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Existence and multiplicity of solutions to a class of elliptic problems involving operators with variable exponent

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BOLSISTA CAPES

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Isaías 55:8-9

Resumo

Neste trabalho estudamos a existência e multiplicidade de soluções não triviais para duas classes de problemas elípticos. O primeiro problema elíptico que estudamos abrange uma classe geral de operadores com expoentes variáveis onde a não linearidade possui crescimento subcrítico. O segundo problema trata de uma equação não local envolvendo uma ampla classe de operadores onde a não linearidade possui crescimento sublinear/superlinear, mais um termo com crescimento crítico.

O primeiro problema que estudamos é

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) &= \lambda f(x,u) \quad \text{em } \Omega, \\ u &= 0 \quad \text{em } \partial\Omega, \end{cases}$$

onde $\lambda > 0$ é um parâmetro real, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, é um domínio limitado com fronteira suave $\partial\Omega$, $p \in C(\overline{\Omega})$ e $1 < p(x) < N$, para todo $x \in \overline{\Omega}$. Os resultados para este problema foram demonstrados através do Teorema do Passo da Montanha, de uma versão simétrica do Teorema do Passo da Montanha, do Teorema de Fountain, do Teorema de Dual Fountain e da Teoria de Gênero introduzida por Krasnoselskii.

O segundo problema que estudamos é modelado pelo problema elíptico envolvendo não linearidade com crescimento crítico:

$$\begin{cases} -M(\mathscr{A}(u))\operatorname{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ em } \Omega, \\ u = 0 \text{ em } \partial\Omega, \end{cases}$$

onde $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, é um domínio limitado com fronteira suave $\partial\Omega$, $\lambda > 0$ é um parâmetro real. Obtemos dois resultados de existência e multiplicidade de soluções, que foram provados através de uma versão do Teorema do Passo da Montanha e pela Teoria de Gênero de Krasnoselskii.

Abstract

We study the existence and multiplicity of nontrivial solutions for two classes of elliptic problems. The first problem covers a general class of operators with variables exponents where the nonlinearity has subcritical growth. The second problem is a nonlocal elliptic problem where the nonlinearity has critical growth.

The first problem is

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) &= \lambda f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a real parameter, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$, and $1 < p(x) < N$, for all $x \in \overline{\Omega}$. In this first problem, we proof some results about the existence and multiplicity of solutions. The results of this problem was demonstrated by the Mountain Pass Theorem, symmetric Mountain Pass Theorem, Fountain Theorem, Dual Fountain Theorem, and the Genus Theory introduced by Krasnoselskii.

The second problem deals with the nonlocal elliptic problem with critítical growth:

$$\begin{cases} -M(\mathscr{A}(u))\operatorname{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a real parameter. We obtain the existence and multiplicity of solutions by using the Mountain Pass of Theorem without the condition of Palais-Smale and the Genus Theory introduced by Krasnoselskii.

Contents

Introdução

Esta tese é dedicada ao estudo de uma ampla classe de problemas elípticos quasilineares com expoentes variáveis. Apresentaremos resultados de existência e multiplicidade de soluções, bem como de alguns aspectos qualitativos, para duas classes de problemas elípticos envolvendo uma ampla classe de operadores diferenciais com expoentes variáveis e outros operadores não locais com expoentes variáveis. Os resultados que apresentaremos serão demonstrados através da aplicação de métodos variacionais e topológicos. Estes métodos são uma poderosa ferramenta para obter a multiplicidade de soluções e estabelecer resultados sobre o comportamento qualitativo das soluções.

Recentemente, problemas elípticos não locais e problemas com expoentes variáveis tem atraído a atenção de muitos pesquisadores, devido aos interessantes argumentos matemáticos, tendo como marco natural os espaços de Sobolev com expoentes variáveis $W^{1,p(\cdot)}(\Omega)$, e sua aplicabilidade em diversos fenômenos físicos, como por exemplo em processamento de imagens, matemática biológica, fluidos electroreologicos, etc (veja [1], [8], [13], [31], [47], [65], [68], [77] e suas referências).

Agora, vamos descrever brevemente os problemas estudados e os progressos obtidos nos **Capítulos** 1, 2 e 3.

No **Capítulo 1**, estudamos a existência e multiplicidade de soluções de uma classe de problemas elípiticos com a condição de Dirichlet envolvendo um operador do tipo $p(x)-$Laplaciano, a saber

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) &= \lambda f(x,u) \quad \text{em } \Omega, \\ u &= 0 \quad \text{em } \partial\Omega, \end{cases} \tag{1}$$

onde $\lambda > 0$ é um parâmetro real , $\Omega \subset \mathbb{R}^N$, $N \geq 3$, é um domínio limitado com fronteira suave $\partial\Omega$, $p \in C(\overline{\Omega})$ e $1 < p(x) < N$, para todo $x \in \overline{\Omega}$. A função $a : \mathbb{R}^+ \to \mathbb{R}$ satisfaz as hipóteses:

(a_0) $a : \mathbb{R}^+ \to \mathbb{R}$ é contínua e a aplicação $\Upsilon : \mathbb{R}^N \to \mathbb{R}$, dada por $\Upsilon(\eta) := A(|\eta|^{p(x)})$, é estritamente convexa, onde A é a primitiva de a, isto é $A(t) = \int_0^t a(\sigma)\,d\sigma$;

(a_1) existem constantes $\alpha, \beta > 0$ com $0 < \alpha \leqslant \beta$, tais que

$$\alpha \leqslant a(\sigma) \leqslant \beta,$$

para todo $\sigma \geqslant 0$.

A não linearidade $f : \Omega \times \mathbb{R} \to \mathbb{R}$ é uma função de Carathéodory com crescimento subcrítico, isto é, nós assumiremos que

(f_0) $$|f(x,t)| \leqslant c_1(1 + |t|^{\nu(x)-1}) \text{ para todo } (x,t) \in \Omega \times \mathbb{R},$$

onde c_1 é uma constante positiva, $\nu \in C(\overline{\Omega})$ é a função subcrítica, no sentido que $\nu(x) \leqslant \nu^+ < p^*(x)$ para todo $x \in \overline{\Omega}$ e $p^*(x) = Np(x)/(N - p(x))$ é o expoente crítico variável.

Consideraremos, também, que a não linearidade f possui as seguintes condições de crescimento no zero e no infinito:

(f_1) $\lim\limits_{|t| \to +\infty} \frac{F(x,t)}{|t|^{p^+}} = +\infty$ uniformemente para quase todo ponto (q.t.p.) $x \in \Omega$, isto é, f é p^+-superlinear no infinito, a função F é a função primitiva de f com respeito a segunda variável, isto é, $F(x,t) := \int_0^t f(x,s)\,ds$;

(f_2) $f(x,t) = o(|t|^{p^+-1})$, quando $t \to 0$, uniformemente em quase todo ponto $x \in \Omega$;

(f_3) Existem constantes positivas $\mathsf{C}_* > 0$ e $\varpi > 1$ tal que

$$G(x,t) \leqslant G(x,s) + \mathsf{C}_*,$$

para qualquer $x \in \Omega$, $0 < t < s$ ou $s < t < 0$, onde $G(x,t) := tf(x,t) - \varpi F(x,t)$.

Com a finalidade de obtermos infinitas soluções, alguns dos nosso resultados assumirão a seguinte condição de simetria:

(f_4) f é ímpar em t, isto é, $f(x,-t) = -f(x,t)$ para todo $x \in \Omega$ e $t \in \mathbb{R}$.

Provaremos nossos resultados de existência e multiplicidade de soluções, bem como algumas propriedades qualitativas, combinando técnicas variacionais e topológicas. Mais precisamente, usaremos o Teorema do Passo da Montanha, uma versão simétrica do Teorema do Passo da Montanha, o Teorema de Fountain, o Teorema do Dual Fountain e a Teoria de Gênero de Krasnoselskii, para obter os pontos críticos do funcional de Euler-Lagrange, $\Psi : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$, associado ao problema (1), sendo

$$\Psi(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\, dx - \lambda \int_\Omega F(u,x)\, dx.$$

Observamos que a condição (f_3) é uma consequência da seguinte condição (f_3'), a qual foi introduzida por Miyagaki e Souto ([62]) quando $p(x) = 2$ e desenvolvido por Li ([58]) no caso quando $p(x) = p > 1$:

(f_3') Existe t_0 tal que para todo $x \in \Omega$,

$$\frac{f(x,t)}{|t|^{p^+-2}t} \text{ é não decrescente em } t \geqslant t_0 \text{ e não crescente em } t \leqslant -t_0.$$

A fim de provar que o funcional de Euler-Lagrange satisfaz a condição de Cerami ($(C)_c$ condição), assumimos que as funções a e A satisfazem a condição:

$(\mathcal{T})$ Para $\varpi > 1$ como em (f_3), temos que

$$\mathcal{T}(t) := \frac{1}{p(x)} A(t) - \frac{1}{\varpi} a(t) t,$$

é não decrescente para todo $t \geqslant 0$ e para todo $x \in \Omega$.

Em 1973, o Teorema do Passo da Montanha (veja [6]) foi proposto por Ambrosetti e Rabinowitz, e eles introduziram pela primeira vez, a conhecida condição de Ambrosetti-Rabinowitz, a saber: existe $\theta > p$ tal que

$$0 < \theta F(x,t) =: \theta \int_0^t f(x,s),\, ds \leqslant f(x,t)t, \tag{2}$$

para todo $x \in \Omega$ e $t \in \mathbb{R}\backslash\{0\}$. Desde então, o Teorema do Passo da Montanha passou a ser uma eficiente ferramenta no estudo de existência de solução para problemas superlineares e a desigualdade (2) é uma condição natural que garante a geometria do Teorema do Passo da Montanha e da condição de Palais-Smale. Note que, integrando a desigualdade (2), obtemos o seguinte comportamento para a primitiva de $f(x,\cdot)$: existem duas constantes positivas k_1, k_2 tais que

$$F(x,t) \geqslant k_1|t|^\theta - k_2, \ \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3}$$

Observamos ainda, que condição (3) implica que

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty. \tag{4}$$

Porém, a recíproca não é verdadeira. No entanto, a condição de Ambrosetti-Rabinowitz é bem natural, no sentido de que existem exemplos básicos, como algumas não linearidades polinomiais, que a satisfazem. Mas, ressaltamos que se trata de uma condição um pouco restritiva e elimina muitas outras não linearidades que, recentemente, tem atraído o interesse de muitos pesquisadores, veja por exemplo [12], [23], [51], [56], [62], [58], [57], [69], [78], e suas referências.

