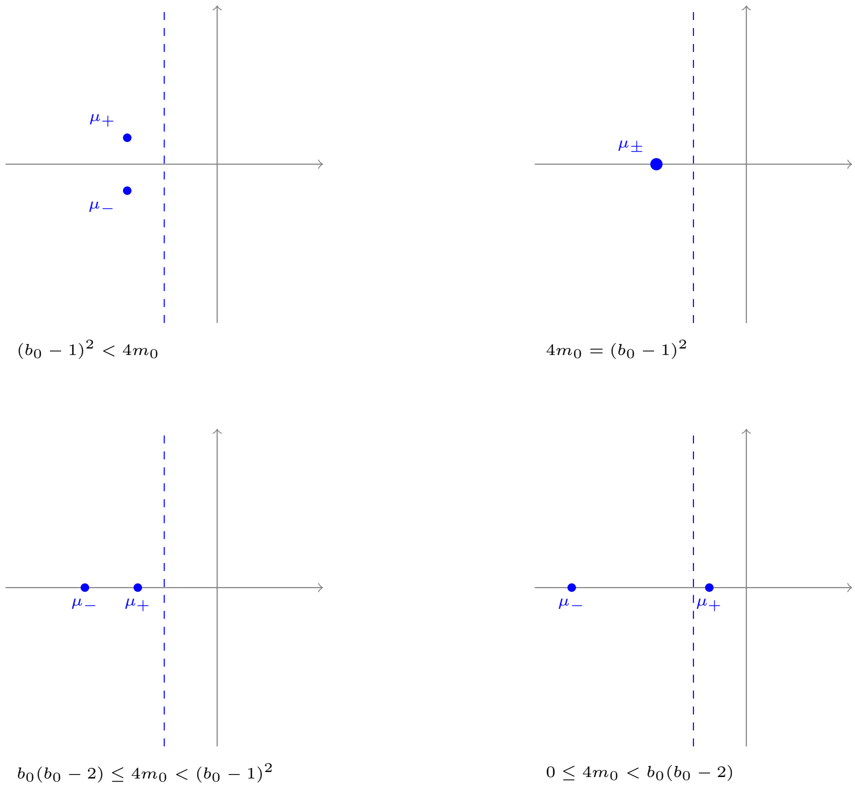

Fig. 4.1: Collecting the estimates

with uniformly bounded matrices $N_k, N_k^{-1} \in S_N^{\ell-k-1,\infty}\{0,0\}$.

The representation is true in the hyperbolic zone for large frequencies. For small frequencies we will use the following: For $0 \le s \le \theta_{|\xi|} \le t$ it holds

$$\mathcal{E}(t,s,\xi) = \frac{1}{\lambda(t)} M^{-1} N_k(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi)\mathcal{Q}_k(t,\theta_{|\xi|},\xi)N_k(\theta_{|\xi|},\xi)^{-1}M\lambda(\theta_{|\xi|})\mathcal{E}(\theta_{|\xi|},s,\xi). \tag{4.72}$$

4.2.2 $L^2 - L^2$ decay estimates

The representation of fundamental solutions obtained so far allows us to conclude estimates for the solution and their derivatives. This section is devoted to derive energy estimates.

The influence of high frequencies is described by a WKB-representation of solutions giving an overall decay estimate based on the function $\lambda(t)$. In Figure 4.1 this corresponds to the dashed line in the complex plane. The two dots correspond to the exponents $\mu_\pm$ arising from the Levinson's Theorem. They are responsible for the small frequency behaviour and the interplay of the relation of these dots and the dashed line will be the major reason for the appearing different cases of final estimates.

The main estimates obtained so far can be seen in Tables 4.1 and 4.2. We shall distinguish between the situation of condition (A1) in Hypothesis 4.4 and the situation of condition (A2) in Hypothesis 4.4. In the latter case we can only treat mass terms satisfying $4m_0 < (b_0-1)^2$.

conditions on m_0 and b_0	behaviour in $Z_{pd}(N)$	behaviour in $Z_{hyp}(N)$
$(b_0-1)^2 < 4m_0$	$(t/s)^{-(b_0+1)/2}$	$(t/s)^{-b_0/2}$
$4m_0 = (b_0-1)^2$	$(t/s)^{-(b_0+1)/2+\varepsilon}$	$(t/s)^{-b_0/2}$
$b_0(b_0-2) \le 4m_0 < (b_0-1)^2$	$(t/s)^{\mu_+}$, $\mu_+ \le -b_0/2$	$(t/s)^{-b_0/2}$
$0 \le 4m_0 < b_0(b_0-2)$	$(t/s)^{\mu_+}$, $\mu_+ > -b_0/2$	$(t/s)^{-b_0/2}$

Tab. 4.1: Estimates of fundamental solutions assuming (A1).

conditions on m_0 and b_0	behaviour in $Z_{pd}(N)$	behaviour in $Z_{hyp}(N)$
$b_0(b_0-2) < 4m_0 < (b_0-1)^2$	$(t/s)^{\mu_+}$, $\mu_+ < -b_0/2$	$(t/s)^{-b_0/2}$
$0 \le 4m_0 < b_0(b_0-2)$	$(t/s)^{\mu_+}$, $\mu_+ > -b_0/2$	$(t/s)^{-b_0/2}$

Tab. 4.2: Estimates of fundamental solutions assuming (A2).

Choice of parameters

The number of diagonalization steps needed in the hyperbolic zone determines the zone constant N and thus the decomposition of the phase space. When proving energy estimates it will be enough to apply one non-trivial step of diagonalization in the hyperbolic zone and for this any choice of N sufficiently large will be good. When proving $L^p - L^q$ estimates several such steps are necessary and N has to be chosen large enough.

The number ℓ of derivatives required in Hypothesis 4.3 depends on the number of diagonalization steps to be used and the needed symbol properties of the matrix function $\mathcal{Q}_k(t, \theta_{|\xi|}, \xi)$. When proving energy estimates, $\ell = 1$ is sufficient.

Theorem 4.5. *Assume Hypothesis 4.3 with $\ell = 1$, Hypothesis 4.4 with $\sigma = 1$ and $b_0(b_0-2) \le 4m_0$. Then the $L^2 - L^2$ estimate*

$$\|\mathcal{E}(t,s,D)\|_{L^2\to L^2} \lesssim \frac{\lambda(s)}{\lambda(t)}$$

holds true, where $\lambda = \lambda(t)$ is defined by (4.5).

Proof. The proof is divided into two steps.

Considerations in the pseudo-differential zone:

Here Theorem 4.3 in combination with Remark 4.4 yields

$$\|\mathcal{E}(t,s,\xi)\| \lesssim \left(\frac{1+t}{1+s}\right)^{\operatorname{Re}\mu_+} \lesssim \frac{\lambda(s)}{\lambda(t)} \tag{4.73}$$

uniformly with respect to $0 \le s \le t \le \theta_{|\xi|}$. Therefore, if $v = v(x) \in L^2(\mathbb{R}^n)$, then

$$\|\mathcal{E}(t,s,\cdot)\varphi_{pd}(t,\cdot)\widehat{v}\|_{L^2} \lesssim \|\mathcal{E}(t,s,\cdot)\varphi_{pd}(t,\cdot)\|_{L^\infty}\|\widehat{v}\|_{L^2} \lesssim \frac{\lambda(s)}{\lambda(t)}\|\widehat{v}\|_{L^2}.$$

Applying the formula of Parseval-Plancharel we have

$$\big\|F^{-1}\big(\mathcal{E}(t,s,\xi)\varphi_{pd}(t,\xi)\widehat{v}(\xi)\big)(t,\cdot)\big\|_{L^2} \lesssim \frac{\lambda(s)}{\lambda(t)}\|v\|_{L^2}.$$

Considerations in the hyperbolic zone:

In the hyperbolic zone we will use the representation of the fundamental solution. Let us consider $v = v(x) \in L^2(\mathbb{R}^n)$. We proved that $N_k(t,\xi)$, $N_k(t,\xi)^{-1}$ and $\mathcal{Q}_k(t,s,\xi)$ are uniformly bounded for all $t \geq s$ and $(s,\xi) \in Z^\ell_{hyp}(N)$. Therefore by (4.71) with $k = 1$ it follows for large frequencies

$$\begin{aligned}&\|\mathcal{E}(t,s,\cdot)\varphi^\ell_{hyp}\widehat{v}\|_{L^2}\\ &= \frac{\lambda(s)}{\lambda(t)}\big\|M^{-1}N_1(t,\cdot)\mathcal{E}_0(t,s,\cdot)\mathcal{Q}_1(t,s,\cdot)N_1(s,\cdot)^{-1}M\varphi^\ell_{hyp}(\cdot)\widehat{v}\big\|_{L^2} \lesssim \frac{\lambda(s)}{\lambda(t)}\|\widehat{v}\|_{L^2}.\end{aligned}$$

Applying the formula of Parseval-Plancharel brings

$$\big\|F^{-1}\big(\mathcal{E}(t,s,\xi)\varphi^\ell_{hyp}(\xi)\widehat{v}(\xi)\big)(t,\cdot)\big\|_{L^2} \lesssim \frac{\lambda(s)}{\lambda(t)}\|v\|_{L^2}.$$

Remark 4.5 implies that $\lambda(\theta_{|\xi|})\mathcal{E}(\theta_{|\xi|},s,\xi)$ is also uniformly bounded for $|\xi| \leq N$. Therefore for small frequencies it follows from (4.72) that

$$\begin{aligned}&\|\mathcal{E}(t,s,\cdot)\varphi^s_{hyp}(t,\cdot)\widehat{v}\|_{L^2}\\ =\ & \frac{1}{\lambda(t)}\|M^{-1}N_1(t,\cdot)\mathcal{E}_0(t,\theta_{|\cdot|},\cdot)\mathcal{Q}_1(t,\theta_{|\cdot|},\cdot)N_1(\theta_{|\cdot|},\cdot)^{-1}M\lambda(\theta_{|\cdot|})\mathcal{E}(\theta_{|\cdot|},s,\cdot)\varphi^s_{hyp}(t,\cdot)\widehat{v}\|_{L^2}\\ \lesssim\ & \frac{1}{\lambda(t)}\|\widehat{v}\|_{L^2}.\end{aligned}$$

Applying the formula of Parseval-Plancharel brings

$$\big\|F^{-1}\big(\mathcal{E}(t,s,\xi)\varphi^s_{hyp}(t,\xi)\widehat{v}(\xi)\big)(t,\cdot)\big\|_{L^2} \lesssim \frac{1}{\lambda(t)}\|v\|_{L^2},$$

which is a better decay in comparison with the statement of the theorem. □

Corollary 4.1. *Assume Hypothesis 4.3 with* $\ell = 1$, *Hypothesis 4.4 with* $\sigma = 1$ *and* $b_0(b_0-2) \leq 4m_0$. *Then the* $L^2 - L^2$ *estimate*

$$\|(1+t)^{-1}u(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \qquad (4.74)$$

holds true for any solution u *of* (4.26) *to initial data* $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$.

Proof. The proof is divided into two steps.

Considerations in the pseudo-differential zone:

The micro-energy in the pseudo-differential zone becomes

$$U(t,\xi) = \Big(\frac{1}{1+t}\widehat{u}(t,\xi), \widehat{u}_t(t,\xi)\Big).$$

From $U(t,\xi) = \mathcal{E}(t,0,\xi)U(0,\xi)$ it follows

$$\begin{aligned}\frac{1}{1+t}\widehat{u}(t,\xi)\varphi_{pd}(t,\xi) &= \mathcal{E}_{11}(t,0,\xi)\varphi_{pd}(t,\xi)\widehat{u}_0 + \mathcal{E}_{12}(t,0,\xi)\varphi_{pd}(t,\xi)\widehat{u}_1,\\ \widehat{u}_t(t,\xi)\varphi_{pd}(t,\xi) &= \mathcal{E}_{21}(t,0,\xi)\varphi_{pd}(t,\xi)\widehat{u}_0 + \mathcal{E}_{22}(t,0,\xi)\varphi_{pd}(t,\xi)\widehat{u}_1,\end{aligned}$$

where $\mathcal{E}_{ij}(t,0,\xi), i,j=1,2$, are the entries of the matrix $\mathcal{E}(t,0,\xi)$. Therefore, Theorem 4.5 and the definition of the pseudo-differential zone imply

$$\begin{aligned}
\big\|F^{-1}\big((1+t)^{-1}\widehat{u}(t,\xi)\varphi_{pd}(t,\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{L^2}+\|u_1\|_{L^2}\big),\\
\big\|F^{-1}\big(|\xi|\widehat{u}(t,\xi)\varphi_{pd}(t,\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{L^2}+\|u_1\|_{L^2}\big),\\
\big\|F^{-1}\big(\widehat{u}_t(t,\xi)\varphi_{pd}(t,\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{L^2}+\|u_1\|_{L^2}\big).
\end{aligned}$$

Considerations in the hyperbolic zone:

The micro-energy in the hyperbolic zone becomes

$$U(t,\xi)=\big(|\xi|\widehat{u}(t,\xi),\widehat{u}_t(t,\xi)\big).$$

We have for large frequencies

$$\begin{aligned}
&(1+t)^{-1}\widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi)=(1+t)^{-1}\mathcal{E}_{11}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{u}_0\\
&\quad+\big((1+t)|\xi|\big)^{-1}\mathcal{E}_{12}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{u}_1,\\
&|\xi|\widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi)=\mathcal{E}_{11}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)|\xi|\widehat{u}_0+\mathcal{E}_{12}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{u}_1,\\
&\widehat{u}_t(t,\xi)\varphi^{\ell}_{hyp}(\xi)=\mathcal{E}_{21}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)|\xi|\widehat{u}_0+\mathcal{E}_{22}(t,0,\xi)\varphi^{\ell}_{hyp}(\xi)\widehat{u}_1.
\end{aligned}$$

Therefore, Theorem 4.5 yields with $\big((1+t)|\xi|\big)^{-1}\lesssim 1$

$$\begin{aligned}
\big\|F^{-1}\big((1+t)^{-1}\widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{L^2}+\|u_1\|_{L^2}\big),\\
\big\|F^{-1}\big(|\xi|\widehat{u}(t,\xi)\varphi^{\ell}_{hyp}(\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{H^1}+\|u_1\|_{L^2}\big),\\
\big\|F^{-1}\big(\widehat{u}_t(t,\xi)\varphi^{\ell}_{hyp}(\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{H^1}+\|u_1\|_{L^2}\big).
\end{aligned}$$

For small frequencies the approach is analogous. We have that

$$\begin{aligned}
&(1+t)^{-1}\widehat{u}(t,\xi)\varphi^{s}_{hyp}(t,\xi)=(1+t)^{-1}\mathcal{E}_{11}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_0\\
&\quad+\big((1+t)|\xi|\big)^{-1}\mathcal{E}_{12}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_1,\\
&|\xi|\widehat{u}(t,\xi)\varphi^{s}_{hyp}(t,\xi)=\mathcal{E}_{11}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)|\xi|\widehat{u}_0+\mathcal{E}_{12}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_1,\\
&\widehat{u}_t(t,\xi)\varphi^{s}_{hyp}(t,\xi)=\mathcal{E}_{21}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)|\xi|\widehat{u}_0+\mathcal{E}_{22}(t,0,\xi)\varphi^{s}_{hyp}(t,\xi)\widehat{u}_1.
\end{aligned}$$

Consequently, we may conclude

$$\begin{aligned}
\big\|F^{-1}\big((1+t)^{-1}\widehat{u}(t,\xi)\varphi^{s}_{hyp}(t,\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{L^2}+\|u_1\|_{L^2}\big),\\
\big\|F^{-1}\big(|\xi|\widehat{u}(t,\xi)\varphi^{s}_{hyp}(t,\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{H^1}+\|u_1\|_{L^2}\big),\\
\big\|F^{-1}\big(\widehat{u}_t(t,\xi)\varphi^{s}_{hyp}(t,\xi)\big)(t,\cdot)\big\|_{L^2} &\lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{H^1}+\|u_1\|_{L^2}\big).
\end{aligned}$$

This completes the proof. □

If we assume Hypothesis 4.4 with $\sigma > 1$ we have to restrict the admissible values of m_0 further. The proof goes in analogy to the above one replacing Theorem 4.3 by Theorem 4.4 for the treatment in the pseudo-differential zone.

Theorem 4.6. *Assume Hypothesis 4.3 with $\ell = 1$, Hypothesis 4.4 with $\sigma \in (1,2]$ and $b_0(b_0-2) \le 4m_0 < (b_0-1)^2$. Then the $L^2 - L^2$ estimate*

$$\|\mathcal{E}(t,s,D)\|_{L^2\to L^2} \lesssim \frac{\lambda(s)}{\lambda(t)}$$

holds true.

Then we may conclude the following energy estimate:

Corollary 4.2. *Assume Hypothesis 4.3 with $\ell = 1$, Hypothesis 4.4 with $\sigma \in (1,2]$ and $b_0(b_0-2) \le 4m_0 < (b_0-1)^2$. Then the $L^2 - L^2$ estimate*

$$\|(1+t)^{-1}u(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \lesssim \frac{1}{\lambda(t)}\big(\|u_0\|_{H^1} + \|u_1\|_{L^2}\big) \qquad (4.75)$$

holds true for any solution u of (4.26) to initial data $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$.

Example 4.3. *Let us consider for $b_0, m_0 \in \mathbb{R}$*

$$b(t) = \frac{b_0}{1+t} + \frac{h_1(t)}{1+t}, \qquad (4.76)$$

$$m(t) = \frac{m_0}{(1+t)^2} + \frac{h_2(t)}{(1+t)^2} \qquad (4.77)$$

with uniformly bounded $h_j(t)$, $j = 1,2$, and uniformly bounded $t\partial_t h_j(t)$ and with the integrability condition

$$\int_0^\infty |h_j(t)|\frac{\mathrm{d}t}{1+t} < \infty, \qquad j = 1,2. \qquad (4.78)$$

Then Hypothesis 4.3 is satisfied with $\ell = 1$ and Hypothesis 4.4 is satisfied with $\sigma = 1$. If we further suppose that $b_0(b_0-2) \le 4m_0$, then the energy estimate

$$\|\big((1+t)^{-1}u(t,\cdot), u_t(t,\cdot), \nabla_x u(t,\cdot)\big)\|_{L^2} \lesssim (1+t)^{-\frac{b_0}{2}}\left(\|u_0\|_{H^1} + \|u_1\|_{L^2}\right) \qquad (4.79)$$

holds true. The decay is independent of m_0 and related to the decay for non-effective wave damped models treated in [59].

Example 4.4. *We consider the same situation as in the previous example, but replace (4.78) by*

$$\int_0^\infty |h_j(t)|^\sigma \frac{\mathrm{d}t}{1+t} < \infty, \qquad j = 1,2, \qquad (4.80)$$

then under the more restrictive condition $b_0(b_0-2) \le 4m_0 < (b_0-1)^2$ on the numbers m_0 and b_0 the same estimate (4.79) holds true. To be more specific, this allows to treat

$$b(t) = \frac{b_0}{1+t} + \frac{b_1}{(\mathrm{e}+t)(\ln(\mathrm{e}+t))^\gamma}, \qquad (4.81)$$

$$m(t) = \frac{m_0}{(1+t)^2} + \frac{m_1}{(\mathrm{e}+t)^2(\ln(\mathrm{e}+t))^\gamma} \qquad (4.82)$$

with arbitrary b_1, m_1 and $\gamma \in (1/2,1]$. It satisfies (4.80) with $\sigma \in (\gamma^{-1},2]$.

Remark 4.7. *The Examples 4.3 and 4.4 show us that small mass terms have no influence on the decay estimates for a suitable energy of solutions to Cauchy problem (4.26).*

4.2.3 Modified scattering result

Now we discuss the sharpness of energy estimates again and formulate a more precise statement. In fact, there is a relation between solutions to the Cauchy problem with mass and dissipation

$$u_{tt} - \Delta u + b(t)u_t + m(t)u = 0, \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{4.83}$$

under our hypotheses and solutions of the free wave equation

$$v_{tt} - \Delta v = 0, \qquad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \tag{4.84}$$

with appropriate related data. We follow some ideas of Wirth [61] and give (in combination with the energy conservation for free waves) a very precise description of sharpness of the above energy estimates.

Theorem 4.7. *Assume Hypotheses 4.3 with $\ell = 1$ and 4.4 with*

$$\sigma = 1 \qquad \textit{and} \qquad b_0(b_0 - 2) \le 4m_0 \tag{4.85}$$

or with

$$\sigma \in (1, 2] \qquad \textit{and} \qquad b_0(b_0 - 2) \le 4m_0 < (b_0 - 1)^2. \tag{4.86}$$

Then there exists a bounded operator

$$W_+ : H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \tag{4.87}$$

such that for Cauchy data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ of (4.83) and associated data $(v_0, v_1) = W_+(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to (4.84) the corresponding solutions $u = u(t,x)$ and $v = v(t,x)$ satisfy

$$\big\| \lambda(t)\big(u_t(t,\cdot), \nabla_x u(t,\cdot)\big) - \big(v_t(t,\cdot), \nabla_x v(t,\cdot)\big) \big\|_{L^2} \to 0,$$

as $t \to \infty$.

Proof. First let us define for any $\varepsilon > 0$ the following closed subset of $L^2 \times L^2$:

$$F_\epsilon := \Big\{ U_0 \in L^2 \times L^2 : \ \widehat{U}_0(\xi) = 0 \ \text{for any } |\xi| \le \epsilon \Big\}.$$

We remember that $\mathcal{L} = \cup_{\epsilon>0} F_\epsilon$ is a dense subset of $L^2 \times L^2$. If we introduce $\mathcal{E}_0 = \mathcal{E}_0(t,s,\xi)$ as in (4.66) and if v solves the free wave equation (4.84), then after defining $V(t,\xi) = (|\xi|\widehat{v}, D_t\widehat{v})^T$ we can write $V(t,\xi) = \tilde{\tilde{\mathcal{E}}}_0(t,s,\xi)V(s,\xi)$, where

$$\tilde{\tilde{\mathcal{E}}}_0(t,s,\xi) = M^{-1}\mathcal{E}_0(t,s,\xi)M.$$

The proof is based on an explicit representation of the modified Moeller wave operator W_+. Our goal is to prove that the limit

$$W_+(\xi) = \lim_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) \tag{4.88}$$

exists uniformly in $|\xi| > \varepsilon$ for all $\varepsilon > 0$. After proving this property we are able to relate the Cauchy data by

$$W_+(\xi)U(0,\xi) = V(0,\xi).$$

From Lemma 4.5 we know that the limit

$$\mathcal{Q}_1(\infty, \theta_{|\xi|}, \xi) = \lim_{t\to\infty} \mathcal{Q}_1(t, \theta_{|\xi|}, \xi)$$

exists uniformly when $|\xi| \geq \epsilon$ for any $\epsilon > 0$. Hence, if we restrict $\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi)$ on $\mathcal{L}$ we obtain

$$\begin{aligned}
& \lim_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi) \\
= & \lim_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,\theta_{|\xi|},\xi)\mathcal{E}(\theta_{|\xi|},0,\xi) \\
= & \lim_{t\to\infty} \lambda(\theta_{|\xi|})\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}M^{-1}N_1(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi)\mathcal{Q}_1(t,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)^{-1}M\mathcal{E}(\theta_{|\xi|},0,\xi) \\
= & \lim_{t\to\infty} \lambda(\theta_{|\xi|})M^{-1}\mathcal{E}_0(0,t,\xi)N_1(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi)\mathcal{Q}_1(t,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)^{-1}M\mathcal{E}(\theta_{|\xi|},0,\xi) \\
= & \lambda(\theta_{|\xi|})M^{-1}\mathcal{E}_0(0,\theta_{|\xi|},\xi)\mathcal{Q}_1(\infty,\theta_{|\xi|},\xi)N_1(\theta_{|\xi|},\xi)^{-1}M\mathcal{E}(\theta_{|\xi|},0,\xi).
\end{aligned}$$

Here we used the fact that

$$\mathcal{E}_0(0,t,\xi)N_1(t,\xi)\mathcal{E}_0(t,\theta_{|\xi|},\xi) = \mathcal{E}_0(0,\theta_{|\xi|},\xi) + \mathcal{E}_0(0,t,\xi)(N_1(t,\xi)-I)\mathcal{E}_0(t,\theta_{|\xi|},\xi)$$

and $N_1(t,\xi) \to I$ uniformly for $|\xi| \geq \epsilon$. In the pseudo-differential zone the boundness of the fundamental solution of the free wave equation and the estimates in Theorem 4.3 and Theorem 4.4 guarantee that the limit (4.88) goes to zero.

According to the estimates for the energy from Theorem 4.5 and Theorem 4.6 we conclude that

$$\lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\xi)^{-1}\mathcal{E}(t,0,\xi)$$

is uniformly bounded on $\mathcal{L}$. Therefore applying Banach-Steinhaus Theorem 7.3 we conclude that

$$W_+(D) = \operatorname*{s-lim}_{t\to\infty} \lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,D)^{-1}\mathcal{E}(t,0,D)$$

exists as strong limit in $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Finally, we study the difference

$$\begin{aligned}
\|\lambda(t)U(t,\cdot) - V(t,\cdot)\|_{L^2} &= \|\lambda(t)\mathcal{E}(t,0,\cdot)U(0,\cdot) - \tilde{\tilde{\mathcal{E}}}_0(t,0,\cdot)V(0,\cdot)\|_{L^2} \\
&= \left\|\Big(\lambda(t)\tilde{\tilde{\mathcal{E}}}_0(t,0,\cdot)^{-1}\mathcal{E}(t,0,\cdot) - W_+(\cdot)\Big)U(0,\cdot)\right\|_{L^2}.
\end{aligned}$$

Under our assumption $(u_0,u_1) \in H^1\times L^2$ and by definition of $W_+(\xi)$ we may conclude that

$$\|\lambda(t)U(t,\cdot) - V(t,\cdot)\|_{L^2} \to 0$$

as t tends to infinity. The proof is completed. □

Remark 4.8. *The modified scattering result involves only the hyperbolic energy terms* $\nabla u(t,\cdot)$ *and* $u_t(t,\cdot)$. *If we are interested in results containing also the solution* $u(t,\cdot)$ *itself, we can not hope for the same kind of (non-weighted) result. Note for this, that the estimate* $\|v(t,\cdot)\|_{L^2} \leq t(\|v_0\|_{L^2} + \|v_1\|_{H^{-1}})$ *is in general sharp for solutions to the Cauchy problem for the free wave equation, nevertheless there are no initial data with* *this precise rate. We only have* $\|v(t,\cdot)\|_{L^2} = o(t)$ *as* $t \to \infty$ *for each (fixed) solution. Similarly one obtains for solutions to* (4.83) *to initial data from* $L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$

$$\lim_{t\to\infty} \frac{\lambda(t)}{1+t}\|u(t,\cdot)\|_{L^2} = 0. \tag{4.89}$$

This rate is sharp for general data and can only by improved by further assumptions on initial data. We omit the proof.

4.2.4 $L^p - L^q$ decay estimates on the conjugate line

Finally, we want to give Strichartz estimates for solutions. These are L^p–L^q estimates for conjugate dual indices. The estimate is again independent of m_0, but the range of admissible b_0 depends on m_0. For this statement we need to use the representations of Subsection 4.2.1 with $k > 1$ and, therefore, we also need higher regularity of the coefficient functions compared to the energy estimates given before.

Theorem 4.8. *Assume Hypothesis 4.3 with* $\ell = n+1$*, Hypothesis 4.4 with*

$$\sigma = 1 \qquad \text{and} \qquad b_0(b_0-2) \le 4m_0 \tag{4.90}$$

or with

$$\sigma \in (1,2] \qquad \text{and} \qquad b_0(b_0-2) \le 4m_0 < (b_0-1)^2. \tag{4.91}$$

Then the $L^p - L^q$ *estimate*

$$\|\mathcal{E}(t,0,D)\|_{L^p\to L^q} \le C_{p,q}\frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

holds true for $p \in (1,2]$*,* p *and* q *from the conjugate line and with regularity* $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Proof. The proof is divided into two steps.

Considerations in the pseudo-differential zone

In the pseudo-differential zone we have the estimate

$$\|\mathcal{E}(t,0,\xi)\varphi_{pd}(t,\xi)\| \lesssim \frac{1}{\lambda(t)}.$$

If $v = v(x) \in L^p(\mathbb{R}^n)$, then

$$\begin{aligned} \left\|F^{-1}\left(\mathcal{E}(t,0,\xi)\varphi_{pd}(t,\xi)\widehat{v}(\xi)\right)(t,\cdot)\right\|_{L^q} &\le \|\mathcal{E}(t,0,\cdot)\varphi_{pd}(t,\cdot)\widehat{v}\|_{L^p} \\ &\le \|\mathcal{E}(t,0,\cdot)\|_{L^\infty}\|\varphi_{pd}(t,\cdot)\|_{L^{\frac{pq}{q-p}}}\|\widehat{v}\|_{L^q} \\ &\lesssim \frac{1}{\lambda(t)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^p}. \end{aligned}$$

This is a better decay estimate than the desired one of the theorem.

Considerations in the hyperbolic zone

Let us consider $v = v(x) \in L^p(\mathbb{R}^n)$. For large frequencies we use the representation of (4.71) to split the propagator into several parts and estimate each of them separately. For this we choose k such that

$$\ell = 2(k-1) \qquad \text{and} \qquad k-1 \geq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{4.92}$$

Then,

$$\mathcal{E}(t,0,D)\varphi^\ell_{hyp}(D) = \frac{1}{\lambda(t)} \underbrace{M^{-1}N_k(t,D)}_{L^q\to L^q} \underbrace{\mathcal{E}_0(t,0,D)}_{L^{p,r}\to L^q} \underbrace{\mathcal{Q}_k(t,0,D)}_{L^{p,r}\to L^{p,r}} \underbrace{N_k(0,D)^{-1}M}_{L^{p,r}\to L^{p,r}} \varphi^\ell_{hyp}(D)$$

with the under braced mapping properties. Indeed,

- we know that $M^{-1}N_k(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$, then $MN_k(t,\xi) \in \dot{S}^0_{\ell-k+1}$, Marcinkiewicz's Theorem 7.1 implies that $MN_k(t,\xi) \in M_q^q$ uniformly in t, here the condition $\ell - k + 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1$ is essential;
- Theorem 3.2 implies $\mathcal{E}_0(t,0,D) : L^{p,r} \to L^q$ with a decay rate $(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$;
- Proposition 4.2 implies that $\mathcal{Q}_k(t,0,D) \in \dot{S}^0_{k-1}$, then Marcinkiewicz's Theorem 7.1 brings $\mathcal{Q}_k(t,0,\xi) \in M_{p,r}^{p,r}$ uniformly in t, here the condition $k-1 \geq \left\lceil \frac{n}{2} \right\rceil + 1$ is essential;
- Finally, $N_k(0,\xi) \in \dot{S}^0_{k-1}$ by construction and $\varphi^\ell_{\mathrm{hyp}} \in \dot{S}^0$.

Therefore, it follows that

$$\left\| F^{-1}\big(\mathcal{E}(t,0,\xi)\varphi^\ell_{hyp}(\xi)\widehat{v}(\xi)\big)(t,\cdot)\right\|_{L^q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|v\|_{L^{p,r}}.$$

For small frequencies we conclude from (4.72) the representation of fundamental solution

$$\begin{aligned}\mathcal{E}(t,0,D)\varphi^s_{hyp}(t,D) = \frac{1}{\lambda(t)} &\underbrace{M^{-1}N_k(t,D)}_{L^q\to L^q} \underbrace{\mathcal{E}_0(t,\theta_{|D|},D)}_{L^{p,r}\to L^q} \underbrace{\mathcal{Q}_k(t,\theta_{|D|},D)}_{L^{p,r}\to L^{p,r}} \underbrace{N_k(\theta_{|D|},D)^{-1}M}_{L^{p,r}\to L^{p,r}} \\ \times &\underbrace{\lambda(\theta_{|D|})\mathcal{E}(\theta_{|D|},0,D)}_{L^{p,r}\to L^{p,r}} \varphi^s_{hyp}(t,D)\end{aligned}$$

with the under braced mapping properties. In fact,

- we know that $M^{-1}N_k(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$, then $MN_k(t,\xi) \in \dot{S}^0_{\ell-k+1}$, Marcinkiewicz's Theorem 7.1 implies that $MN_k(t,\xi) \in M_q^q$ uniformly in t, here the condition $\ell - k + 1 \geq \left\lceil \frac{n}{2} \right\rceil + 1$ is essential;
- Theorem 3.2 and the property $\mathcal{E}_0(t,\theta_{|\xi|},\xi) = \mathcal{E}_0(t,0,\xi)\mathcal{E}_0(0,\theta_{|\xi|},\xi)$ with $\mathcal{E}_0(0,\theta_{|\xi|},\xi) \in \dot{S}^0_\infty$ imply that $\mathcal{E}_0(t,\theta_{|D|},D) : L^{p,r} \to L^q$ with decay rate $(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$;
- Proposition 4.2 implies that $\mathcal{Q}_k(t,\theta_{|\xi|},\xi) \in \dot{S}^0_{k-1}$, then Marcinkiewicz's Theorem 7.1 gives $\mathcal{Q}_k(t,\theta_{|\xi|},\xi) \in M_{p,r}^{p,r}$ uniformly in t, here the condition $k-1 \geq \left\lceil \frac{n}{2} \right\rceil + 1$ is essential;

- By Lemma 4.4 and the properties of $\theta_{|\xi|}$ we know that $N_k(\theta_{|\xi|},\xi) \in \dot{S}^0$;
- Remark 4.5 implies $\lambda(\theta_{|\xi|})\mathcal{E}(\theta_{|\xi|},0,\xi) \in \dot{S}^0_{\ell+1}$.