O principal objetivo do **Capítulo** 1 é desenvolver as idéias dadas por Miyagaki e Souto (ver [62]) para a classe de problemas elípticos com expoentes variáveis e sem a condição de Ambrosetti-Rabinowitz dada pelo problema (1). Como não consideraremos a condição de Ambrosetti-Rabinowitz, teremos algumas dificuldades adicionais para verificar as condições geométricas exigidas nos teoremas variacionais, e, principalmente, no estudo da convergência da sequência minimizante. Devido a estas dificuldades utilizaremos a sequência de Cerami ao contrário da usual sequência de Palais Smale. Além disso, no estudo da sequência de Cerami, foi crucial a prova de que nosso operador, Ψ', satisfaz a propriedade (S_+) (Veja o Lema 4.7).

O operador $p(x)-$Laplaciano é um importante exemplo de operador abordado pelos resultados que provaremos. Ele é obtido quando consideramos $a(t) \equiv 1$. Observamos, ainda, que esse operador coincide com o operador $p-$Laplaciano quando $p(x) = p$ e com o operador Laplaciano quando $p(x) = 2$. Além disso, por um simples cálculo, podemos verificar que a função $\mathcal{T}(t) = \frac{t}{p(x)} - \frac{t}{\varpi}$, $t \in \mathbb{R}^+$, $x \in \Omega$, satisfaz a hipótese $(\mathcal{T})$ com $\varpi = p^+$.

Destacamos que problemas envolvendo o operador $p(x)-$Laplaciano exige uma formulação mais cuidadosa em virtude de termos que trabalhar em espaços com expoentes variáveis e pelo fato de que estes operadores não são homogêneos. Em particular, não podemos utilizar, por exemplo, o Teorema de Multiplicadores de Lagrange em vários problemas envolvendo este tipo de operador.

Outro exemplo interessante, é a generalização do operador Capillary, conhecido como $p(x)$-Laplacian like, a saber

$$div\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right).$$

Obtemos este operador considerando $a(t)=1+\frac{t}{\sqrt{1+t^2}}$, $t\in\mathbb{R}^+$. Além disso, neste caso, também temos que a função $\mathcal{T}$ satisfaz a hipótese $(\mathcal{T})$ com $\varpi=2p^+$ (Veja [12] e [78]).

Em particular, estes dois exemplos têm sido intensamente estudados nos últimos anos, veja, por exemplo [3], [5], [12], [26], [51], [58], [57], [67] e [78], onde os autores provaram resultados de existência e multiplicidade de soluções, nos casos quando p é constante e $p(x)$ é uma função real.

Agora, enuciaremos os principais resultados do Capítulo 1.

Teorema 0.1 (Theorem 1.1)**.** *Suponha* (a_0), (a_1), $(\mathcal{T})$ *e que* f *satisfaz* $(f_0)-(f_3)$*. Então o problema (1) possui ao menos uma solução fraca em* $W_0^{1,p(\cdot)}(\Omega)$ *para todo* $\lambda>0$*.*

Teorema 0.2 (Theorem 1.2)**.** *Suponha* (a_0), (a_1), $(\mathcal{T})$ *e que* f *satisfaz* $(f_0)-(f_4)$*. Então o problema (1) tem infinitas soluções em* $W_0^{1,p(\cdot)}(\Omega)$ *para todo* $\lambda>0$*.*

Teorema 0.3 (Theorem 1.3)**.** *Suponha* (a_0), (a_1), $(\mathcal{T})$ *e que* f *satisfaz* (f_0), (f_1), (f_3) *e* (f_4)*. Então, para cada* $\lambda\in\left(0,\frac{\alpha}{p^+}\right)$*, o problema (1) tem infinitas soluções fracas* $\{u_n\}_{n\in\mathbb{N}}$ *em* $W_0^{1,p(\cdot)}(\Omega)$ *tais que* $\Phi_\lambda(u_n)\to+\infty$ *quando* $n\to+\infty$*.*

Teorema 0.4 (Theorem 1.4)**.** *Suponha* (a_0), (a_1), $(\mathcal{T})$ *e que* f *satisfaz* (f_0), (f_1), (f_3) *e* (f_4)*. Então, para cada* $\lambda\in\left(0,\frac{\alpha}{p^+}\right)$*, o problema (1) tem uma sequência de soluções fracas* $\{v_n\}_{n\in\mathbb{N}}$ *em* $W_0^{1,p(\cdot)}(\Omega)$ *tais que* $\Phi_\lambda(v_n)<0$ *e* $\Phi_\lambda(v_n)\to 0$ *quando* $n\to\infty$*.*

Teorema 0.5 (Theorem 1.5)**.** *Suponha* (a_0), (a_1) *e* $(\mathcal{T})$*. Se* f *satisfaz* (f_0), (f_1), (f_3), $f(x,0)=0$, $f(x,t)\geqslant 0$ *q.t.p.* $x\in\Omega$, *e todo* $t\geqslant 0$*. Então existe uma constante positiva* $\overline{\lambda}$ *tal que o problema (1) possui ao menos uma solução para todo* $\lambda\in(0,\overline{\lambda})$*. Além disso*

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=+\infty.$$

Teorema 0.6 (Theorem 1.6)**.** *Suponha* (a_0), (a_1) *e* (f_4)*. Além disso, vamos supor a seguinte condição:*

(f_5) *$f : \Omega \times \mathbb{R} \to \mathbb{R}$ é uma função contínua e existem constantes positivas C_0, C_1 tais que*

$$C_0|t|^{q(x)-1} \leqslant f(x,t) \leqslant C_1|t|^{q(x)-1},$$

para todo $x \in \overline{\Omega}$ e $t \geqslant 0$, onde $q \in C(\overline{\Omega})$ é tal que $1 < q(x) < p^*(x)$ para todo $x \in \overline{\Omega}$, com $q^+ < p^-$. Então o problema (1) tem infinitas soluções em $W_0^{1,p(\cdot)}(\Omega)$ para todo $\lambda > 0$.

Estes resultados melhoram, estendem e complementam alguns resultados obtidos em [12], [26], [34], [62], [58], [67], [72], [75] e [78], no sentido de que os autores consideraram o caso quando $p(x)$ é constante, e os operadores considerados estão incluídos em nossa classe de operadores. Nós mostramos que os principais resultados de [12], [34], [51], [62], [58], [67] e [78] permanecem válidos para uma classe mais ampla de operadores envolvendo a função real $p(x)$.

No Teorema 0.1, melhoramos e estendemos alguns resultados recentes da existência de soluções para o problema (1). No caso em que $p = 2$, Miyagaki e Souto [62], através da combinação de alguns argumentos usados por Struwe e Tarantello (veja [71]) e Schechter e Zou (ver [69]), obtiveram a existência de pelo menos uma solução usando o Teorema do Passo da Montanha. No caso $p(x) = p > 1$, em [26] os autores com a condição de Ambrosetti-Rabinowitz, provaram a existência e multiplicidade de soluções usando o Teorema do Passo da Montanha e o Princípio Variacional de Ekeland.

No caso em que o expoente é variável, gostaríamos de mencionar os recentes trabalhos [34] e [67], em que os autores provaram a existência e multiplicidade soluções usando o Teorema do Passo da Montanha e o Teorema de Fountain, considerando a condição Ambrosetti-Rabinowitz.

Quando $p(x)$ é uma função real, nas obras [12], [51], [58], [57], [72], [75] e [78] os autores obtiveram a existência e multiplicidade de soluções, usando o Teorema do Passo da Montanha, Teorema de Fountain, uma variante do Teorema de Fountain, e Teoria de Morse, com a condição de Ambrosetti-Rabinowitz e as hipóteses de não linearidade ligeiramente diferentes das nossas.

No Teorema 0.2, usando a versão $\mathbb{Z}_2$ simétrica do Teorema do Passo da Montanha, obtemos um número infinito de soluções com algumas pequenas diferenças nas hipóteses de não linearidade $f(x,\cdot)$, em relação as consideradas em [62]. Além disso, gostaríamos de mencionar que em [58] e [62] os autores obtiveram resultados de existência de pelo menos uma solução para o $p-$Laplaciano sem a condição de Ambrosetti-Rabinowitz usando o Teorema do Passo da Montanha. Em [34] e [67] foi considerada uma não linearidade que satisfaz a condição de Ambrosetti-Rabinowitz nos casos superlinear e sublinear, onde foi obtido resultados de existência e multiplicidade de soluções usando o Teorema do Passo da Montanha e o Teorema de Fountain. Em [12] e [78], os autores obtiveram resultados de existência de soluções para o $p(x)-$Laplaciano sem a condição de Ambrosetti-Rabinowitz usando o Teorema do Passo da Montanha.

No Teorema 0.3, complementamos e ampliamos alguns resultados obtidos em [12], [34], [51], [62], [58], [67] e [78]. Usamos o Teorema de Fountain estabelecido por T. Bartsch [11] e M. Willem [73], esta é uma poderosa ferramenta para o estudo da existência de um número infinito de pontos críticos, este teorema foi usado para estudar a existência de infinitas soluções com alta energia e nos fornece uma informação qualitativa do funcional de Euler-Lagrange.

No Teorema 0.4, complementamos alguns resultados dos trabalhos [12], [36], [58], [62], [67], [73] e [78], para uma ampla classe de operadores, utilizando o Teorema de Dual Fountain. Provamos a existência de uma sequência de soluções não triviais com valores críticos negativos e energia muito pequena.

No Teorema 0.5, fomos inspirados pelo trabalho [63]. Complementamos alguns resultados para uma ampla classe de operadores, provando a existência de pelo menos uma solução não trivial, usando o Teorema do Passo da Montanha. Além disso, obtivemos informação sobre o comportamento assintótico da solução; a norma Sobolev da solução tende ao infinito quando o parâmetro λ tende a 0^+.

Finalmente, no Teorema 0.6, complementamos o estudo para esta classe de problemas com não linearidade sublinear. Obtemos infinitas soluções utilizando a Teoria de Gênero de Krasnoselskii. Ao impor a condição (f_5) sobre a não-linearidade, provamos que o funcional de Euler-Lagrange, Φ_λ, satisfaz a condição de Palais-Smale.

Nos **Capítulos** 2 e 3, estudaremos o problema elíptico não linear com não linearidade envolvendo o expoente crítico

$$\begin{cases} -M(\mathscr{A}(u))\,\mathrm{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ em } \Omega, \\ u = 0 \text{ em } \partial\Omega, \end{cases} \tag{5}$$

onde $\Omega \subset \mathbb{R}^N$, $N \geq 3$, é um domínio limitado com fronteira suave e com condições de contorno de Dirichlet sobre $\partial\Omega$, a não linearidade $f : \Omega \times \mathbb{R} \to \mathbb{R}$ é uma função contínua, $a : \mathbb{R}^+ \to \mathbb{R}^+$ é uma função de classe C^1, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ é uma função contínua, cujas propriedades serão introduzidas depois, e λ é um parâmetro positivo. Suponha que

$\mathscr{C} = \{x \in \Omega : s(x) = \gamma^*(x)\} \neq \emptyset$, onde $\gamma^*(x) = N\gamma(x)/(N - \gamma(x))$ é o expoente crítico de Sobolev. As funções $p(x)$, $q(x)$, $r(x)$ e $s(x)$ são contínuas em $\overline{\Omega}$, e definimos a função $\gamma(x) = (1 - \mathcal{H}(\kappa_3))p(x) + \mathcal{H}(\kappa_3)q(x)$, e o expoente variável crítico de Sobolev

$$\gamma^*(x) = \begin{cases} \frac{N\gamma(x)}{N-\gamma(x)}, & \text{se} \quad \gamma(x) < N, \\ +\infty, & \text{se} \quad \gamma(x) \geqslant N, \end{cases}$$

para todo $x \in \overline{\Omega}$, onde $\mathcal{H} : \mathbb{R}_0^+ \to \{0, 1\}$ é

$$\mathcal{H}(\kappa) = \begin{cases} 1, & \text{se} \quad \kappa > 0, \\ 0, & \text{se} \quad \kappa = 0. \end{cases}$$

O operador $\mathscr{A} : X \to \mathbb{R}$ é dado por

$$\mathscr{A}(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\, dx,$$

onde $A(t) = \int_0^t a(\kappa) d\kappa$, a função $a(\cdot)$ é de classe C^1, e X é o espaço de Banach

$$X := W_0^{1,p(\cdot)}(\Omega) \cap W_0^{1,\gamma(\cdot)}(\Omega),$$

munido da norma

$$\|u\| = \|\nabla u\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u\|_{q(\cdot)}. \tag{6}$$

Consideramos $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfazendo as hipóteses:

(b_1) existem constantes positivas, κ_0, κ_1, κ_2 e um constante não negativa κ_3, tais que

$$\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2 \tau^{\frac{q(x)-p(x)}{p(x)}} \leqslant a(\tau) \leqslant \kappa_1 + \kappa_3 \tau^{\frac{q(x)-p(x)}{p(x)}},$$

para todo $\tau \geqslant 0$ e para todo $x \in \overline{\Omega}$.

(b_2) existe $c > 0$ tais que

$$\min\left\{a(\tau^{p(x)})\tau^{p(x)-2}, a(\tau^{p(x)})\tau^{p(x)-2} + \tau \frac{\partial (a(\tau^{p(x)})\tau^{p(x)-2})}{\partial \tau}\right\} \geqslant c\tau^{p(x)-2},$$

para quase todo $x \in \Omega$ e para todo $\tau > 0$;

Sobre a função contínua $M : \mathbb{R}_0^+ \to \mathbb{R}$, vamos supor amplas condições de crescimento, a saber:

$(\mathcal{M}_1)$ existe $m_0 > 0$ tal que $M(t) \geqslant m_0 = M(0) > 0$ para todo $t \in \mathbb{R}_0^+$;

$(\mathcal{M}_2)$ M é crescente em $\mathbb{R}_0^+$.

Em virtude da presença do termo $M(\mathscr{A}(\cdot))$ o problema (5) é dito não local, uma vez que sua presença implica que a equação não é mais uma identidade pontual, e por isso, será necessário um argumento de truncamento apropriado para o funcional de Euler-Lagrange associado. Esse argumento foi inspirado no trabalho [37] e desempenha um papel fundamental na prova do nossos resultados principais. Também, provaremos que a solução do problema truncado é uma solução para o problema (5) fazendo algumas estimativas para adequados valores de λ.

No melhor do nosso conhecimento, essa classe de problemas, conhecidas como problemas do tipo Kirchhoff, foi motivada pela equação hiperbólica introduzida em 1883 por Kirchhoff em [55], a saber,

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0.$$

Esta equação foi proposta como uma extensão da clássica equação de D'Alembert para a vibração das cordas elásticas. O modelo leva em conta as pequenas vibrações verticais de uma corda elástica esticada quando a tensão é variável, mas as extremidades da corda estão fixas, onde L é o comprimento da corda, h é a área da secção transversal, E é o módulo de Young do material (também referido como o módulo de elasticidade) que mede a resistência das cordas a ser deformado elasticamente, ρ é a densidade da massa, e ρ_0 é a tensão inicial.

Quase um século mais tarde, Jacques-Louis Lions [61] voltou para a equação e propôs a equação geral de Kirchhoff em dimensão arbitrária com força externa, a saber

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \left(a + b\displaystyle\int_\Omega |\nabla u|^2\, dx\right)\triangle u = f(x,u) \text{ em } \Omega, \\ u = 0 \text{ em } \partial\Omega. \end{cases}$$

Este problema é usado para introduzir modelos para fenômenos físicos, como sistemas biológicos ou densidade populacional, onde u descreve um processo que depende da média de si mesmo (ver [20], [28]).

Considerando no problema (5) o caso particular em que

$$M(s) = as + m_0, \text{ com } a, m_0 \geqslant 0,$$

o problema é dito degenerado se $m_0 = 0$ e $a > 0$, e não degenerado quando $m_0 > 0$ e $a \geqslant 0$. Além disso, quando $m_0 > 0$ e $a = 0$, a função de Kirchhoff M é simplesmente uma constante e o problema reduz-se a um problema elíptico quasilinear.

Para o caso particular $M \equiv 1$, citamos o trabalhos [10], [15], [25], [39], [41], [43], [49] e [60]; para problemas não locais envolvendo o operador de Laplace, podemos citar [4], [2], [28], [37] e [40] para domínio limitado, e [27] e [74] em todo o espaço $\mathbb{R}^N$; e para problemas não locais que envolvem o operador $p-$Laplaciano, citamos [17], [20], [48] e suas referências. Além disso, também gostaríamos de mencionar os artigos [9], [21], [22], para equações não degeneradas do tipo Kirchhoff, e os trabalhos [19], [48], e suas referências, para as equações do tipo Kirchhoff degeneradas.

Outro ponto interessante, é a ampla classe de operadores elípticos com expoentes variáveis abrangidos pelo problema (5). Em particular, nosso trabalho engloba os seguintes exemplos:

(i) Se $a \equiv 1$, temos o $p(x)-$Laplaciano:

$$-div(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = -div(|\nabla u|^{p(x)-2}\nabla u),$$

o qual coincide com o conhecido operador $p-$Laplaciano quando $p(x) = p$, e com o Laplaciano quando $p(x) = 2$.

(ii) Se $a(\tau) = 1 + \tau^{\frac{q(x)-p(x)}{p(x)}}$, obtemos o $p\&q-$Laplaciano:

$$-div(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = -\Delta_{p(x)}u - \Delta_{q(x)}u.$$

(iii) Se $a(\tau) = 1 + \frac{\tau}{\sqrt{1+\tau^2}}$, obtemos o conhecido operador $p(x)-$Laplacian like:

$$-div(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = -div\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right).$$

Nos últimos anos, há um crescente interesse nos problemas envolvendo o operador $p(x)-$Laplaciano em domínio limitado, veja por exemplo [19], [21], [22], [42],

[43] e [59], sendo que nos dois últimos temos a não linearidade com crescimento crítico. No caso, $p(x) \equiv p$ constante, citamos, por exemplo, [10], [37], [40], [48] e [49]. Em particular, quando $p(x) \equiv 2$, $a(t) \equiv 1$, e com uma ligeira diferença das nossas hipóteses sobre M e sobre a não linearidade, o autor em [37] provou a existência de uma solução para o problema (5) usando o Teorema do Passo da Montanha sem a condição de Palais-Smale.

O problema do tipo Kirchhoff envolvendo o operador $p\&q$-Laplaciano e não linearidade com crescimento crítico foi estudado, por exemplo, em [17], onde foi demonstrado a existência de pelo menos uma solução não trivial através do Teorema do Passo da Montanha Multidimensional. Também, gostaríamos de citar [25] para o caso $p\&q$-Laplaciano em $\mathbb{R}^N$.

Ressaltamos, ainda, que uma novidade nesta tese é que englobamos operadores do tipo $p(x)-$Laplaciano para $p(x) > 1$ para $x \in \overline{\Omega}$, embora, geralmente, na literatura que conhecemos, os operadores do tipo $p(x)-$Laplaciano são considerados apenas quando $p(x) \geqslant 2$ para todo $x \in \overline{\Omega}$, veja por exemplo, [38] e [67]. Conseguimos isto impondo, no Capítulo 1, uma condição de que o operador é estritamente convexo; e nos Capítulos 2 e 3, impondo condições de crescimento na função $a(\cdot)$, as quais nos permitem obter uma desigualdade muito importante no marco de equações não lineares a qual nos referimos como a "desigualdade do tipo Simon" (veja [70]) para expoentes variáveis.

Outra questão importante sobre o problema (5) é o crescimento crítico da não linearidade. Como sabemos, problemas elípticos que envolvem não linearidades com crescimento crítico atrai a atenção de muitos pesquisadores e um dos trabalhos pioneiros sobre o assunto é o conhecido artigo de Brezis e Nirenberg [15]. A principal dificuldade decorre da falta de compacidade do mergulho $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$, e para superarmos esta dificuldade, usamos uma versão do princípio de concentração de compacidade para espaços com expoentes variáveis provado por Bonder e Silva em [14], que é uma versão mais geral do resultado provado em [42], no sentido de que a ideia de criticidade apresentada por Bonder e Silva [14] é a de que o conjunto $\mathscr{C} = \{x \in \Omega : s(x) = \gamma^*(x)\}$ é não vazio.

Nos artigos [10], [25], [37], [40], [41], [48] e [60] foram estudados problemas semelhantes ao problema (5), porém com as hipóteses ligeiramente diferentes das consideradas em nosso trabalho. Nestes artigos foram combinados métodos variacionais com o princípio de concentração e compacidade para estabelecer uma solução.

Os resultados provados nos Capítulos 2 e 3 estendem e complementam alguns dos artigos para problemas não locais envolvendo uma classe geral de operadores com expoentes variáveis e não linearidade com crescimento crítico. Em particular, completamos o estudo de [67] no sentido de que nosso problema envolve um operador não local e não linearidade com crescimento crítico.