Hence, it follows that

$$\big\|F^{-1}\big(\mathcal{E}(t,0,\xi)\varphi^s_{hyp}(t,\xi)\widehat{v}(\xi)\big)(t,\cdot)\big\|_{L^q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|v\|_{L^{p,r}},$$

for small frequencies in the hyperbolic zone. □

Taking account of Theorem 4.8 allows to conclude the following Strichartz' estimates for a suitable energy of solutions to damped Klein-Gordon models (4.26).

Corollary 4.3. *Assume Hypothesis 4.3 with $\ell = n+1$, Hypothesis 4.4 with*

$$\sigma = 1 \qquad \text{and} \qquad b_0(b_0-2) \le 4m_0 \tag{4.93}$$

or with

$$\sigma \in (1,2] \qquad \text{and} \qquad b_0(b_0-2) \le 4m_0 < (b_0-1)^2. \tag{4.94}$$

Then the $L^p - L^q$ estimate

$$\big\|\big((1+t)^{-1}u(t,\cdot),u_t(t,\cdot),\nabla_x u(t,\cdot)\big)\big\|_{L^q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(\|u_0\|_{L^{p,r+1}}+\|u_1\|_{L^{p,r}}\big)$$

hold true for $p \in (1,2]$, p and q from the conjugate line and with regularity $r = n\big(\frac{1}{p}-\frac{1}{q}\big)$.

Proof. Analogous to the proof of Corollary 4.1. □

We will conclude this chapter with significant concluding remarks.

Remark 4.9. *The estimates of the solution $u(t,\cdot)$ itself following from Theorems 4.5, 4.6 and 4.8 are not optimal in the present form. Indeed in Chapters 2 and 3 we established better decay estimates for the solution itself in case where $b \equiv 0$. This is due to the attempted $\sigma-$dependent formulation of the results. Under Hypothesis 4.4 with $\sigma = 1$ it is possible to improve the estimate for the solution in the following way*

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{1+\operatorname{Re}\mu_+}\big(\|u_0\|_{H^1}+\|u_1\|_{L^2}\big).$$

We will give the essential argument behind this improvement. The improvement is based on (4.29). Within $Z_{pd}(N)$ the construction gave the estimate

$$\Big|\frac{1}{1+t}\widehat{u}(t,\xi)\Big| \lesssim (1+t)^{\operatorname{Re}\mu_+}\big(\widehat{u}_0(t,\xi)+\widehat{u}_1(t,\xi)\big).$$

If we consider the hyperbolic zone $Z_{hyp}(N)$, we obtain in analogy

$$||\xi|\widehat{u}(t,\xi)| = \begin{cases} (1+t)^{-\frac{b_0}{2}}\left(|\xi|\widehat{u}_0(t,\xi)+\widehat{u}_1(t,\xi)\right), & |\xi| > N, \\ \left(\frac{1+t}{1+\theta_{|\xi|}}\right)^{-\frac{b_0}{2}}(1+\theta_{|\xi|})^{\operatorname{Re}\mu_+}\left(\widehat{u}_0(t,\xi)+\widehat{u}_1(t,\xi)\right), & |\xi| \le N, \end{cases}$$

and together with $|\xi|(1+\theta_{|\xi|}) = N$, the positivity of $\delta = 1+\frac{b_0}{2}+\operatorname{Re}\mu_+ > 0$ and the monotonicity of t^δ the desired estimate follows. For $\sigma > 1$ the latter needs not to be valid any more and improvements depend on the behavior of the quotient of the right-hand side of (4.45) and $\lambda(t)$.

Remark 4.10. *The restriction of Assumption $(A2)$ to the range $\sigma \in (1,2]$ is due to just one application of the Hartmann-Wintner transform of Theorem 7.6. Applying finitely many such transformations in an iterative way allows to extend Hypothesis 4.4 to arbitrary $\sigma > 1$. The price to pay for this is a series of correction terms in* (4.45) *instead of just one. The estimates of Theorems 4.6, 4.7 and 4.8 still depend on the hyperbolic zone and (as long as the right-hand side of* (4.45) *with $s = 0$ is still majored by $\lambda(t)^{-1}$) are valid unchanged.*

Our last remark will give some comments on the relation of the results in this chapter to the known treatments:

Remark 4.11. *For $m(t) \equiv 0$ we are in the non-effective damped wave equation case, i.e., Wirth [59]. If $m_0 = 0$ and $b_0 \in [0,1) \cup (1,2)$, then we are in the setting of Wirth [59] (or [57] for the particular case $b(t) = b_0(1+t)^{-1}$) and the estimates of Theorem 4.8 reduce to results from these papers.*
If $b_0 = 0$, then we can treat arbitrary m_0 and obtain from Theorem 4.8 with $\sigma = 1$, uniform bounds on the energy as well as the standard wave type $L^p - L^q$ decay estimates. The scale-invariant case was considered in [5] with similar observations.

5 Wave models with structural properties of the time-dependent potential

In this chapter we will apply C^m theory and a stabilization condition for Klein-Gordon equations with non-effective time-dependent potential. The idea of C^m theory together with the introduction of some stabilization was at first developed by Hirosawa to investigate the asymptotic behavior for the total energy of wave equation with time-dependent propagation speed in the paper [27], note that without stabilization condition the oscillations on the coefficient of speed of propagation may have a very deteriorating influence on the energy behavior (see [10, 46]). In 2009 Hirosawa/Wirth extended the result to wave equations with speed of propagations having non-trivial shape functions in [28]. In 2008 the C^m theory and stabilization condition was applied to wave equations with non-effective time-dependent dissipation in [29]. In 2010 Böhme/Hirosawa used C^m theory to prove generalized energy conservation for Klein-Gordon equations with effective time-dependent potential, see [6], where a stabilization condition is not required. In the semi-linear theory the C^m theory and stabilization condition was applied to wave models with smooth time-dependent propagation speeds by Hirosawa/Inooka/Pham in [26] to prove globally (in time) existence of solutions.

5.1 Idea of stabilization

Basically what we have done in Chapter 2 was to prove both-sided or generalized energy estimates for Klein-Gordon equations with non-effective time-dependent mass term. The basic hypothesis for the derivative of the time-dependent coefficient of potential term was

$$\Big|\frac{d}{dt}m(t)^2\Big| \lesssim \frac{1}{(1+t)^3}.$$

This assumption is a sort of "very slow oscillations". We are interested in the behavior of the energy as $t \to \infty$ for the coefficients bearing "very fast oscillations", in the classification of Reissig and Yagdjian [46] and [47]. Indeed, very fast oscillations are allowed under C^m properties and stabilization condition. Roughly speaking we are interested in the interplay between stabilization and behavior of the derivatives.

Under C^m properties and some stabilization condition the hypothesis for the related symbol-like estimates are thought to be weaker than the ones from Chapter 2. The stabilization allows to use weaker assumptions on derivatives by shrinking the hyperbolic zone, details are given in Section 5.4. We pay for this by using more steps of diagonalization requesting in that way more regularity of the time-dependent coefficient of potential term. The number of steps for diagonalization will be the number m that describes the regularity of the coefficient of potential term. Performing m steps of diagonalization we guarantee that the remainder terms are uniformly integrable over the hyperbolic zone. The basic ideas follow from the consideration made in Chapter 2 and in the papers about C^m theory and stabilization condition [27], [28] and [29].

5.2 Motivation

Consider the following Cauchy problem

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{5.1}$$

with $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $m(t)^2 \in C^m(\mathbb{R})$ and $m(t)^2 > 0$. Like in Chapter 2 we want to have some feeling about the behavior of solutions to (5.1). For a general mass term $m(t)^2$ we may transform the time-dependent potential to a time-dependent damping and a new potential. If we introduce the change of variables given by $u(t,x) = \eta(t)v(t,x)$, then the Cauchy problem (5.1) takes the form

$$v_{tt} - \triangle v + 2\frac{\eta'(t)}{\eta(t)} v_t + \Big(\frac{\eta''(t)}{\eta(t)} + m(t)^2\Big) v = 0, \;\; v(0,x) = \frac{u_0(x)}{\eta(0)}, \;\; v_t(0,x) = v_1(x) \tag{5.2}$$

with $v_1(x) = \frac{u_1(x) - \frac{\eta'(0)}{\eta(0)} u_0(x)}{\eta(0)}$. Therefore if we can take η such that

$$\eta''(t) + m(t)^2 \eta(t) = 0, \tag{5.3}$$

then we may apply the results of [29]. Indeed, we shall solve the previous ordinary differential equation with time-dependent coefficient of the non-effective potential term (see Definition 5.1). We will give more details in the next section.

5.3 Models with structural properties

First we will prove statements for a general non-effective mass term and later we shall consider a special structure for the time-dependent coefficient $m(t)^2$.

Definition 5.1. *(Non-effective mass) Consider the Cauchy problem* (5.1). *We say that $m(t)^2 u$ is non-effective if the time-dependent coefficient $m(t)^2$ satisfies:*

$$\limsup_{t\to\infty} (1+t) \int_t^\infty m(s)^2 \, ds < \frac{1}{4} \tag{5.4}$$

and if the derivatives satisfy the following estimates:

$$\Big| \frac{d^k}{dt^k} m(t)^2 \Big| \lesssim (1+t)^{-(k+2)\gamma},$$

for some $0 < \gamma \leq 1$, $k = 1, 2$ in the case $\gamma = 1$ and $k = 1, 2, \cdots, m$ otherwise.

A solution of equation (5.3) can be represented formally by

$$\eta = \eta(t) = \exp\left(\sum_{j=1}^{\infty} \int_0^t \int_\tau^\infty q_j(s) ds d\tau\right), \tag{5.5}$$

where

$$q_k = q_k(t) = \begin{cases} m(t)^2 \quad \text{if } k = 1 \\ \sum_{j=1}^{k-1} \Big(\int_t^\infty q_j(s) ds\Big)\Big(\int_t^\infty q_{k-j}(s) ds\Big) & \text{if } k = 2, 3, 4, \cdots . \end{cases} \tag{5.6}$$

Note that $q_k \geq 0$ for $k = 1, 2, \cdots$. If the coefficient $m(t)^2$ satisfies the non-effective condition, then the series converges. Actually we have the following proposition:

Proposition 5.1. *We define the sequence* $\{\eta_N(t)\}_{N=1}^\infty$ *by*

$$\eta_N(t) := \exp\Big(\sum_{j=1}^{N} \int_0^t \int_\tau^\infty q_j(s)\, ds\, d\tau\Big). \tag{5.7}$$

If $m(t)^2 u$ *is non-effective, then* $\{\eta_N(t)\}_{N=1}^\infty$ *is a uniformly converging sequence on* $[0, \infty)$ *and* $\lim_{N\to\infty} \eta_N(t) =: \eta(t) \in C^{m+2}([0, \infty))$. *Moreover, we have*

$$\frac{\eta''(t)}{\eta(t)} + m(t)^2 = 0. \tag{5.8}$$

Proof. By the definition of $\eta_N(t)$ we can verify that η solves the equation (5.8) if $\{\eta_N(t)\}_{N=1}^\infty$ converges uniformly. Indeed,

$$\frac{\eta''(t)}{\eta(t)} + m(t)^2 = \left(\sum_{j=1}^{\infty} \int_t^\infty q_j(s) ds\right)^2 - \sum_{j=2}^{\infty} q_j(t).$$

The Cauchy product formula

$$\Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{m=0}^{\infty} b_m\Big) = \sum_{j=0}^{\infty} c_j, \quad \text{where} \quad c_j = \sum_{k=0}^{j} a_k b_{j-k}$$

implies that

$$\Big(\sum_{j=1}^{\infty} \int_t^\infty q_j(s) ds\Big)^2 = \sum_{j=2}^{\infty} q_j(t).$$

Therefore,

$$\frac{\eta''(t)}{\eta(t)} + m(t)^2 = 0.$$

We shall investigate the uniform convergence of $\{\eta_N(t)\}_N$. Let us denote the k-th Catalan number by $\gamma_k = \frac{(2k)!}{(k+1)!k!}$. Then we have

$$\gamma_0 = 1, \quad \gamma_{k+1} = \sum_{j=0}^{k} \gamma_j \gamma_{k-j}, \tag{5.9}$$

and

$$\lim_{k\to\infty}\frac{\widetilde{\mu}^{k+1}\gamma_k}{\widetilde{\mu}^k\gamma_{k-1}}=4\widetilde{\mu} \tag{5.10}$$

for any $\widetilde{\mu}>0$. It follows that the series $\sum_{k=0}^{\infty}\widetilde{\mu}^k\gamma_{k-1}$ converges since $\widetilde{\mu}<1/4$. By (5.4) there exists a constant $\widetilde{\mu}\in(0,1/4)$ such that

$$0<\int_t^{\infty}q_1(s)\,ds=\int_t^{\infty}m(s)^2\,ds\le\widetilde{\mu}(1+t)^{-1}=\widetilde{\mu}\gamma_0(1+t)^{-1} \tag{5.11}$$

for $t\gg 1$. From now on we suppose that t is large enough, so (5.11) is valid. Therefore, we have

$$q_2(t)=\Big(\int_t^{\infty}q_1(s)\,ds\Big)^2\le\widetilde{\mu}^2\gamma_0^2(1+t)^{-2}=\widetilde{\mu}^2\gamma_1(1+t)^{-2}.$$

Here we assume that the following estimates are established:

$$q_k(t)\le\widetilde{\mu}^k\gamma_{k-1}(1+t)^{-2} \tag{5.12}$$

for $k=2,\cdots,j$. It follows that

$$\int_t^{\infty}q_k(s)\,ds\le\widetilde{\mu}^k\gamma_{k-1}(1+t)^{-1}. \tag{5.13}$$

Then we have

$$\begin{aligned}q_{j+1}(t)=&\sum_{k=1}^{j}\Big(\int_t^{\infty}q_k(s)\,ds\Big)\Big(\int_t^{\infty}q_{j-k+1}(s)\,ds\Big)\\ \le&\sum_{k=1}^{j}\big(\widetilde{\mu}^k\gamma_{k-1}(1+t)^{-1}\big)\big(\widetilde{\mu}^{j-k+1}\gamma_{j-k}(1+t)^{-1}\big)=\widetilde{\mu}^{j+1}\sum_{k=1}^{j}\gamma_{k-1}\gamma_{j-k}(1+t)^{-2}\\ =&\widetilde{\mu}^{j+1}\gamma_j(1+t)^{-2}.\end{aligned}$$

Thus (5.12) and (5.13) are valid for any $k\ge 2$. Consequently, the sequence $\{\sum_{j=1}^{N}q_j(t)\}_{N=1}^{\infty}$ converges uniformly in $C^0([0,\infty))$, and thus

$$\lim_{N\to\infty}\sum_{j=1}^{N}\int_0^t\int_\tau^{\infty}q_j(s)\,d\tau\,ds\in C^2([0,\infty)). \tag{5.14}$$

This implies

$$\eta(t)\in C^2([0,\infty)). \tag{5.15}$$

Recalling that the estimate (5.12) is valid, thus we have

$$q_k(t)\le\widetilde{\mu}^k\gamma_{k-1}(1+t)^{-2\beta}, \tag{5.16}$$

for $0<\beta\le 1$. Let us assume that for $\ell=1,\cdots,j$ there exist positive constants C_l such that the following estimates are established:

$$\Big|q_k^{(\ell)}(t)\Big|\le C_\ell\widetilde{\mu}^k\gamma_{k-1}(1+t)^{-(\ell+2)\beta}. \tag{5.17}$$

Then, noting the equalities

$$\begin{aligned}
q_k^{(j+1)}(t) =& \sum_{r=1}^{k-1}\sum_{h=0}^{j+1}\binom{j+1}{h}\left(\frac{d^h}{dt^h}\int_t^\infty q_r(s)\,ds\right)\left(\frac{d^{j-h+1}}{dt^{j-h+1}}\int_t^\infty q_{k-r}(s)\,ds\right)\\
=& -\sum_{r=1}^{k-1}\left(\int_t^\infty q_r(s)\,ds\right)q_{k-r}^{(j)}(t) - \sum_{r=1}^{k-1} q_r^{(j)}(t)\left(\int_t^\infty q_{k-r}(s)\,ds\right)\\
&+\sum_{r=1}^{k-1}\sum_{h=1}^{j}\binom{j+1}{h}q_r^{(h-1)}(t)q_{k-r}^{(j-h)}(t),
\end{aligned}$$

the estimates (5.12) and (5.13), we have

$$\begin{aligned}
\left|q_k^{(j+1)}(t)\right| \le& \sum_{r=1}^{k-1}\left(\widetilde{\mu}^r\gamma_{r-1}(1+t)^{-1}\right)\left(C_j\widetilde{\mu}^{k-r}\gamma_{k-r-1}(1+t)^{-(j+2)\beta}\right)\\
&+\sum_{r=1}^{k-1}\left(C_j\widetilde{\mu}^r\gamma_{r-1}(1+t)^{-(j+2)\beta}\right)\left(\widetilde{\mu}^{k-r}\gamma_{k-r-1}(1+t)^{-1}\right)\\
&+\sum_{r=1}^{k-1}\sum_{h=1}^{j-1}\binom{j+1}{h}\left(C_{h-1}\widetilde{\mu}^r\gamma_{r-1}(1+t)^{-(h+1)\beta}\right)\left(C_{j-h}\widetilde{\mu}^{k-r}\gamma_{k-r-1}(1+t)^{-(j-h+2)\beta}\right)\\
=&2C_j\widetilde{\mu}^k\sum_{r=1}^{k-1}\gamma_{r-1}\gamma_{k-r-1}(1+t)^{-1-(j+2)\beta}+\sum_{h=1}^{j-1}\binom{j+1}{h}C_{h-1}C_{j-h}\widetilde{\mu}^k\\
&\times\sum_{r=1}^{k-1}\gamma_{r-1}\gamma_{k-r-1}(1+t)^{-(j+3)\beta}\\
=&\widetilde{\mu}^k\gamma_{k-1}\left(2C_j(1+t)^{-1+\beta}+\sum_{h=1}^{j-1}\binom{j+1}{h}C_{h-1}C_{j-h}\right)(1+t)^{-(j+3)\beta}\\
\le& C_{j+1}\widetilde{\mu}^k\gamma_{k-1}(1+t)^{-((j+1)+2)\beta},
\end{aligned}$$

where $C_{j+1} = 2C_j + \sum_{h=1}^{j-1}\binom{j+1}{h}C_{h-1}C_{j-h}$. Consequently, (5.17) is valid for any $\ell = 0,\cdots,m$. Moreover, we may conclude by (5.5) that $\eta(t)\in C^{m+2}([0,\infty))$. □

Now we shall investigate the Cauchy problem

$$v_{tt} - \Delta v + 2\frac{\eta'(t)}{\eta(t)}v_t = 0$$

with the time-dependent coefficient $2\frac{\eta'(t)}{\eta(t)}$ in the dissipative term. This coefficient satisfies the following estimates:

Lemma 5.1. *Assume that $m(t)^2$ satisfies the non-effective condition. Let us define $b_1(t)$ by*

$$b_1(t) := \frac{\eta'(t)}{\eta(t)} = \sum_{j=1}^\infty\int_t^\infty q_j(s)\,ds. \tag{5.18}$$

Then we have the following estimates

$$\left|b_1^{(k)}(t)\right| \lesssim \begin{cases}(1+t)^{-1}, & k=0,\\ (1+t)^{-(k+1)\beta}, & k=1,\cdots,m.\end{cases} \tag{5.19}$$

Proof. The inequalities (5.13) and (5.17) imply this lemma. In fact, for $k=0$ we have

$$\begin{aligned} |b_1(t)| &= \frac{\eta'(t)}{\eta(t)} = \sum_{j=1}^{\infty}\int_t^{\infty} q_j(s)ds \\ &\le \sum_{j=1}^{\infty}\widetilde{\mu}^j\gamma_{j-1}(1+t)^{-1} \le C(1+t)^{-1} \end{aligned}$$

once that $\widetilde{\mu}<\frac{1}{4}$. For $k=1,\cdots,m$ we have

$$\begin{aligned} |b_1^{(k)}(t)| &= \Big|\sum_{j=1}^{\infty}\frac{d^k}{dt^k}\int_t^{\infty} q_j(t)\Big| \\ &= \sum_{j=1}^{\infty} q_j^{(k-1)}(t) \lesssim \sum_{j=1}^{\infty}\widetilde{\mu}^j\gamma_{j-1}(1+t)^{-(k+1)\beta} \le C(1+t)^{-(k+1)\beta}. \end{aligned}$$

So, the desired inequalities (5.19) are proved. □

If the potential term is non-effective, then $2\frac{\eta'(t)}{\eta(t)}u_t$ is a non-effective dissipation, this means that

$$\limsup_{t\to\infty} 2(1+t)\frac{\eta'(t)}{\eta(t)} < 1.$$

Indeed we have the following result:

Proposition 5.2. *Assume that $m(t)^2u$ is a non-effective potential. Then*

$$\limsup_{t\to\infty} 2(1+t)\frac{\eta'(t)}{\eta(t)} < 1.$$

Proof. If we consider the $k-$th Catalan number γ_k, then recalling (5.13) there exists a constant $\widetilde{\mu}<\frac{1}{4}$ such that

$$(1+t)\int_t^{\infty} q_k(s)ds \le \widetilde{\mu}^k\gamma_{k-1}$$

for $k=1,2,\cdots$. Therefore

$$2(1+t)\frac{\eta'(t)}{\eta(t)} = 2(1+t)\sum_{j=1}^{\infty}\int_t^{\infty} q_j(s)ds \le 2\sum_{j=1}^{\infty}\widetilde{\mu}^j\gamma_{j-1}. \tag{5.20}$$

Let us denote $\sigma=\sum_{j=1}^{\infty}\widetilde{\mu}^j\gamma_{j-1}$. On the one hand the Cauchy product formula implies that

$$\sigma^2 = \left(\sum_{j=1}^{\infty}\widetilde{\mu}^j\gamma_{j-1}\right)^2 = \sum_{j=2}^{\infty}\widetilde{\mu}^j\gamma_{j-1} = \sigma-\widetilde{\mu},$$

therefore

$$\sigma_{\pm} = \frac{1}{2}\pm\frac{1}{2}\sqrt{1-4\widetilde{\mu}}.$$

On the other hand $\widetilde{\mu} < \frac{1}{4}$. Thus

$$\sigma \leq \sum_{j=1}^{\infty} 4^{-j}\gamma_{j-1} = \frac{1}{2}.$$

We can conclude that $\sigma = \sigma_- < \frac{1}{2}$. From (5.20) follows that

$$\limsup_{t\to\infty} 2(1+t)\frac{\eta'(t)}{\eta(t)} \leq 2\sigma < 1,$$

what we wanted to prove. □

From now on we will consider a special structure of the coefficient $m(t)^2$, namely,

$$m(t)^2 = \frac{\mu^2}{(1+t)^2 g(t)^2} + \delta(t), \quad \mu^2 \neq 0, \tag{5.21}$$

where $\frac{\mu^2}{(1+t)^2 g(t)^2}$ is the shape function (see Chapter 2) and $\delta = \delta(t)$ is a bounded oscillating function with

$$|\delta(t)| \lesssim (1+t)^{-2}. \tag{5.22}$$

Remark 5.1. *The goal is to deal with $m(t)^2$ bearing very fast oscillations. The following time-dependent potential term is a perturbation of the scale-invariant potential and it is an example of a mass term $m(t)^2 u$ that we want to deal within this chapter:*

$$m(t)^2 = \frac{\mu^2 + \mu^2 \sin(t^\sigma)}{(1+t)^2}$$

with $\sigma \in (0,1)$. Note that this example shows us that the choice of μ^2 in the non-effective potential will be smaller than in the scale-invariant case [5]. Indeed, when $\sin(t^\sigma) = 1$ we expect that $\mu^2 < \frac{1}{8}$.

We shall enforce hypothesis for δ such that the oscillations have no contributions to the energy estimates. For our purposes let us take g and δ satisfying, for a non-negative integer m, the following hypothesis:

Hypothesis 5.1. *Let $g = g(t) \in C^m([0,\infty))$ be a positive, non-decreasing function with $g(0) = 1$ and*

$$\left|g^{(k)}(t)\right| \leq c_k g(t)(1+t)^{-k},$$

where c_k is a constant depending on k.

Hypothesis 5.2. *For a real number β from the interval $(1/(m+1), 1)$ and $\delta \in C^m([0,\infty))$ we suppose the following estimates:*

$$\left|\delta^{(k)}(t)\right| \lesssim (1+t)^{-(k+2)\beta} \text{ for } k = 1, \cdots, m. \tag{5.23}$$

Here the parameter β describes the asymptotic behavior of derivatives of $m(t)^2$ for large t.

Hypothesis 5.3. *(Stabilization condition) There exists a constant $\alpha \in [0,1)$ such that the perturbation function δ satisfies*

$$\Big|\int_t^\infty \delta(s)ds\Big| \le \nu(1+t)^{\alpha-2} \tag{5.24}$$

with $\nu \le \mu^2$ and $\beta = \alpha + \frac{1-\alpha}{m+1}$.

Remark 5.2. *The condition for the function g implies the following estimates for the derivatives of the shape function:*

$$\left|\frac{d^k}{dt^k}\left(\frac{\mu^2}{(1+t)^2 g(t)^2}\right)\right| \lesssim (1+t)^{-(k+2)} \lesssim (1+t)^{-(k+2)\beta} \text{ for } k = 1, \cdots, m. \tag{5.25}$$

Therefore the derivatives of the shape function satisfy better estimates than the perturbation function $\delta(t)$.

Remark 5.3. *The exponent $\alpha - 2$ inspires us to call Hypothesis 5.3 stabilization condition because it is a more restrictive assumption for the potential term. Indeed, for $\beta > \frac{2}{3}$ the Hypothesis 5.2 implies that*

$$\Big|\int_t^\infty \Big(\int_s^\infty \delta'(\tau)d\tau\Big)ds\Big| \lesssim \int_t^\infty \Big(\int_s^\infty (1+\tau)^{-3\beta}d\tau\Big)ds \approx (1+t)^{2-3\beta},$$

which is, in general, a worse decay than $(1+t)^{\alpha-2}$ because the condition $\beta = \alpha + \frac{1-\alpha}{m+1}$ implies $\alpha - 2 < 2 - 3\beta$, since $\beta < 1$. For $\beta \le \frac{2}{3}$, we have from (5.22)

$$\Big|\int_t^\infty \delta(s)ds\Big| \le (1+t)^{-1},$$

which is a worse decay than $(1+t)^{\alpha-2}$ because of $\alpha - 2 = \beta + \frac{\beta-1}{m} - 2 \le -\frac{4}{3}$.

Remark 5.4. *We do not suppose a sign condition for $\delta(t)$ although we require $m(t)^2 > 0$.*

Remark 5.5. *The mass term $m(t)^2 u$ satisfies under Hypotheses 5.1 and 5.2 the assumptions we asked so far if $\mu^2 < \frac{1}{4}$. Then from now on we will use for Klein-Gordon models with potential $m(t)^2 u$ all the results we have proved for Klein-Gordon models with a general non-effective potential term $M(t)u$.*

5.4 Representation of solutions

We perform the partial Fourier transformation with respect to x in (**??**) . If we denote by $\widehat{u}(t,\xi)$ the partial Fourier transform $F_{x\to\xi}(u)(t,\xi)$, then we obtain

$$\widehat{u}_{tt} + |\xi|^2\widehat{u} + m(t)^2\widehat{u} = 0, \;\; \widehat{u}(0,\xi) = \widehat{u}_0(\xi), \;\; \widehat{u}_t(0,\xi) = \widehat{u}_1(\xi). \tag{5.26}$$

We divide the extended phase space $[0,\infty) \times \mathbb{R}^n$ into three zones: the *pseudo-differential zone* $Z_{pd}(N)$, the *hyperbolic zone* $Z_{H,m}(N)$ for $m = 1, 2, \cdots$ and the *intermediate zone* $Z_I(N)$. The zones $Z_{pd}(N)$ and $Z_{H,m}(N)$ are defined by

$$\begin{aligned} Z_{pd}(N) &= \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : (1+t)|\xi| \le N\}, \\ Z_{H,m}(N) &= \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : |\xi|(1+t)^\alpha \ge N\}. \end{aligned}$$

Here we note that $\alpha = \beta - \frac{1-\beta}{m} < 1$ since $m \geq 1$ and $\beta < 1$. If we consider the zone $Z_{hyp}(N)$ (hyperbolic zone as defined in Chapter 2) it follows that $Z_{hyp}(N) \supset Z_{H,m}(N)$ for $m \geq 1$. The gap between the zones $Z_{hyp}(N)$ and $Z_{H,m}(N)$ we define as intermediate zone, i.e.,

$$\begin{aligned} Z_I(N) &= Z_{hyp}(N) \setminus Z_{H,m}(N) \\ &= \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : (1+t)^\alpha |\xi| \leq N \leq (1+t)|\xi|\}. \end{aligned}$$

The *separating curve* between the pseudo-differential zone and the intermediate zone is given by

$$\theta^{(1)}_{|\xi|} : (0,N] \to [0,\infty), \quad |\xi|\big(1+\theta^{(1)}_{|\xi|}\big) = N.$$

We put $\theta^{(1)}_0 = \infty$, and $\theta^{(1)}_{|\xi|} = 0$ for any $|\xi| \geq N$. The pair (t,ξ) from the extended phase space belongs to $Z_{pd}(N)$ if and only if $t \leq \theta^{(1)}_{|\xi|}$. The *separating curve* between the intermediate zone and the hyperbolic zone $Z_{H,m}(N)$ is given by

$$\theta^{(2)}_{|\xi|} : (0,N] \to [0,\infty), \quad |\xi|\big(1+\theta^{(2)}_{|\xi|}\big)^\alpha = N.$$

We put $\theta^{(2)}_0 = \infty$, and $\theta^{(2)}_{|\xi|} = 0$ for any $|\xi| \geq N$. The pair (t,ξ) from the extended phase space belongs to $Z_{H,m}(N)$ if and only if $t \geq \theta^{(2)}_{|\xi|}$.

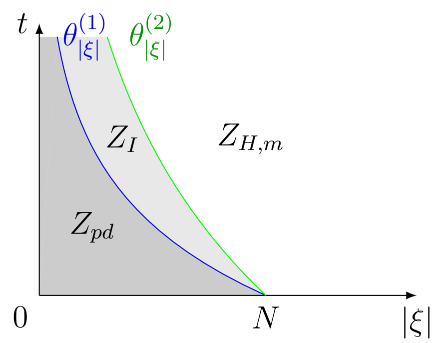

Fig. 5.1: Sketch of the zones.

We define the micro-energy

$$U(t,\xi) = \Big(h(t,\xi)\widehat{u}, \widehat{u}_t - \frac{\eta'(t)}{\eta(t)}\widehat{u}\Big)^T, \tag{5.27}$$

where

$$h(t,\xi) = \frac{1}{1+t}\phi_1(t,\xi) + i|\xi|\phi_2(t,\xi).$$

Here $\phi_1(t,\xi)$ is a characteristic function related to the pseudo-differential zone and $\phi_2(t,\xi)$ is a characteristic function related to the hyperbolic $Z_{hyp}(N)$. We introduce $\phi_2(t,\xi) = \chi\big(\frac{(1+t)|\xi|}{N}\big)$ with $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(t) = 1$ for $t \leq \frac{1}{2}$, $\chi(t) = 0$ for $t \geq 2$ and $\chi'(t) \leq 0$ together with $\phi_1(t,\xi) + \phi_2(t,\xi) = 1$.

5.4.1 Considerations in the pseudo-differential zone

In the pseudo-differential zone we estimate solutions by brute force reformulating the system related to the Cauchy problem (??) as a system of integral equations. The ansatz is similar as in the paper [19]. In the pseudo-differential zone $Z_{pd}(N)$ the micro-energy (5.27) reduces to

$$U=\Big(\frac{\widehat{u}}{1+t},\widehat{u}_t-\frac{\eta'(t)}{\eta(t)}\widehat{u}\Big)^T,\;\; U_0(\xi)=\Big(\widehat{u}_0(\xi),\widehat{u}_1(\xi)-\frac{\eta'(0)}{\eta(0)}\widehat{u}_0(\xi)\Big)^T,\;\text{ and }\; U=\eta(t)\widetilde{U}.$$

So we have

$$\partial_t\widetilde{U}(t,\xi)=\mathcal{A}(t,\xi)\widetilde{U}:=\begin{pmatrix}-\frac{1}{1+t} & \frac{1}{1+t}\\ -(1+t)|\xi|^2 & -2\frac{\eta'(t)}{\eta(t)}\end{pmatrix}\widetilde{U}. \tag{5.28}$$

Let us consider the fundamental solution $E=E(t,s,\xi)$ to (5.28), that is, the solution to

$$\partial_t E=\mathcal{A}(t,\xi)E\,,\quad E(s,s,\xi)=I.$$

Lemma 5.2. *Assume Hypotheses 5.1 and 5.2. The fundamental solution $E(t,0,\xi)$ satisfies the estimate*

$$\|E(t,0,\xi)\|\lesssim\eta(t)^{-2},$$

for all $(t,\xi)\in Z_{pd}(N)$.

Proof. If we put $E=(E_{ij})_{i,j=1,2}$, then we can write for $j=1,2$, the following system of coupled integral equations of Volterra type:

$$E_{1j}(t,0,\xi) = (1+t)^{-1}\Big(\delta_{1j}+\int_0^t E_{2j}(\tau,0,\xi)d\tau\Big), \tag{5.29}$$

$$E_{2j}(t,0,\xi) = \eta(t)^{-2}\Big(\delta_{2j}-\int_0^t(1+\tau)\eta(\tau)^2|\xi|^2E_{1j}(\tau,0,\xi)d\tau\Big). \tag{5.30}$$

By replacing (5.30) into (5.29) and after integration by parts we get

$$\begin{aligned}E_{1j}(t,0,\xi)&=(1+t)^{-1}\Big(\delta_{1j}+\delta_{2j}\int_0^t\eta(\tau)^{-2}d\tau\Big)-(1+t)^{-1}\\ &\quad\times\int_0^t(1+\tau)\eta(\tau)^2|\xi|^2E_{1j}(\tau,0,\xi)\int_\tau^t\eta(s)^{-2}dsd\tau.\end{aligned} \tag{5.31}$$

From Proposition 5.2 together with Proposition 7 of [59] we have

$$\int_0^t\eta(s)^{-2}ds\approx\frac{t}{\eta(t)^2}, \tag{5.32}$$

and $\frac{t}{\eta(t)^2}$ is increasing for large t. Introducing

$$h_j(t,\xi):=\|E_{1j}(t,0,\xi)\|\eta(t)^2$$

and by using $\eta(t)^2\le 1+t$ (see Proposition 5.2) for large t we conclude from (5.31) and (5.32) that

$$h_j(t,\xi)\le C+C\int_0^t(1+\tau)|\xi|^2h_j(\tau,\xi)d\tau.$$

Applying Gronwall's type inequality we conclude

$$h_j(t,\xi) \le C \exp\Big(C\int_0^t (1+\tau)|\xi|^2 d\tau\Big).$$

In $Z_{pd}(N)$ we have $(1+t)|\xi| \le C$. So, from the last estimate we get

$$h_j(t,\xi) \lesssim 1.$$

Therefore we get $\|E_{1j}(t,0,\xi)\| \lesssim \eta(t)^{-2}$. From the boundedness of $\|E_{1j}(t,0,\xi)\|\eta(t)^2$ we can estimate $\|E_{2j}(t,0,\xi)\| \lesssim \eta(t)^{-2}$. Summarizing we proved $\|E(t,0,\xi)\| \lesssim \eta(t)^{-2}$ for all $t \in [0,\theta_{|\xi|}]$. □

This lemma implies

$$\|U(t,\xi)\| \le C\eta(t)^{-1}\|U_0(\xi)\| \text{ for all } t \in (0,\theta_{|\xi|}]. \tag{5.33}$$

5.4.2 Considerations in the hyperbolic zone

In the hyperbolic zone $Z_{H,m}(N)$ we follow basically the approach of [45], in particular, the diagonalization procedure. However, to cope with the stronger oscillating behaviour of $b_1(t)$ we need in our approach more diagonalization steps and we shall restrict the considerations to a smaller hyperbolic zone in the phase space. The basic ideas are taken from [28] and [29].

First of all let us introduce the symbol class $S_N^\ell\{m_1,m_2\}$ in the zone $Z_{H,m}(N)$.

Definition 5.2. *The time-dependent amplitude function $a = a(t,\xi)$ belongs to the symbol class $S_N^\ell\{m_1,m_2\}$ with restricted smoothness ℓ if it satisfies the symbol-like estimates*

$$\big|D_t^k D_\xi^\alpha a(t,\xi)\big| \le C_{K,\alpha}|\xi|^{m_1-|\alpha|}\Big(\frac{1}{1+t}\Big)^{(m_2+k)\beta} \tag{5.34}$$

for all $(t,\xi) \in Z_{H,m}(N)$, all non-negative integers $k \le \ell$ and all multi-indices $\alpha \in \mathbb{N}^n$.

We will denote by $S_N\{m_1,m_2\}$ the symbol class when $\ell = \infty$, that is, $S_N\{m_1,m_2\} = S_N^\infty\{m_1,m_2\}$.

The rules of the symbolic calculus are collected in the following proposition.

Proposition 5.3. (1) $S_N^\ell\{m_1,m_2\}$ *is a vector space for all non-negative integers ℓ.*
(2) $S_N^{\ell'}\{m_1-k,m_2+\ell\} \subset S_N^{\ell_1}\{m_1,m_2\}$ *for all* $\ell \ge k \ge 0, \ell' \ge \ell_1$.
(3) $S_N^\ell\{m_1,m_2\}\cdot S_N^{\ell'}\{m_1',m_2'\} \subset S_N^{\widetilde{\ell}}\{m_1+m_1',m_2+m_2'\}$ *for all non-negative integers ℓ and ℓ' with $\widetilde{\ell} = \min\{\ell,\ell'\}$.*
(4) $D_t^k D_\xi^\alpha S_N^\ell\{m_1,m_2\} \subset S_N^{\ell-k}\{m_1-|\alpha|,m_2+k\}$ *for all non-negative integers ℓ with $k \le \ell$.*
(5) $S_N^\ell\{-1,2\} \subset L_\xi^\infty L_t^1(Z_{H,m})$ *for all non-negative integers ℓ.*

In the zone $Z_{H,m}(N)$ the micro-energy (5.27) reduces to

$$U = \Big(i|\xi|\widehat{u}, \widehat{u}_t - \frac{\eta'(t)}{\eta(t)}\widehat{u}\Big)^T, \quad U_0(\xi) = \Big(i|\xi|\widehat{u}(\theta_{|\xi|}^{(2)},\xi), \widehat{u}_t(\theta_{|\xi|}^{(2)},\xi) - \frac{\eta'(\theta_{|\xi|}^{(2)})}{\eta(\theta_{|\xi|}^{(2)})}\widehat{u}(\theta_{|\xi|}^{(2)},\xi)\Big)^T,$$

and $U=\eta(t)\widetilde{U}$, so that

$$\partial_t\widetilde{U}=\begin{pmatrix}0&1\\1&0\end{pmatrix}i|\xi|\widetilde{U}+\begin{pmatrix}0&0\\0&-2\frac{\eta'(t)}{\eta(t)}\end{pmatrix}\widetilde{U} \tag{5.35}$$

for $t\geq\theta^{(2)}_{|\xi|}$ with initial datum $\widetilde{U}(\theta^{(2)}_{|\xi|},\xi)=\eta(\theta^{(2)}_{|\xi|})^{-1}U_0(\xi)$.

Let M be the diagonalizer of the principal part (with respect to powers of $|\xi|$) of (5.35) given by

$$M=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix},\qquad M^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}.$$

If we put $V^{(0)}(t,\xi):=M^{-1}\widetilde{U}(t,\xi)$, then we get

$$D_tV^{(0)}=\big(\mathcal{D}_0(t,\xi)+R_0(t,\xi)\big)V^{(0)} \tag{5.36}$$

with

$$\mathcal{D}_0(t,\xi)=\begin{pmatrix}-|\xi|+i\frac{\eta'(t)}{\eta(t)}&0\\0&|\xi|+i\frac{\eta'(t)}{\eta(t)}\end{pmatrix},\ R_0(t,\xi)=i\frac{\eta'(t)}{\eta(t)}\begin{pmatrix}0&-1\\-1&0\end{pmatrix}.$$

Note that $R_0\in S^m_N\{0,1\}$ and $\frac{\eta'(t)}{\eta(t)}\in C^{m+1}([0,\infty))$. Now we apply an iterative diagonalization procedure.

Lemma 5.3. *Let us assume the Hypotheses 5.1 and 5.2. There exists a zone constant $N>0$ such that for any $k=0,1,\cdots,m$ there exist matrices with the following properties:*

- *the matrices $N_k=N_k(t,\xi)\in S^{m-k}_N\{0,0\}$ are invertible and $N_k^{-1}\in S^{m-k}_N\{0,0\}$. Furthermore, the matrices tend to the identity as $t\to\infty$ for all fixed $\xi\neq0$;*
- *the matrices $R_k=R_k(t,\xi)\in S^{m-k}_N\{-k,k+1\}$ are antidiagonal;*
- *the matrices $\mathcal{D}_k=\mathcal{D}_k(t,\xi)\in S^{m-k}_N\{1,0\}$ are diagonal and*

  $$\mathcal{D}_k(t,\xi)=\operatorname{diag}(\tau_k^+(t,\xi),\tau_k^-(t,\xi))$$

  with $|\tau_k^+(t,\xi)-\tau_k^-(t,\xi)|\geq C_k|\xi|$;

all these matrices are defined in $Z_{H,m}$ such that the operator identity

$$\big(D_t-\mathcal{D}_k-R_k\big)N_k=N_k\big(D_t-\mathcal{D}_{k+1}-R_{k+1}\big) \tag{5.37}$$

is valid for $k=0,1,\cdots,m-1$.

Proof. The proof goes by direct construction. Let us denote the difference of the diagonal entries by

$$\delta_k(t,\xi)=\tau_k^+(t,\xi)-\tau_k^-(t,\xi).$$

Assume that we have given a system $D_t V^{(k)} = (\mathcal{D}_k(t,\xi) + R_k(t,\xi))\, V^{(k)}$ with $\mathcal{D}_k(t,\xi) = \operatorname{diag}(\tau_k^+(t,\xi), \tau_k^-(t,\xi)) \in S_N^{m-k}\{1,0\}$ satisfying $|\delta_k(t,\xi)| = |\tau_k^+(t,\xi) - \tau_k^-(t,\xi)| \geq C_k|\xi|$ and antiagonal remainder $R_k(t,\xi) \in S_N^{m-k}\{-k, k+1\}$. Set

$$N_k(t,\xi) = I + \begin{pmatrix} 0 & -\frac{(R_k)_{12}}{\delta_k} \\ \frac{(R_k)_{21}}{\delta_k} & 0 \end{pmatrix} \tag{5.38}$$

such that $[\mathcal{D}_k, N_k] + R_k = 0$ and, therefore,

$$\begin{aligned} B^{(k+1)} &= (D_t - \mathcal{D}_k - R_k)\, N_k - N_k\,(D_t - \mathcal{D}_k) = D_t N_k - [\mathcal{D}_k, N_k] - R_k N_k \\ &= D_t N_k - R_k\,(N_k - I) \in S_N^{m-k-1}\{-k-1, k+2\}. \end{aligned}$$

The matrix N_k is invertible if we choose the zone constant N sufficiently large. Indeed, we have that $N_k - I \in S_N^{m-k}\{-k-1, k+1\}$. Therefore

$$\|N_k - I\| \lesssim \left(|\xi|(1+t)^{\beta}\right)^{-k-1} \lesssim \left(|\xi|(1+t)^{\beta - \frac{1-\beta}{m}}\right)^{-k-1} \lesssim N^{-k-1} \to 0$$

as $N \to \infty$. Thus by defining

$$\begin{aligned} \mathcal{D}_{k+1}(t,\xi) &= \mathcal{D}_k(t,\xi) - \operatorname{diag}\left(N_k(t,\xi)^{-1} B^{(k+1)}(t,\xi)\right), \\ R_{k+1}(t,\xi) &= \operatorname{diag}\left(N_k(t,\xi)^{-1} B^{(k+1)}(t,\xi)\right) - N_k(t,\xi)^{-1} B^{(k+1)}(t,\xi) \end{aligned}$$

we obtain the operator equation

$$\big(D_t - \mathcal{D}_k - R_k\big) N_k = N_k \big(D_t - \mathcal{D}_{k+1} - R_{k+1}\big)$$

with $\mathcal{D}_{k+1}(t,\xi) \in S_N^{m-k-1}\{1,0\}$, $R_{k+1}(t,\xi) \in S_N^{m-k-1}\{-k-1, k+2\}$. The estimate for $B^{(k+1)}$ implies that

$$|\tau_{k+1}^+(t,\xi) - \tau_{k+1}^-(t,\xi)| \geq |\tau_k^+(t,\xi) - \tau_k^-(t,\xi)| - |\xi|\frac{C}{N}.$$

If we choose N large enough, then the statement is proved with $C_{k+1} := C_k - \frac{C}{N}$. □

Finally, we obtain for $k = m$ that the remainder $R_m(t,\xi) \in S_N^0\{-m, m+1\}$ is uniformly integrable over the hyperbolic zone,

$$\int_{\theta_{|\xi|}^{(2)}}^{\infty} |R_m(s,\xi)|\, ds \lesssim (1 + \theta_{|\xi|}^{(2)})^{1-(m+1)\beta} |\xi|^{-m} = \left((1+\theta_{|\xi|}^{(2)})^{\beta - \frac{1-\beta}{m}} |\xi|\right)^{-m} \approx 1. \tag{5.39}$$

Lemma 5.4. *Assume the Hypotheses 5.1 and 5.2. Then the difference of the diagonal entries of $\mathcal{D}_k$ is real for all $k = 0, 1, \cdots, m-1$.*

Proof. Let us proceed by induction over k following the diagonalization scheme. We will show that the above statement and the following hypothesis

(H_k) R_k has the form $R_k = i \begin{pmatrix} 0 & \overline{\beta_k} \\ \beta_k & 0 \end{pmatrix}$ with complex-valued $\beta_k(t,\xi)$

are valid. For $k = 0$ the assertion (H_0) is satisfied. Suppose that (H_k) is true. The construction gives $N_k(t,\xi) = I + \begin{pmatrix} 0 & -\frac{(R_k)_{12}}{\delta_k} \\ \frac{(R_k)_{21}}{\delta_k} & 0 \end{pmatrix}$ with $\det N_k = 1 - \frac{|\beta_k|^2}{\delta_k^2} \neq 0$ after a suitable choice of the zone constant N. Following the diagonalization scheme of [27, 28, 29] and set $d_k = \frac{|\beta_k|^2}{\delta_k^2}$, then

$$N_k^{-1}\left(\mathcal{D}_k + R_k\right)N_k = \frac{1}{1-d_k}\left(\operatorname{diag}(\tau_k^+ - d_k\tau_k^+ - \delta_k d_k, \tau_k^- - d_k\tau_k^- + \delta_k d_k) + d_k R_k\right)$$

and

$$N_k^{-1}\left(D_t N_k\right) = \frac{1}{1-d_k}\left[\begin{pmatrix} i\frac{\overline{\beta_k}}{\delta_k}\partial_t\frac{\beta_k}{\delta_k} & 0 \\ 0 & i\frac{\beta_k}{\delta_k}\partial_t\frac{\overline{\beta_k}}{\delta_k}\end{pmatrix} + \begin{pmatrix} 0 & -\partial_t\frac{\overline{\beta_k}}{\delta_k} \\ \partial_t\frac{\beta_k}{\delta_k} & 0\end{pmatrix}\right]$$

such that

$$\begin{aligned} Re\left(\frac{\beta_k}{\delta_k}\partial_t\frac{\overline{\beta_k}}{\delta_k}\right) &= \frac{1}{2}\left(\frac{\beta_k}{\delta_k}\partial_t\frac{\overline{\beta_k}}{\delta_k} + \frac{\overline{\beta_k}}{\delta_k}\partial_t\frac{\beta_k}{\delta_k}\right) \\ &= \frac{1}{2}\partial_t d_k = Re\left(\frac{\overline{\beta_k}}{\delta_k}\partial_t\frac{\beta_k}{\delta_k}\right) \end{aligned}$$

implies

$$\tau_{k+1}^{\pm} = \tau_k^{\pm} \mp \frac{1}{1-d_k}\left(d_k\delta_k + Im\left(\frac{\beta_k}{\delta_k}\partial_t\frac{\overline{\beta_k}}{\delta_k}\right)\right) - i\frac{\partial_t d_k}{2(d_k-1)}.$$

Hence δ_{k+1} is real again and R_{k+1} satisfies (H_{k+1}) and, therefore, both statements are true for all $k = 0, 1, \cdots, m-1$. □

Now we want to construct the fundamental solution $\mathcal{E}(t,s,\xi)$ for the system $D_t - D_0 - R_0$. For this purpose it is sufficient to construct the fundamental solution for the diagonalized system $D_t - D_m - R_m$. At first we devote to the diagonal operator $D_t - \mathcal{D}_m(t,\xi)$. Its fundamental solution is given by

$$\exp\left(i\int_s^t \mathcal{D}_m(\theta,\xi)d\theta\right) = \operatorname{diag}\left(e^{i\int_s^t \tau_m^+(\theta,\xi)d\theta}, e^{i\int_s^t \tau_m^-(\theta,\xi)d\theta}\right).$$

Since $\delta_m = \tau_m^+ - \tau_m^-$ is real, it follows that $Im\,\tau_m^+ = Im\,\tau_m^- := \tau_m$ and thus the matrix

$$\exp\left(\int_s^t Im\,\tau_m(\theta,\xi)d\theta\right)\exp\left(i\int_s^t \mathcal{D}_m(\theta,\xi)d\theta\right)$$

is unitary. The integrability of the remainder term $R_m(t,\xi)$ over the hyperbolic zone implies as in Chapter 3 that the fundamental solution of $D_t - \mathcal{D}_m - R_m$ is given by

$$\exp\left(i\int_s^t \mathcal{D}_m(\theta,\xi)d\theta\right)\mathcal{Q}_m(t,s,\xi)$$

with a uniformly bounded and invertible matrix $\mathcal{Q}_m(t,s,\xi)$ that can be represented as Peano-Baker series

$$\mathcal{Q}_m(t,s,\xi) = I + \sum_{k=1}^{\infty}\int_s^t \widetilde{R}_m(t_1,s,\xi)\cdots\widetilde{R}_m(t_k,s,\xi)dt_k\cdots dt_1,$$

where

$$\widetilde{R}_m(t,s,\xi) = \exp\left(-i\int_s^t \mathcal{D}_m(\theta,\xi)d\theta\right)R_m(t,\xi)\exp\left(i\int_s^t \mathcal{D}_m(\theta,\xi)d\theta\right)$$

is an auxiliary function. We obtain the following statement:

Lemma 5.5. *Assume Hypotheses 5.1 and 5.2. The fundamental solution $\mathcal{E}(t,s,\xi)$ is representable in the following form:*

$$\mathcal{E}(t,s,\xi) = M^{-1}\Big(\prod_{k=0}^{m-1} N_k(t,\xi)^{-1}\Big)\exp\Big(i\int_s^t \mathcal{D}_m(\theta,\xi)d\theta\Big)\mathcal{Q}_m(t,s,\xi)\Big(\prod_{k=0}^{m-1} N_k(s,\xi)\Big)M$$

for all $(t,\xi),(s,\xi)\in Z_{H,m}(N)$, where

- *the matrices $N_k = N_k(t,\xi)$ and $N_k^{-1} = N_k(t,\xi)^{-1}$ are uniformly bounded and invertible;*
- *the matrices $\mathcal{Q}_m = \mathcal{Q}_m(t,s,\xi)$ and $\mathcal{Q}_m^{-1} = \mathcal{Q}_m^{-1}(t,s,\xi)$ are uniformly bounded and invertible.*

Proof. The representation as Peano-Baker series implies the uniform estimate

$$\begin{aligned}\|\mathcal{Q}_m(t,s,\xi)\| &\le \exp\Big(\int_s^t \|\widetilde{R}_m(\theta,s,\xi)\|\Big)\\ &\lesssim \exp\Big(\Big(1+\theta^{(2)}_{|\xi|}\Big)^{1-(m+1)\beta}|\xi|^{-m}\Big)\lesssim 1.\end{aligned}$$

Additionally, we know that $\mathcal{Q}_m(t,s,\xi)$ satisfies

$$D_t\mathcal{Q}_m(t,s,\xi) = \widetilde{R}_m(t,s,\xi)\mathcal{Q}_m(t,s,\xi),\quad \mathcal{Q}_m(s,s,\xi) = I.$$

Then after applying Liouville theorem and the invariance of the trace under multiplication we get

$$\det\mathcal{Q}_m(t,s,\xi) = \exp\Big(i\int_s^t tr\widetilde{R}_m(\theta,s,\xi)d\theta\Big) = \exp\Big(i\int_s^t trR_m(\theta,\xi)d\theta\Big) = 1$$

and $\|\mathcal{Q}_m^{-1}(t,s,\xi)\|\lesssim 1.$ □

The asymptotic behavior of the fundamental solution $\mathcal{E}(t,s,\xi)$ is given by the following corollary:

Corollary 5.1. *Assume the Hypotheses 5.1 and 5.2. Then the fundamental solution $\mathcal{E}(t,s,\xi)$ satisfies*

$$\|\mathcal{E}(t,s,\xi)\|\lesssim\frac{\eta(s)}{\eta(t)}$$

uniformly in $(t,\xi),(s,\xi)\in Z_{H,m}(N)$.

Proof. The statement of Lemma 5.5 implies that

$$\|\mathcal{E}(t,s,\xi)\|\lesssim\exp\Big(-\int_s^t Im\,\tau_m(\theta,\xi)d\theta\Big)\ \text{for}\ t\to\infty$$

uniformly in $(t,\xi),(s,\xi)\in Z_{H,m}(N)$. We can use our representation of $\tau_m(t,\xi)$ to deduce

$$Im\,\tau_m(t,\xi) = \frac{\eta'(t)}{\eta(t)} + \sum_{j=1}^{m-1}\frac{\partial_t d_j}{2(d_j-1)}$$

such that

$$\exp\Big(-\int_s^t Im\tau_m(\theta,\xi)d\theta\Big) = \exp\Big(-\int_s^t\frac{\eta'(\theta)}{\eta(\theta)}d\theta\Big)\prod_{j=1}^{m-1}\Big(\frac{d_j(t,\xi)-1}{d_j(s,\xi)-1}\Big)^{-\frac{1}{2}}\lesssim\frac{\eta(s)}{\eta(t)}.$$

This completes the proof. □

5.4.3 Considerations in the intermediate zone

In the intermediate zone of the extended phase space we will use for the first time the special structure of $m(t)^2$ given in (5.21). We relate the fundamental solution $\mathcal{E}(t,s,\xi)$ to the fundamental solution $\mathcal{E}_{sf}(t,s,\xi)$ to the corresponding Klein-Gordon model with non-effective time-dependent potential and without any perturbation, i.e., with $\delta \equiv 0$. We remark that in Chapter 2 we did estimates for the fundamental solution $\mathcal{E}_{sf}(t,s,\xi)$.

The key idea is to rewrite $\frac{\eta'(t)}{\eta(t)}$ as a sum of two functions $\mu(t)$ and $\sigma(t)$, where $\mu(t)$ is a shape function and $\sigma(t)$ is a perturbation function (see [29]). Denote

$$Q_j(t) = \int_t^\infty q_j(s)ds, \quad j = 1, 2, \cdots, \tag{5.40}$$

where q_j is defined in (5.6) and $\gamma_j = \frac{(2j)!}{(j+1)!j!}$ are the Catalan numbers. We have that

$$\frac{\eta'(t)}{\eta(t)} = \sum_{j=1}^\infty Q_j(t). \tag{5.41}$$

Investigating $Q_j(t)$ for each $j = 1, 2, \cdots$ we arrive at the following proposition:

Lemma 5.6. *If the stabilization condition of Hypothesis 5.3 is satisfied and* $\mu^2 < \frac{2-\alpha}{12}$*, then there exist a positive shape function* $\mu = \mu(t)$ *and a perturbation function* $\sigma = \sigma(t)$ *such that*

$$\frac{\eta'(t)}{\eta(t)} = \mu(t) + \sigma(t)$$

with

$$\mu(t) \lesssim (1+t)^{-1} \quad \text{and} \quad |\sigma(t)| \lesssim (1+t)^{\alpha-2} \tag{5.42}$$

for $t \geq 0$ *large.*

Proof. Let us construct the functions μ and σ step by step. For $j = 1$, we have

$$Q_1(t) = \int_t^\infty m(s)^2 ds = \int_t^\infty \frac{\mu^2}{(1+s)^2 g(s)^2} ds + \int_t^\infty \delta(s) ds.$$

Denoting the shape function

$$\mu_1(t) := \int_t^\infty \frac{\mu^2}{(1+s)^2 g(s)^2} ds$$

and the perturbation

$$\sigma_1(t) := \int_t^\infty \delta(s) ds$$

of $Q_1(t)$ after using properties of g and the stabilization condition we arrive at

$$\mu_1(t) \leq \frac{2\gamma_1 \mu^2}{(2-\alpha)(1+t)g(t)^2} \quad \text{and} \quad |\sigma_1(t)| \leq \mu^2 \gamma_1 (1+t)^{\alpha-2}. \tag{5.43}$$

By definition $q_2 = Q_1 Q_1$. So, it follows

$$Q_2(t) = \int_t^\infty \big(\mu_1(s)^2 + 2\mu_1(s)\sigma_1(s) + \sigma_1(s)^2\big)\,ds.$$

Denoting the shape function

$$\mu_2(t) := \int_t^\infty \mu_1(s)^2 ds$$

and the perturbation

$$\sigma_2(t) := \int_t^\infty \big(2\mu_1(s)\sigma_1(s) + \sigma_1(s)^2\big)\,ds$$

of $Q_2(t)$ we get

$$\mu_2(t) \le \frac{4\gamma_2\mu^4}{(2-\alpha)^2(1+t)g(t)^4} \quad \text{and} \quad |\sigma_2(t)| \le \frac{5}{2-\alpha}\mu^4\gamma_2(1+t)^{\alpha-2}.$$

Indeed, recalling $\alpha < 1$ we obtain

$$|\sigma_2(t)| \le \mu^4\gamma_2 \int_t^\infty \big(4(1+s)^{\alpha-3} + (1+s)^{2\alpha-4}\big)\,ds \le \frac{5}{2-\alpha}\mu^4\gamma_2(1+t)^{\alpha-2}.$$

By the representation of q_3 we have

$$q_3(t) = \sum_{k=1}^{2} Q_k(t)Q_{k-3}(t) = \sum_{k=1}^{2}(\mu_k(t)+\sigma_k(t))(\mu_{k-3}(t)+\sigma_{k-3}(t)).$$

The shape function and the perturbation of $Q_3(t)$ will be defined by

$$\mu_3(t) := \sum_{k=1}^{2}\int_t^\infty \mu_k(s)\mu_{3-k}(s)ds,$$

and by

$$\sigma_3(t) := \sum_{k=1}^{2}\int_t^\infty \left(\mu_k(s)\sigma_{3-k}(s) + \mu_{3-k}(s)\sigma_k(s) + \sigma_k(s)\sigma_{3-k}(s)\right)ds.$$

Therefore,

$$\mu_3(t) \le \frac{6\gamma_3\mu^6}{(2-\alpha)^3(1+t)g(t)^6} \quad \text{and} \quad |\sigma_3(t)| \le \frac{19}{(2-\alpha)^2}\mu^6\gamma_3(1+t)^{\alpha-2}.$$

Indeed, the first inequality is trivial to conclude. The second one follows by

$$|\sigma_3(t)| \le \gamma_3\mu^6\left(\frac{14}{(2-\alpha)^3}(1+t)^{\alpha-2} + \frac{5}{(2-\alpha)^2}(1+t)^{2\alpha-3}\right) \le \frac{19\gamma_3\mu^6}{(2-\alpha)^2}(1+t)^{\alpha-2}.$$

By the same way the shape functions $\mu_j(t)$ and the perturbations $\sigma_j(t)$ will be defined by

$$\mu_j(t) := \sum_{k=1}^{j-1}\int_t^\infty \mu_k(s)\mu_{j-k}(s)ds,$$

$$\sigma_j(t) := \sum_{k=1}^{j-1} \int_t^\infty \left(\mu_k(s)\sigma_{j-k}(s) + \mu_{j-k}(s)\sigma_k(s) + \sigma_k(s)\sigma_{j-k}(s)\right) ds.$$

Let us suppose that

$$\mu_k(t) \le \frac{2^k \gamma_k \mu^{2k}}{(2-\alpha)^k (1+t) g(t)^{2k}} \quad \text{and} \quad |\sigma_k(t)| \le \frac{3^k - 2^k}{(2-\alpha)^{k-1}} \mu^{2k} \gamma_k (1+t)^{\alpha-2}, \qquad (5.44)$$

for $k = 1, 2, \cdots, j$. Then we my conclude for μ_{j+1} and for σ_{j+1} the estimates

$$\begin{aligned}
\mu_{j+1}(t) &\le \sum_{k=1}^{j} \int_t^\infty \frac{2^k \gamma_k \mu^{2k}}{(2-\alpha)^k (1+s) g(s)^{2k}} \frac{2^{j+1-k} \gamma_{j+1-k} \mu^{2(j+1-k)}}{(2-\alpha)^{j+1-k} (1+s) g(s)^{2(j+1-k)}} ds \\
&= \frac{2^{j+1}}{(2-\alpha)^{j+1}} \sum_{k=1}^{j} \frac{\gamma_k \gamma_{j+1-k} \mu^{2(j+1)}}{g(t)^{2(j+1)}} \int_t^\infty \frac{ds}{(1+s)^2} \\
&= \frac{2^{j+1}}{(2-\alpha)^{j+1}} \frac{\gamma_{j+1} \mu^{2(j+1)}}{(1+t) g(t)^{2(j+1)}},
\end{aligned}$$

and

$$\begin{aligned}
&|\sigma_{j+1}(t)| \\
&\le \mu^{2j+1} \gamma_{j+1} \frac{2^k \cdot 3^{j+1-k} + 2^{j+1-k} \cdot 3^k - 2 \cdot 2^{j+1} + \left(3^k - 2^k\right)\left(3^{j+1-k} - 2^{j+1-k}\right)}{(2-\alpha)^{j-1}} (1+t)^{\alpha-2} \\
&= \mu^{2j+1} \gamma_{j+1} \frac{3^{j+1} - 2^{j+1}}{(2-\alpha)^{j-1}} (1+t)^{\alpha-2}.
\end{aligned}$$

Consequently, (5.44) is valid for all $j = 1, 2, 3, \cdots$. Therefore, we choose $\mu(t)$ and $\sigma(t)$ as follows:

$$\mu(t) = \sum_{j=1}^\infty \mu_j(t) \quad \text{and} \quad \sigma(t) = \sum_{j=1}^\infty \sigma_j(t). \qquad (5.45)$$

Both functions satisfy the desired estimates. Indeed,

$$\mu(t) \le \sum_{j=1}^\infty \frac{2^j \gamma_j \mu^{2j}}{(2-\alpha)^j} (1+t)^{-1}$$

and taking into consideration

$$\lim_{j\to\infty} \frac{2^{j+1} \gamma_{j+1} \mu^{2j+1}}{(2-\alpha)^{j+1}} \frac{(2-\alpha)^j}{2^j \gamma_j \mu^{2j}} = \frac{4\mu^2}{2-\alpha} < \frac{1}{3} < 1$$

it follows $\mu(t) \lesssim (1+t)^{-1}$. Furthermore,

$$|\sigma(t)| \le \sum_{j=1}^\infty \frac{(3^j - 2^j) \gamma_j \mu^{2j}}{(2-\alpha)^{j-1}} (1+t)^{\alpha-2}$$

and from

$$\lim_{j\to\infty} \frac{(3^{j+1} - 2^{j+1}) \gamma_{j+1} \mu^{2(j+1)}}{(2-\alpha)^j} \frac{(2-\alpha)^{j-1}}{(3^j - 2^j) \gamma_j \mu^{2j}} = \frac{12\mu^2}{(2-\alpha)} < 1,$$

it follows immediately $\sigma(t) \lesssim (1+t)^{\alpha-2}$. The proposition is proved. □

Remark 5.6. *Let us denote $\eta(t) = \eta_{sf}(t)$ when we have no perturbation in the mass term, i.e., $\delta \equiv 0$. We can conclude that $\eta(t) \approx \eta_{sf}(t)$. Indeed, the stabilization condition ensures that*

$$\eta(t) \approx \exp\Big(\int_0^t \frac{\eta'(s)}{\eta(s)}ds\Big) = \exp\Big(\int_0^t (\mu(s)+\sigma(s))ds\Big) \approx \exp\Big(\int_0^t \mu(s)ds\Big) \approx \eta_{sf}(t).$$

The following proposition is important to estimate the fundamental solution in the intermediate zone $Z_I(N)$.