A fim de enunciarmos o resultado provado no **Capítulo 2**, vamos considerar, além das hipóteses já apresentadas para o problema (5), as seguintes condições:

(p_1)
$$\begin{cases} 1 < p^- \leqslant p(x) \leqslant p^+ < q^- \leqslant q(x) \leqslant q^+ < N, \\ \gamma^- \leqslant \gamma(x) \leqslant \gamma^+ < r^- \leqslant r(x) \leqslant r^+ < s^- \leqslant s(x) \leqslant \gamma^*(x) < +\infty, \end{cases}$$

para todo $x \in \overline{\Omega}$, where $p^- := \min_{x\in\overline{\Omega}} p(x)$, $p^+ := \max_{x\in\overline{\Omega}} p(x)$, e analogamente para r^-, r^+, q^-, q^+, γ^-, γ^+, s^- e s^+.

(b_3) existem constantes positivas α e θ tal que

$$A(\tau) \geqslant \frac{1}{\alpha} a(\tau)\tau \quad \text{com} \quad \gamma^+ < \theta < s^- \quad \text{e} \quad \frac{q^+}{p^+} \leqslant \alpha < \frac{\theta}{p^+},$$

para todo $\tau \geqslant 0$.

Assumimos que a não linearidade $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ é uma função contínua, satisfazendo as seguintes propriedades:

(e_1)
$$\lim_{|t|\to 0} \frac{f(x,t)}{t^{q^+-1}} = 0 \text{ uniformemente em } x \in \Omega;$$

(e_2)
$$\lim_{|t|\to +\infty} \frac{f(x,t)}{t^{r^--1}} = 0 \text{ uniformemente em } x \in \Omega;$$

(e_3) para $\gamma^+ < \theta < s^-$ dado em (a_3),

$$0 < \theta F(x,t) = \theta \int_0^t f(x,s)ds \leqslant t f(x,t),$$

para todo $x \in \Omega$ e para todo $t > 0$.

Observamos que em virtude das hipóteses (e_1) e (e_2), temos um crescimento subcrítico para f uma vez que $r^+ < s^-$. A hipótese (e_3) é conhecida como a condição superlinear de Ambrosetti-Rabinowitz (veja [6]). Além disso, a condição (e_3) assegura que o funcional de Euler-Lagrange associado ao problema (5) possui a geometria do Passo da Montanha e, também, garante a limitação das sequências de Palais-Smale correspondente ao funcional de Euler-Lagrange.

Um típico exemplo de uma função que satisfaz as condições (e_1), (e_2) e (e_3) é dada por

$$f(x,t) = \vartheta(x)|t|^{r_1-2}t,$$

com $\gamma^+ < r_1 < r^-$, $\vartheta \in C(\overline{\Omega})$, e $\vartheta(x) \geqslant 0$ para todo $x \in \Omega$.

O principal resultado do Capítulo 2 estabelece a existência de pelo menos uma solução para o problema (5) sob as hipóteses anteriores.

Teorema 0.7. *(Theorem 2.1) Suponha as hipóteses $(b_1)-(b_3)$, (p_1), $(\mathcal{M}_1)-(\mathcal{M}_2)$ e $(e_1)-(e_3)$. Então, existe $\overline{\lambda} > 0$, tal que o problema (5) possui pelo menos uma solução fraca não trivial em X para cada $\lambda \geqslant \overline{\lambda}$. Além disso, a solução, u_λ, satisfaz*

$$\lim_{\lambda \to +\infty} \|u_\lambda\| = 0.$$

Em que a norma $\|\cdot\|$ é definida como em (6).

Agora, para podermos enunciar o principal resultado do **Capítulo 3**, vamos supor que

(p_2) $$\begin{cases} 1 < r^- \leqslant r(x) \leqslant r^+ < p^- \leqslant p(x) \leqslant p^+ < q^- \leqslant q(x) \leqslant q^+ < N, \\ 1 < r^- \leqslant r(x) \leqslant r^+ < \gamma^- \leqslant \gamma(x) \leqslant \gamma^+ < s^- \leqslant s(x) \leqslant \gamma^*(x) < +\infty, \end{cases}$$

para todo $x \in \overline{\Omega}$, onde $p^- := \min\limits_{x\in\overline{\Omega}} p(x)$, $p^+ := \max\limits_{x\in\overline{\Omega}} p(x)$, e analogamente para r^-, r^-, q^-, q^+, γ^-, γ^+, s^- e s^+;

(b_3') existe uma constante positiva α tal que

$$A(\tau) \geqslant \frac{1}{\alpha} a(\tau)\tau \text{ with } \frac{r^+}{p^+} < \frac{\gamma^+}{p^+} \leqslant \alpha < \frac{s^-}{p^+},$$

para todo $\tau \geqslant 0$.

A não linearidade $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ é uma função contínua satisfazendo:

(e_1') f é ímpar na segunda variável, isto é,

$$f(x,-t) = -f(x,t) \text{ para todo } (x,t) \in \overline{\Omega} \times \mathbb{R};$$

(e_2') existe uma constante positiva a_1, a_2 e uma função $r \in C^+(\overline{\Omega})$ tais que

$$a_1 t^{r(x)-1} \leqslant f(x,t) \leqslant a_2 t^{r(x)-1} \text{ para todo } (x,t) \in \overline{\Omega} \times \mathbb{R}_0^+.$$

Observe que a hipótese (e_1') implica que a função $u \equiv 0$ é uma solução do problema $(\mathscr{P}_\lambda)$. Também, um simples exemplo para a não linearidade f é $f(x,t) = \vartheta(x)|t|^{r(x)-2}t$, onde $\vartheta : \overline{\Omega} \to \mathbb{R}$ é uma função contínua não negativa.

O principal resultado do **Capítulo 3** demonstra a existência de um número infinito de soluções para o problema $(\mathscr{P}_\lambda)$ e a técnica usada se baseia na Teoria de Gênero de Krasnoselski como em [10].

Teorema 0.8. *(Theorem 3.1) Suponha que as funções a, M e f satisfazem as condições $(b_1)-(b_2)$, (b_3'), (p_2), $(\mathcal{M}_1)-(\mathcal{M}_2)$ e $(e_1')-(e_2')$. Então, existe $\overline{\lambda} > 0$, tal que o problema $(\mathscr{P}_\lambda)$ possui infinitas soluções fracas em X para cada $\lambda \in (0, \overline{\lambda})$. Além disso, se u_λ é uma solução do problema $(\mathscr{P}_\lambda)$ então*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

Em que a norma $\|\cdot\|$ é definida como em (6).

Por fim, no **Apêndice**, apresentamos alguns resultados e definições importantes que utilizamos no decorrer deste trabalho.

Gostaríamos de esclarecer que a formtação dos capítulos desta tese é baseada nos artigos que escrevemos e que foram submetidos a publicação. Os Capítulos 1, 2 e 3 estão distribuídos da seguinte forma:

Capítulo 1. Artigo [44]- *Existence and multiplicity of solutions for a class of elliptic equations without Ambrosetti-Rabinowitz type conditions*, aceito para publicação na revista Journal of Dynamics and Differential Equations.

Capítulo 2. Artigo [45]- *Existence and asymptotic behaviour of solution for a quasilinear Kirchhoff type equation with variable critical growth exponent*, submetido a publicação.

Capítulo 3. Artigo [46]- *Existence of solutions for a class of nonlocal elliptic problems with critical growth exponent,* submetido a publicação.

Por esse motivo, a identificação das hipóteses em cada um dos capítulos é a descrita no início do respectivo capítulo.

Chapter

1

Existence and multiplicity of solutions for a class of elliptic equations without Ambrosetti-Rabinowitz type conditions

The purpose of this chapter is to show the existence of infinitely many weak solutions for the class of quasilinear elliptic problems

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) & = & \lambda f(x,u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a real parameter, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$, $1 < p(x) < N$, for all $x \in \overline{\Omega}$, and $p^- := \min_{x\in\overline{\Omega}} p(x)$, $p^+ := \max_{x\in\overline{\Omega}} p(x)$.

We consider the function $a : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following hypotheses:

(a_0) The function $a : \mathbb{R}^+ \to \mathbb{R}$ is continuous and the mapping $\Upsilon : \mathbb{R} \to \mathbb{R}$, given by $\Upsilon(\eta) := A(|\eta|^{p(x)})$ is strictly convex, where A is the primitive of a, that is $A(t) = \int_0^t a(s)\,ds$;

(a_1) There are constants α, $\beta > 0$, $0 < \alpha \leqslant \beta$ such that

$$\alpha \leqslant a(\sigma) \leqslant \beta,$$

for all $\sigma \geqslant 0$.

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies:

(f_0) There exists a positive constant c_1 such that f satisfies the subcritical growth condition

$$|f(x,t)| \leqslant c_1(1+|t|^{\nu(x)-1}),$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where $\nu \in C(\overline{\Omega})$, $1 < p^+ < \nu^- \leqslant \nu(x) \leqslant \nu^+ < p^*(x)$ for $x \in \overline{\Omega}$, and $p^*(x)$ denote the critical variable exponent related to $p(x)$, which is defined for all $x \in \overline{\Omega}$ by the pointwise relation

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for} \quad p(x) < N, \\ +\infty & \text{for} \quad p(x) \geqslant N; \end{cases}$$

(f_1) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^{p^+}} = +\infty$ uniformly for almost everywhere (a.e.) $x \in \Omega$, that is, f is p^+-superlinear at infinity, the function F is the primitive of f with respect to the second variable, that is, $F(x,t) := \int_0^t f(x,s)\,ds$;

(f_2) $f(x,t) = o(|t|^{p^+-1})$, as $t \to 0$, uniformly a.e. $x \in \Omega$;

(f_3) There are positive constants $\mathsf{C}_* > 0$ and $\varpi > 0$ such that

$$G(x,t) \leqslant G(x,s) + \mathsf{C}_*$$

for any $x \in \Omega,\ 0 < t < s$ or $s < t < 0$, where $G(x,t) := tf(x,t) - \varpi F(x,t)$.

With the aim of finding infinitely many solutions is natural impose certain symmetry condition on the nonlinearity. In the sequel we will assume the following assumption on f :

(f_4) f is odd in t, that is, $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.

Our main results are the following theorems.

Theorem 1.1. *Assume* (a_0), (a_1), $(\mathcal{T})$, *and* f *satisfies* $(f_0) - (f_3)$. *Then problem (1.1) has at least one nontrivial weak solution in* $W_0^{1,p(\cdot)}(\Omega)$ *for all* $\lambda > 0$.

Theorem 1.2. *Assume* (a_0), (a_1), $(\mathcal{T})$, *and* f *satisfies* $(f_0) - (f_4)$. *Then problem (1.1) has infinitely many solutions in* $W_0^{1,p(\cdot)}(\Omega)$ *for all* $\lambda > 0$.

Theorem 1.3. *Assume* (a_0), (a_1), $(\mathcal{T})$, *and that* f *satisfies* (f_0), (f_1), (f_3), *and* (f_4). *Then, for each* $\lambda \in \left(0, \frac{\alpha}{p^+}\right)$, *the problem* (1.1) *has infinitely many weak solutions* $\{u_n\}_{n\in\mathbb{N}}$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *such that* $\Phi_\lambda(u_n) \to +\infty$ *as* $n \to +\infty$.

Theorem 1.4. *Assume* (a_0), (a_1), $(\mathcal{T})$, *and that* f *satisifies* (f_0), (f_1), (f_3), *and* (f_4). *Then, for each* $\lambda \in \left(0, \frac{\alpha}{p^+}\right)$, *the problem (1.1) has a sequence of weak solutions* $\{v_n\}_{n\in\mathbb{N}}$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *such that* $\Phi_\lambda(v_n) < 0$, $\Phi_\lambda(v_n) \to 0$ *as* $n \to \infty$.

Theorem 1.5. *Assume* (a_0), (a_1), *and* $(\mathcal{T})$. *If* f *satisfies* (f_0), (f_1), (f_3), $f(x, 0) = 0$, $f(x, t) \geqslant 0$ *a.e.* $x \in \Omega$, *and all* $t \geqslant 0$. *Then there exists a positive constant* $\overline{\lambda}$ *such that problem (1.1) possesses at least one solution for all* $\lambda \in (0, \overline{\lambda})$. *Moreover*

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = +\infty.$$

Where $\|\cdot\|$ is the norm in $W_0^{1,p(\cdot)}(\Omega)$.

Theorem 1.6. *Assume* (a_0), (a_1), *and* (f_4). *In addition we will assume the following condition:*

(f_5) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and there exist positive constants* C_0, C_1 *such that*

$$C_0|t|^{q(x)-1} \leqslant f(x,t) \leqslant C_1|t|^{q(x)-1},$$

for all $x \in \overline{\Omega}$ *and* $t \geqslant 0$, *where* $q \in C(\overline{\Omega})$ *such that* $1 < q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *with* $q^+ < p^-$. *Then the problem (1.1) has infinitely many solutions in* $W_0^{1,p}(\Omega)$ *for all* $\lambda > 0$.

1.1 Variational framework

We start with the definition of the Lebesgue and Sobolev spaces with variable exponent, $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, and some properties of them, for more details, see [32], [34], and [35].

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We denote by $|\Omega|$ the Lebesgue measure of the set $\Omega \subset \mathbb{R}^N$. For each $p \in C(\overline{\Omega})$, we define

$$p^+ := \max_{x\in\overline{\Omega}} p(x),\ p^- := \min_{x\in\overline{\Omega}} p(x),$$

and

$$C^+(\overline{\Omega}) = \Big\{p \in C(\overline{\Omega}) : p^- > 1\Big\}.$$

Note that $p^- \leq p^+ < +\infty$.

Let p be a fixed function in $C^+(\overline{\Omega})$. The variable exponent Lebesgue space, denoted by $L^{p(\cdot)}(\Omega)$, is the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u(x)|^{p(x)}\, dx$ is finite, that is,

$$L^{p(\cdot)}(\Omega) = \left\{ u \text{ is a measurable real valued function with } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\}.$$

We consider this space endowed with the so-called Luxemburg norm:

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leqslant 1 \right\}.$$

Note that, if p is a constant function, the Luxemburg norm $\|\cdot\|_{p(\cdot)}$ coincide with the standard norm $\|\cdot\|_p$ of the Lebesgue space $L^p(\Omega)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

The corresponding norm for this space is

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^{+\infty}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$.

From Proposition 4.2, the spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces when $p^- > 1$. Moreover, the norms $\|\nabla u\|_{p(\cdot)}$ and $\|u\|$ are equivalents on $W_0^{1,p(\cdot)}(\Omega)$. We will use $\|\nabla u\|_{p(\cdot)}$ to replace $\|u\|$ in the following discussions.

Now we introduce the Euler Lagrange functional $\Phi_\lambda : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ associated to the problem (1.1) defined by

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\, dx - \lambda \int_\Omega F(u, x)\, dx \tag{1.2}$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$. The functional Φ_λ is well defined on $W_0^{1,p(\cdot)}(\Omega)$ and $\Phi_\lambda \in$

$C^1(W_0^{1,p(\cdot)}(\Omega),\mathbb{R})$ with Fréchet derivative given by

$$\langle \Phi_\lambda'(u), v\rangle = \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,dx - \lambda\int_\Omega f(x,u)u\cdot v\,dx,$$

for all $u, v \in W_0^{1,p(\cdot)}(\Omega)$.

Definition 1.1. *Let X be a real Banach space and $\Phi \in C^1(X,\mathbb{R})$. We say that Φ satisfies the Cerami condition, denoted by $(C)_c$ condition, if for every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\Phi(u_n)\to c$ and $\|\Phi'(u_n)\|_{X^*}(1+\|u_n\|)\to 0$, as $n\to+\infty$, has a convergent subsequence in X.*

1.2 Proof of Theorems 1.1, 1.2, 1.3, 1.4, 1.5, and 1.6.

To prove Theorem 1.1 we need some Lemmas and results presented below.

Lemma 1.1. *Assume that $(f_0)-(f_2)$ are satisfied. Then, we have the following assertions:*

(i) There exist $\phi \in W_0^{1,p(\cdot)}(\Omega)$, $\phi > 0$, such that $\Phi_\lambda(t\phi)\to-\infty$ as $t\to+\infty$.

(ii) There are $r > 0$ and $\mathcal{R} > 0$ such that $\Phi_\lambda(u) \geqslant \mathcal{R}$ for any $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| = r$.

Proof.

(i) From (f_1), given $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that

$$F(x,t) \geqslant M|t|^{p^+} - C_M,\ \forall x\in\Omega,\ \forall t\in\mathbb{R}. \tag{1.3}$$

Take $\phi \in W_0^{1,p(\cdot)}(\Omega)$ with $\phi > 0$, for $t > 1$, by (1.3) and (a_1), we have

$$\begin{aligned}
\Phi_\lambda(t\phi) &= \int_\Omega \frac{1}{p(x)}A(|\nabla(t\phi)|^{p(x)})\,dx - \lambda\int_\Omega F(x,t\phi)\,dx\\
&\leqslant \int_\Omega \frac{\beta}{p(x)}|t\nabla\phi|^{p(x)}\,dx - \lambda\int_\Omega F(x,t\phi)\,dx\\
&\leqslant t^{p^+}\Big(\int_\Omega \frac{\beta|\nabla\phi|^{p(x)}}{p(x)}\,dx - \lambda M\int_\Omega \phi^{p^+}\,dx\Big) + \lambda C_M|\Omega|,
\end{aligned} \tag{1.4}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. From (1.4) and taking M large enough such that

$$\int_\Omega \frac{\beta|\nabla\phi|^{p(x)}}{p(x)} - \lambda M\int_\Omega \phi^{p^+}\,dx < 0,$$

we have

$$\lim_{t\to+\infty} \Phi_\lambda(t\phi) = -\infty,$$

which completes the proof of (i).

(ii) Since the embeddings $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\nu(x)}(\Omega)$ are continuous (Proposition 4.2), there exist positive constants c_2, c_3 such that

$$\|u\|_{p^+} \leqslant c_2\|u\| \text{ and } \|u\|_{\nu(x)} \leqslant c_3\|u\|,\ \forall u \in W^{1,p(\cdot)}(\Omega). \tag{1.5}$$

Let $0 < \varepsilon < \frac{\alpha}{\lambda c_2^{p^+}}$. From (f_0) and (f_2), it follows that, for all given $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$F(x,t) \leqslant \frac{\varepsilon}{p^+}|t|^{p^+} + C_\varepsilon|t|^{\nu(x)},\ \forall (x,t) \in \Omega\times\mathbb{R}. \tag{1.6}$$

Thus, for $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| < 1$ sufficiently small, we have, from (1.5) – (1.6) and (a_1), that

$$\begin{aligned}\Phi_\lambda(u) &= \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\,dx - \lambda\int_\Omega F(x,u)\,dx\\ &\geqslant \int_\Omega \frac{\alpha}{p(x)}|\nabla u|^{p(x)}\,dx - \frac{\lambda\varepsilon}{p^+}\int_\Omega |u|^{p^+}\,dx - \lambda C_\varepsilon\int_\Omega |u|^{\nu(x)}\,dx\\ &\geqslant \frac{\|u\|^{p^+}}{p^+}\left(\alpha - \lambda\varepsilon c_2^{p^+}\right) - \lambda C_\varepsilon c_3^{\nu^-}\|u\|^{\nu^-}.\end{aligned} \tag{1.7}$$

From (1.7) and since $\nu^- > p^+$, we can choose $\mathcal{R} > 0$ and $r > 0$ such that $\Phi_\lambda(u) \geqslant \mathcal{R} > 0$ for every $u \in W_0^{1,p(\cdot)}(\Omega)$ and $\|u\| = r$. This completes the proof of (ii).

□

Lemma 1.2. *Assume that* (a_0), (a_1), $(\mathcal{T})$ *hold and* f *satisfies* (f_0), (f_1), *and* (f_3). *Then the functional* Φ_λ *satisifies the* $(C)_c$ *condition at any level* $c > 0$.

Proof. Let $c \in \mathbb{R}$ and let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(\Omega)$ be a (C_c) sequence for Φ_λ, that is,

$$\Phi_\lambda(u_n) \to c > 0 \text{ and } \|\Phi'_\lambda(u_n)\|_{X^*}(1 + \|u_n\|) \to 0 \text{ as } n \to +\infty. \tag{1.8}$$

First of all, we prove that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$.