Proposition 5.4. *If $\zeta(t) \in C^1(\mathbb{R})$ and $\zeta(t)$ is bounded, then there exists a constant $C > 0$ such that*

$$\int_0^t |\exp(\zeta(s)) - 1|ds \le C\int_0^t |\zeta(s)|ds$$

for all $t > 0$.

Proof. From the continuity of ζ it follows

$$\begin{aligned}\int_0^t |\exp(\zeta(s)) - 1|ds &\le \int_0^t \sum_{k=1}^{\infty} \frac{|\zeta(s)|^k}{k!}ds \\ &\le \frac{1}{C_0}\sum_{k=1}^{\infty}\frac{C_0^k}{k!}\int_0^t |\zeta(s)|ds \\ &\le C\int_0^t |\zeta(s)|ds,\end{aligned}$$

where $C = \frac{\exp(C_0)}{C_0}$. The statement is proved. □

Now we are able to describe the asymptotic behavior of the fundamental solution $\mathcal{E}(t,s,\xi)$. In the intermediate zone $Z_I(N)$ the micro-energy (5.27) reduces to

$$U = \Big(i|\xi|\widehat{u}, \widehat{u}_t - \frac{\eta'(t)}{\eta(t)}\widehat{u}\Big)^T, \quad U_0(\xi) = \Big(i|\xi|\widehat{u}(\theta^{(1)}_{|\xi|},\xi), \widehat{u}_t(\theta^{(1)}_{|\xi|},\xi) - \frac{\eta'(\theta^{(1)}_{|\xi|})}{\eta(\theta^{(1)}_{|\xi|})}\widehat{u}(\theta^{(1)}_{|\xi|},\xi)\Big)^T,$$

and $U = \eta(t)\widetilde{U}$, so that

$$D_t\widetilde{U} = \begin{pmatrix} 0 & |\xi| \\ |\xi| & 2i\left(\mu(t)+\sigma(t)\right)\end{pmatrix}\widetilde{U} = A(t,\xi)\widetilde{U} \tag{5.46}$$

for $\theta^{(1)}_{|\xi|} \le t \le \theta^{(2)}_{|\xi|}$. Let us consider the fundamental solution $\mathcal{E} = \mathcal{E}(t,s,\xi)$ to (5.46), i.e., the solution of

$$D_t\mathcal{E}(t,s,\xi) = A(t,\xi)\mathcal{E}(t,s,\xi) \quad \text{and} \quad \mathcal{E}(s,s,\xi) = I.$$

If $\delta \equiv 0$, then $\frac{\eta'_{sf}}{\eta_{sf}}(t) = \mu(t)$. Hence,

$$A_{sf}(t,\xi) = \begin{pmatrix} 0 & |\xi| \\ |\xi| & 2i\mu(t)\end{pmatrix}. \tag{5.47}$$

We denote by $\mathcal{E}_{sf} = \mathcal{E}_{sf}(t,s,\xi)$ the fundamental solution to $(D_t - A_{sf})\,\widetilde{U} = 0$, i.e., the solution to

$$D_t\mathcal{E}_{sf}(t,s,\xi) = A_{sf}(t,\xi)\mathcal{E}_{sf}(t,s,\xi) \quad \text{and} \quad \mathcal{E}_{sf}(s,s,\xi) = I.$$

In $Z_I(N)$ we relate $\mathcal{E}(t,s,\xi)$ to $\mathcal{E}_{sf}(t,s,\xi)$ and use the stabilization condition. For this reason we solve

$$D_t\Lambda(t,s,\xi) = (A(t,\xi) - A_{sf}(t,\xi))\,\Lambda(t,s,\xi) \quad \text{and} \quad \Lambda(s,s,\xi) = I$$

which gives

$$\Lambda(t,s,\xi) = \operatorname{diag}\Big(1, \exp\Big(-\int_s^t \sigma(\tau)d\tau\Big)\Big).$$

We make the ansatz $\mathcal{E}(t,s,\xi) = \Lambda(t,s,\xi)\mathcal{R}(t,s,\xi)$. It follows that the matrix $\mathcal{R}(t,s,\xi)$ satisfies

$$D_t\mathcal{R}(t,s,\xi) = \Lambda(s,t,\xi)A_{sf}(t,\xi)\Lambda(t,s,\xi)\mathcal{R}(t,s,\xi) \quad \text{and} \quad \mathcal{R}(s,s,\xi) = I,$$

where the coefficient matrix $\widetilde{A}_{sf} = \widetilde{A}_{sf}(t,s,\xi)$ has the form

$$\begin{aligned}\widetilde{A}_{sf}(t,s,\xi) &:= \Lambda(s,t,\xi)A_{sf}(t,\xi)\Lambda(t,s,\xi)\\ &= \begin{pmatrix} 0 & \exp\Big(-\int_s^t \sigma(\tau)d\tau\Big)|\xi| \\ \exp\Big(\int_s^t \sigma(\tau)d\tau\Big)|\xi| & 2i\mu(t)\end{pmatrix}.\end{aligned}$$

Note that

$$\Big|\int_0^\infty \sigma(s)ds\Big| \le \int_0^\infty |\sigma(s)|ds \lesssim \frac{(1+s)^{\alpha-1}}{\alpha-1}\Big|_0^\infty < \infty,$$

so $\exp\big(\int_0^\infty \sigma(s)ds\big) = \omega < \infty$, where ω is a non-negative constant. Define $\widetilde{\omega}(s) := \omega\exp\big(-\int_0^s \sigma(\theta)d\theta\big)$, the stabilization condition implies $0 < c \le \widetilde{\omega}(s) \le C < \infty$ with suitable constants c and C.

Denote by $W_{sf} = W_{sf}(t,s,\xi)$ the matrix

$$W_{sf}(t,s,\xi) = \begin{pmatrix} 0 & \widetilde{\omega}(s)^{-1}|\xi| \\ \widetilde{\omega}(s)|\xi| & 2i\mu(t)\end{pmatrix}.$$

The diagonalizer of the $|\xi|-$homogeneous part of W_{sf} is given by

$$\widetilde{M}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ \widetilde{\omega}(s) & \widetilde{\omega}(s)\end{pmatrix}, \qquad \widetilde{M}^{-1}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \widetilde{\omega}(s)^{-1} \\ -1 & \widetilde{\omega}(s)^{-1}\end{pmatrix}.$$

So,

$$\widetilde{M}(s)^{-1}W_{sf}(t,s,\xi)\widetilde{M}(s) = \begin{pmatrix} |\xi| & 0 \\ 0 & -|\xi|\end{pmatrix} + i\mu(t)\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}.$$

This means

$$M\widetilde{M}(s)^{-1}W_{sf}(t,s,\xi)\widetilde{M}(s)M^{-1} = A_{sf}(t,\xi).$$

Then we can conclude that the solution $\widetilde{\mathcal{E}}_{sf}(t,s,\xi)$ to the auxiliary problem

$$D_t\widetilde{\mathcal{E}}_{sf}(t,s,\xi) = W_{sf}(t,s,\xi)\widetilde{\mathcal{E}}_{sf}(t,s,\xi), \qquad \widetilde{\mathcal{E}}_{sf}(s,s,\xi) = I,$$

satisfies $\widetilde{\mathcal{E}}_{sf}(t,s,\xi) = \widetilde{M}(s)M^{-1}\mathcal{E}_{sf}(t,s,\xi)M\widetilde{M}(s)^{-1}$.

Corollary 5.2. *Assume the Hypothesis 5.1 and* $0 < \mu^2 < \frac{2-\alpha}{12}$*. Then the fundamental solutions* $\widetilde{\mathcal{E}}_{sf}(t,s,\xi)$ *and* $\mathcal{E}_{sf}(t,s,\xi)$ *satisfy*

$$\|\widetilde{\mathcal{E}}_{sf}(t,s,\xi)\|, \|\mathcal{E}_{sf}(t,s,\xi)\| \lesssim 1$$

uniformly in $(t,\xi),(s,\xi) \in Z_I(N)$.

Proof. First note that when $\delta \equiv 0$ we are in the same position as in Chapter 2. It follows from Hypothesis 5.1 and the condition for μ^2 that the mass is non-effective. We will prove this result in a larger zone $Z_{hyp}(N)$ which is defined by:

$$Z_{hyp}(N) \quad = \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : (1+t)|\xi| \geq N\}.$$

The *separating curve* is given by

$$\theta_{|\xi|} : (0,N] \to [0,\infty), \quad (1+\theta_{|\xi|})|\xi| = N.$$

We put $\theta_0 = \infty$, and $\theta_{|\xi|} = 0$ for any $|\xi| \geq N$. In the zone $Z_{hyp}(N)$ we define the micro-energy

$$U = \Big(i|\xi|\widehat{u}, \widehat{u}_t - \frac{\eta'(t)}{\eta(t)}\widehat{u}\Big)^T,$$

and $U = \eta(t)\widetilde{U}$. When $\delta \equiv 0$, then we can write $\frac{\eta'(t)}{\eta(t)} = \mu(t)$, where $\mu(t) \lesssim (1+t)^{-1}$ and $|d_t\mu(t)| \lesssim (1+t)^{-2}$ (remember that for very slow oscillations we only need regularity C^1). Then

$$\partial_t\widetilde{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i|\xi|\widetilde{U} + \begin{pmatrix} 0 & 0 \\ 0 & -2\mu(t) \end{pmatrix} \widetilde{U} \tag{5.48}$$

for $t \geq \theta_{|\xi|}$. Let P be the diagonalizer of the principal part (with respect to powers of $|\xi|$) of (5.48). It is given by

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

If we put $V(t,\xi) := P^{-1}\widetilde{U}(t,\xi)$, then we get

$$\partial_t V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i|\xi| V + B_0(t,\xi)\, V, \tag{5.49}$$

where

$$B_0(t) := -\mu(t)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Now we define the *second diagonalizer* that depends on the anti-diagonal entries of $B_0(t)$:

$$K(t,\xi) := \begin{pmatrix} 1 & \frac{q(t)}{2i|\xi|} \\ -\frac{q(t)}{2i|\xi|} & 1 \end{pmatrix}, \qquad q(t) = \mu(t). \tag{5.50}$$

Using $\limsup_{t\to\infty}(1+t)\mu(t) < \frac{1}{2}$ it follows

$$\frac{|q(t)|}{|\xi|} \leq \frac{C}{(1+t)|\xi|} \leq \frac{C}{N}$$

for $t \geq \theta_{|\xi|}$. Hence, $|\det K| \geq 1 - C^2/(4N^2)$. Therefore, $K(t,\xi)$ and $K^{-1}(t,\xi)$ are uniformly bounded in $Z_{hyp}(N)$ for a sufficiently large N. We replace $V(t,\xi) =: K(t,\xi)W(t,\xi)$. We get

$$\partial_t W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i|\xi| W - \mu(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W + J(t,\xi)W, \tag{5.51}$$

where $J(t,\xi) = K^{-1}(t,\xi)R(t,\xi)$ with $D_0(t,\xi) = \operatorname{diag}(-i|\xi|, i|\xi|)$, $H(t,\xi) = K(t,\xi) - I$ and

$$\begin{aligned} R &= D_0K + B_0K - \partial_t K - KD_0 - K\text{diag}B_0 \\ &= B_0 + D_0H - HD_0 - \text{diag}B_0 - H\text{diag}B_0 - \partial_t H + B_0H. \end{aligned}$$

By construction the sum of the first four terms of $R(t,\xi)$ vanishes. Thanks to the non-effectiveness of the dissipative term the matrix $R(t,\xi)$, and therefore $J(t,\xi)$, satisfies the following estimate in $Z_{hyp}(N)$:

$$\|J(t,\xi)\| \leq \frac{C}{|\xi|(1+t)^2}. \tag{5.52}$$

After substituting $W(t,\xi) =: \frac{\eta(s)}{\eta(t)} D(t,\xi)Z(t,\xi)$, where

$$D(t,\xi) = \operatorname{diag}\Big(\exp(-i|\xi|(t-s), \exp(i|\xi|(t-s)\Big),$$

we obtain the following Cauchy problem in $Z_{hyp}(N)$:

$$\begin{cases} \partial_t Z = \widetilde{J}(t,\xi)\, Z, & t \geq s, \\ Z(s,\xi) = K^{-1}(s,\xi)P^{-1}\,\widetilde{U}(s,\xi), \end{cases} \tag{5.53}$$

where the matrix $\widetilde{J}(t,\xi) = D^{-1}(t,\xi)J(t,\xi)D(t,\xi)$ satisfies (5.52), too. For any $s,t \geq \theta_{|\xi|}$ we have

$$\int_s^t \|\widetilde{J}(\tau,\xi)\|\, d\tau \leq C \int_{\theta_{|\xi|}}^{\infty} \frac{1}{|\xi|(1+\tau)^2}\, d\tau \leq \frac{C'}{|\xi|(1+\theta_{|\xi|})} = \frac{C'}{N}.$$

Hence $\|Z(t,\xi)\| \leq C\|Z(s,\xi)\|$, i.e.,

$$\|\mathcal{E}_{sf}(t,s,\xi)\| \lesssim 1 \tag{5.54}$$

for all $s,t \geq \theta_{|\xi|}$. □

Now we use the stabilization condition to find $\mathcal{R}(t,s,\xi) = \widetilde{\mathcal{E}}_{sf}(t,s,\xi)\mathcal{Q}_{\mathcal{R}}(t,s,\xi)$. The coefficient matrix of the Cauchy problem

$$\begin{aligned} D_t\mathcal{Q}_{\mathcal{R}}(t,s,\xi) &= \widetilde{\mathcal{E}}_{sf}(s,t,\xi)\Big(\widetilde{A}_{sf}(t,s,\xi) - W_{sf}(t,s,\xi)\Big)\widetilde{\mathcal{E}}_{sf}(t,s,\xi)\mathcal{Q}_{\mathcal{R}}, \\ \mathcal{Q}_{\mathcal{R}}(s,s,\xi) &= I, \end{aligned}$$

satisfies the estimate

$$
\begin{aligned}
& \int_s^t \Big\| \widetilde{\mathcal{E}}_{sf}(s,\theta,\xi)\Big(\widetilde{A}_{sf}(\theta,s,\xi) - W_{sf}(\theta,s,\xi)\Big)\widetilde{\mathcal{E}}_{sf}(\theta,s,\xi)\Big\| d\theta \\
\lesssim \;& |\xi| \int_s^t \Big| \exp\Big(\int_s^\tau \sigma(\theta)d\theta\Big) - \widetilde{\omega}(s)\Big| d\tau \lesssim |\xi| \int_s^t \Big| \exp\Big(\int_0^\tau \sigma(\theta)d\theta\Big) - \omega\Big| d\tau \\
= \;& |\xi| \int_s^t \Big| \omega \exp\Big(-\int_\tau^\infty \sigma(\theta)d\theta\Big) - \omega\Big| d\tau \lesssim |\xi| \int_s^t \Big| \exp\Big(-\int_\tau^\infty \sigma(\theta)d\theta\Big) - 1\Big| d\tau \\
\lesssim \;& |\xi| \int_s^t \Big| \int_\tau^\infty \sigma(\theta)d\theta\Big| d\tau \lesssim |\xi| \int_s^t \int_\tau^\infty |\sigma(\theta)| d\theta d\tau \\
\lesssim \;& |\xi|(1+t)^\alpha \lesssim 1.
\end{aligned}
$$

Now the standard construction of $\mathcal{Q}_{\mathcal{R}}$ in terms of a Peano-Baker series gives uniform bounds for this matrix and for its inverse within the intermediate zone $Z_I(N)$. Thus we arrive at the following lemma.

Lemma 5.7. *Assume the Hypotheses 5.1 to 5.3. Then the fundamental solution $\mathcal{E}(t,s,\xi)$ can be represented in $Z_I(N)$ as*

$$\mathcal{E}(t,s,\xi) = \Lambda(t,s,\xi)\widetilde{\mathcal{E}}_{sf}(t,s,\xi)\mathcal{Q}_{\mathcal{R}}(t,s,\xi),$$

where $\Lambda(t,s,\xi)$ and $\mathcal{Q}_{\mathcal{R}}(t,s,\xi)$ are uniformly bounded in $(t,\xi),(s,\xi) \in Z_I(N)$.

Corollary 5.3. *Assume the Hypotheses 5.1 to 5.3 and $0 < \mu^2 < \frac{2-\alpha}{12}$. Then the fundamental solution $\mathcal{E}(t,s,\xi)$ satisfies*

$$\|\mathcal{E}(t,s,\xi)\| \lesssim 1$$

uniformly in $(t,\xi),(s,\xi) \in Z_I(N)$.

Remark 5.7. *We have a decay for the fundamental solution within the hyperbolic zone. In the intermediate zone, we have that the fundamental solution is bounded. This allows to conclude our results using our special micro-energy. In [19] we proved that the micro-energy is bounded in $Z_I(N) \cup Z_{H,m}(N)$.*

5.5 Energy estimates

Consider the following Cauchy problem

$$u_{tt} - \Delta u + m(t)^2 u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{5.55}$$

with $(t,x) \in [0,\infty) \times \mathbb{R}^n$ and

$$m(t)^2 = \frac{\mu^2}{(1+t)^2 g(t)^2} + \delta(t), \quad \mu^2 \neq 0, \tag{5.56}$$

where $\frac{\mu^2}{(1+t)^2 g(t)^2}$ is the shape function and $\delta = \delta(t)$ is a bounded oscillating function with

$$|\delta(t)| \lesssim (1+t)^{-2}. \tag{5.57}$$

If

$$\eta(t) = \exp\left(\sum_{j=1}^{\infty}\int_0^t\int_\tau^\infty q_j(s)dsd\tau\right), \tag{5.58}$$

where

$$q_k(t) = \begin{cases} m(t)^2 \text{ if } k=1 \\ \sum_{j=1}^{k-1}\Big(\int_t^\infty q_j(s)ds\Big)\Big(\int_t^\infty q_{k-j}(s)ds\Big) \text{ if } k=2,3,4,\cdots, \end{cases} \tag{5.59}$$

then we can conclude the following theorem:

Theorem 5.1. *Let us consider the Cauchy problem (5.55) with* $(u_0,u_1) \in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *and the mass term satisfying Hypotheses 5.1 to 5.3 and* $0<\mu^2<\frac{2-\alpha}{12}$. *Then the solution* $u=u(t,x)$ *to (5.55) satisfies the following energy estimate*

$$\|\left(u_t(t,\cdot), \nabla_x u(t,\cdot), p(t)u(t,\cdot)\right)\|_{L^2} \lesssim \left(\|u_0\|_{H^1} + \|u_1\|_{L^2}\right) \tag{5.60}$$

with $p(t)=\frac{\eta(t)}{1+t}$.

Proof. This theorem basically follows from Lemma 5.2, Corollary 5.1, Corollary 5.3 and the estimates made in the verification of the proof of Theorem 2.1. □

Remark 5.8. *We shall prove the sharpness of the choice of the function* η. *For this we will compare the behavior of* η *with the behavior of the function* ψ *chosen in Chapter* 2. *We already know from Remark 5.6 that the function* η *in (5.5) has the same asymptotic behavior as* η_{sf} *and that* $\psi(t) \approx \exp\big(\sum_{k=1}^\infty \mu^{2k}\gamma_k \int_0^t \frac{d\tau}{(1+\tau)g(\tau)^{2k}}\big)$, *where* γ_k *are the Catalan numbers. We will prove at least for* $\mu<\frac{1}{12}$ *that*

$$\frac{\psi'(t)}{\psi(t)} - \frac{\eta_{sf}'(t)}{\eta_{sf}(t)} \in L^1(\mathbb{R}). \tag{5.61}$$

From (5.61) we can conclude

$$\psi(t) = \exp\left(\int_0^t \frac{\psi'(s)}{\psi(s)}ds\right) \approx \exp\left(\int_0^t \frac{\eta'(s)}{\eta(s)}ds\right) = \eta(t).$$

To prove (5.61) we pose further assumptions to the function $g(t)$. *Let us assume that there exists an increasing function* $a(t)$ *such that*

$$\int_t^\infty \frac{2g'(t)}{(1+s)g(s)^3}ds \le \frac{1}{(1+t)a(t)g(t)^2}, \quad \text{with} \quad \frac{1}{(1+t)a(t)g(t)^2} \in L^1(\mathbb{R}). \tag{5.62}$$

We have

$$\frac{\psi'(t)}{\psi(t)} - \frac{\eta_{sf}'(t)}{\eta_{sf}(t)} = \sum_{k=1}^\infty \Big(\frac{\mu^{2k}\gamma_k}{(1+t)g(t)^{2k}} - \int_t^\infty q_k(s)ds\Big). \tag{5.63}$$

By induction we will prove that for all $k=1,2,\cdots$ *the following statement is valid:*

$$\int_t^\infty q_k(s)ds = \frac{\mu^{2k}\gamma_k}{(1+t)g(t)^{2k}} - h_k(t), \tag{5.64}$$

with non-negative functions $h_k(t)$ satisfying

$$h_k(t) \leq \frac{\mu^{2k}\gamma_k(3^k-1)}{2(1+t)a(t)g(t)^2}.$$

Indeed, for $k=1$ using integration by parts we will arrive at

$$\int_t^\infty q_1(s)ds = \int_t^\infty \frac{\mu^2\gamma_1}{(1+t)^2 g(s)^2}ds = \frac{\mu^2\gamma_1}{(1+t)g(t)^2} - h_1(t),$$

where $h_1(t) = \mu^2\gamma_1 \int_t^\infty \frac{2g'(s)}{(1+s)g(s)^3}ds$ is a non-negative function once that g is positive and non-decreasing. Let us suppose that (5.64) it is true for $k = 1, 2, \cdots, \ell-1$. Then we have for $k=\ell$:

$$\begin{aligned}
\int_t^\infty q_\ell(s)ds &= \int_t^\infty \sum_{j=1}^{\ell-1}\left(\int_s^\infty q_j(\tau)d\tau\right)\left(\int_s^\infty q_{\ell-j}(\tau)d\tau\right)ds \\
&= \int_t^\infty \frac{\mu^{2\ell}\gamma_\ell}{(1+s)^2 g(s)^{2\ell}}ds + \sum_{j=1}^{\ell-1}\int_t^\infty h_j(s)h_{\ell-j}(s)ds \\
&- \sum_{j=1}^{\ell-1}\left(\int_t^\infty \frac{\mu^{2j}\gamma_j}{(1+s)g(s)^{2j}}h_{\ell-j}(s)ds + \int_t^\infty \frac{\mu^{2(\ell-j)}\gamma_{\ell-j}}{(1+s)g(s)^{2(\ell-j)}}h_j(s)ds\right).
\end{aligned}$$

Performing integration by parts we arrive at

$$\int_t^\infty \frac{\mu^{2\ell}\gamma_\ell}{(1+s)^2 g(s)^{2\ell}}ds = \frac{\mu^{2\ell}\gamma_\ell}{(1+t)g(t)^{2\ell}} - \mu^{2\ell}\gamma_\ell\int_t^\infty \frac{2\ell g'(s)}{(1+s)g(s)^{2\ell+1}}ds.$$

If we take

$$\begin{aligned}
h_\ell(t) &= \mu^{2\ell}\gamma_\ell\int_t^\infty \frac{2\ell g'(s)}{(1+s)g(s)^{2\ell+1}}ds - \sum_{j=1}^{\ell-1}\int_t^\infty h_j(s)h_{\ell-j}(s)ds \\
&+ \sum_{j=1}^{\ell-1}\left(\int_t^\infty \frac{\mu^{2j}\gamma_j}{(1+s)g(s)^{2j}}h_{\ell-j}(s)ds + \int_t^\infty \frac{\mu^{2(\ell-j)}\gamma_{\ell-j}}{(1+s)g(s)^{2(\ell-j)}}h_j(s)ds\right),
\end{aligned}$$

then

$$\int_t^\infty q_\ell(s)ds = \frac{\mu^{2\ell}\gamma_\ell}{(1+t)g(t)^{2\ell}} - h_\ell(t).$$

From the previous equality we can conclude that $h_\ell(t) \geq 0$ once

$$\int_t^\infty q_\ell(s)ds \leq \frac{\mu^{2\ell}\gamma_\ell}{(1+t)g(t)^{2\ell}}.$$

Moreover, after using the induction hypothesis we arrive at

$$\begin{aligned} h_\ell(t) &\leq \mu^{2\ell}\gamma_\ell \int_t^\infty \frac{2\ell g'(s)}{(1+s)g(s)^{2\ell+1}}ds + \sum_{j=1}^{\ell-1}\int_t^\infty \frac{\mu^{2j}\gamma_j}{(1+s)g(s)^{2j}}h_{\ell-j}(s)ds \\ &+ \sum_{j=1}^{\ell-1}\int_t^\infty \frac{\mu^{2(\ell-j)}\gamma_{\ell-j}}{(1+s)g(s)^{2(\ell-j)}}h_j(s)ds \\ &\leq \frac{\ell\mu^{2\ell}\gamma_\ell}{(1+t)a(t)g(t)^2} + \frac{\mu^{2\ell}}{2(1+t)a(t)g(t)^2}\sum_{j=1}^{\ell-1}\gamma_j\gamma_{\ell-j}\left(3^{\ell-j}+3^j-2\right) \\ &\leq \mu^{2\ell}\gamma_\ell\frac{(2\ell-1)+2\cdot 3^{\ell-1}-1}{2(1+t)a(t)g(t)^3} \leq \frac{\mu^{2\ell}\gamma_\ell\left(3^\ell-1\right)}{2(1+t)a(t)g(t)^2}. \end{aligned}$$

Therefore (5.64) is true for all $k = 1, 2, \cdots$, and (5.63) is reduced to

$$\frac{\psi'(t)}{\psi(t)} - \frac{\eta'_{sf}(t)}{\eta_{sf}(t)} = \sum_{k=1}^\infty h_k(t).$$

Once

$$\lim_{t\to\infty}\frac{\mu^{2(k+1)}\gamma_{k+1}(3^{k+1}-1)}{\mu^{2k}\gamma_k(3^k-1)} = 12\mu^2,$$

it follows that the series $\sum_{k=1}^\infty \mu^{2k}\gamma_k(3^k-1)$ convergence for $\mu^2 < \frac{1}{12}$. Finally, from the estimates for h_k we may conclude

$$\frac{\psi'(t)}{\psi(t)} - \frac{\eta'_{sf}(t)}{\eta_{sf}(t)} \in L^1(\mathbb{R}),$$

for $\mu^2 < \frac{1}{12}$. This completes the proof.

5.6 Examples

Let us conclude this chapter with some examples:

Example 5.1. *First we set $g(t) \equiv 1$. Then the hypothesis for g are fulfilled. If we choose*

$$\delta(t) = \frac{\nu}{(1+t)^2}\sin(t^\sigma), \quad \sigma \in (0,1),$$

then the hypothesis for the derivatives of the perturbation is satisfied taking $\beta = 1 - \frac{\sigma}{3}$. Furthermore,

$$\begin{aligned} \Big|\int_t^\infty \frac{\nu}{(1+s)^2}\sin(s^\sigma)ds\Big| &\leq \frac{\nu}{1+t}\Big|\int_t^\infty \frac{\sin(s^\sigma)}{1+s}ds\Big| \\ &= \frac{1}{\sigma}\frac{\nu}{1+t}\Big|\int_{t^\sigma}^\infty \frac{\sin(\theta)}{\theta^{\frac{\sigma-1}{\sigma}}+\theta}d\theta\Big| \\ &\leq \frac{1}{\sigma}\frac{\nu}{1+t}\Big|\int_{t^\sigma}^\infty \frac{\sin(\theta)}{\theta}d\theta\Big| \\ &= \frac{1}{\sigma}\frac{\nu}{1+t}|si(t^\sigma)|. \end{aligned}$$

If $\gamma < 1$, then $\lim_{t\to\infty} t^{\sigma\gamma} si(t^\sigma) = 0$. For more details see [25]. Therefore, for t large we have

$$\Big| \int_t^\infty \frac{\nu}{(1+s)^2} \sin(s^\sigma) ds \Big| \le \nu (1+t)^{-\sigma\gamma-1}.$$

Take $\gamma = \frac{2}{3}$. Then the stabilization condition of Hypothesis 5.3 is satisfied after choosing $\nu \le \mu^2$, $m = 1$ and $\alpha = 1 - \frac{2\sigma}{3}$. Indeed,

$$\beta = \alpha + \frac{1-\alpha}{2},$$

thus Theorem 5.1 is applicable. Note that the condition (5.62) *is trivially satisfied.*

Remark 5.9. *In the previous examples it is allowed to choose $g(t)^2$ like in Chapter 2, i.e., we can consider $g(t)^2 = \ln(e+t)\cdots\ln^{[m]}(e^{[m]}+t)$, $g(t)^2 = (\ln(e+t))^\gamma$ for $0 < \gamma < 1$ and $g(t)^2 = \ln(\ln(e^e+t))$. Naturally the decay estimate for the solution itself will depend on the function g. Note that all the above choices for g satisfy the condition* (5.62). *Indeed we shall take $a(t) = \ln(e+t)\cdots\ln^{[m]}(e^{[m]}+t)$, $a(t) = \ln(e+t)$ and $a(t) = \ln(e^e+t)\ln(\ln(e^e+t))$, respectively.*

6 Semi-linear wave models with scale-invariant time-dependent mass and dissipation

In this chapter we will consider the following semi-linear Cauchy problem with scale-invariant mass and dissipation:

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t}u_t + \frac{\mu_2^2}{(1+t)^2}u = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{6.1}$$

with $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $p > 1$ and $\mu_1 > 0, \mu_2$ are real constants. Our goal is to understand the interplay between μ_1 and μ_2 to prove global existence in time of small data energy solutions.

If μ_1 and μ_2 are zero, then we are in the situation of the semi-linear Cauchy problem for the free wave equation. The critical exponent p_{crit} is the Strauss exponent $p_0(n)$ which is the positive solution to

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Critical exponent means that for small initial data in a suitable functional space there exist global in time energy solutions for some range of admissible $p > p_{crit}$ and it is possible to find suitable small data such that there exist no global in time energy solutions if $1 < p \le p_{crit}$. For $p_0(n) < p \le \frac{n-3}{n-1}$, $n > 1$, it was proved global existence in time for small data energy solution, see [54, 32, 24, 22, 65] for $n = 1$ solutions for the semi-linear Cauchy problem for the free wave equation blow-up for any $p > 1$, see [24], hence we put $p_0(1) = \infty$. If $1 < p \le p_0(n)$ and $n > 1$, then the energy solutions for the semi-linear Cauchy problem for the free wave equation blow-up for a suitable choice of small initial data, see [64, 32, 31, 48, 24, 49].

For the classical semi-linear Klein-Gordon equation Lindblad-Sogge proved in 1996 global existence in time for small data energy solution for $n \le 3$ and $p > p_{Fuj} = 1 + \frac{2}{n}$, see [38]. Note that the critical exponent p_{Fuj} is related with the heat equation, see [21]. For $1 < p \le p_{Fuj}$ and $n = 1,2,3$ blow-up results are established, see [30] for a one-dimensional counterexample due to B. Iordanov and [33] for the higher-dimensional case. In the paper [33] Keel-Tao conjectured that for sufficiently large dimensions the solution for the semi-linear Cauchy problem has a blow-up for $p = 1 + \frac{2}{n} + \varepsilon$ with $\varepsilon > 0$, i.e., for sufficiently large dimensions we do not expect p_{Fuj} as the critical exponent.