Indeed, arguing by contradiction, up to a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, we suppose that $\{u_n\}_{n\in\mathbb{N}}$ is unbounded in $W_0^{1,p(\cdot)}(\Omega)$. Define $\omega_n := \frac{u_n}{\|u_n\|}$, for any $n \in \mathbb{N}$. Thus, $\{\omega_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(\cdot)}$ and $\|\omega_n\| = 1$. By the *Eberlein – Šmulian* Theorem ([16], Theorem 3.19), we can extract a subsequence, still denoted by $\{\omega_n\}_{n\in\mathbb{N}}$, and $\omega \in W_0^{1,p(\cdot)}(\Omega)$ such that $\omega_n \rightharpoonup \omega$ weakly in $W_0^{1,p(\cdot)}(\Omega)$. By the Sobolev's embedding theorem (see Proposition 4.2), we have

$$\omega_n(x) \to \omega(x) \text{ a.e. in } \Omega, \omega_n \to \omega \text{ strongly in } L^{p^+}(\Omega), \text{ and } \omega_n \to \omega \text{ strongly in } L^{\nu(\cdot)}(\Omega). \tag{1.9}$$

Let $\Omega_{\neq} := \{x \in \Omega : \omega(x) \neq 0\}$. If $x \in \Omega_{\neq}$, by (1.9), we have

$$|u_n(x)| = |\omega_n(x)|\|u_n\| \to +\infty \text{ a.e. in } \Omega_{\neq} \text{ as } n \to \infty.$$

Therefore, from (f_1), we have for each $x \in \Omega_{\neq}$ that

$$\lim_{n\to+\infty} \frac{F(x,u_n(x))}{|u_n(x)|^{p^+}} \frac{|u_n(x)|^{p^+}}{\|u_n\|^{p^+}} = \lim_{n\to+\infty} \frac{F(x,u_n(x))}{|u_n(x)|^{p^+}} |\omega_n(x)|^{p^+} = +\infty. \tag{1.10}$$

Also, by (f_1) there exists $K > 0$ such that

$$\frac{F(x,t)}{|t|^{p^+}} > 1, \ \forall (x,t) \in \Omega \times \mathbb{R} \text{ with } |t| \geq K. \tag{1.11}$$

Since $F(x,t)$ is continuous on $\overline{\Omega} \times [-K,K]$, there exists a positive constant c_3 such that

$$|F(x,t)| \leqslant c_3, \ \forall (x,t) \in \overline{\Omega} \times [-K,K]. \tag{1.12}$$

From (1.11) and (1.12), we see that there is a constant c_4 such that

$$F(x,t) \geqslant c_4, \ \forall (x,t) \in \overline{\Omega} \times \mathbb{R},$$

which shows that

$$\frac{F(x,u_n) - c_4}{\|u_n\|^{p^+}} \geqslant 0, \ \forall x \in \Omega, \ \forall n \in \mathbb{N},$$

that is

$$\frac{F(x,u_n(x))}{|u_n(x)|^{p^+}} |\omega_n|^{p^+} - \frac{c_4}{\|u_n\|^{p^+}} \geqslant 0, \ \forall x \in \Omega, \ \forall n \in \mathbb{N}. \tag{1.13}$$

By (1.8) and (a_1), we have

$$\begin{aligned} c &= \Phi_\lambda(u_n) + o_n(1) \\ &\geqslant \frac{\alpha}{p^+}\|u_n\|^{p^-} - \lambda \int_\Omega F(x, u_n(x)) + o_n(1). \end{aligned}$$

So we see that

$$\int_\Omega F(x, u_n(x))\, dx \geqslant \frac{\alpha}{\lambda p^+}\|u_n\|^{p^-} - \frac{c}{\lambda} + o_n(1) \to +\infty \text{ as } n \to +\infty. \tag{1.14}$$

As before, we also have

$$\begin{aligned} c &= \Phi_\lambda(u_n) + o_n(1) \\ &\leqslant \frac{\beta}{p^-}\|u_n\|^{p^+} - \lambda \int_\Omega F(x, u_n)\, dx + o_n(1). \end{aligned}$$

Then, by using (1.14), we achieve

$$\|u_n\|^{p^+} \geqslant \frac{p^- c}{\beta} + \frac{\lambda p^-}{\beta} \int_\Omega F(x, u_n)\, dx - o_n(1) > 0, \tag{1.15}$$

for n large enough.

We claim that $|\Omega_{\neq}| = 0$. Indeed, if $|\Omega_{\neq}| \neq 0$, then by (1.10), (1.13), (1.15), and Fatou's Lemma, we have

$$\begin{aligned} +\infty &= \int_{\Omega_{\neq}} \liminf_{n\to+\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} |\omega_n(x)|^{p^+}\, dx - \int_{\Omega_{\neq}} \limsup_{n\to+\infty} \frac{c_5}{\|u_n\|^{p^+}}\, dx \\ &= \int_{\Omega_{\neq}} \liminf_{n\to+\infty} \left(\frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} |\omega_n(x)|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}} \right) dx \\ &\leqslant \liminf_{n\to+\infty} \int_{\Omega_{\neq}} \left(\frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} |\omega_n(x)|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}} \right) dx \\ &\leqslant \liminf_{n\to+\infty} \int_{\Omega} \left(\frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} |\omega_n(x)|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}} \right) dx \\ &= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} |\omega(x)|^{p^+}\, dx - \limsup_{n\to+\infty} \int_\Omega \frac{c_5}{\|u_n\|^{p^+}}\, dx \\ &= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x, u_n(x))}{\|u_n\|^{p^+}}\, dx \\ &\leqslant \liminf_{n\to+\infty} \frac{\int_\Omega F(x, u_n(x))\, dx}{\frac{\lambda p^-}{\beta} \int_\Omega F(x, u_n(x))\, dx - o_n(1)}. \end{aligned} \tag{1.16}$$

Therefore, by (1.15) and (1.16), we obtain that $+\infty \leqslant \frac{\beta}{\lambda p^-}$, which is a contradiction. This proves that $|\Omega_{\neq}| = 0$ and thus $\omega(x) = 0$ a.e. in Ω.

As in [50], define the continuous function: $\kappa_n : [0,1] \longrightarrow \mathbb{R}$, $\kappa_n(t) := \Phi_\lambda(tu_n)$. Since $\kappa_n(t) := \Phi_\lambda(tu_n)$ is continuous in $[0,1]$, we can say that for each $n \in \mathbb{N}$ there exists $t_n \in [0,1]$ such that

$$\Phi_\lambda(t_n u_n) := \max_{t\in[0,1]} \kappa_n(t). \tag{1.17}$$

(If, for $n \in \mathbb{N}$, t_n is not unique, we choose the smaller possible value). It is easily seen that $t_n > 0$ for all $n \in \mathbb{N}$. Indeed, passing to a subsequence if necessary, we have $\Phi_\lambda(u_n) \geq \dfrac{c}{2}$, for all $n \in \mathbb{N}$. So, if $t_n = 0$ for any $n \in \mathbb{N}$ follows that

$$\Phi_\lambda(t_n u_n) = \Phi_\lambda(0) = 0,$$

however

$$0 < \frac{c}{2} \leqslant \Phi_\lambda(u_n) \leqslant \max_{t\in[0,1]} \Phi_\lambda(tu_n) = \Phi_\lambda(t_n u_n),$$

which is a contradicton.

If $t_n \in (0,1)$, by (1.17), we have that

$$\frac{d}{dt}_{|t=t_n} \Phi_\lambda(tu_n) = 0, \ \forall n \in \mathbb{N}.$$

Moreover, if $t_n = 1$, then, from (1.8) we have $\langle \Phi'_\lambda(u_n), u_n \rangle = o_n(1)$. So we always have

$$\langle \Phi'_\lambda(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt}_{|t=t_n} \Phi_\lambda(tu_n) = o_n(1).$$

Let $\{r_k\}_{k\in\mathbb{N}}$ be a positive sequence of real numbers such that $r_k > 1$ and $\lim_{k\to+\infty} r_k = +\infty$. Then $\|r_k\omega_n\| = r_k > 1$ for any k and n. Fix k, since $\omega_n \to 0$ strongly in $L^{\nu(\cdot)}(\Omega)$ and $\omega_n(x) \to 0$ a.e. $x \in \Omega$, as $n \to +\infty$, by using the condition (f_0), there exists a constant c_6 such that

$$\left| F(x, r_k\omega_k) \right| \leqslant c_6 \left(r_k|\omega_n(x)| + r_k^{\nu(x)}|\omega_n|^{\nu(x)} \right). \tag{1.18}$$

Moreover, by (1.17) and the continuity of the function F, we get

$$F(x, r_k\omega_k(x)) \to F(x, r_k\omega(x)) = 0, \ \text{a.e. } x \in \Omega, \ \text{as } n \to \infty, \tag{1.19}$$

for each $k \in \mathbb{N}$. Hence, by (1.18), (1.19), and the Dominated Convergence Theorem, we obtain that

$$\lim_{n\to+\infty} \int_\Omega F(x, r_k\omega_n)\, dx = 0, \tag{1.20}$$

for any $k \in \mathbb{N}$.

Recall that $\|u_n\| \to +\infty$ as $n \to +\infty$. So, we have either $\|u_n\| > r_k$ or $\frac{r_k}{\|u_n\|} \in (0,1)$ for n large enough. Hence, from (1.17), (1.20), and (a_1), we deduce that

$$\begin{aligned}
\Phi_\lambda(t_n u_n) &\geqslant \Phi_\lambda\left(\frac{r_k}{\|u_n\|}u_n\right) \\
&= \Phi_\lambda(r_k\omega_k) \\
&\geqslant \int_\Omega \frac{\alpha}{p(x)}|\nabla(r_k\omega_n)|^{p(x)}\, dx - \lambda\int_\Omega F(x, r_k\omega_n)\, dx \\
&\geqslant \frac{\alpha r_k^{p^-}}{p^+} - \lambda\int_\Omega F(x, r_k\omega_n)\, dx,
\end{aligned} \tag{1.21}$$

for any n large enough. From (1.21), letting $n, k \to +\infty$ we have

$$\limsup_{n\to+\infty} \Phi_\lambda(t_n u_n) = +\infty. \tag{1.22}$$

On the other hand, we affirm that

$$\limsup_{n\to+\infty} \Phi_\lambda(t_n u_n) \leqslant \delta,$$

for a suitable positive constant δ. Indeed, by the condition (f_3), $(\mathcal{T})$, (1.8), and all n large enough, we have

$$\begin{aligned}
\Phi_\lambda(t_nu_n) &= \Phi_\lambda(t_nu_n) - \tfrac{1}{\varpi}\langle \Phi'_\lambda((t_nu_n)), t_nu_n\rangle + o_n(1)\\
&= \int_\Omega \left(\frac{1}{p(x)}A(|\nabla(t_nu_n)|^{p(x)}) - \frac{1}{\varpi}a(|\nabla(t_nu_n)|^{p(x)})|\nabla(t_nu_n)|^{p(x)}\right) dx\\
&\quad + \frac{\lambda}{\varpi}\int_\Omega G(x,t_nu_n)\,dx + o_n(1)\\
&= \int_\Omega \mathcal{T}(|\nabla(t_nu_n(x))|^{p(x)})\,dx + \frac{\lambda}{\varpi}\int_\Omega (G(x,t_nu_n))\,dx + o_n(1)\\
&\leqslant \int_\Omega \mathcal{T}(|\nabla u_n(x)|^{p(x)})\,dx + \frac{\lambda}{\varpi}\int_\Omega (G(x,u_n) + \mathsf{C}_*)\,dx + o_n(1)\\
&= \int_\Omega \frac{1}{p(x)}A(|\nabla u_n|^{p(x)})\,dx - \lambda\int_\Omega F(x,u_n)\,dx\\
&\quad - \frac{1}{\varpi}\left[\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx - \lambda\int_\Omega f(x,u_n)u_n\right] + \frac{\lambda \mathsf{C}_*}{\varpi}|\Omega| + o_n(1)\\
&= \Phi_\lambda(u_n) - \tfrac{1}{\varpi}\langle \Phi'_\lambda(u_n), u_n\rangle + \frac{\lambda\mathsf{C}_*}{p^+}|\Omega| + o_n(1),
\end{aligned}$$

so, we get

$$\Phi_\lambda(t_nu_n) \longrightarrow c + \frac{\lambda\mathsf{C}_*}{\varpi}|\Omega| \text{ as } n \longrightarrow +\infty. \tag{1.23}$$

By (1.22) and (1.23) we obtain a contradiction. Therefore the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$.