If $\mu_2 = 0$, then D'Abbicco has recently shown in [14] that if $p > p_{Fuj}$ and

- if $n = 1$ and $\mu_1 > \frac{5}{3}$,
- if $n = 2$ and $\mu_1 > 3$,

- if $n \geq 3$ and $\mu_1 > n+2$,

then there exist global (in time) small data energy solutions. Note that the assumptions for μ_1 imply that the damping term is not non-effective according to the classification of Wirth [60]. If the coefficient μ_1 of the damping term is small, then it is expected a shift in the Strauss critical exponent. Indeed, D'Abbicco-Lucente-Reissig have proved in [16] that the critical exponent for $\mu_1 = 2$ is a shift of the Strauss exponent $p_0(n)$ to $p_0(n+2)$.

We will show that the presence of the mass term, for suitable choices of μ_1 and μ_2, allows us to prove for $n \leq 4$ global (in time) solutions for $p \geq 2$ even for a smaller range of μ_1 compared with those of the paper [14].

Let us define

$$\Delta = (\mu_1 - 1)^2 - 4\mu_2^2.$$

It is convenient to consider two cases: The case, where the mass term is predominant, i.e., when $\Delta < 1$ and the case where the dissipative term is predominant, i.e., when $\Delta \geq 0$. Note that in the overlapping case $\Delta \in [0,1)$ it is possible to choose which term is going to be predominant. The case when the dissipative term is predominant was studied by Palmieri in his Master thesis [42]. In this chapter we will prove global existence in time of small data energy solutions in a suitable function space for $\Delta \leq 0$, i.e., when the mass term is predominant excluding the case $\Delta \in (0,1)$. We will also prove blow-up behavior for solutions in the case $\Delta = 1$.

6.1 Motivation: Duhamel's principle

Let us consider the family of linear parameter dependent Cauchy problem

$$v_{tt} - \Delta v + \frac{\mu_1}{1+t} v_t + \frac{\mu_2^2}{(1+t)^2} v = 0, \quad v(s,x) = v_0(x), \quad v_t(s,x) = v_1(x), \tag{6.2}$$

and denote by $E_0(t,s,x)$, $E_1(t,s,x)$ the fundamental solution to the linear homogeneous Cauchy problem (6.2) with the initial data $(v_0, v_1) = (\delta_x, 0)$ and $(v_0, v_1) = (0, \delta_x)$, respectively, where δ_x is the Dirac distribution in the x-variable.

The solution $v(t,x)$ to the linear Cauchy problem (6.2) is given by

$$v(t,x) = E_0(t,s,x) *_{(x)} v_0(x) + E_1(t,s,x) *_{(x)} v_1(x). \tag{6.3}$$

By Duhamel's principle we get that

$$v^{nl}(t,x) = \int_0^t E_1(t,s,x) *_{(x)} |v(s,x)|^p ds \tag{6.4}$$

is the solution to the inhomogeneous problem

$$v_{tt}^{nl} - \Delta v^{nl} + \frac{\mu_1}{1+t} v_t^{nl} + \frac{\mu_2^2}{(1+t)^2} v^{nl} = |v(t,x)|^p, \quad v^{nl}(s,x) = 0, \quad v_t^{nl}(s,x) = 0. \tag{6.5}$$

Hence, the solution to the Cauchy problem (6.1) can be written in the following form:

$$u(t,x) = E_0(t,s,x) *_{(x)} u_0(x) + E_1(t,s,x) *_{(x)} u_1(x) + \int_0^t E_1(t,s,x) *_{(x)} |u(s,x)|^p ds.$$

Therefore, in order to apply Duhamel's principle we shall derive estimates for the solutions to the family of linear parameter dependent Cauchy problems (6.2).

6.2 Linear decay estimates

Performing the dissipative transformation $v(t,x) = (1+t)^\gamma w(t,x)$ the Cauchy problem (6.2) becomes

$$w_{tt} - \Delta w + \frac{2\gamma+\mu_1}{1+t} w_t + \frac{\gamma(\gamma-1)+\gamma\mu_1+\mu_2^2}{(1+t)^2} w = 0, \quad w(s,x) = w_0(x), \quad w_t(s,x) = w_1(x), \tag{6.6}$$

where $w_0(x) = (1+s)^{-\gamma} v_0(x)$ and $w_1(x) = (1+s)^{-\gamma} v_1(x) - \gamma(1+s)^{-\gamma-1} v_0(x)$. If we choose $\gamma = -\frac{\mu_1}{2}$, then the Cauchy problem (6.6) takes the form

$$w_{tt} - \Delta w + \frac{\mu}{(1+t)^2} w = 0, \quad w(s,x) = w_0(x), \quad w_t(s,x) = w_1(x), \tag{6.7}$$

where $\mu = \frac{\mu_1}{2} - \frac{\mu_1^2}{4} + \mu_2^2$. Note that $\Delta = 1 - 4\mu \leq 0$ if, and only if, $\mu \geq \frac{1}{4}$, i.e., we are dealing with not non-effective masses.

Performing the partial Fourier transform with respect to x in the Cauchy problem (6.7) and denoting by $\widehat{w} = \widehat{w}(t,\xi)$ the partial Fourier transform $F_{x\to\xi}(u)(t,\xi)$ we obtain

$$\widehat{w}_{tt} + |\xi|^2 \widehat{w} + \frac{\mu}{(1+t)^2} \widehat{w} = 0. \tag{6.8}$$

The scale-invariant property allows us to derive explicit representations of solutions in terms of known special functions. In this case we will perform, like in [5], a change of variables to reduce the ordinary differential equation (6.8) in a confluent hypergeometric equation. If

$$\tau = |\xi|(1+t), \quad \widehat{w}(t,\xi) = \tau^\rho \widetilde{v}(\tau),$$

then choosing $2\rho = 1 + \sqrt{1-4\mu}$ we have that $\widetilde{v} = \widetilde{v}(\tau)$ satisfies

$$\tau \widetilde{v}_{\tau\tau} + 2\rho \widetilde{v}_\tau + \tau \widetilde{v} = 0. \tag{6.9}$$

We can reduce the equation (6.9) to a confluent hypergeometric equation if we perform the change of variables

$$z = 2i\tau, \quad v(z) = e^{i\tau} \widetilde{v}(\tau).$$

Then $v = v(z)$ solves the following equation

$$z v_{zz} + (2\rho - z) v_z - \rho v = 0. \tag{6.10}$$

This describes a confluent hypergeometric equation with $\alpha = 2\rho$ and $\beta = \rho$. Its fundamental solutions are called confluent hypergeometric functions and depend on the parameter ρ. If $\mu > \frac{1}{4}$, then we are in the situation where $\operatorname{Re}\rho = \frac{1}{2}$ and $\operatorname{Im}\rho \neq 0$. If $\mu = \frac{1}{4}$, then $\rho = \frac{1}{2}$. For more details about confluent hypergeometric functions see [3, 1]. In the next proposition we will list two important properties of confluent hypergeometric functions.

Proposition 6.1. *Let $\widetilde{\Phi} = \widetilde{\Phi}(\alpha,\beta;z)$ be a confluent hypergeometric function, where α and β are complex parameters. Then,*

1. *$\widetilde{\Phi}$ is an entire function;*

2. $\partial_z \widetilde{\Phi}(\alpha, \beta; z) = \frac{\alpha}{\beta} \widetilde{\Phi}(\alpha+1, \beta+1; z)$.

If $\rho \neq \frac{1}{2}$, then the fundamental solutions $v_{1,\Phi}(z)$, $v_{2,\Phi}(z)$ to (6.10) are given by

$$\begin{aligned} v_{1,\Phi}(z) &= \Phi(\rho, 2\rho; z), \\ v_{2,\Phi}(z) &= z^{1-2\rho}\Phi(1-\rho, 2\rho; z) = z^{1-2\rho}e^z\Phi(1-\rho, 2-2\rho; -z). \end{aligned}$$

If $\rho = \frac{1}{2}$, then the fundamental solutions $v_{1,\Psi}(z)$, $v_{2,\Psi}(z)$ to (6.10) are given by

$$\begin{aligned} v_{1,\Psi}(z) &= \Psi(\rho, 2\rho; z), \\ v_{2,\Psi}(z) &= z^{1-2\rho}\Psi(\rho, 2\rho; -z) = z^{1-2\rho}e^z\Psi(1-\rho, 2-2\rho; -z), \end{aligned}$$

where Φ and Ψ denote the confluent hypergeometric functions. Note that in both cases for the second solution we used Kummer's transformation.

If we define

$$\Theta_0(\alpha, \beta; z) = \begin{cases} \Phi(\alpha, \beta; z), & \beta \notin \mathbb{Z}, \\ \Psi(\alpha, \beta; z), & \beta \in \mathbb{Z}, \end{cases}$$

then we can write the two fundamental solutions of the general confluent hypergeometric equation as

$$v_1(z) = \Theta_0(\rho, 2\rho; z), \tag{6.11}$$
$$v_2(z) = z^{1-2\rho}e^z\Theta_0(1-\rho, 2-2\rho; -z). \tag{6.12}$$

Then we can state two fundamental solutions e_1, e_2 to the problem (6.8) as follows:

$$\begin{aligned} e_1(t,\xi) &= ((1+t)|\xi|)^\rho\, e^{-i(1+t)|\xi|}\Theta_0(\rho, 2\rho; z), \\ e_2(t,\xi) &= ((1+t)|\xi|)^\rho\, z^{1-2\rho}e^{i(1+t)|\xi|}\Theta_0(1-\rho, 2-2\rho; -z). \end{aligned}$$

Denoting $\partial_t e_1 = e_{1,t}$, differentiation with respect to t gives

$$\begin{aligned} e_{1,t}(t,\xi) &= ((1+t)|\xi|)^{\rho-1}\, |\xi| e^{-i(1+t)|\xi|}\Theta_1(\rho, 2\rho; z), \\ e_{2,t}(t,\xi) &= ((1+t)|\xi|)^{\rho-1}\, z^{1-2\rho}|\xi| e^{i(1+t)|\xi|}\Theta_1(1-\rho, 2-2\rho; -z), \end{aligned}$$

where

$$\Theta_1(\alpha, \beta; z) = \begin{cases} \frac{z}{2}\Phi(\alpha, \beta; z) + (\beta-\alpha)\Phi(\alpha-1, \beta; z), & \beta \notin \mathbb{Z}, \\ \frac{z}{2}\Psi(\alpha, \beta; z) - \Psi(\alpha-1, \beta; z), & \beta \in \mathbb{Z}. \end{cases}$$

The solution of (6.8) can be represented by

$$\widehat{w}(t,\xi) = c_1(s,\xi)e_1(t,\xi) + c_2(s,\xi)e_2(t,\xi)$$

with the fundamental solutions e_1, e_2 depending on t and ξ, and the coefficients c_1, c_2 depending on ξ and the initial time $s \geq 0$. The coefficients can be found after imposing the initial conditions as follows:

$$\begin{aligned} c_1(s,\xi) &= \frac{e_{2,t}(s,\xi)\widehat{w}(s,\xi) - e_2(s,\xi)\widehat{w}_t(s,\xi)}{e_{2,t}(s,\xi)e_1(s,\xi) - e_{1,t}(s,\xi)e_2(s,\xi)}, \\ c_2(s,\xi) &= \frac{e_1(s,\xi)\widehat{w}_t(s,\xi) - e_{1,t}(s,\xi)\widehat{w}(s,\xi)}{e_{2,t}(s,\xi)e_1(s,\xi) - e_{1,t}(s,\xi)e_2(s,\xi)}. \end{aligned}$$

Let us write the solution of (6.8) as

$$\widehat{w}(t,\xi) = H_{1,0}(t,s,\xi)\widehat{w}(s,\xi) + H_{2,0}(t,s,\xi)\widehat{w}_t(s,\xi), \tag{6.13}$$

where

$$\begin{aligned} H_{1,0}(t,s,\xi) &= \frac{e_{2,t}(s,\xi)e_1(t,\xi) - e_{1,t}(s,\xi)e_2(t,\xi)}{e_{2,t}(s,\xi)e_1(s,\xi) - e_{1,t}(s,\xi)e_2(s,\xi)}, \\ H_{2,0}(t,s,\xi) &= \frac{e_1(s,\xi)e_2(t,\xi) - e_2(s,\xi)e_1(t,\xi)}{e_{2,t}(s,\xi)e_1(s,\xi) - e_{1,t}(s,\xi)e_2(s,\xi)}, \end{aligned}$$

with $t \geq s \geq 0$. Therefore, the derivative of $\widehat{w} = \widehat{w}(t,\xi)$ with respect to t is

$$\widehat{w}_t(t,\xi) = H_{1,1}(t,s,\xi)\widehat{w}(s,\xi) + H_{2,1}(t,s,\xi)\widehat{w}_t(s,\xi), \tag{6.14}$$

where $H_{k,1} = \partial_t H_{k,0}(t,s,\xi)$, $k = 1,2$. These fundamental solutions satisfy $H_{k,\ell}(s,s,\xi) = (\delta_{k,\ell+1})$, $k = 1,2$, $\ell = 0,1$. Moreover, due to the formulas for the Wronskian of confluent hypergeometric functions we can calculate the denominator of $H_{k,0}$ for $k = 1,2$ by

$$e_{2,t}(t,\xi)e_1(t,\xi) - e_{1,t}(t,\xi)e_2(t,\xi) = C_\rho(2i)^{1-2\rho}|\xi|,$$

where $C_\rho = (1-2\rho)$ for $2\rho \neq 1$, $C_\rho = e^{i\pi\rho}$ for $2\rho = 1$. The representations for $H_{k,\ell}(t,s,\xi)$ are given in the following lemma.

Lemma 6.1. *Denote by $z = z(t) = 2i(1+t)|\xi|$ and $z_0 = z(s)$. Then we have for $k = 1,2$, $\ell = 0,1$, the representations*

$$H_{k,\ell}(t,s,\xi) = C_\rho(-2i|\xi|)^{1-k+\ell}\det G_{k,\ell}(t,s,\xi), \tag{6.15}$$

where $C_\rho = (1-2\rho)^{-1}$ for $2\rho \neq 1$, $C_\rho = e^{-i\pi\rho}$ for $2\rho = 1$ and

$$G_{k,\ell}(t,s,\xi) = \begin{pmatrix} z^{\rho-\ell}e^{-\frac{z}{2}}\Theta_\ell(\rho,2\rho;z) & z^{-\rho+1-\ell}e^{\frac{z}{2}}\Theta_\ell(1-\rho,2-2\rho;-z) \\ z_0^{\rho-2+k}e^{-\frac{z_0}{2}}\Theta_{2-k}(\rho,2\rho;z_0) & z_0^{-\rho-1+k}e^{\frac{z_0}{2}}\Theta_{2-k}(1-\rho,2-2\rho;-z_0) \end{pmatrix}.$$

To study the behavior of the solution and their derivatives, according to (6.15), it is necessary to analyze the behavior of the function Θ_0 for small and large arguments.

Proposition 6.2. *Let α and β fixed parameters in $\mathbb{C}$ and $k = 0,1$.*

1. *For $\beta = 1$ and small $|z|$ it holds*

$$\Theta_k(\alpha,\beta;z) \sim sgn(\Gamma(\alpha-k))\ln z,$$

where we suppose that z is a pure imaginary number and, therefore, $\ln z = \ln|z| + i\frac{\pi}{2}sgn\operatorname{Im} z$ and $\Gamma(\cdot)$ is the Gamma function.

2. *For $\beta \notin \mathbb{Z}$ and small arguments $|z|$ we have*

$$|\Theta_k(\alpha,\beta;z)| \leq C.$$

3. *For $\beta = 1$ and large arguments $|z|$ we have*

$$|\Theta_k(\alpha,\beta;z)| \leq C|z|^{k-\operatorname{Re}\alpha}.$$

4. For $\beta \notin \mathbb{Z}$ and large arguments $|z|$ we have

$$|\Theta_k(\alpha,\beta;z)| \leq C_{\alpha,\beta}|z|^{\max\{k+\mathrm{Re}(\alpha-\beta),k-\mathrm{Re}\,\alpha\}}.$$

Proof. See [3]. □

In the next result we will see that additional regularity for the initial data brings better estimates for the solution itself, but for the derivatives of the solution we have no further improvement of the estimates. Let us define the function space

$$\mathcal{D}_m = \left(H^1 \cap L^m\right) \times \left(L^2 \cap L^m\right),$$

with $m \in [1,2)$ and the norm $\|(u,v)\|^2_{\mathcal{D}_m} = \|u\|^2_{L^m} + \|u\|^2_{H^1} + \|v\|^2_{L^m} + \|v\|^2_{L^2}$.

Theorem 6.1. *Suppose that $(v_0,v_1) \in \mathcal{D}_m$ and $\Delta \leq 0$. Then the solution $v \in \mathcal{C}([0,\infty),H^1) \cap \mathcal{C}^1([0,\infty),L^2)$ for the Cauchy problem*

$$v_{tt} - \Delta v + \frac{\mu_1}{1+t}v_t + \frac{\mu_2^2}{(1+t)^2}v = 0, \quad v(s,x) = v_0(x), \quad v_t(s,x) = v_1(x),$$

satisfies the following estimates:

$$\begin{aligned}
\|(v_t(t,\cdot),\nabla_x v(t,\cdot))\|_{L^2} &\lesssim (1+t)^{-\frac{\mu_1}{2}}\left(1+\ln\left(\frac{1+t}{1+s}\right)\right)^{\gamma}\left(\|v_0\|_{H^1} + (1+s)^{\frac{1}{2}}\|v_1\|_{L^2}\right),\\
\|v(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{\mu_1}{2}} q_\Delta(t,s)\left(\|v_0\|_{H^1\cap L^m} + (1+s)\|v_1\|_{L^2\cap L^m}\right),
\end{aligned}$$

for all $t \geq s \geq 0$, where $\gamma = 1$ if $\Delta = 0$, $\gamma = 0$ if $\Delta < 0$ and

$$q_0(t,s) = \begin{cases} 1+\ln\left(\frac{1+t}{1+s}\right) & \text{for } n > \frac{m}{2-m},\\ \left(\ln\left(\frac{1+t}{1+s}\right)\right)^{\frac{2-m}{2}}\left(1+\ln\left(\frac{1+t}{1+s}\right)\right) & \text{for } n = \frac{m}{2-m},\end{cases}$$

and

$$q_\Delta(t,s) = \begin{cases} 1 & \text{for } n > \frac{m}{2-m},\\ \left(\ln\left(\frac{1+t}{1+s}\right)\right)^{\frac{2-m}{2m}} & \text{for } n = \frac{m}{2-m},\end{cases}$$

for $\Delta < 0$.

Proof. The aim is to estimate the fundamental solutions $H_{k,\ell}$, $k = 1,2$, $\ell = 0,1$, where the representation is given by (6.15). In that way we can derive estimates for the solution w of the Cauchy problem (6.8) and its derivatives. For this reason we shall divide the extended phase space into three zones: For $0 \leq s \leq t$, we introduce the zones

$$\begin{aligned}
Z_1(N) &= \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : (1+t)|\xi| \leq N\},\\
Z_2(N,s) &= \left\{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : \frac{N}{1+t} \leq |\xi| \leq \frac{N}{1+s}\right\},\\
Z_3(N,s) &= \{(t,\xi) \in [0,\infty) \times \mathbb{R}^n : (1+s)|\xi| \geq N\}.
\end{aligned}$$

Furthermore, the *separating curve* between $Z_1(N)$, $Z_2(N,s)$ and $Z_1(N)$, $Z_3(N,s)$ is given by

$$\theta^{(1)}_{|\xi|} : (0,N] \to [0,\infty), \quad (1+\theta^{(1)}_{|\xi|})|\xi| = N.$$

We put also $\theta^{(1)}_0 = \infty$, and $\theta^{(1)}_{|\xi|} = 0$ for any $|\xi| \geq N$.

The *separating curve* between $Z_2(N,s)$ and $Z_3(N,s)$ is given by

$$(1+s)\theta^{(2)}_{|\xi|} = N \quad \text{for} \quad t \geq \theta^{(1)}_{|\xi|}.$$

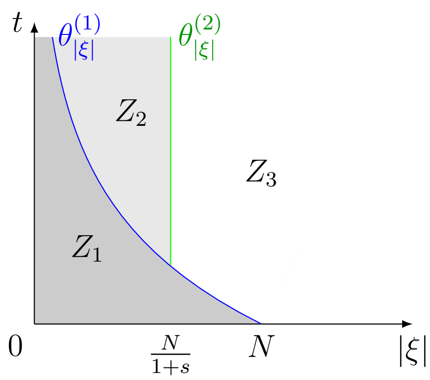

Fig. 6.1: Sketch of the zones.

In order to separate the extended phase space into three parts we introduce the function $\chi \in C^\infty(\mathbb{R}_+)$ such that $\chi(t) = 1$ for $t \leq \frac{1}{2}$, $\chi(t) = 0$ for $t \geq 2$ and $\chi'(t) \leq 0$. We can define the characteristic functions φ_1, φ_2 and φ_3 of the zones $Z_1(N)$, $Z_2(N,s)$ and $Z_3(N,s)$, respectively, by

$$\begin{aligned} \varphi_1(t,s,\xi) &= \chi\left((1+s)|\xi|N^{-1}\right)\chi\left((1+t)|\xi|N^{-1}\right), \\ \varphi_2(t,s,\xi) &= \chi\left((1+s)|\xi|N^{-1}\right)\left(1-\chi\left((1+t)|\xi|N^{-1}\right)\right), \\ \varphi_3(s,\xi) &= 1-\chi\left((1+s)|\xi|N^{-1}\right) \end{aligned}$$

such that $\varphi_1+\varphi_2+\varphi_3 = 1$, where $\varphi_1 = \varphi_1(t,s,\xi)$, $\varphi_2 = \varphi_2(t,s,\xi)$ and $\varphi_3 = \varphi_3(s,\xi)$. The proof is divided into three steps:

Considerations in $Z_1(N)$:

If $(t,\xi) \in Z_1(N)$, then $|z_0|$ and $|z|$ are small. If $\Delta < 0$, then from Lemma 6.1 and Proposition 6.2 we obtain the estimates

$$\|H_{k,\ell}(t,s,\xi)\varphi_1(t,s,\xi)\| \lesssim (1+t)^{\frac{1}{2}-\ell}(1+s)^{-\frac{3}{2}+k} \tag{6.16}$$

for all $t \geq s \geq 0$ and $(t,\xi) \in Z_1(N)$.

The following proposition is useful for the analysis of the case $\Delta = 0$.

Proposition 6.3. *For all times* $s \leq t \leq \theta^{(1)}_{|\xi|}$ *we have*

$$\left|\ln(-z)\ln(z_0)e^{\frac{z-z_0}{2}} - \ln z\ln(-z_0)e^{-\frac{z-z_0}{2}}\right| \lesssim 1+\ln\left(\frac{1+t}{1+s}\right),$$

where $z = z(t) = 2i(1+t)|\xi|$ *and* $z_0 = z(s)$.

Proof. See [4]. □

If $\Delta = 0,$ then from Lemma 6.1, Proposition 6.2, 6.3 it follows

$$\|H_{k,\ell}(t,s,\xi)\varphi_1(t,s,\xi)\| \lesssim (1+t)^{\frac{1}{2}-\ell}(1+s)^{-\frac{3}{2}+k}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big). \tag{6.17}$$

Case $\Delta < 0$: From the representations (6.13) and (6.14) we can estimate the behavior of $\widehat{w}$ and $\widehat{w}_t$. Therefore for the elastic energy we have,

$$\begin{aligned}
|\xi||\widehat{w}(t,\xi)\varphi_1| &\lesssim |\xi|\Big(\|H_{1,0}(t,s,\xi)\varphi_1\||\widehat{w}(s,\xi)| + \|H_{2,0}(t,s,\xi)\varphi_1\||\widehat{w}_t(s,\xi)|\Big)\\
&\lesssim |\xi|(1+t)^{\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)| + (1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)|\Big)\\
&\lesssim (1+t)^{-\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)| + (1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)|\Big).
\end{aligned}$$

Therefore, applying Parseval's equation we deduce the following $L^2 - L^2$ estimate:

$$\|F^{-1}\left(|\xi|\widehat{w}(t,\xi)\varphi_1\right)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^2} + (1+s)^{\frac{1}{2}}\|w_1\|_{L^2}\Big).$$

For the kinetic energy we have

$$\begin{aligned}
|\widehat{w}_t(t,\xi)\varphi_1| &\lesssim \|H_{1,1}(t,s,\xi)\varphi_1\||\widehat{w}(s,\xi)| + \|H_{2,1}(t,s,\xi)\varphi_1\||\widehat{w}_t(s,\xi)|\\
&\lesssim (1+t)^{-\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)| + (1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)|\Big).
\end{aligned}$$

Therefore, applying Parseval's equation we deduce the following $L^2 - L^2$ estimate:

$$\|F^{-1}\left(\widehat{w}_t(t,\xi)\varphi_1\right)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^2} + (1+s)^{\frac{1}{2}}\|w_1\|_{L^2}\Big).$$

For the potential energy we have

$$\begin{aligned}
|\widehat{w}(t,\xi)\varphi_1| &\lesssim \|H_{1,0}(t,s,\xi)\varphi_1^{\frac{1}{2}}\||\widehat{w}(s,\xi)\varphi_1^{\frac{1}{2}}| + \|H_{2,0}(t,s,\xi)\varphi_1^{\frac{1}{2}}\||\widehat{w}_t(s,\xi)\varphi_1^{\frac{1}{2}}|\\
&\lesssim (1+t)^{\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)\varphi_1^{\frac{1}{2}}| + (1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)\varphi_1^{\frac{1}{2}}|\Big).
\end{aligned}$$

Let us denote by m' the conjugate to m. Then using L^m regularity on the data, Hölder and Hausdorff-Young inequalities we get

$$\begin{aligned}
\int_{\mathbb{R}^n}|\widehat{w}(s,\xi)|^2\varphi_1(t,s,\xi)d\xi &\le \Big(\int_{\mathbb{R}^n}\varphi_1(t,s,\xi)^{\frac{m}{2-m}}d\xi\Big)^{\frac{2-m}{m}}\|\widehat{w}(s,\xi)\|^2_{L^{m'}}\\
&\le \Big(\int_{(1+t)|\xi|\le N}d\xi\Big)^{\frac{2-m}{m}}\|w_0\|^2_{L^m}\\
&\le (1+t)^{-\frac{n(2-m)}{m}}\|w_0\|^2_{L^m}.
\end{aligned}$$

Analogously,

$$\int_{\mathbb{R}^n}|\widehat{w}_t(s,\xi)|^2\varphi_1(t,s,\xi)d\xi \le (1+t)^{-\frac{n(2-m)}{m}}\|w_1\|^2_{L^m}.$$

Applying Parseval's equation we arrive at

$$\|F^{-1}\left(\widehat{w}(t,\xi)\varphi_1\right)(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{1}{2}-\frac{n(2-m)}{2m}}\Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^m}+(1+s)^{\frac{1}{2}}\|w_1\|_{L^m}\Big).$$

Case $\Delta=0$: For the elastic energy we have,

$$\begin{aligned}|\xi||\widehat{w}(t,\xi)\varphi_1| &\lesssim |\xi|\Big(\|H_{1,0}(t,s,\xi)\varphi_1\||\widehat{w}(s,\xi)|+\|H_{2,0}(t,s,\xi)\varphi_1\||\widehat{w}_t(s,\xi)|\Big)\\ &\lesssim |\xi|(1+t)^{\frac{1}{2}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)|+(1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)|\Big)\\ &\lesssim (1+t)^{-\frac{1}{2}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)|+(1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)|\Big).\end{aligned}$$

Therefore, applying Parseval's equation we deduce the following L^2-L^2 estimate:

$$\|F^{-1}\left(|\xi|\widehat{w}(t,\xi)\varphi_1\right)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)\Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^2}+(1+s)^{\frac{1}{2}}\|w_1\|_{L^2}\Big).$$

For the kinetic energy we have

$$\begin{aligned}|\widehat{w}_t(t,\xi)\varphi_1| &\lesssim \|H_{1,1}(t,s,\xi)\varphi_1\||\widehat{w}(s,\xi)|+\|H_{2,1}(t,s,\xi)\varphi_1\||\widehat{w}_t(s,\xi)|\\ &\lesssim (1+t)^{-\frac{1}{2}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)|+(1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)|\Big).\end{aligned}$$

Therefore, applying Parseval's equation we deduce the following L^2-L^2 estimate:

$$\|F^{-1}\left(\widehat{w}_t(t,\xi)\varphi_1\right)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)\Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^2}+(1+s)^{\frac{1}{2}}\|w_1\|_{L^2}\Big).$$

For the potential energy we have

$$\begin{aligned}|\widehat{w}(t,\xi)\varphi_1| &\lesssim \|H_{1,0}(t,s,\xi)\varphi_1^{\frac{1}{2}}\||\widehat{w}(s,\xi)\varphi_1^{\frac{1}{2}}|+\|H_{2,0}(t,s,\xi)\varphi_1^{\frac{1}{2}}\||\widehat{w}_t(s,\xi)\varphi_1^{\frac{1}{2}}|\\ &\lesssim (1+t)^{\frac{1}{2}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)\varphi_1^{\frac{1}{2}}|+(1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)\varphi_1^{\frac{1}{2}}|\Big).\end{aligned}$$

Let us denote by m' the conjugate to m. Then using L^m regularity of the data, Hölder and Hausdorff-Young inequalities we get

$$\begin{aligned}\int_{\mathbb{R}^n}|\widehat{w}(s,\xi)|^2\varphi_1(t,s,\xi)d\xi &\leq \Big(\int_{\mathbb{R}^n}\varphi_1(t,s,\xi)^{\frac{m}{2-m}}d\xi\Big)^{\frac{2-m}{m}}\|\widehat{w}(s,\xi)\|^2_{L^{m'}}\\ &\leq \Big(\int_{(1+t)|\xi|\leq N}d\xi\Big)^{\frac{2-m}{m}}\|w_0\|^2_{L^m}\\ &\leq (1+t)^{-\frac{n(2-m)}{m}}\|w_0\|^2_{L^m}.\end{aligned}$$

Analogously,

$$\int_{\mathbb{R}^n}|\widehat{w}_t(s,\xi)|^2\varphi_1(t,s,\xi)d\xi \leq (1+t)^{-\frac{n(2-m)}{m}}\|w_1\|^2_{L^m}.$$

Applying Parseval's equation we arrive at

$$\begin{aligned}\|F^{-1}\left(\widehat{w}(t,\xi)\varphi_1\right)(t,\cdot)\|_{L^2} &\lesssim (1+t)^{\frac{1}{2}-\frac{n(2-m)}{2m}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)\\ &\quad\times\Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^m}+(1+s)^{\frac{1}{2}}\|w_1\|_{L^m}\Big).\end{aligned}$$

Considerations in $Z_2(N,s)$:

If $(s,\xi),(t,\xi)\in Z_2(N,s)$, then $|z_0|$ is small and $|z|$ is large. Suppose $\Delta<0$. Then from Lemma 6.1 and Proposition 6.2 we obtain the estimates

$$\|H_{k,\ell}(t,s,\xi)\varphi_2(t,s,\xi)\|\lesssim|\xi|^{-\frac{1}{2}+\ell}(1+s)^{-\frac{3}{2}+k} \tag{6.18}$$

for all $(t,\xi)\in Z_2(N,s)$.