Now, with standard arguments, we prove that any $(C)_c$ sequence has a convergent subsequence. Since since $W_0^{1,p(\cdot)}(\Omega)$ is a Banach reflexive space, by the *Eberlein* $-$ *Šmulian* Theorem, there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ such that, up to a subsequence still denoted by $\{u_n\}_{n\in\mathbb{N}}$,

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(\cdot)}(\Omega),$$

and by the Sobolev embedding theorem, we have

$$u_n(x) \to u(x) \text{ a.e. } x\in\Omega,\ \ u_n \to u \text{ in } L^{\nu(\cdot)}(\Omega), \text{ and } u_n \to u \text{ in } L^{p(\cdot)}(\Omega).$$

Hence, by the Hölder inequality, we have

$$\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| \leqslant C\|1+|u_n|^{\nu(x)-1}\|_{\nu'(\cdot)}\|u_n-u\|_{\nu(\cdot)} \to 0 \text{ as } n\to+\infty.$$

Thus, by using (1.8), we have

$$\langle \Phi'_\lambda(u_n), u_n-u\rangle \to 0,$$

and, therefore

$$\langle \Psi'(u_n), u_n - u\rangle = \lambda \int_\Omega f(x,u_n)(u_n-u)\,dx + \langle \Phi'_\lambda(u_n), u_n - u\rangle \to 0 \text{ as } n \to +\infty.$$

Since the operator Ψ' is of type (S_+) (see Lemma 4.7), we deduce that $u_n \to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$. This proves that Φ_λ satisfies the $(C)_c$ condition on $W_0^{1,p(\cdot)}(\Omega)$. The proof is complete. □

1.2.1 Proof Theorem 1.1

Proof. By Lemma 1.1 and Lemma 1.2, the functional Φ_λ satisfies the geometric conditions the Mountain Pass Theorem. Therefore, by Lemma 4.4, the fucntional Φ_λ has a critical value $c \geqslant \mathcal{R} > 0$. Hence problem (1.1) has at least one nontrivial weak solution in $W_0^{1,p(\cdot)}(\Omega)$. □

To prove the Theorem 1.2, we need the following "$\mathbb{Z}_2 - symmetric$" version (for even functionals) of the Mountain Pass Theorem (see Theorem 4.2). In order to prove this result, we need of the following Lemmas.

Lemma 1.3. *There exist $r, \rho > 0$ such that $\Phi_\lambda(u) \geqslant \rho$ for every $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| = r$.*

Proof. The proof is as in the Lemma 1.1. □

Lemma 1.4. *For every finite dimensional subspace $\widehat{X} \subset W_0^{1,p(\cdot)}(\Omega)$ there exists $\mathcal{R}_0 = \mathcal{R}_0(\widehat{X})$ such that*

$$\Phi_\lambda(u) \leqslant 0 \text{ for all } u \in \widehat{X} \setminus B_{\mathcal{R}_0}(\widehat{X}),$$

where $B_{\mathcal{R}_0}(\widehat{X}) = \{u \in \widehat{X} : \|u\| < \mathcal{R}_0\}$.

Proof. Let $\lambda > 0$ fixed. From (f_1), it follows that for any $M > 0$ there exists a constant $C_M > 0$, such that

$$F(x,t) \geqslant M|t|^{p^+} - C_M, \ \forall (x,t) \in \Omega \times \mathbb{R}. \tag{1.24}$$

Let $\widehat{X}$ be a finite dimensional subspace of $W_0^{1,p(\cdot)}(\Omega)$ and let $u \in \widehat{X}$ with $\|u\| = 1$ fixed. For all $t \geqslant 1$ and by Proposition 4.3,

$$\int_\Omega \frac{1}{p(x)} A(|\nabla tu|^{p(x)})\,dx \leqslant \frac{\beta t^{p^+}}{p^-} \int_\Omega |\nabla u|^{p(x)}\,dx < \beta t^{p^+},$$

we have, if M is large enough, by (a_1) and (1.24), we have

$$\begin{aligned}\Phi_\lambda(tu) &= \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\,dx - \lambda\int_\Omega F(x,u)\,dx \\ &\leqslant \beta t^{p^+} - \lambda M t^{p^+} - \lambda C_M|\Omega|.\end{aligned}$$

We have $\Pi(t) = \beta t^{p^+} - \lambda M t^{p^+} - \lambda C_M|\Omega| \to -\infty$ as $t \to +\infty$. Therefore, we have

$$\sup\{\Phi_\lambda(u) : u \in \widehat{X}, \|u\| = \mathcal{R}\} = \sup\{\Phi_\lambda(\mathcal{R}u) : u \in \widehat{X}, \|u\| = 1\} \to -\infty \text{ as } \mathcal{R} \to +\infty.$$

Hence, there exists $\mathcal{R}_0 > 0$ sufficiently large such that $\Phi_\lambda(u) \leqslant 0$ for all $u \in \widehat{X}$ with $\|u\| = \mathcal{R}$ and $\mathcal{R} \geqslant \mathcal{R}_0$. □

1.2.2 Proof of Theorem 1.2

Proof. By using (f_4), we obtain that Φ_λ is even. Since $\Phi_\lambda(0) = 0$ and Φ_λ satisfies the $(C)_c$ condition, by the Lemma 1.3, Lemma 1.4, and Theorem 4.2 we deduce the existence of an unbounded sequence of weak solutions to problem (1.1). □

In order to prove the Theorem 1.3 we will use the Fountain Theorem of Bartsch ([11], [73]).

Let X be a real, reflexive, and separable Banach space, it is known ([33, Chapter 4] or [76, Section 17]) that for a separable and reflexive Banach space there exist $\{e_j\}_{j\in\mathbb{N}} \subset X$ and $\{e_j^*\}_{j\in\mathbb{N}} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2, \ldots\}},\ X^* = \overline{span\{e_j^* : j = 1, 2, \ldots\}}^{\omega^*},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1 & \text{se} \quad i = j, \\ 0 & \text{se} \quad i \neq j. \end{cases}$$

We denote

$$X_j = span\{e_j\},\ Y_k = \bigoplus_{j=1}^{k} X_j = span\{e_1, \ldots, e_k\},\ \text{and } Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j} = \overline{span\{e_k, e_{k+1}\ldots\}}.$$

Lemma 1.5. *([34, Lemma 4.9]) If* $\nu \in C^+(\overline{\Omega})$, $p^+ < \nu(x) < (p^*)^-$ *for any* $x \in \overline{\Omega}$, *denote*

$$\beta_k := \sup\{\|u\|_{\nu(\cdot)} : \|u\| = 1,\, u \in Z_k\},$$

then $\lim_{k\to+\infty} \beta_k = 0$.

Lemma 1.6. *([36, Lemma 3.3]) Assume that* $\Xi : X \to \mathbb{R}$ *is weakly-strongly continuous and* $\Xi(0) = 0$. *Then for each* $\tau > 0$ *and* $k \in \mathbb{N}$ *there exists*

$$\alpha_k := \sup\{|\Xi(u)| : u \in Z_k,\ \|u\| < \tau\} < \infty.$$

Moreover, $\lim_{k\to+\infty} \alpha_k = 0$.

To prove the Theorem 1.3 we will need of the following Lemma 1.7.

Lemma 1.7. *Assume that* (f_0) *and* (f_2) *are satisfied. Then there exists* $\rho_k > r_k > 0$ *such that*

(A_2) $b_k := \inf\{\Phi_\lambda(u) : u \in Z_k,\, \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$.

(A_3) $a_k := \max\{\Phi_\lambda(u) : u \in Y_k,\, \|u\| = \rho_k\} \leqslant 0$.

Proof. To prove (A_2), first, notice that by (f_0) there exists $c_1 > 0$ such that

$$|F(x,t)| \leqslant c_1(|t| + |t|^{\nu(x)})$$

for all $(x,t) \in \Omega \times \mathbb{R}$, $p^+ < \nu(x) < (p^*)^-$. Then, for any $u \in Z_k$ with $\|u\| > 1$, by (a_1) and (f_0), we have

$$\begin{aligned} \Phi_\lambda(u) &= \int_\Omega A(|\nabla u|^{p(x)})\,dx - \lambda \int_\Omega F(x,u)\,dx \\ &\geqslant \frac{\alpha}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda c_1 \int_\Omega (|u| + |u|^{\nu(x)})\,dx \\ &\geqslant \frac{\alpha}{p^+}\|u\|^{p^-} - \lambda c_1 \beta_k^{\nu^+}\|u\|^{\nu^+} - \lambda c_8\|u\|, \end{aligned} \tag{1.25}$$

where $\beta_k := \sup\{\|u\|_{L^{\nu(x)}(\Omega)} : \|u\| = 1,\, u \in Z_k\}$.

Note that, since $p^- \leqslant p^+ < \nu^+$, by using the Lemma 1.5, it is easy see that $r_k := (c_1\beta_k^{\nu^+})^{\frac{1}{p^- - \nu^+}} \to +\infty$ as $k \to +\infty$. Then, for k sufficiently large, $u \in Z_k$ with

$\|u\| = r_k > 1$, and by (1.25), we get

$$\begin{aligned}\Phi_\lambda(u) &\geqslant \frac{\alpha}{p^+}\|u\|^{p^-} - \lambda c_1\beta_k^{\nu^+}\|u\|^{\nu^+} - \lambda c_8\|u\| \\ &= \frac{\alpha}{p^+}(c_1\beta_k^{\nu^+})^{\frac{p^-}{p^- - \nu^+}} - \lambda c_1\beta_k^{\nu^+}(c_1\beta_k^{\nu^+})^{\frac{\nu^+}{p^- - \nu^+}} - \lambda c_2(c_1\beta_k^{\nu^+})^{\frac{1}{p^- - \nu^+}} \\ &= \left(\frac{\alpha}{p^+} - \lambda\right) r_k^{p^-} - \lambda c_2 r_k.\end{aligned}$$

Therefore, since $p^- > 1$ and $\frac{\alpha}{p^+} > \lambda$, we obtain that $b_k := \inf\{\Phi_\lambda(u) : u \in Z_k, \|u\| = r_k\} \to +\infty$ as $k \to +\infty$.