If $\Delta=0$, then

$$\|H_{k,\ell}(t,s,\xi)\varphi_2(t,s,\xi)\|\lesssim|\xi|^{-\frac{1}{2}+\ell}(1+s)^{-\frac{3}{2}+k}\Big|\ln(z_0)e^{\frac{z-z_0}{2}}-\ln(-z_0)e^{-\frac{z-z_0}{2}}\Big|. \tag{6.19}$$

Observe that

$$\begin{aligned}&\ln(z_0)e^{\frac{z-z_0}{2}}-\ln(-z_0)e^{-\frac{z-z_0}{2}}=\\&\Big(\ln(2(1+s)|\xi|)+\frac{\pi}{2}i\Big)e^{i(t-s)|\xi|}-\Big(\ln(2(1+s)|\xi|)-\frac{\pi}{2}i\Big)e^{-i(t-s)|\xi|}=\\&\Big(2\ln(2(1+s)|\xi|)\sin((t-s)|\xi|)+\pi\cos((t-s)|\xi|)\Big)i.\end{aligned}$$

Therefore,

$$|\ln(z_0)e^{\frac{z-z_0}{2}}-\ln(-z_0)e^{-\frac{z-z_0}{2}}|\lesssim 1+|\ln(2(1+s)|\xi|)\sin((t-s)|\xi|)|.$$

Since $|\xi|(1+s)\le N$ in $Z_2(N,s)$, then for $|\xi|\neq 0$ the second term in the last inequality is bounded. While for small frequencies we have

$$\lim_{|\xi|\to 0}\ln(2(1+s)|\xi|)\sin((t-s)|\xi|)=0.$$

Thus we may conclude that

$$\|H_{k,\ell}(t,s,\xi)\varphi_2(t,s,\xi)\|\lesssim|\xi|^{-\frac{1}{2}+\ell}(1+s)^{-\frac{3}{2}+k}. \tag{6.20}$$

If $(t,\xi)\in Z_2(N,s)$, then $\theta^{(1)}_{|\xi|}\ge s$, i.e., it is necessary to apply the "gluing procedure". Therefore, for the elastic energy we have

$$\begin{aligned}|\xi||\widehat{w}(t,\xi)\varphi_2| &\lesssim |\xi|\Big(\|H_{1,0}(t,\theta^{(1)}_{|\xi|},\xi)\varphi_2\||\widehat{w}(\theta^{(1)}_{|\xi|},\xi)|+\|H_{2,0}(t,\theta^{(1)}_{|\xi|},\xi)\varphi_2\||\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)|\Big)\\&\lesssim |\xi|^{\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(\theta^{(1)}_{|\xi|},\xi)|+(1+s)^{\frac{1}{2}}|\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)|\Big).\end{aligned}$$

For the kinetic energy,

$$\begin{aligned}|\widehat{w}_t(t,\xi)\varphi_2| &\lesssim \|H_{1,1}(t,\theta^{(1)}_{|\xi|},\xi)\varphi_2\||\widehat{w}(\theta^{(1)}_{|\xi|},\xi)|+\|H_{2,1}(t,\theta^{(1)}_{|\xi|},\xi)\varphi_2\||\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)|\\&\lesssim |\xi|^{\frac{1}{2}}\Big((1+s)^{-\frac{1}{2}}|\widehat{w}(\theta^{(1)}_{|\xi|},\xi)|+(1+s)^{\frac{1}{2}}|\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)|\Big).\end{aligned}$$

Now we use the estimates derived in the zone $Z_1(N)$ to estimate $|\xi|^{\frac{1}{2}}(1+s)^{-\frac{1}{2}}|\widehat{w}(\theta^{(1)}_{|\xi|},\xi)|$ and $|\xi|^{\frac{1}{2}}(1+s)^{\frac{1}{2}}|\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)|$. We have the following

$$\begin{aligned}&|\xi|^{\frac{1}{2}}(1+s)^{-\frac{1}{2}}|\widehat{w}(\theta^{(1)}_{|\xi|},\xi)|\\\lesssim\ &|\xi|^{\frac{1}{2}}(1+s)^{-\frac{1}{2}}\Big(\|H_{1,0}(\theta^{(1)}_{|\xi|},s,\xi)\||\widehat{w}(s,\xi)|+\|H_{2,0}(\theta^{(1)}_{|\xi|},s,\xi)\||\widehat{w}_t(s,\xi)|\Big)\\\lesssim\ &\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\gamma}\Big((1+s)^{-1}|\widehat{w}(s,\xi)|+|\widehat{w}_t(s,\xi)|\Big)\end{aligned}$$

and

$$\begin{aligned}
&|\xi|^{\frac{1}{2}}(1+s)^{\frac{1}{2}}|\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)| \\
\lesssim\ & |\xi|^{\frac{1}{2}}(1+s)^{\frac{1}{2}}\Big(\|H_{1,1}(\theta^{(1)}_{|\xi|},s,\xi)\||\widehat{w}(s,\xi)| + \|H_{2,1}(\theta^{(1)}_{|\xi|},s,\xi)\||\widehat{w}_t(s,\xi)|\Big) \\
\lesssim\ & \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\gamma}\Big(|\xi||\widehat{w}(s,\xi)| + (1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)|\Big).
\end{aligned}$$

Therefore, applying Parseval's equation we deduce the following L^2-L^2 estimates:

$$\begin{aligned}
&\|F^{-1}\left(|\xi|\widehat{w}(t,\xi)\varphi_2(t,s,\xi)\right)(t,\cdot)\|_{L^2} \\
\lesssim\ & \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\gamma}\Big((1+s)^{-1}\|w_0\|_{L^2} + (1+s)^{\frac{1}{2}}\|w_1\|_{L^2}\Big)
\end{aligned}$$

$$\begin{aligned}
&\|F^{-1}\left(\widehat{w}_t(t,\xi)\varphi_2(t,s,\xi)\right)(t,\cdot)\|_{L^2} \\
\lesssim\ & \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\gamma}\Big((1+s)^{-1}\|w_0\|_{L^2} + (1+s)^{\frac{1}{2}}\|w_1\|_{L^2}\Big).
\end{aligned}$$

For the potential energy we get

$$\begin{aligned}
|\widehat{w}(t,\xi)\varphi_2| \lesssim\ & \|H_{1,0}(t,\theta^{(1)}_{|\xi|},\xi)\varphi_2^{\frac{1}{2}}\||\widehat{w}(\theta^{(1)}_{|\xi|},\xi)\varphi_2^{\frac{1}{2}}| + \|H_{2,0}(t,\theta^{(1)}_{|\xi|},\xi)\varphi_2^{\frac{1}{2}}\||\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)\varphi_2^{\frac{1}{2}}| \\
\lesssim\ & |\widehat{w}(\theta^{(1)}_{|\xi|},\xi)\varphi_2^{\frac{1}{2}}| + |\xi|^{-\frac{1}{2}}(1+s)^{\frac{1}{2}}|\widehat{w}_t(\theta^{(1)}_{|\xi|},\xi)\varphi_2^{\frac{1}{2}}| \\
\lesssim\ & (1+\theta^{(1)}_{|\xi|})^{\frac{1}{2}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\gamma} \\
\times\ & \Big((1+s)^{-\frac{1}{2}}|\widehat{w}(s,\xi)\varphi_2^{\frac{1}{2}}| + (1+s)^{\frac{1}{2}}|\widehat{w}_t(s,\xi)\varphi_2^{\frac{1}{2}}|\Big).
\end{aligned}$$

Using the L^m regularity of the data, Hölder and Hausdorff-Young inequalities we get for $n > \frac{m}{2-m}$ the estimates

$$\begin{aligned}
\int_{\mathbb{R}^n}(1+\theta^{(1)}_{|\xi|})|\widehat{w}(s,\xi)|^2\varphi_2(t,s,\xi)d\xi \leq\ & \Big(\int_{\mathbb{R}^n}\Big((1+\theta^{(1)}_{|\xi|})\varphi_2(t,s,\xi)\Big)^{\frac{m}{2-m}}d\xi\Big)^{\frac{2-m}{m}}\|\widehat{w}(s,\xi)\|^2_{L^{m'}} \\
\lesssim\ & \Big(\int_{\frac{N}{1+t}}^{\frac{N}{1+s}}|\xi|^{-\frac{m}{2-m}}d\xi\Big)^{\frac{2-m}{m}}\|w_0\|^2_{L^m} \\
\lesssim\ & (1+s)^{\frac{m-n(2-m)}{m}}\|w_0\|^2_{L^m}.
\end{aligned}$$

Analogously,

$$\int_{\mathbb{R}^n}(1+\theta^{(1)}_{|\xi|})|\widehat{w}_t(s,\xi)|^2\varphi_2(t,s,\xi)d\xi \lesssim (1+s)^{\frac{m-n(2-m)}{m}}\|w_1\|^2_{L^m}.$$

Applying Parseval's equation we arrive at

$$\begin{aligned}
\|F^{-1}\left(\widehat{w}(t,\xi)\varphi_2\right)(t,\cdot)\|_{L^2} \lesssim\ & \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\gamma}(1+s)^{\frac{m-n(2-m)}{2m}} \\
\times\ & \Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^m} + (1+s)^{\frac{1}{2}}\|w_1\|_{L^m}\Big).
\end{aligned}$$

For $n=\frac{m}{2-m}$ we have

$$\begin{aligned}\int_{\mathbb{R}^n}(1+\theta^{(1)}_{|\xi|})|\widehat{w}(s,\xi)|^2\varphi_2(t,s,\xi)d\xi &\leq \Big(\int\Big((1+\theta^{(1)}_{|\xi|})\varphi_2(t,s,\xi)\Big)^{\frac{m}{2-m}}d\xi\Big)^{\frac{2-m}{m}}\|\widehat{w}(s,\xi)\|^2_{L^{m'}}\\ &\lesssim \Big(\int_{\frac{N}{1+t}}^{\frac{N}{1+s}}|\xi|^{-\frac{m}{2-m}}d\xi\Big)^{\frac{2-m}{m}}\|w_0\|^2_{L^m}\\ &\sim \Big(\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\frac{2-m}{m}}\|w_0\|^2_{L^m}.\end{aligned}$$

Analogously,

$$\int_{\mathbb{R}^n}(1+\theta^{(1)}_{|\xi|})|\widehat{w}_t(s,\xi)|^2\varphi_2(t,s,\xi)d\xi \lesssim \Big(\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\frac{2-m}{m}}\|w_1\|^2_{L^m}.$$

Therefore we can conclude for $n=\frac{m}{2-m}$ the following estimate:

$$\begin{aligned}\|F^{-1}\left(\widehat{w}(t,\xi)\varphi_2\right)(t,\cdot)\|_{L^2} &\lesssim \Big(\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\frac{2-m}{2m}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{\gamma}\\ &\times \Big((1+s)^{-\frac{1}{2}}\|w_0\|_{L^m}+(1+s)^{\frac{1}{2}}\|w_1\|_{L^m}\Big).\end{aligned}$$

Considerations in $Z_3(N,s)$:

If $(s,\xi),(t,\xi)\in Z_3(N,s)$, then $|z|$ and $|z_0|$ are large and $s\geq\theta^{(1)}_{|\xi|}$. So, we do not need any "gluing procedure". From Proposition 6.2 we see that the estimates for $H_{k,\ell}$ coincide for the cases $\Delta=0$ and $\Delta<0$. Therefore from Lemma 6.1 we obtain the following estimate:

$$\|H_{k,\ell}(t,s,\xi)\varphi_3(s,\xi)\|\lesssim|\xi|^{1-k+\ell}, \tag{6.21}$$

for all $t\geq s\geq 0$, $(t,\xi)\in Z_3(N,s)$ and $\Delta\leq 0$. Then for the elastic energy we have

$$\begin{aligned}|\xi||\widehat{w}(t,\xi)\varphi_3| &\lesssim |\xi|\left(\|H_{1,0}(t,s,\xi)\varphi_3\||\widehat{w}(s,\xi)|+\|H_{2,0}(t,s,\xi)\varphi_3\||\widehat{w}_t(s,\xi)|\right)\\ &\lesssim |\xi||\widehat{w}(s,\xi)|+|\widehat{w}_t(s,\xi)|.\end{aligned}$$

Applying Parseval's equation we deduce the following estimate:

$$\|F^{-1}\left(|\xi|\widehat{w}(t,\xi)\varphi_3(s,\xi)\right)(t,\cdot)\|_{L^2}\lesssim\|w_0\|_{H^1}+\|w_1\|_{L^2}.$$

For the kinetic energy,

$$\begin{aligned}|\widehat{w}_t(t,\xi)\varphi_3| &\lesssim \|H_{1,1}(t,s,\xi)\varphi_3\||\widehat{w}(s,\xi)|+\|H_{2,1}(t,s,\xi)\varphi_3\||\widehat{w}_t(s,\xi)|\\ &\lesssim |\xi||\widehat{w}(s,\xi)|+|\widehat{w}_t(s,\xi)|.\end{aligned}$$

Applying Parseval's equation we deduce the following estimate:

$$\|F^{-1}\left(\widehat{w}_t(t,\xi)\varphi_3(s,\xi)\right)(t,\cdot)\|_{L^2}\lesssim\|w_0\|_{H^1}+\|w_1\|_{L^2}.$$

Finally, for the potential energy,

$$\begin{aligned}|\widehat{w}(t,\xi)\varphi_3| &\lesssim \left(\|H_{1,0}(t,s,\xi)\varphi_3\||\widehat{w}(s,\xi)|+\|H_{2,0}(t,s,\xi)\varphi_3\||\widehat{w}_t(s,\xi)|\right)\\ &\lesssim |\widehat{w}(s,\xi)|+|\xi|^{-1}|\widehat{w}_t(s,\xi)|\\ &\lesssim |\widehat{w}(s,\xi)|+(1+s)|\widehat{w}_t(s,\xi)|.\end{aligned}$$

Applying Parseval's equation we deduce the following $L^2 - L^2$ estimate:

$$\|F^{-1}\left(\widehat{w}(t,\xi)\varphi_3(s,\xi)\right)(t,\cdot)\|_{L^2} \lesssim \|w_0\|_{L^2} + (1+s)\|w_1\|_{L^2}.$$

Using the relation $v(t,x) = (1+t)^{-\frac{\mu_1}{2}} w(t,x)$ the proof is completed. □

Remark 6.1. *In the case where the potential term is non-effective, i.e., when $0 < \mu < \frac{1}{4}$, D'Abbicco has proved in the paper [14] that additional regularity L^1 on the initial data brings better estimates for the potential energy. This is a similar result obtained in the last theorem for the not non-effective case.*

In the previous theorem, the best decay behavior appears when $m = 1$. Actually with additional regularity L^1 on the data we have the following decay for the solution and its derivatives:

Corollary 6.1. *Suppose that $(v_0, v_1) \in \mathcal{D}_1$ and $\Delta \le 0$. Then the solution $v \in \mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$ for the Cauchy problem (6.2) satisfies*

$$\begin{aligned}
\|(v_t(t,\cdot), \nabla_x v(t,\cdot))\|_{L^2} &\lesssim (1+t)^{-\frac{\mu_1}{2}}\left(1+\ln\left(\frac{1+t}{1+s}\right)\right)^{\gamma}\left(\|v_0\|_{H^1} + (1+s)^{\frac{1}{2}}\|v_1\|_{L^2}\right), \\
\|v(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{\mu_1}{2}} q_\Delta(t,s)\big(\|v_0\|_{H^1\cap L^1} + (1+s)\|v_1\|_{L^2\cap L^1}\big)
\end{aligned}$$

for all $t \ge s \ge 0$, where $\gamma = 1$ if $\Delta = 0$, $\gamma = 0$ if $\Delta < 0$ and

$$q_0(t,s) = \begin{cases} 1+\ln\left(\frac{1+t}{1+s}\right) & \text{for } n > 1, \\ \left(\ln\left(\frac{1+t}{1+s}\right)\right)^{\frac{1}{2}}\left(1+\ln\left(\frac{1+t}{1+s}\right)\right) & \text{for } n = 1, \end{cases}$$

and

$$q_\Delta(t,s) = \begin{cases} 1 & \text{for } n > 1, \\ \left(\ln\left(\frac{1+t}{1+s}\right)\right)^{\frac{1}{2}} & \text{for } n = 1, \end{cases}$$

for $\Delta < 0$.

Remark 6.2. *Indeed, L^1 regularity improves the estimate of the solution. If the initial data $(v_0, v_1) \in H^1 \times L^2$, then we can only prove*

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\mu_1}{2}+\frac{1}{2}}(1+\ln(1+t))^{\gamma}\big(\|v_0\|_{H^1} + (1+s)\|v_1\|_{L^2}\big),$$

where $\gamma = 1$ if $\Delta = 0$ and $\gamma = 0$ if $\Delta < 0$. This is a worse estimate than the one derived in the previous theorem. For the derivatives we can not use L^1 regularity because the integral $\int_N^\infty \omega_n r^{n-1} dr$ is infinity, i.e., for large frequencies we have no benefit of the additional L^1 regularity.

6.3 Global existence in time and decay behavior

Here we follow the techniques for semi-linear problems contained in the papers [14], [15] and in the PhD thesis [2].

6.3.1 Application of Banach's fixed-point theorem

The goal is to prove global existence (in time) of small data energy solution for the Cauchy problem (6.1) and decay estimates for the solution and its derivatives. For this purpose we introduce for $t > 0$ the family of spaces

$$X(t) = \{u \in \mathcal{C}([0,t], H^1) \cap \mathcal{C}^1([0,t], L^2)\}$$

with the norm

$$\|u\|_{X(t)} = \sup_{0\leq\tau\leq t}\Big[(1+\tau)^{\frac{\mu_1}{2}}\left(\widetilde{q}_\Delta(\tau)^{-1}\|u(\tau,\cdot)\|_{L^2} + (1+\ln(1+\tau))^{-\gamma}\|\left(\nabla_x u(\tau,\cdot), u_t(\tau,\cdot)\right)\|_{L^2}\right)\Big],$$

where $\gamma = 1$ if $\Delta = 0$, $\gamma = 0$ if $\Delta < 0$ and

$$\widetilde{q}_0(\tau) = \begin{cases} 1+\ln(1+\tau) \quad \text{for} & n > 1, \\ \Big(\ln(1+\tau)\Big)^{\frac{1}{2}}(1+\ln(1+\tau)) \quad \text{for} & n = 1, \end{cases}$$

and

$$\widetilde{q}_\Delta(\tau) = \begin{cases} 1 \quad \text{for} & n > 1, \\ (\ln(1+\tau))^{\frac{1}{2}} \quad \text{for} & n = 1, \end{cases}$$

for $\Delta < 0$.

We remark that the norm of the space $X(t)$ is defined according to the linear estimates. Therefore, if we show the existence of solutions for the semi-linear Cauchy problem in $X(t)$, then automatically this solution will have the same decay estimates as the solutions in the linear case.

Let $u \in X(t)$ and define the following operator

$$Nu(t,x) = E_0(t,0,x) *_{(x)} u_0(x) + E_1(t,0,x) *_{(x)} u_1(x) + \int_0^t E_1(t,s,x) *_{(x)} |u(s,x)|^p ds.$$

Our goal is reduced to prove the existence of a fixed point for the operator N. We know that $(X(t), \|\cdot\|_{X(t)})$ is a Banach space, so to use Banach's fixed-point theorem we shall prove the following two estimates:

$$\|Nu\|_{X(t)} \leq C\|(u_0,u_1)\|_{\mathcal{D}_1} + C\|u\|^p_{X(t)}, \tag{6.22}$$

$$\|Nu - Nv\|_{X(t)} \leq C\|u-v\|_{X(t)}\Big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\Big) \tag{6.23}$$

for $u, v \in X(t)$, uniformly with respect to $t \in [0,\infty)$.

Theorem 6.2. *Let $n \leq 4$, $\Delta \leq 0$ and suppose that $\mu_1 > 2$ and*

$$\begin{cases} p \geq 2 & \text{if} \quad n = 1,2, \\ 2 \leq p \leq 3 = p_{GN}(3) & \text{if} \quad n = 3, \\ p = 2 = p_{GN}(4) & \text{if} \quad n = 4. \end{cases} \tag{6.24}$$

There exists $\varepsilon_0 > 0$ such that for all $(u_0,u_1) \in \mathcal{D}_1$ with

$$\|(u_0,u_1)\|_{\mathcal{D}_1} \leq \varepsilon_0$$

there exists a unique solution to (6.1) *in* $\mathcal{C}([0,\infty),H^1)\cap\mathcal{C}^1([0,\infty),L^2)$. *Moreover, there exists a constant* $C>0$ *such that the solution satisfies the decay estimates*

$$\begin{aligned}\|(u_t(t,\cdot),\nabla_x u(t,\cdot))\|_{L^2} &\le C(1+t)^{-\frac{\mu_1}{2}}(1+\ln(1+t))^{\gamma}\|(u_0,u_1)\|_{\mathcal{D}_1},\\ \|u(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(t)\|(u_0,u_1)\|_{\mathcal{D}_1},\end{aligned}$$

for all $t\ge 0$, *where* $\gamma=1$ *if* $\Delta=0$ *and* $\gamma=0$ *if* $\Delta<0$.

After these considerations we know that to show global existence in time for small data solution is equivalent to show the inequalities (6.22) and (6.23). More precisely, we put

$$\|u\|_{X_0(t)}=\sup_{0\le\tau\le t}\Big[(1+\tau)^{\frac{\mu_1}{2}}\left(\widetilde{q}_\Delta(\tau)^{-1}\|u(\tau,\cdot)\|_{L^2}+(1+\ln(1+\tau))^{-\gamma}\|\nabla_x u(\tau,\cdot)\|_{L^2}\right)\Big] \tag{6.25}$$

where $\gamma=1$ if $\Delta=0$, $\gamma=0$ if $\Delta<0$ and we prove two stronger inequalities than (6.22) and (6.23), that are

$$\|Nu\|_{X(t)} \le C\|(u_0,u_1)\|_{\mathcal{D}_1}+C\|u\|^p_{X_0(t)}, \tag{6.26}$$

$$\|Nu-Nv\|_{X(t)} \le C\|u-v\|_{X_0(t)}\Big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\Big) \tag{6.27}$$

uniformly with respect to $t\in[0,\infty)$. The motivation to introduce the space $X_0(t)$ comes from Gagliardo-Nirenberg inequality (see Lemma 7.7). These conditions will follow from the next proposition in which the restriction on the power p and on the dimension n will appear.

Proposition 6.4. *Let us assume the condition* (6.24) *for* p. *Let* $(u_0,u_1)\in\mathcal{D}_1$ *and* $u,v\in X(t)$. *For* $j+\ell=0,1$ *it holds*

$$\begin{aligned}&(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{\ell+j-1}\|\nabla_x^j\partial_t^\ell Nu\|_{X(t)}\\ \le\;& C\|(u_0,u_1)\|_{\mathcal{D}_1}+C\|u\|^p_{X_0(t)},\end{aligned} \tag{6.28}$$

$$\begin{aligned}&(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{\ell+j-1}\|\nabla_x^j\partial_t^\ell\left(Nu-Nv\right)\|_{X(t)}\\ \le\;& C\|u-v\|_{X_0(t)}\Big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\Big),\end{aligned} \tag{6.29}$$

where $\gamma=1$ *if* $\Delta=0$ *and* $\gamma=0$ *if* $\Delta<0$.

Proof. We first prove (6.28). Basically we use the definition of the norm in $X_0(t)$, the estimates for the linear Cauchy problem in Corollary 6.1 and Gagliardo-Nirenberg inequality (see Remark 7.1). First we have that

$$\begin{aligned}\|\nabla_x^j\partial_t^\ell Nu(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac{\mu_1}{2}}(1+\ln(1+t))^{\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{1-\ell-j}\|(u_0,u_1)\|_{\mathcal{D}_1}\\ &\quad+\int_0^t\left\|\nabla_x^j\partial_t^\ell\left(E_1(t,s,x)*_{(x)}|u(s,x)|^p\right)\right\|_{L^2}ds\end{aligned}$$

for $j+\ell=0,1$. Then,

$$\begin{aligned}&(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{\ell+j-1}\|\nabla_x^j\partial_t^\ell Nu(t,\cdot)\|_{L^2}\le C\|(u_0,u_1)\|_{\mathcal{D}_1}\\ +\;& C\int_0^t(1+s)^{\frac{2-(j+\ell)}{2}}\||u(s,\cdot)|^p\|_{L^2\cap L^1}ds.\end{aligned}$$

We have that

$$\||u(s,\cdot)|^p\|_{L^2\cap L^1} \lesssim \||u(s,\cdot)|^p\|_{L^1} + \||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|^p_{L^p} + \|u(s,\cdot)\|^p_{L^{2p}}.$$

Applying Gagliardo-Nirenberg inequality (see Remark 7.1) for $q = p$ and $q = 2p$ we get

$$\|u(s,\cdot)\|^p_{L^p} \lesssim \|u(s,\cdot)\|^{p(1-\theta(p))}_{L^2}\|\nabla_x u(s,\cdot)\|^{p\theta(p)}_{L^2}, \tag{6.30}$$

$$\|u(s,\cdot)\|^p_{L^{2p}} \lesssim \|u(s,\cdot)\|^{p(1-\theta(2p))}_{L^2}\|\nabla_x u(s,\cdot)\|^{p\theta(2p)}_{L^2}, \tag{6.31}$$

where

$$\theta(p) = \frac{n}{2}\frac{(p-2)}{p}, \quad \theta(2p) = \frac{n}{2}\frac{(p-1)}{p}.$$

We note that the requisite $\theta(p) \geq 0$ implies that $p \geq 2$ and the requisite $\theta(2p) \leq 1$ implies that $p \leq p_{GN}(n) = \frac{n}{n-2}$ for $n \geq 3$. Now we are able to estimate $\||u(s,\cdot)|^p\|_{L^2\cap L^1}$ using (6.30), (6.31) and the definition of $\|\cdot\|_{X_0(t)}$:

$$\||u(s,\cdot)|^p\|_{L^2\cap L^1} \lesssim \|u\|^p_{X_0(t)}(1+s)^{-\frac{\mu_1}{2}p}\widetilde{q}_\Delta(s)^{p(1-\theta(p))}(1+\ln(1+s))^{\gamma p\theta(2p)} \tag{6.32}$$

$$= \|u\|^p_{X_0(t)}(1+s)^{-\frac{\mu_1}{2}p}\widetilde{q}_\Delta(s)^{p\left(1-\frac{n}{2}\right)+n}(1+\ln(1+s))^{\gamma n\frac{p-1}{2}}, \tag{6.33}$$

since $\theta(p) \leq \theta(2p)$. Therefore, if $n > 1$, then we have for all $\varepsilon > 0$

$$\begin{aligned}
&(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\|\nabla^j_x\partial^\ell_t Nu(t,\cdot)\|_{L^2}\\
\leq\ & C\|(u_0,u_1)\|_{\mathcal{D}_1} + C\|u\|^p_{X_0(t)}\int_0^t (1+s)^{\frac{2-(j+\ell)}{2}-\frac{\mu_1}{2}p}(1+\ln(1+s))^{\gamma\left(p+\frac{n}{2}\right)}ds\\
\leq\ & C\|(u_0,u_1)\|_{\mathcal{D}_1} + C\|u\|^p_{X_0(t)}\int_0^t (1+s)^{1-\frac{\mu_1}{2}p+\varepsilon}ds.
\end{aligned}$$

Our assumption implies that $4 < \mu_1 p$, then we can conclude

$$(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\|\nabla^j_x\partial^\ell_t Nu(t,\cdot)\|_{L^2} \leq C\|(u_0,u_1)\|_{\mathcal{D}_1} + C\|u\|^p_{X_0(t)}.$$

If $n = 1$, then for all $\varepsilon > 0$

$$\begin{aligned}
&(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{\ell+j-1}\|\nabla^j_x\partial^\ell_t Nu(t,\cdot)\|_{L^2}\\
\leq\ & C\|(u_0,u_1)\|_{\mathcal{D}_1}\\
+\ & C\|u\|^p_{X_0(t)}\int_0^t (1+s)^{\frac{2-(j+\ell)}{2}-\frac{\mu_1}{2}p}(\ln(1+s))^{\frac{p+2}{4}}(1+\ln(1+s))^{\gamma\left(p+\frac{1}{2}\right)}ds\\
\leq\ & C\|(u_0,u_1)\|_{\mathcal{D}_1} + C\|u\|^p_{X_0(t)}\int_0^t (1+s)^{1-\frac{\mu_1}{2}p+\epsilon}ds.
\end{aligned}$$

Therefore we conclude

$$(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{\ell+j-1}\|\nabla^j_x\partial^\ell_t Nu(t,\cdot)\|_{L^2} \leq C\|(u_0,u_1)\|_{\mathcal{D}_1} + C\|u\|^p_{X_0(t)}.$$

Now we prove (6.29). We remark that

$$\|Nu - Nv\|_{X(t)} = \left\|\int_0^t E_1(t,s,x) *_{(x)} \left(|u(s,x)|^p - |v(s,x)|^p\right)ds\right\|_{X(t)}.$$

Thanks to the linear estimates for the solutions to the family of parameter dependent Cauchy problems we can estimate

$$(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{\ell+j-1}\left\|\nabla_x^j\partial_t^\ell E_1(t,s,x)*_{(x)}\left(|u(s,x)|^p-|v(s,x)|^p\right)\right\|_{L^2}$$
$$\lesssim (1+s)^{\frac{2-(j+\ell)}{2}}\,\||u(s,x)|^p-|v(s,x)|^p\|_{L^2\cap L^1}$$

for $j+\ell=0,1$. Now

$$||u(s,x)|^p-|v(s,x)|^p| \lesssim |u(s,x)-v(s,x)|\left(|u(s,x)|^{p-1}-|v(s,x)|^{p-1}\right).$$

Applying Hölder's inequality we can arrive at

$$\begin{aligned}
\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{L^1} &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^p}\left(\|u(s,\cdot)\|_{L^p}^{p-1}+\|v(s,\cdot)\|_{L^p}^{p-1}\right),\\
\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{L^2} &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{2p}}\left(\|u(s,\cdot)\|_{L^{2p}}^{p-1}+\|v(s,\cdot)\|_{L^{2p}}^{p-1}\right).
\end{aligned}$$

Using Gagliardo-Nirenberg inequality we get

$$\begin{aligned}
\|u(s,\cdot)-v(s,\cdot)\|_{L^p} &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^2}^{1-\theta(p)}\|\nabla_x\left(u(s,\cdot)-v(s,\cdot)\right)\|_{L^2}^{\theta(p)}\\
&\lesssim (1+s)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(s)^{1-\frac{n(p-2)}{2p}}(\ln(1+s))^{\gamma\frac{n(p-2)}{2p}}\|u(s,\cdot)-v(s,\cdot)\|_{X_0(t)},\\
\|u(s,\cdot)-v(s,\cdot)\|_{L^{2p}} &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^2}^{1-\theta(2p)}\|\nabla_x\left(u(s,\cdot)-v(s,\cdot)\right)\|_{L^2}^{\theta(2p)}\\
&\lesssim (1+s)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(s)^{1-\frac{n(p-1)}{2p}}(\ln(1+s))^{\gamma\frac{n(p-1)}{2p}}\|u(s,\cdot)-v(s,\cdot)\|_{X_0(t)},
\end{aligned}$$

and

$$\begin{aligned}
\|u(s,\cdot)\|_{L^p} &\lesssim \|u(s,\cdot)\|_{L^2}^{1-\theta(p)}\|\nabla_x u(s,\cdot)\|_{L^2}^{\theta(p)}\\
&\lesssim (1+s)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(s)^{1-\frac{n(p-2)}{2p}}(\ln(1+s))^{\gamma\frac{n(p-2)}{2p}}\|u(s,\cdot)\|_{X_0(t)},\\
\|u(s,\cdot)\|_{L^{2p}} &\lesssim \|u(s,\cdot)\|_{L^2}^{1-\theta(2p)}\|\nabla_x u(s,\cdot)\|_{L^2}^{\theta(2p)}\\
&\lesssim (1+s)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(s)^{1-\frac{n(p-1)}{2p}}(\ln(1+s))^{\gamma\frac{n(p-1)}{2p}}\|u(s,\cdot)\|_{X_0(t)}.
\end{aligned}$$

Therefore, after using $4<\mu_1 p$ we may conclude

$$\begin{aligned}
&(1+t)^{\frac{\mu_1}{2}}(1+\ln(1+t))^{-\gamma(j+\ell)}\widetilde{q}_\Delta(t)^{\ell+j-1}\|\nabla_x^j\partial_t^\ell\left(Nu-Nv\right)\|_{X(t)}\\
\lesssim\ &\|u-v\|_{X_0(t)}\left(\|u\|_{X_0(t)}^{p-1}+\|v\|_{X_0(t)}^{p-1}\right)\left(\int_0^t(1+s)^{1-\frac{\mu_1}{2}p+\varepsilon}ds\right)\\
\lesssim\ &\|u-v\|_{X_0(t)}\left(\|u\|_{X_0(t)}^{p-1}+\|v\|_{X_0(t)}^{p-1}\right),
\end{aligned}$$

what we wanted to prove. □

Remark 6.3. *It is possible to prove global existence (in time) of small data energy solutions for $\mu_1<2$ for $n=1,2,3$ imposing new hypothesis for p. Indeed, let us suppose that*

$$p>1+\frac{4-\mu_1}{\mu_1}, \tag{6.34}$$

and $0<\mu_1<2$ for $n=1,2$, $\frac{3}{4}<\mu_1<2$ for $n=3$ and $\Delta\le 0$. There is a constant $\varepsilon_0>0$ such that for all $(u_0,u_1)\in\mathcal{D}_1$ with $\|(u_0,u_1)\|_{\mathcal{D}_1}<\varepsilon_0$ there exists a unique *solution to (6.1) in $\mathcal{C}([0,\infty),H^1)\cap\mathcal{C}^1([0,\infty),L^2)$. Moreover, there exists a constant $C>0$ such that the solution satisfies the decay estimates*

$$\begin{aligned}\|(u_t(t,\cdot),\nabla_x u(t,\cdot))\|_{L^2} &\leq C(1+t)^{-\frac{\mu_1}{2}}(1+\ln(1+t))^{\gamma}\|(u_0,u_1)\|_{\mathcal{D}_1},\\ \|u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(t)\|(u_0,u_1)\|_{\mathcal{D}_1}.\end{aligned}$$

Remark 6.4. *When $\mu_2=0$ and $0<\mu_1\leq 1$, then Wakasuki has proved blow-up results for*

$$1<p\leq 1+\frac{2}{n+(\mu_1-1)}$$

in his PhD thesis [56]. Note that the only case that we can choose $\mu_2=0$ is when $\Delta=0$ and $\mu_1=1$. From Remark 6.3 we can conclude that there exist a global solution in time for $p>4$ and Wakasuki has proved blow-up for $1<p\leq 1+\frac{2}{n}$. There is a gap for $1+\frac{2}{n}<p\leq 4$.

Remark 6.5. *Note that for $n=2,3,4$ we proved global existence (in time) of small data energy solutions for $\mu_1>2$, which is a large set of choices for μ_1 compare with the results in [14]. For $n=1$ we improve the choice of the power of non-linearity for $p\geq 2$, but we pay a price choosing $\mu_1>2$. In the paper [14], D'Abbicco proved global existence of small data energy solutions for $p>3$ and $\mu_1\geq\frac{5}{3}$, if we restrict ourselves to $p>3$ it is possible to prove global existence (in time) for $\mu_1\geq\frac{4}{3}$ which is also a large set of choices for μ_1. So, in general, the presence of the mass term allows us to consider smaller μ_1. We will write this information in the next corollary.*

Corollary 6.2. *Let $n=1$ and suppose $\mu_1\geq\frac{4}{3}$, $\Delta\leq 0$ and $p>p_{Fuj}(1)=3$. There exists $\varepsilon_0>0$ such that for all $(u_0,u_1)\in\mathcal{D}_1$ with*

$$\|(u_0,u_1)\|_{\mathcal{D}_1}\leq\varepsilon_0$$

there exists a unique solution to (6.1) in $\mathcal{C}([0,\infty),H^1)\cap\mathcal{C}^1([0,\infty),L^2)$. Moreover, there exists a constant $C>0$ such that the solution satisfies the decay estimates

$$\begin{aligned}\|(u_t(t,\cdot),\nabla_x u(t,\cdot))\|_{L^2} &\leq C(1+t)^{-\frac{\mu_1}{2}}(1+\ln(1+t))^{\gamma}\|(u_0,u_1)\|_{\mathcal{D}_1},\\ \|u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{\mu_1}{2}}\widetilde{q}_\Delta(t)\|(u_0,u_1)\|_{\mathcal{D}_1},\end{aligned}$$

where $\gamma=1$ if $\Delta=0$ and $\gamma=0$ if $\Delta<0$.

6.4 Expectations for $\Delta=1$

The goal in this section is to prove blow-up results and to show that in the case when $\Delta=1$ we expect a shift for the critical Strauss exponent as observed in [16].

Let us consider the following semi-linear scaling-invariant Cauchy problem for the wave equation with time-dependent mass and dissipation

$$u_{tt}-\Delta u+\frac{\mu_1}{1+t}u_t+\frac{\mu_2^2}{(1+t)^2}u=|u|^p,\ \ u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x), \tag{6.35}$$

with $(t,x)\in[0,\infty)\times\mathbb{R}^n$, $p>1$ and $\mu_1>0$, μ_2 are real constants. Let us suppose that

$$\Delta=(\mu_1-1)^2-4\mu_2^2=1. \tag{6.36}$$

Performing the change of variables

$$u(t,x) = (1+t)^{-\frac{\mu_1}{2}} v(t,x),$$

we arrive at the following Cauchy problem:

$$v_{tt} - \Delta v + \frac{\mu_1(2-\mu_1)+4\mu_2^2}{4(1+t)^2} v = (1+t)^{-\frac{\mu_1}{2}(p-1)}|v|^p, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x), \tag{6.37}$$

with $v_0(x) = u_0(x)$ and $v_1(x) = u_1(x) + \frac{\mu_1}{2}u_0(x)$. Note that $\Delta = 1$ implies that $\mu_1(2-\mu_1) = -4\mu_2^2$, then the Cauchy problem (6.37) becomes the Cauchy problem for the semi-linear wave equation with non-linearity $(1+t)^{-\frac{\mu_1}{2}(p-1)}|v|^p$, i.e.,

$$v_{tt} - \Delta v = (1+t)^{-\frac{\mu_1}{2}(p-1)}|v|^p, \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x). \tag{6.38}$$

Therefore we can expect that the critical value of p is a shift of the Strauss exponent and we can apply the blow-up methods developed in the paper [64] to the Cauchy problem (6.35). The following theorem was conjectured by D'Abbicco-Lucente-Reissig in the paper [16] and we will give a formal proof in this section.

Theorem 6.3. *Assume that $u \in \mathcal{C}^2([0,T)\times\mathbb{R}^n)$ is a solution to (6.35) with $\Delta = (\mu_1-1)^2 - 4\mu_2^2 = 1$ and initial data $(u_0,u_1) \in \mathcal{C}_0^2(\mathbb{R}^n)\times\mathcal{C}_0^1(\mathbb{R}^n)$ such that $u_0, u_1 > 0$. If*

$$p \in (1, p_{\mu_1}(n)],$$

then $T < \infty$, where

$$p_{\mu_1}(n) = \max\left\{p_{Fuj}\left(n-1+\frac{\mu_1}{2}\right); p_0(n+\mu_1)\right\}. \tag{6.39}$$

Remark 6.6. *In the paper [16] the number $p_{\mu_1}(n)$ was clarified as follows:*

1. $p_{\mu_1}(1) = p_{Fuj}\left(\frac{\mu_1}{2}\right)$,
2. $p_{\mu_1}(2) = \begin{cases} p_{Fuj}\left(1+\frac{\mu_1}{2}\right) & \text{if } \mu_1 \geq 2, \\ p_0(2+\mu_1) & \text{if } \mu_1 \in [0,2], \end{cases}$
3. $p_{\mu_1}(n) = p_0(n+\mu_1)$ *if* $n \geq 3$.

For the proof of this theorem we will use the following lemma on the blow-up dynamics for ordinary differential inequalities with polynomial non-linearity.

Lemma 6.2. *(Kato's Lemma) Let $p > 1$, $q \in \mathbb{R}$ and $F \in \mathcal{C}^2([0,T))$ be positive, satisfying*

$$\frac{d^2}{dt^2}F(t) \geq k_1(t+R)^{-q}(F(t))^p, \tag{6.40}$$

for any $t \in [T_1,T)$, for some $k_1, R > 0$ and $T_1 \in [0,T)$. If

$$F(t) \geq k_0(t+R)^a \tag{6.41}$$

for any $t \in [T_0,T)$, for some $a \geq 1$ satisfying $a > \frac{q-2}{p-1}$ and for some k_0 and $T_0 \in [0,T)$, then $T < \infty$. Moreover, let $q \geq p+1$ in (6.40) and suppose that the constant $k_0 = k_0(k_1) > 0$ is sufficiently large such that, if (6.41) holds with $a = \frac{q-2}{p-1}$ for some $T_0 \in [0,T)$, then $T < \infty$.

Proof. See [16] and [49]. □

Transforming the problem (6.35) in (6.38) with $\Delta = 1$, Theorem 6.3 follows as a consequence of the next proposition. In the proof we follow the techniques of the papers [64, 63, 16].

Proposition 6.5. *Let $f \in \mathcal{C}^2(\mathbb{R}^n)$ and $g \in \mathcal{C}^1(\mathbb{R}^n)$ be positive and compactly supported. Assume $u \in \mathcal{C}^2([0,T) \times \mathbb{R}^n)$ is the maximal, with respect to the time interval, solution to*

$$u_{tt} - \Delta u = (1+t)^{-\frac{\mu_1}{2}(p-1)}|u|^p, \;\; u(0,x) = f(x), \;\; u_t(0,x) = g(x). \tag{6.42}$$

If $1 < p \le p_{\mu_1}(n)$, with $p_{\mu_1}(n)$ as in (6.39), then $T < \infty$.

Proof. In the proof we choose $R > 0$ such that $\operatorname{supp} f, \operatorname{supp} g \subset B(R)$, where $B(R)$ is the ball centered in the origin with radius R. Therefore, $\operatorname{supp} u(t,\cdot) \subset B(R+t)$. Define

$$F(t) := \int_{\mathbb{R}^n} u(t,x)dx.$$

Thanks to the finite speed of propagation of u we have

$$\frac{d^2}{dt^2}F(t) = \int_{\mathbb{R}^n} u_{tt}(t,x)dx = (1+t)^{-\frac{\mu_1}{2}(p-1)} \int_{B(t+R)} |u(t,x)|^p dx. \tag{6.43}$$

Hölder's inequality implies that

$$\begin{aligned} |F(t)|^p &\le \Big(\int_{B(t+R)} |u(t,x)|dx\Big)^p \\ &\le \int_{B(t+R)} |u(t,x)|^p dx \Big(\int_{B(t+R)} dx\Big)^{p-1} \\ &\sim (t+R)^{n(p-1)} \int_{B(t+R)} |u(t,x)|^p dx. \end{aligned}$$

Therefore, we can conclude from (6.43) the following relation:

$$\frac{d^2}{dt^2}F(t) \gtrsim (1+t)^{-(\frac{\mu_1}{2}+n)(p-1)}|F(t)|^p. \tag{6.44}$$

We want to apply Lemma 6.2 and for this reason we need to show that $F(t)$ is positive. So let us consider the functions

$$\phi_1(x) = \int_{\mathcal{S}^{n-1}} e^{x\cdot\omega}d\omega, \quad \psi_1(t,x) = \phi_1(x)e^{-t},$$

where $\mathcal{S}^{n-1}$ is the $n-1$ dimensional sphere and

$$F_1(t) := \int_{\mathbb{R}^n} u(t,x)\psi_1(t,x)dx.$$

Applying Hölder inequality once more we have

$$\begin{aligned} |F_1(t)|^p &\le \Big(\int_{B(t+R)} |u(t,x)\psi_1(t,x)|dx\Big)^p \\ &\le \int_{B(t+R)} |u(t,x)|^p dx \Big(\int_{B(t+R)} |\psi_1(t,x)|^{\frac{p}{p-1}}dx\Big)^{p-1}. \end{aligned}$$

Therefore, from (6.43) it follows

$$\frac{d^2}{dt^2}F(t) \gtrsim (1+t)^{-\frac{\mu_1}{2}(p-1)}|F_1(t)|^p \left(\int_{|x|\le R+t} |\psi_1(t,x)|^{\frac{p}{p-1}}\right)^{-(p-1)}. \tag{6.45}$$

Note that $\psi_1(t,x) > 0$. Let us estimate the last integral. Recalling that $\psi_1(t,x) = e^{-t}\phi_1(x)$ we see that

$$\int_{B(K)} (\psi_1(t,x))^{\frac{p}{p-1}} dx \le C(K,A,p)(t+R)^{-A}$$

for any fixed $K < t+R$ and $A > 0$. Using that (see [12])

$$\phi_1(x) \lesssim |x|^{-\frac{n-1}{2}} e^{|x|} \quad \text{as} \quad |x| \to \infty$$

we get for large t and K the estimate

$$\int_{B(t+R)\setminus B(K)} (\psi_1(t,x))^{\frac{p}{p-1}} dx \lesssim \int_K^{t+R} (1+\rho)^{n-1-\frac{(n-1)p}{2(p-1)}} e^{\frac{p}{p-1}(\rho-t)} d\rho.$$

After integration by parts we have

$$\begin{aligned} &\int_K^{t+R} (1+\rho)^{n-1-\frac{(n-1)p}{2(p-1)}} e^{\frac{p}{p-1}(\rho-t)} d\rho \\ \lesssim\ & (1+t)^{n-1-\frac{(n-1)p}{2(p-1)}} - \left(n-1-\frac{(n-1)p}{2(p-1)}\right) \int_K^{t+R} e^{\frac{p}{p-1}(\rho-t)} (1+\rho)^{n-2-\frac{(n-1)p}{2(p-1)}} d\rho. \end{aligned}$$

If $n-1-\frac{(n-1)p}{2(p-1)} > 0$, i.e., $p \ge 2$ we may immediately conclude

$$\int_K^{t+R} (1+\rho)^{n-1-\frac{(n-1)p}{2(p-1)}} e^{\frac{p}{p-1}(\rho-t)} d\rho \lesssim (1+t)^{n-1-\frac{(n-1)p}{2(p-1)}}. \tag{6.46}$$

The same estimate holds if $n-1-\frac{(n-1)p}{2(p-1)} < 0$. Indeed, we may write

$$\left(1+\left(n-1-\frac{(n-1)p}{2(p-1)}\right)\frac{1}{1+K}\right) \int_K^{t+R} (1+\rho)^{n-1-\frac{(n-1)p}{2(p-1)}} e^{\frac{p}{p-1}(\rho-t)} d\rho \lesssim (1+t)^{n-1-\frac{(n-1)p}{2(p-1)}}$$

and for large K and t we recover (6.46).
Thus we can conclude from (6.45)

$$\frac{d^2}{dt^2}F(t) \gtrsim (1+t)^{-\frac{n-1+\mu_1}{2}p+n-1+\frac{\mu_1}{2}}|F_1(t)|^p. \tag{6.47}$$

The sign of the non-linearity comes into play to estimate $|F_1(t)|^p$. More precisely, the following result, which proof can be found in the paper [64], holds for any smooth solution to $u_{tt} - \Delta u = G(t,x,u)$ with positive G.

Lemma 6.3. *(Lemma 2.2 in [64]) It holds*

$$F_1(t) \gtrsim \frac{1}{2}(1-e^{-2t}) \int_{\mathbb{R}^n} (f(x)+g(x))\,\phi_1(x)dx + e^{-2t}\int_{\mathbb{R}^n} f(x)\phi_1(x)dx \tag{6.48}$$

for $t \ge 0$.

In particular, our assumptions for the initial data imply that $F_1(t) \geq c > 0$, for all $t \geq 0$. Then,

$$\frac{d^2}{dt^2}F(t) \gtrsim (1+t)^{-\frac{n-1+\mu_1}{2}p+n-1+\frac{\mu_1}{2}}, \tag{6.49}$$

for all $t \geq 0$. Integrating twice we arrive at

$$\begin{aligned} F(t) &\gtrsim (1+t)^{\max\left\{-\frac{n-1+\mu_1}{2}p+n+1+\frac{\mu_1}{2},1\right\}} + t\frac{d}{dt}F(0) + F(0) \\ &\gtrsim (1+t)^{\max\left\{-\frac{n-1+\mu_1}{2}p+n+1+\frac{\mu_1}{2},1\right\}}, \end{aligned} \tag{6.50}$$

once that our assumptions for the initial data also imply that $F(0) \geq 0$ and $\frac{d}{dt}F(0) \geq 0$.

The subcritical case:

From (6.44) and (6.50) we can apply the Lemma 6.2 under the following conditions:

$$-\frac{n-1+\mu_1}{2}p+n+1+\frac{\mu_1}{2} > n+\frac{\mu_1}{2}-\frac{2}{p-1}, \tag{6.51}$$

$$1 > n+\frac{\mu_1}{2}-\frac{2}{p-1}. \tag{6.52}$$

The condition (6.51) holds if, and only if, $p < p_0\left(n+\mu_1\right)$ and the condition (6.52) holds if, and only if, $p < p_{Fuj}\left(n-1+\frac{\mu_1}{2}\right)$. Then the proposition is true for the subcritical case

$$p < \max\Big\{p_{Fuj}\Big(n-1+\frac{\mu_1}{2}\Big), p_0(n+\mu_1)\Big\}.$$

The critical case if $n = 1$:

For $n = 1$ we have that $p_{\mu_1}(1) = 1 + \frac{4}{\mu_1}$. By (6.44) it follows that $q = 2 + \frac{4}{\mu_1}$. We note that in this case the maximum of the right-hand side of (6.50) is 1. Thus, from (6.44)

$$\frac{d^2}{dt^2}F(t) \gtrsim (1+t)^{-1}.$$

Integrating twice we arrive at

$$F(t) \gtrsim (1+t)\ln(1+t).$$

Note that $\frac{q-2}{p-1} = 1$ and $q = p+1$, then the result follows from the application of Lemma 6.2 with $a = 1$.

The critical case if $n = 2$:

Let us suppose that $\mu_1 \geq 2$, then for $n = 2$ we have that $p_{\mu_1}(2) = 1 + \frac{4}{2+\mu_1}$. By (6.44) it follows that $q = \frac{2\mu_1+8}{2+\mu_1}$. We note that in this case the maximum of the right-hand side of (6.50) is 1. Thus, from (6.44)

$$\frac{d^2}{dt^2}F(t) \gtrsim (1+t)^{-1}.$$

Integrating twice we arrive at

$$F(t) \gtrsim (1+t)\ln(1+t).$$

Note that $\frac{q-2}{p-1} = 1$ and $q = p+1$, then the result follows from the application of Lemma 6.2 with $a = 1$.

If $\mu_1 \in [0,2)$, then $p = p_{\mu_1}(2) = p_0(2+\mu_1)$. This case will be treated together with the critical case for $n \geq 3$.

The critical case if $n \geq 3$ or $n = 2$ with $\mu_1 \in [0,2)$:

For $n \geq 3$ and for $n = 2$ with $\mu_1 \in [0,2)$, we have that $p_{\mu_1}(n) = p_0(n+\mu_1)$. We can suppose, without loss of generality, that $u(t,\cdot)$ is radial. This is so because one can use Darboux's identity to transform the problem into a suitable one for the radial case. Let us define

$$\widetilde{u}(t,r) = \frac{1}{\omega_n}\int_{|\omega|=1} u(t,r\omega)d\sigma_\omega,$$

where $\omega_n = \int_{|\omega|=1} d\sigma_\omega$.
From Hölder's inequality it follows that $\widetilde{u}$ satisfies that following problem:

$$\begin{aligned}
\widetilde{u}_{tt} - \Delta\widetilde{u} &= \widetilde{|u|^p}(1+t)^{-\frac{\mu_1}{2}(p-1)} \\
&= (1+t)^{-\frac{\mu_1}{2}(p-1)}\frac{1}{\omega_n}\int_{|\omega|=1}|u(t,r\omega)|^p d\sigma_\omega \\
&\geq (1+t)^{-\frac{\mu_1}{2}(p-1)}|\widetilde{u}|^p.
\end{aligned}$$

Following the technique of [64] we consider the Radon transform of u with respect to the space variable defined by

$$Ru(t,\rho) := \int_{x\cdot\omega=\rho} u(t,x)d\sigma_x, \tag{6.53}$$

where $d\sigma_x$ is the Lebesgue measure of the hyperplane $\{x : x\cdot\omega = \rho\}$ and $\omega \in \mathbb{R}^n$ is a unitary vector. Next we show that Ru is independent of ω. From (6.53) and the assumption that $u(t,\cdot)$ is radial it follows that

$$\begin{aligned}
Ru(t,\rho) &= \int_{\{x':x'\cdot\omega=0\}} u(t,\rho\omega+x')d\sigma_x \\
&= c_n\int_0^\infty u(t,\sqrt{\rho^2+|x'|^2})|x'|^{n-2}d|x'|.
\end{aligned}$$

Using the change of variables $r^2 = \rho^2+|x'|^2$, we have

$$Ru(t,\rho) = c_n\int_{|\rho|}^\infty u(t,r)(r^2-\rho^2)^{\frac{n-3}{2}}rdr, \tag{6.54}$$

this shows that $Ru(t,\rho)$ is independent of ω. Now let us derive a lower bound for $Ru(t,\rho)$.

Since u is a solution of (6.42), then Ru satisfies the one-dimensional wave equation

$$\partial_t^2 Ru(t,\rho) - \partial_\rho^2 Ru(t,\rho) = (1+t)^{-\frac{\mu_1}{2}(p-1)}R|u|^p(t,\rho).$$

From the D'Alembert's formula and the assumptions for the initial data it follows

$$Ru(t,\rho) \geq \frac{1}{2}\int_0^t (1+s)^{-\frac{\mu_1}{2}(p-1)} \int_{\rho-(t-s)}^{\rho+(t-s)} R|u|^p(s,\rho_1)d\rho_1 ds. \tag{6.55}$$

Note that $\operatorname{supp} u(s,\cdot) \subset B(s+R)$. Therefore, if $|\rho_1| > s+R$, then, for any vector y which is perpendicular to a unit vector ω, it holds

$$|\rho_1\omega + y| = \sqrt{|\rho_1|^2 + |y|^2} \geq |\rho_1| > s+R.$$

Thus

$$R|u|^p(s,\rho_1) = \int_{\{y:y\cdot\omega=0\}} |u(s,\rho_1\omega+y)|d\sigma_y = 0.$$

This shows that

$$\operatorname{supp} R|u|^p(s,\cdot) \subset B(s+R). \tag{6.56}$$

Assume $\rho \geq 0$. If $s \leq \frac{t-\rho-R}{2}$, then

$$\rho + (t-s) \geq s+R, \;\; \rho - (t-s) \leq -(s+R).$$

By this, (6.55) and (6.56) we deduce

$$\begin{aligned} Ru(t,\rho) &\geq \frac{1}{2}\int_0^{\frac{t-\rho-R}{2}} (1+s)^{-\frac{\mu_1}{2}(p-1)} \int_{\rho-(t-s)}^{\rho+(t-s)} R|u|^p(s,\rho_1)d\rho_1 ds \\ &= \frac{1}{2}\int_0^{\frac{t-\rho-R}{2}} (1+s)^{-\frac{\mu_1}{2}(p-1)} \int_{-\infty}^{\infty} R|u|^p(s,\rho_1)d\rho_1 ds \\ &= \frac{1}{2}\int_0^{\frac{t-\rho-R}{2}} (1+s)^{-\frac{\mu_1}{2}(p-1)} \int_{\mathbb{R}^n} |u(s,y)|^p dy ds \\ &= \frac{1}{2}\int_0^{\frac{t-\rho-R}{2}} \frac{d^2}{dt^2}F(s)ds. \end{aligned}$$

Recalling (6.49) we have

$$Ru(t,\rho) \geq \frac{1}{2}\int_0^{\frac{t-\rho-R}{2}} (1+s)^{-\frac{n-1+\mu_1}{2}p+n-1+\frac{\mu_1}{2}} ds.$$

Now note that

$$-\frac{n-1+\mu_1}{2}p + n - 1 + \frac{\mu_1}{2} \neq -1.$$

Indeed, observe that

$$-\frac{n-1+\mu_1}{2}p + n - 1 + \frac{\mu_1}{2} = -1 \text{ if, and only if, } \; p = p_0(n+\mu_1) = \frac{2n+\mu_1}{n-1+\mu_1}.$$

Then recalling that

$$(n+\mu_1-1)p^2 - (n+\mu_1+1)p - 2 = 0$$

we arrive at

$$2n^2 + (\mu_1 - 4)n - (3\mu_1 - 2) = 0$$

which is a contradiction once $n \geq 3$. Note that this is also a contradiction for $n = 2$ and $\mu_1 < 2$.

So, we arrive at

$$Ru(t, \rho) \gtrsim (R + t - \rho)^{-\frac{n-1+\mu_1}{2}p + n + \frac{\mu_1}{2}}. \tag{6.57}$$

Note that for any $f \in L^p$ the operator $T : L^p \to L^p$ defined by

$$T(f)(\tau) = \frac{1}{|t - \tau + R|^{\frac{n-1}{2}}} \int_\tau^{t+R} f(r)|r - \tau|^{\frac{n-3}{2}} dr$$

is bounded. Indeed,

$$\begin{aligned} |T(f)(\tau)| &\leq \frac{1}{|t - \tau + R|} \int_\tau^{t+R} |f(r)| dr \\ &\leq \frac{2}{2|t - \tau + R|} \int_{-(t+R)+2\tau}^{t+R} |f(r)| dr \\ &\leq 2M(|f|)(\tau), \end{aligned}$$

where $M(|f|)$ is the maximal function of f. Therefore,

$$\|T(f)\|_{L^p} \leq c\|f\|_{L^p}. \tag{6.58}$$

Applying (6.58) to the function

$$f(r) = \begin{cases} |u(t, r)| r^{\frac{n-1}{p}} & \text{if} \quad r \geq 0, \\ 0 & \text{if} \quad r < 0, \end{cases}$$

we have

$$\begin{aligned} & \int_0^{t+R} \Big(\frac{1}{(t - \rho + R)^{\frac{n-1}{2}}} \int_\rho^{t+R} |u(t, r)| r^{\frac{n-1}{p}} (r - \rho)^{\frac{n-3}{2}} dr \Big)^p d\rho \\ \leq \; & C \int_0^\infty |u(t, r)|^p r^{n-1} dr \\ = \; & C \int_{\mathbb{R}^n} |u(t, x)|^p dx. \end{aligned}$$

When $r \geq \rho$ and $1 < p \leq 2$, we observe that

$$r^{\frac{n-1}{p}} \geq r^{\frac{n-1}{2}} \rho^{(n-1)\left(\frac{1}{p} - \frac{1}{2}\right)}.$$

Hence,

$$\begin{aligned} & \int_0^{t+R} \Big(\frac{1}{(t - \rho + R)^{\frac{n-1}{2}}} \int_\rho^{t+R} |u(t, r)| r^{\frac{n-1}{2}} (r - \rho)^{\frac{n-3}{2}} dr \Big)^p \rho^{n-1-(n-1)\frac{p}{2}} d\rho \\ \leq \; & C \int_{\mathbb{R}^n} |u(t, x)|^p dx. \end{aligned} \tag{6.59}$$

From (6.54) and the fact that $\operatorname{supp} u(t,\cdot) \subset B(t+R)$ we know that

$$\begin{aligned} R|u|(t,\rho) &= c_n \int_\rho^{t+R} |u(t,r)| r (r^2-\rho^2)^{\frac{n-3}{2}} dr \\ &= c_n \int_\rho^{t+R} |u(t,r)| r (r-\rho)^{\frac{n-3}{2}} (r+\rho)^{\frac{n-3}{2}} dr \\ &\le c \int_\rho^{t+R} |u(t,r)| r^{\frac{n-1}{2}} (r-\rho)^{\frac{n-3}{2}} dr. \end{aligned} \tag{6.60}$$

Substituting (6.60) to (6.59), we reach

$$\int_0^{t+R} \frac{(R|u|(t,\rho))^p}{(t-\rho+R)^{(n-1)\frac{p}{2}}} \rho^{n-1-(n-1)\frac{p}{2}} d\rho \le c \int_{\mathbb{R}^n} |u(t,x)|^p dx. \tag{6.61}$$

Using the lower bound of $R|u|$ in (6.57) and (6.61), we arrive at

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \ge C_R \int_0^{t-R-1} \frac{\rho^{n-1-(n-1)\frac{p}{2}}}{(t-\rho+R)^{\frac{(n-1+\mu_1)p^2-(n+1+\mu_1)p}{2}}} d\rho.$$

Recalling

$$(n-1+\mu_1)p^2 - (n+1+\mu_1)p - 2 = 0$$

it follows that

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \ge C_R \int_0^{t-R-1} \frac{\rho^{n-1-(n-1)\frac{p}{2}}}{t-\rho+R} d\rho.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^n} |u(t,x)|^p dx &\ge C_R \int_{\frac{t-R-1}{2}}^{t-R-1} \frac{\rho^{n-1-(n-1)\frac{p}{2}}}{t-\rho+R} d\rho \\ &\ge C_R (t-R-1)^{n-1-(n-1)\frac{p}{2}} \int_{\frac{t-R-1}{2}}^{t-R-1} \frac{1}{t-\rho+R} d\rho, \end{aligned}$$

and we obtain

$$\int_{\mathbb{R}^n} |u(t,x)|^p dx \ge C_R (t-R)^{n-1-(n-1)\frac{p}{2}} \ln(t-R).$$

Thus

$$\frac{d^2}{dt^2} F(t) \gtrsim (1+t)^{-\frac{n-1+\mu_1}{2}p+n-1+\frac{\mu_1}{2}} \ln(t-R).$$

Integrating twice we arrive at

$$F(t) \gtrsim (1+t)^{-\frac{n-1+\mu_1}{2}p+n+1+\frac{\mu_1}{2}} \ln(t-R),$$

and the result follows for sufficiently large t after applying again Lemma 6.2 with $a = -\frac{n-1+\mu_1}{2}p + n + 1 + \frac{\mu_1}{2}$. □

Summarizing, we expected for $\Delta = 1$

$$p_{crit}(n) = p_{\mu_1}(n) = \max\Big\{p_{Fuj}\Big(n-1+\frac{\mu_1}{2}\Big); p_0(n+\mu_1)\Big\}.$$

We collect in the following table the results obtained in this chapter for $\mu_1 > 2$:

	$\Delta \le 0$	$\Delta = 1$
$n = 1$	Global existence in time for $p \ge 2$.	Blow-up for $1 < p \le 1 + \frac{4}{\mu_1}$.
$n = 2$	Global existence in time for $p \ge 2$.	Blow-up for $1 < p \le 1 + \frac{4}{2+\mu_1}$.
$n = 3$	Global existence in time for $2 \le p \le 3$.	Blow-up for $1 < p \le p_0(3 + \mu_1)$.
$n = 4$	Global existence in time for $p = 2$.	Blow-up for $1 < p \le p_0(4 + \mu_1)$.
$n \ge 5$	No result for global existence in time.	Blow-up for $1 < p \le p_0(n + \mu_1)$.

Tab. 6.1: Interplay between global existence in time and blow-up result for the solution depending on the choice of Δ.

If $\Delta = 1$, then we feel a shift of Strauss exponent $p_0(n) \to p_0(n + \mu_1)$ for $n \ge 3$ (cf. with [16]).

7 Notation-Guide to the reader

7.0.1 Preliminaries

$\langle \cdot \rangle$	which stands for $\langle x \rangle = \sqrt{1+\|x\|^2}$,
$\| \cdot \|$	denotes the absolute value,
$[\xi]$	with definition $[\xi] = \frac{\|\xi\|}{\langle \xi \rangle}$
$\lceil \cdot \rceil$	denotes the smallest integer then a given number, i.e., $\lceil x \rceil = \min\{m \in \mathbb{Z}; x \leq m\}$,
$\\| \cdot \\|$	denotes the norms for a vector or a matrix.,
$\\| \cdot \\|_{L^p}$	norm in L^p spaces,
$L^p \to L^q$	for $\mathcal{L}(L^p, L^q)$, endowed with the norm topology
$\\| \cdot \\|_{L^{p,r} \to L^q}$	for operator norm in $L^{p,r} \to L^q$,
$f \lesssim g$	if there exists a constant $c > 0$ such that for all arguments $f \leq cg$.
$f \gtrsim g$	if there exists a constant $c > 0$ such that for all arguments $cf \geq g$.
$f \approx g$	if $f \gtrsim g$ and $f \lesssim g$.
$f \sim g$	if $\lim\limits_{t\to\infty} \frac{f(t)}{g(t)} = 1$, i.e., f and g have the same asymptotic behavior.
D_t	denotes $D_t = \frac{1}{i}\partial_t$.
∂_x^α	denotes the partial derivatives $\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}$ with a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where α_i is non-negative for all $i = 1, 2, \cdots, n$.
Δ	denotes the Laplace operator with respect to $x \in \mathbb{R}^n$, ie, $\Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$.
$f(t) = o(g(t))$	if $\limsup\limits_{t\to\infty} \frac{f(t)}{g(t)} = 0$.

7.0.2 Frequently used function spaces

We collect function spaces with are frequently used within this thesis.

$L^p(\mathbb{R}^n)$	L^p spaces , $1 \leq p \leq \infty$,
$L^pL^r(\mathbb{R}^n \times \mathbb{R}^m)$	mixed space $L^p(\mathbb{R}^n, L^r(\mathbb{R}^m))$,
$L^{p,\alpha}(\mathbb{R}^n)$	Bessel potential space, $L^{p,\alpha}(\mathbb{R}^n) = \langle D \rangle^{-\alpha} L^p(\mathbb{R}^n)$
$C^k(\mathbb{R}^n)$	space of k times continuously differentiable function,
$C^\infty(\mathbb{R}^n)$	space of infinitely continuously differentiable functions,

$C_0^\infty(\mathbb{R}^n)$	space of infinitely continuously differentiable functions with compact support,
$H^s(\mathbb{R}^n)$	Sobolev space based on $L^2(\mathbb{R}^n)$,
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space of rapidly decay functions,
$\mathcal{D}'(\mathbb{R}^n)$	space of distributions,
$\mathcal{S}'(\mathbb{R}^n)$	space of tempered distributions,
$B^s_{p,q}(\mathbb{R}^n)$	Besov space,
$M_p^q(\mathbb{R}^n)$	space of multipliers inducing bounded translation invariant operators $L^p \to L^q$.

7.0.3 Symbols used throughout the thesis

$h(t,\xi)$	$h(t,\xi) = \frac{1}{1+t}\phi_{pd,N}(t,\xi) + i\lvert\xi\rvert\phi_{hyp,N}(t,\xi)$, with the characteristic functions $\phi_{pd,N}(t,\xi)$ and $\phi_{hyp,N}(t,\xi)$ of the zones,
$\widetilde{h}(t,\xi)$	$\widetilde{h}(t,\xi) = \left(\lvert\xi\rvert^2 + \frac{N^2}{(1+t)^2}\right)^{\frac{1}{2}}$
$U(t,\xi)$	micro-energy $U = (h(t,\xi)\widehat{u}, D_t\widehat{u})^T$, satisfies $D_tU = A(t,\xi)U$,
$\mathcal{E}(t,s,\xi)$	fundamental solution of the $D_t - A(t,\xi)$,
$\mathcal{E}_0(t,s,\xi)$	fundamental solution of the free wave equation,
$\mathcal{E}_k(t,s,\xi)$	fundamental solution of the system after k steps of diagonalization, $k \geq 1$,
$W_+(\xi)$	multiplier corresponding to the Moeller wave operator.

7.1 Basic tools

7.1.1 Fourier multipliers and multiplier spaces

The next theorem is very important for we state L^p-L^q estimates.

Definition 7.1. *Let $f \in \mathcal{S}'$. Define the following operator*

$$m(D)f = \mathcal{F}^{-1}\left[m(\xi)\mathcal{F}(f)\right],$$

for a suitably regular function or distribution $m(\xi)$. These operator are so-called Fourier multipliers.

Definition 7.2. *Denote by $p \leq q$*

$$M_p^q = \{m(\xi); m(D) : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)\},$$

that so-called multiplier space.

The multiplier space M_p^q is a Banach space endowed with the corresponding operator norm. Holds

Proposition 7.1. *1.* $M_2^2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$,
2. $M_p^p(\mathbb{R}^n) \subset M_2^2(\mathbb{R}^n)$*, for all* $p \in [1,\infty]$,
3. $M_p^p(\mathbb{R}^n) = M_q^q(\mathbb{R}^n)$*, for dual* p *and* q*,*
4.$L^1(\mathbb{R}^n) \subset M_1^\infty(\mathbb{R}^n)$,
5. $M_1^\infty(\mathbb{R}^n) \cap M_2^2(\mathbb{R}^n) \subset M_p^q(\mathbb{R}^n)$*, for dual* p *and* q*.*

Let us enunciate the Marcinkiewicz multiplier theorem.

Theorem 7.1. *Assume that* $m(\xi) \in C^k(\mathbb{R}^n - \{0\})$ *for* $k = \lceil \frac{n}{2} \rceil + 1$ *and*

$$\left| D_\xi^\alpha m(\xi) \right| \leq C_\alpha |\xi|^{-|\alpha|}, \forall\ |\alpha| \leq k$$

in other words, $m(\xi) \in \dot{S}_k^0$*. Then* $m(\xi) \in M_p^p(\mathbb{R}^n)$ *for all* $p \in [1,\infty]$*.*

Proof. Look [51]. □

7.1.2 Further lemmas of importance

According to the papers [43, 7] we can conclude the following estimate.

Lemma 7.1. *Let us assume that* $K = K(t)$ *is a real-valued function and* $a = a(t,\xi) \in C_0^\infty(\mathbb{R}_\xi^n)$*. Then there exists a positive integer* M *such that*

$$\left\| F^{-1}\left(e^{iK(t)|\xi|} a(t,\xi)\right) \right\|_{L^\infty} \leq C\left(1 + K(t)\right)^{-\frac{n-1}{2}} \sum_{|\alpha| \leq M} \|D_\xi^\alpha a(t,\xi)\|_{L^\infty},$$

with a constant C *independent of* t *and* ξ*.*

In order to handle with $L^1 - L^\infty$ and $L^2 - L^2$ estimates, we have the following lemma:

Lemma 7.2. *Let* $a \in L^1$*.*

1. *If* $\|F^{-1}(a)\|_{L^\infty} \leq C_0$*, then* $\|F^{-1}(aF(u))\|_{L^\infty} \leq C_0 \|u\|_{L^1}$*.*

2. *If* $\|a\|_{L^\infty} \leq C_1$*, then* $\|F^{-1}(aF(u))\|_{L^\infty} \leq C_1 \|u\|_{L^2}$*.*

In order to handle the $L^p - L^q$ estimates we state the important Riesz-Thorin interpolation theorem.

Theorem 7.2. *(Riesz-Thorin Interpolation Theorem) Lets* $p_i, q_i \in [1,\infty]$*, for* $i = 0, 1$ *and if* $0 < \theta < 1$*, defines* p *and* q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T *is a linear operator of* $(L^{p_0}, L^{p_1}) \longmapsto (L^{q_0}, L^{q_1})$*, such that*

$$\|Tu\|_{L^{q_0}} \leq M_0 \|u\|_{L^{p_0}},$$

$$\|Tu\|_{L^{q_1}} \leq M_1 \|u\|_{L^{p_1}},$$

thus,

$$\|Tu\|_{L^q} \leq M_0^{1-\theta} M_1^\theta \|u\|_{L^p}.$$

Lemma 7.3. *Let $a \in L^\infty(\mathbb{R}^n)$ and assume that*

$$\|F^{-1}(a\phi_j F(u))\|_{L^q} \leq C\|u\|_{L^p}$$

uniformly for all $j \in \mathbb{Z}$ with $1 < p \leq 2$ and p, q conjugate line. Then there exists a constant A independent of the function a such that

$$\|F^{-1}(aF(u))\|_{L^q} \leq AC\|u\|_{L^p}.$$

The next theorem was strongly used to define Moeller wave operator with the goal to prove modified scattering theorem in the Chapters 2 and 4.

Theorem 7.3. *(Banach-Steinhaus Theorem) Let A and B Banach spaces and suppose that $\{F_n\}$ is a sequence of continuous linear operators from A to B. Then F_n converges pointwise to a continuous linear operator $F : A \to B$, i.e., $F_n(x)$ converges to $F(x)$ for all $x \in A$* or $F = \text{s-lim}\, F_n$, *if and only if*

1. *the sequence of operator norms $\|F_n\|$ is bounded;*
2. *the sequence $F_n(x)$ converges to $F(x)$ for all $x \in \mathcal{L}$, where $\mathcal{L}$ is a dense subset of A.*

7.1.3 The Peano-Baker formula

Theorem 7.4. *Let $A(t) \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution $\mathcal{E}(t,s)$ to*

$$\begin{cases} \frac{d\mathcal{E}}{dt}(t,s) = A(t)\mathcal{E}(t,s) \\ \mathcal{E}(s,s) = I \end{cases} . \tag{7.1}$$

is given by the Peano-Baker formula,

$$\mathcal{E}(t,s) = I + \sum_{k=1}^{\infty} \int_s^t A(t_1) \int_s^{t_1} A(t_2) \cdots \int_s^{t_{k-1}} A(t_k) dt_k \cdots dt_1. \tag{7.2}$$

Proposition 7.2. *Assume $r \in L^1_{loc}(\mathbb{R})$. Then*

$$\left| \int_s^t r(t_1) \int_s^{t_1} r(t_2) \cdots \int_s^{t_{k-1}} r(t_k) dt_k \cdots dt_1 \right| \leq \frac{1}{k!} \left(\int_s^t |r(\tau)| d\tau \right)^k , \tag{7.3}$$

for all $k \in \mathbb{N}$.

7.1.4 Faà di Bruno's formula

In this section we will write two well-known form of Faà di Bruno's formula. The most simple one can be founded in [8] and [9] and says

Lemma 7.4. *Let $f(g(x)) = (f \circ g)(x)$ with $x \in \mathbb{R}$. Then we have*

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{m_1!1!^{m_1} m_2!2!^{m_2} \cdots m_n!n!^{m_n}} f^{(m_1+\cdots+m_n)}(g(x)) \prod_{j=1}^n \left(g^{(j)}(x)\right)^{m_j},$$

where the sum is taken over all $n-$tuples of non-negative integers $(m_1, \cdots, m_n)$ satisfying the condition

$$1m_1 + 2m_2 + \cdots + nm_n = n.$$

A multivariate version of Faà di Bruno's formula can be founded in [11], [37] and is given in the next statement.

Lemma 7.5. *Let $y = g(x_1, \cdots, x_n)$. Then the following identity holds regardless of whether the n variables are all distinct, or all identical, or partitioned into several distinguishable classes of indistinguishable variables*

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} f(y) = \sum_{\pi \in \Pi} f^{(|\pi|)}(y) \prod_{B \in \pi} \frac{\partial^{|B|} y}{\prod_{j \in B} \partial x_j},$$

where,

- *π runs through the set Π of all partitions of the set $\{1. \cdots, n\}$,*
- *$B \in \pi$ means the variable B runs through the list of all "blocks" of the partition π and*
- *$|A|$ denotes the cardinality of the set A (so that $|\pi|$ is the number of blocks in the partition π and $|B|$ is the size of the block B).*

Let us give some generalizations of Faà di Bruno's formula for a composite function with a vector-valued argument, see [40].

Lemma 7.6. *If f and t are scalars, $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_r(t)]^T$ is an $r-$vector and $f(\mathbf{x}(t))$ is a composite function for which all the necessary derivatives are defined, then*

$$D^n f(\mathbf{x}(t)) = \sum_0 \sum_1 \cdots \sum_n C(n, k_i, q_{ij}) \frac{\partial^k f}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_r^{p_r}} \prod_{i=1}^n (x_1^i)^{q_{i1}} (x_2^i)^{q_{i2}} \cdots (x_r^i)^{q_{ir}},$$

where the respective sums are taken over all non-negative integer solution of the Diophantine equation as follows:

$$\begin{aligned} \sum_0 &\rightarrow k_1 + 2k_2 + \cdots + nk_n = n \\ \sum_1 &\rightarrow q_{11} + q_{12} + \cdots + q_{1r} = k_1 \\ &\vdots \\ \sum_n &\rightarrow q_{n1} + q_{n2} + \cdots + q_{nr} = k_n, \end{aligned}$$

and the differential operator $D = \frac{d}{dt}$, p_j is the order of the partial derivative with respect to x_j, and k is the order of the partial derivative, more precisely

$$\begin{aligned} p_j &= q_{1j} + q_{2j} + \cdots + q_{nj}, j = 1, 2, \cdots, r \\ k &= p_1 + p_2 + \cdots + p_r = k_1 + k_2 + \cdots + k_n. \end{aligned}$$

7.1.5 Gagliardo-Nirenberg inequality

Here we write Gagliardo-Nirenberg inequalities which come into play in the semi-linear theory to prove global existence of small energy data solutions for wave models with scale-invariant mass and dissipation.

Lemma 7.7. *Let $j, m \in \mathbb{N}$ with $j < m$, and let $u \in \mathcal{C}_c^m(\mathbb{R}^n)$, i.e., $u \in \mathcal{C}^m$ with compact support. Let $\frac{j}{m} \leq a \leq 1$, and let $p, q, r \in [1, \infty]$ such that*

$$j - \frac{n}{q} = \left(m - \frac{n}{r}\right) a - \frac{n}{p}(1-a).$$

Then

$$\|D^j u\|_{L^q} \leq C_{n,m,j,p,r,a} \|D^m u\|_r^a \|u\|_{L^p}^{1-a} \tag{7.4}$$

provided that

$$\left(m - \frac{n}{r}\right) - j \notin \mathbb{N}, \tag{7.5}$$

i.e., $\frac{n}{r} > m - j$ or $\frac{n}{r} \notin \mathbb{N}$. If (7.5) is not satisfied, then (7.4) holds provided that $\frac{j}{m} \leq a < 1$.

Proof. See Theorem 9.3 in [20] part 1. □

Remark 7.1. *If $j = 0$, $m = 1$ and $r = p = 2$, then (7.4) reduces to*

$$\|u\|_{L^q} \lesssim \|\nabla u\|_{L^2}^{\theta(q)} \|u\|_{L^2}^{1-\theta(q)}, \tag{7.6}$$

where $\theta(q)$ is given from

$$-\frac{n}{q} = \left(1 - \frac{n}{2}\right)\theta(q) - \frac{n}{2}(1 - \theta(q)) = \theta(q) - \frac{n}{2}, \tag{7.7}$$

that is,

$$\theta(q) = \frac{n}{2} - \frac{n}{q} = n\left(\frac{1}{2} - \frac{1}{q}\right).$$

It is clear that $\theta(q) \geq 0$ if and only if $q \geq 2$. Analogously $\theta(q) \leq 1$ if and only if

$$\text{either } n = 1, 2 \text{ or } q \leq q_{GN} = \frac{2n}{n-2}. \tag{7.8}$$

Applying a density argument the inequality (7.6) holds for any $u \in H^1$. Assuming $q < \infty$ the condition (7.5) can be neglected also for $n = 2$. Summarizing the estimate (7.6) holds for any finite $q \geq 2$ if $n = 1, 2$ and for any $q \in [2, q_{GN}]$ if $n \geq 3$.

7.2 Asymptotic integration lemma

In this appendix we collect some theorems on the asymptotic integration of ordinary differential equations, which are particularly useful for the treatment on the Chapter 4 of the pseudo-differential zone. We formulate them in more general form than used in the Chapter 4. They follow [18, Sections 1.3 and 1.4] adapted to systems of Fuchs type.

7.2.1 Levinson type theorems

We consider the following system of ordinary differential equations

$$t\partial_t V(t,\nu) = \big(D(t,\nu) + R(t,\nu)\big)V(t,\nu), \qquad t \ge 1, \tag{7.9}$$

depending on a parameter $\nu \in \Upsilon$. The matrix

$$D(t,\nu) = \operatorname{diag}\big(\mu_1(t,\nu),\dots,\mu_d(t,\nu)\big) \tag{7.10}$$

is diagonal and $R(t,\nu) \in \mathbb{C}^{d\times d}$ denotes a remainder term.

Under a dichotomy condition imposed on D and appropriate smallness conditions on the remainder R the diagonal matrix D determines asymptotic properties of solutions to (7.9). We denote by e_k the k-th basis vector of $\mathbb{C}^d$.

Theorem 7.5. *Assume that for $i \ne j$*

$$\limsup_{t\to\infty}\sup_{\nu\in\Upsilon}\Re\int_1^t \big(\mu_i(s,\nu)-\mu_j(s,\nu)\big)\frac{\mathrm{d}s}{s} < +\infty$$

$$\text{or} \quad \liminf_{t\to\infty}\inf_{\nu\in\Upsilon}\Re\int_1^t \big(\mu_i(s,\nu)-\mu_j(s,\nu)\big)\frac{\mathrm{d}s}{s} > -\infty \tag{7.11}$$

together with

$$\sup_{\nu\in\Upsilon}\int_1^\infty \|R(t,\nu)\|\frac{\mathrm{d}t}{t} < \infty. \tag{7.12}$$

Then there exist solutions $V_k(t,\nu)$ to (7.9) satisfying

$$V_k(t,\nu) = \big(e_k + o(1)\big)\exp\left(\int_1^t \mu_k(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}\right) \tag{7.13}$$

uniformly in the parameter $\nu \in \Upsilon$.

Proof. This is a reformulation of Theorem 1.3.1 from [18] with the substitution $t = \mathrm{e}^x$. For the convenience of the reader we sketch the main idea of the proof. We may replace the dichotomy condition (7.11) by an 'either-or' statement assuming in the first case that in addition

$$\liminf_{t\to\infty}\inf_{\nu\in\Upsilon}\Re\int_1^t \big(\mu_i(s,\nu)-\mu_j(s,\nu)\big)\frac{\mathrm{d}s}{s} = -\infty \tag{7.14}$$

holds true. This yields an ordering of the diagonal entries according to their strength and we may assume without loss of generality that for $i<j$ the first alternative holds true. Furthermore, if we write

$$V(t,\nu) = Z(t,\nu)\exp\left(\int_1^t \mu_k(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}\right) \tag{7.15}$$

for a fixed index k then the function $Z(t,\nu)$ satisfies the transformed equation

$$t\partial_t Z(t,\nu) = (D(t,\nu) - \mu_k(t,\nu)I + R(t,\nu))Z(t,\nu) \tag{7.16}$$

and we have to show that there exists a solution to that equation tending to e_k uniformly with respect to $\nu \in \Upsilon$. Thus it is sufficient to prove the original theorem for the case $\mu_k = 0$. Let $\Phi(t) = \Phi_-(t,\nu) + \Phi_+(t,\nu)$ be the fundamental solution to the diagonal part, split as

$$\Phi_-(t,\nu) = \mathrm{diag}(\exp\left(\int_1^t \mu_1(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}\right),\dots,\exp\left(\int_1^t \mu_{k-1}(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}\right),0,\dots) \quad (7.17)$$

and

$$\Phi_+(t,\nu) = \mathrm{diag}(0,\dots,0,1,\exp\left(\int_1^t \mu_{k+1}(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}\right),\dots,\exp\left(\int_1^t \mu_d(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}\right)) \quad (7.18)$$

according to the asymptotics of the entries. Then (7.9) rewrites as an integral equation

$$V(t,\nu) = e_k + \Phi_-(t,\nu)\int_{t_0}^t \Phi^{-1}(\tau,\nu)R(\tau,\nu)V(\tau,\nu)\frac{\mathrm{d}\tau}{\tau} - \Phi_+(t,\nu)\int_t^\infty \Phi^{-1}(\tau,\nu)R(\tau,\nu)V(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}. \quad (7.19)$$

By construction we obtain $\|\Phi_-(t,\nu)\Phi(\tau,\nu)^{-1}\| \le C_-$ uniformly on $1 \le \tau \le t$ and $\|\Phi_+(t,\nu)\Phi(\tau,\nu)^{-1}\| \le C_+$ uniformly on $t \le \tau < \infty$. Thus, this equation can be solved uniquely in $L^\infty([1,\infty))$ by the contraction mapping principle as

$$\left|\Phi_-(t,\nu)\int_1^t \Phi^{-1}(\tau,\nu)R(\tau,\nu)V(\tau,\nu)\frac{\mathrm{d}\tau}{\tau} - \Phi_+(t,\nu)\int_t^\infty \Phi^{-1}(\tau,\nu)R(\tau,\nu)V(\tau,\nu)\frac{\mathrm{d}\tau}{\tau}\right| \le (C_- + C_+)\int_{t_0}^\infty \|R(\tau,\nu)\|\frac{\mathrm{d}\tau}{\tau}\|V(\cdot,\nu)\|_{L^\infty} \quad (7.20)$$

is contractive for t_0 sufficiently large. Thus, solutions to (7.19) are uniformly bounded. To show that they tend to e_k for $t \to \infty$ uniformly with respect to $\nu \in \Upsilon$ one uses the stronger form (7.11)–(7.14) of the dichotomy condition. Indeed, writing (7.19) for $t > T$ as

$$V(t,\nu) = e_k + \Phi_-(t,\nu)\int_{t_0}^T \Phi^{-1}(\tau,\nu)R(\tau,\nu)V(\tau,\nu)\frac{\mathrm{d}\tau}{\tau} + \Psi(t,\nu) \quad (7.21)$$

with

$$\Psi(t,\nu) = \Phi_-(t,\nu)\int_T^t \Phi^{-1}(\tau,\nu)R(\tau,\nu)V(\tau,\nu)\frac{\mathrm{d}\tau}{\tau} - \Phi_+(t,\nu)\int_t^\infty \Phi^{-1}(\tau,\nu)R(\tau,\nu)V(\tau,\nu)\frac{\mathrm{d}\tau}{\tau} \quad (7.22)$$

we obtain

$$\|\Psi(t,\nu)\| \le (C_- + C_+)\int_T^\infty \|R(\tau,\nu)\|\frac{\mathrm{d}\tau}{\tau}\|V(\cdot,\nu)\|_{L^\infty} \quad (7.23)$$

uniformly in $t \geq T$ and $\nu \in \Upsilon$. Hence, we can choose T large enough such that $\|\Psi(t,\nu)\| \leq \varepsilon$. But then the dichotomy condition implies $\Phi_-(t,\nu) \to 0$ uniformly in ν and thus

$$\|V(t,\nu) - e_k\| \leq 2\varepsilon \tag{7.24}$$

holds true uniformly in $\nu \in \Upsilon$ and $t > T$ sufficiently large. As ε was arbitrary, the statement is proven. □

Remark 7.2. *We will use a special form of the previous theorem, where the diagonal matrices D are constant and independent of ν,*

$$D = \operatorname{diag}(\mu_1, \ldots, \mu_d). \tag{7.25}$$

In this case the dichotomy condition (7.11) *is trivially satisfied as the appearing integrals are all logarithmic functions in t which can't approach both infinities. Hence,* (7.12) *is sufficient to conclude the existence of solutions*

$$V_k(t,\nu) = (e_k + o(1))t^{\mu_k} \tag{7.26}$$

for all k and if in addition *it is known that $\mu_i \neq \mu_j$ for $i \neq j$ this yields a fundamental system of solutions. If the diagonal entries coincide, one has to make further assumptions on lower order terms to get precise asymptotic properties, in particular* (7.12) *has to be replaced by adding logarithmic terms.*

Levinson's theorem yields a corresponding statement for the fundamental solution-valued solution to (7.9)*. This follows immediately from the following variant of Liouville theorem. We assume for simplicity that D is constant and that the entries are distinct. Then we take the solutions V_k constructed above as fundamental system. Their Wronskian satisfies*

$$\mathcal{W}_{V_1,\ldots,V_d}(t) = \det\big(V_1(t,\nu)|\cdots|V_d(t,\nu)\big) = t^{\mu_1+\mu_2+\cdots+\mu_k}. \tag{7.27}$$

If we denote by $\mathcal{E}_V(t,1,\nu)$ the matrix valued solution to

$$t\partial_t\mathcal{E}_V(t,1,\nu) = \big(D + R(t,\nu)\big)\mathcal{E}_V(t,1,\nu), \qquad t \geq 1, \tag{7.28}$$

combined with $\mathcal{E}_V(1,1,\nu) = \mathrm{I}$, it follows that

$$\mathcal{E}_V(t,1,\nu) = \big(V_1(t,\nu)|\cdots|V_d(t,\nu)\big)\big(V_1(1,\nu)|\cdots|V_d(1,\nu)\big)^{-1} \tag{7.29}$$

and the norm of the inverse matrix can be estimated by Cramer's rule combined with Hadamard's inequality as

$$\|\big(V_1(1,\nu)|\cdots|V_d(1,\nu)\big)^{-1}\| \leq d\big(\max_{1\leq k\leq d}\|V_k(1,\nu)\|\big)^{d-1} \tag{7.30}$$

and thus

$$\|\mathcal{E}_V(t,1,\nu)\| \leq Ct^{\max_j \Re\mu_j} \tag{7.31}$$

uniformly in ν.

Remark 7.3. *We can use scaling properties of Fuchs type equations. If $V(t,\nu)$ solves* (7.9)*, then $\tilde{V}(t,\nu) = V(\lambda t,\nu)$ solves the re-scaled equation*

$$t\partial_t \tilde{V}(t,\nu) = \big(D(\lambda t,\nu) + R(\lambda t,\nu)\big)\tilde{V}(t,\nu). \tag{7.32}$$

If $\lambda > 1$ then

$$\int_1^\infty \|R(\lambda t,\nu)\|\frac{\mathrm{d}t}{t} = \int_\lambda^\infty \|R(t,\nu)\|\frac{\mathrm{d}t}{t} \le \int_1^\infty \|R(t,\nu)\|\frac{\mathrm{d}t}{t} \tag{7.33}$$

and similarly for the integrals in (7.11)*. Hence, the conditions of Levinson's theorem are uniform in λ and thus are the constructed solutions. Therefore, any estimate of the fundamental solution given in Remark 7.2 is also uniform and therefore of the form*

$$\|\mathcal{E}_V(\lambda t,\lambda,\nu)\| = \|\mathcal{E}_{\tilde{V}}(t,1,\nu)\| \le C t^{\max_j \Re\mu_j} \tag{7.34}$$

uniformly in $\lambda > 1$ and $\nu \in \Upsilon$.

7.2.2 Hartman–Wintner type theorems

Now we discuss improvements of Theorem 7.5 based on a diagonalization procedure. They allow to handle remainders satisfying

$$\int_1^\infty \|R(t,\nu)\|^\sigma \frac{\mathrm{d}t}{t} < C \tag{7.35}$$

for some constant $1 < \sigma < \infty$. They are constructive and give precise asymptotics similar to the above theorem. We formulate it in more general form with diagonal matrix $D(t,\nu)$ with entries satisfying the stronger form of the dichotomy condition

$$\Re\big(\mu_i(t,\nu) - \mu_j(t,\nu)\big) \le C_- \quad \text{or} \quad \Re\big(\mu_i(t,\nu) - \mu_j(t,\nu)\big) \ge C_+ \tag{7.36}$$

uniform in $t \ge t_0$ and $\nu \in \Upsilon$. It implies (7.11).

Theorem 7.6. *Assume* (7.36) *in combination with* (7.35)*. Let further*

$$F(t,\nu) = \operatorname{diag} R(t,\nu) \tag{7.37}$$

denote the diagonal part of $R(t,\nu)$. Then we find a matrix-valued function $N(t,\nu)$ satisfying

$$\int_1^\infty \|N(t,\nu)\|^\sigma \frac{\mathrm{d}t}{t} < C' \tag{7.38}$$

uniformly in $\nu \in \Upsilon$ such that the differential expression

$$\big(t\partial_t - D(t,\nu) - R(t,\nu)\big)\big(\mathrm{I} + N(t,\nu)\big) - \big(\mathrm{I} + N(t,\nu)\big)\big(t\partial_t - D(t,\nu) - F(t,\nu)\big) = B(t,\nu) \tag{7.39}$$

satisfies

$$\int_1^\infty \|B(t,\nu)\|^{\max\{\sigma/2,1\}} \frac{\mathrm{d}t}{t} < \infty. \tag{7.40}$$

Furthermore, $N(t,\nu) \to 0$ as $t \to \infty$ such that the matrix $\mathrm{I} + N(t,\nu)$ is invertible for $t \ge t_0$. Hence, $\tilde{V} = (\mathrm{I} + N(t,\nu))^{-1}V$ solves the transformed problem

$$t\partial_t \tilde{V} = \big(D(t,\nu) + F(t,\nu) + R_1(t,\nu)\big)\tilde{V} \tag{7.41}$$

with $R_1(t,\nu) = (\mathrm{I} + N(t,\nu))^{-1}B(t,\nu)$ also satisfying (7.40).

Proof. This follows [18] Section 1.5 and is a version of the diagonalization scheme we applied earlier on. We set $D_1(t,\nu) = D(t,\nu) + F(t,\nu)$, $F(t,\nu) = \operatorname{diag} R(t,\nu)$ and denote $\tilde{R}(t,\nu) := R(t,\nu) - F(t,\nu)$. We construct $N(t,\nu)$ as solution to

$$t\partial_t N(t,\nu) = D(t,\nu)N(t,\nu) - N(t,\nu)D(t,\nu) + \tilde{R}(t,\nu), \qquad \lim_{t\to\infty} N(t,\nu) = 0, \quad (7.42)$$

such that equation (7.39) becomes

$$B(t,\nu) = N(t,\nu)F(t,\nu) - R(t,\nu)N(t,\nu). \quad (7.43)$$

In a first step we estimate $N(t,\nu)$. Considering individual matrix entries (7.42) reads as

$$t\partial_t n_{jj}(t,\nu) = 0, \quad (7.44)$$
$$t\partial_t n_{ij}(t,\nu) = (\mu_i(t,\nu) - \mu_j(t,\nu))n_{ij}(t,\nu) + r_{ij}(t,\nu) \quad (7.45)$$

such that the diagonal entries are given by $n_{jj}(t,\nu) = 0$. For the off-diagonal entries we formulate integral representations and use the auxiliary function

$$\delta_{ij}(t,\nu) = \int_1^t (\mu_i(s,\nu) - \mu_j(s,\nu))\frac{\mathrm{d}s}{s}. \quad (7.46)$$

Then the off-diagonal entries are given by Duhamel integrals

$$n_{ij}(t,\nu) = -\mathrm{e}^{\delta_{ij}(t,\nu)} \int_t^\infty \mathrm{e}^{-\delta_{ij}(s,\nu)} r_{ij}(s,\nu)\frac{\mathrm{d}s}{s} \quad (7.47)$$

for those i,j where $\Re(\mu_i - \mu_j) \geq C_+ > 0$ and

$$n_{ij}(t,\nu) = \mathrm{e}^{\delta_{ij}(t,\nu)} \int_1^t \mathrm{e}^{-\delta_{ij}(s,\nu)} r_{ij}(s,\nu)\frac{\mathrm{d}s}{s} \quad (7.48)$$

for those with $\Re(\mu_i - \mu_j) \leq C_- < 0$. It follows in particular that $n_{ij}(t,\nu) \to 0$ as $t \to \infty$ and with $\pm C_\pm \geq \delta > 0$ the estimates

$$|n_{ij}(t,\nu)| \leq \int_1^\infty s^{-\delta} |r_{ij}(ts^{\pm 1},\nu)| \frac{\mathrm{d}s}{s}, \quad (7.49)$$

the $\pm$-sign depending on the case of the Dichotomy condition. Therefore, the L^σ-property of r_{ij} implies by Minkowski inequality

$$\left(\int_1^\infty |n_{ij}(t,\nu)|^\sigma \frac{\mathrm{d}t}{t}\right)^{1/\sigma} \leq \int_1^\infty s^{-\delta} \left(\int_1^\infty |r_{ij}(ts^{\pm 1},\nu)|^\sigma \frac{\mathrm{d}t}{t}\right)^{1/\sigma} \frac{\mathrm{d}s}{s}, \quad (7.50)$$

and thus

$$\int_1^\infty \|N(t,\nu)\|^\sigma \frac{\mathrm{d}t}{t} < \infty. \quad (7.51)$$

uniformly in $\nu \in \Upsilon$. Similarly, by Hölder's inequality and with $\sigma\sigma' = \sigma + \sigma'$.

$$\begin{aligned} \sup_t |n_{ij}(t,\nu)| &\leq \int_1^\infty s^{-\delta} |r_{ij}(ts^{\pm 1},\nu)| \frac{\mathrm{d}s}{s} \\ &\leq \left(\int_1^\infty s^{-\delta\sigma'} \frac{\mathrm{d}s}{s}\right)^{1/\sigma'} \left(\int_1^\infty |r_{ij}(ts^{\pm 1},\nu)|^\sigma \frac{\mathrm{d}s}{s}\right)^{1/\sigma} \quad (7.52) \end{aligned}$$

uniformly in $\nu \in \Upsilon$. Hence, the matrix N belongs to $L^r([1,\infty), \mathrm{d}t/t)$ for all $\sigma \le r \le \infty$ uniformly in ν. If $\sigma \ge 2$ then equation (7.43) implies that $B(t,\xi)$ product of two L^σ-functions and thus in $L^{\sigma/2}$. If $\sigma \in [1,2)$, then $\sigma' > \sigma$ and thus $B(t,\xi)$ is product of an L^σ-function with an $L^{\sigma'}$-function and thus in L^1. □

We distinguish two cases. If $\sigma \in (1,2]$ the transformation reduces the system to Levinson form and Theorem 7.5 applies. If σ is larger, than one application of the transform gives a new remainder satisfying (7.35) with σ replaced by $\sigma/2$.

In the first case one conclusion of Theorem 7.6 is the existence of solutions

$$V_k(t,\nu) = \big(e_k + o(1)\big) t^{\mu_k} \exp\left(\int_0^t r_{kk}(s,\nu)\frac{\mathrm{d}s}{s}\right), \qquad k = 1, \dots, d, \tag{7.53}$$

uniformly in the parameter, provided $D = \mathrm{diag}(\mu_1, \dots, \mu_d)$ with distinct entries and $R \in L^\sigma([1,\infty), \mathrm{d}t/t)$.

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