Now, we will prove (A_3). By (a_1), for any $\psi \in Y_k$ with $\|\psi\| = 1,\ t > 1$, and (f_1), we have

$$\begin{aligned}\Phi_\lambda(t\psi) &\leqslant \frac{\beta}{p^-}\|t\psi\|^{p^+} - \lambda M\int_\Omega |t\psi|^{p^+}\,dx + \lambda C_M|\Omega| \\ &= t^{p^+}\left(\frac{\beta}{p^-}\|\psi\|^{p^+} - \lambda M\int_\Omega |\psi|^{p^+}\,dx\right) + \lambda C_M|\Omega|\,.\end{aligned} \tag{1.26}$$

It is clear that we can choose $M > 0$ large enough such that

$$\frac{\beta}{p^-}\|\psi\|^{p^+} - \lambda M\int_\Omega |\psi|^{p^+}\,dx < 0.$$

So, we obtain by (1.26) that

$$\lim_{t\to+\infty}\Phi_\lambda(t\psi) = -\infty.$$

Hence, there exists $t_0 > r_k > 1$ large enough such that $\Phi_\lambda(t_0\psi) \leqslant 0$ and thus, if set $\rho_k = t_0$, we conclude that

$$a_k := \max\{\Phi_\lambda(u) : u \in Y_k;\ \|u\| = \rho_k\} \leqslant 0.$$

□

1.2.3 Proof of Theorem 1.3

Proof. We have by (f_4) that the Euler Lagrange functional Φ_λ is an even, and by Lemma 1.2, Φ_λ satisfies the $(C)_c$ condition for every $c > 0$. By the Lemma 1.7 it was proven that if k is large enough, there exists $\rho_k > \gamma_k > 0$ such that (A_2) and (A_3) hold. This way, we have satisfied all the conditions of the Fountain Theorem 4.3. Hence, we obtain a sequence of critical points $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p(\cdot)}(\Omega)$ such that $\Phi_\lambda(u_n) \to +\infty$ as $n \to +\infty$. $\square$

For the proof of Theorem 1.4, we need some definitions and results.

Definition 1.2. *Let X be a separable and reflexive Banach space, $\Phi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We say that Φ satisfies the $(C)_c^*$ condition (with respect to (Y_n)), if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ for which $u_n \in Y_n$, for any $n \in \mathbb{N}$, $\Phi(u_n) \to c$ and $\|(\Phi_{|Y_n})'(u_n)\|_{X^*}(1 + \|u_n\|) \to 0$, as $n \to \infty$, contain a subsequence converging to a critical point of Φ.*

Lemma 1.8. *Suposse that the hypotheses in Theorem 1.4 hold, then Φ_λ satisfied the $(C)_c^*$ condition.*

Proof. Let $c \in \mathbb{R}$ and the sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(\Omega)$ be such that $u_n \in Y_n$, for all $n \in \mathbb{N}$, $\Phi_\lambda(u_n) \to c > 0$ and $\|\Phi'_{\lambda_{|Y_n}}(u_n)\|(1 + \|u_n\|) \to 0$, as $n \to +\infty$. Therefore, we have

$$c = \Phi_\lambda(u_n) + o_n(1) \text{ and } \langle \Phi'_\lambda(u_n), u_n\rangle = o_n(1),$$

where $o_n(1) \to 0$ as $n \to +\infty$.

Analogously to the proof of Lemma 1.2, we can prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Since $W_0^{1,p(\cdot)}(\Omega)$ is a reflexive space, by the *Eberlein – Šmulian* Theorem, we can extract a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, denoted for $\{u_{n_k}\}_{k\in\mathbb{N}}$, such that $u_{n_k} \rightharpoonup u$ weakly in X.

On the other hand, as $W_0^{1,p(\cdot)}(\Omega) = \overline{\cup_n Y_n} = \overline{span\{e_n : n \geqslant 1\}}$, we can choose $v_n \in Y_n$ such that $v_n \to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$. Hence, we obtain

$$\langle \Phi'_\lambda(u_{n_k}), u_{n_k} - u\rangle = \langle \Phi'_\lambda(u_{n_k}), u_{n_k} - v_{n_k}\rangle + \langle \Phi'_\lambda(u_{n_k}), v_{n_k} - u\rangle.$$

Since $(\Phi_{\lambda|Y_{n_k}})'(u_{n_k}) \to 0$ and $u_{n_k} - v_{n_k} \rightharpoonup 0$ in Y_{n_k}, as $k \to +\infty$, see [16, Proposition 3.5], we achieve

$$\lim_{k\to+\infty} \langle \Phi'_\lambda(u_{n_k}), u_{n_k} - u\rangle = 0.$$

Therefore, we get

$$\langle \Psi'(u_{n_k}), u_{n_k} - u\rangle = \lambda \int_\Omega f(x, u_{n_k})(u_{n_k} - u)\, dx + \langle \Phi'_\lambda(u_{n_k}), u_{n_k} - u\rangle \to 0 \text{ as } k \to +\infty.$$

Since Ψ' is of type (S_+), follows that $u_{n_k} \to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$. Then, we conclude that Φ_λ satisfies the $(C)_c^*$ condition. Furthermore, we have $\Phi'_\lambda(u_{n_k}) \to \Phi'_\lambda(u)$ as $k \to +\infty$.

Let us prove $\Phi'_\lambda(u) = 0$. Indeed, taking $\omega_j \in Y_j$, notice that when $n_k \geqslant j$, we have

$$\begin{aligned}\langle \Phi'_\lambda(u), \omega_j\rangle &= \langle \Phi'_\lambda(u) - \Phi'_\lambda(u_{n_k}), \omega_j\rangle + \langle \Phi'_\lambda(u_{n_k}), \omega_j\rangle \\ &= \langle \Phi'_\lambda(u) - \Phi'_\lambda(u_{n_k}), \omega_j\rangle + \langle (\Phi_{\lambda|_{Y_{n_k}}})'(u_{n_k}), \omega_j\rangle,\end{aligned}$$

so, passing the limit on the right side of the equation above, as $k \to +\infty$, we obtain

$$\langle \Phi'_\lambda(u), w_j\rangle = 0, \text{ for all } \omega_j \in Y_j.$$

Thus $\Phi'_\lambda(u) = 0$ in $(W_0^{1,p(\cdot)}(\Omega))'$, this show that Φ_λ satisfies the $(C)_c^*$ condition for every $c \in \mathbb{R}$. □

1.2.4 Proof of Theorem 1.4

Proof. From (f_3) and Lemma 1.8, f it is odd in the second variable and Φ_λ is even and satisfies the $(C)_c^*$ condition for all $c \in \mathbb{R}$.

Now, we will prove that the conditions (B_1), (B_2), (B_3), and (B_4) of the Dual Fountain Theorem are satisfied:

(B_1) : Claim. There exist $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geqslant k_0$, there exists $\rho_k > r_k > 0$ for which

$$a_k = \inf\{\Phi_\lambda(u) : u \in Z_k, \ \|u\| = \rho_k\} \geqslant 0.$$

Indeed. Notice that, as $\lambda < \frac{\alpha}{p^+}$, we have

$$\lim_{k\to+\infty} \left(\frac{\alpha}{p^+} - \lambda\right) (c_4\beta_k^{\nu^+})^{\frac{p^-}{p^- - \nu^+}} = +\infty,$$

then, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\frac{\alpha}{p^+} - \lambda\right) (c_4\beta_k^{\nu^+})^{\frac{p^-}{p^- - \nu^+}} - \lambda c_4|\Omega| \geqslant 0, \ \forall k \geqslant k_0.$$

Choosing $\rho_k = (c_4\beta_k^{\nu^+})^{\frac{1}{p^- - \nu^+}}$ for $k \geqslant k_0$, it is clear that $\rho_k > 1$ for all $k \in \mathbb{N}, \ k \geqslant k_0$, since $\lim_{k\to+\infty} \rho_k = +\infty$.

As in Theorem 1.3, for all $u \in Z_k$, with $\|u\| = \rho_k$, and (a_1), we get

$$\begin{aligned}
\Phi_\lambda(u) &\geqslant \frac{\alpha}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \int_\Omega \lambda c_4\left(|u|^{\nu(x)}+1\right)\,dx \\
&\geqslant \frac{\alpha}{p^+}\|u\|^{p^-} - \lambda c_4\beta_k^{\nu^+}\|u\|^{\nu^+} - \lambda c_4|\Omega| \\
&\geqslant \left(\frac{\alpha}{p^+} - \lambda\right)(c_4\beta_k^{\nu^+})^{\frac{p^-}{p^- - \nu^+}} - \lambda c_4|\Omega| \\
&\geqslant 0,
\end{aligned}$$

so, we obtain

$$\inf\{\Phi_\lambda(u) : u \in Z_k,\ \|u\| = \rho_k\} \geqslant 0.$$

(B_2) : Claim. For each $k \in \mathbb{N}$ there exists $r_k > 0$ such that

$$\max\{\Phi_\lambda(u) : u \in Y_k,\ \|u\| = r_k\} < 0.$$

Indeed. Since Y_k is finite dimensional all the norms are equivalent, so there exists a constant $c > 0$ such that $\|u\|_{p^+} \geqslant c\|u\|$ for all $u \in Y_k$. Then, we have

$$\Phi_\lambda(u) \leqslant \frac{\beta}{p^-}\|u\|^{p^+} - \lambda c M^{p^+}\|u\|^{p^+} + \lambda c_M|\Omega|,\ \text{for all } u \in Y_k \text{with } \|u\| \geqslant 1.$$

$$\lim_{t\to+\infty} \mathscr{N}(t) = -\infty.$$

Therefore, exists $\bar{t} \in (1, +\infty)$ such that

$$\mathscr{N}(t) < 0,\ \text{for all } t \in [\bar{t}, +\infty).$$

Hence, $\Phi_\lambda(u) < 0$ for all $u \in Y_k$ with $\|u\| = \bar{t}$.

Choosing $r_k = \bar{t}$ for all $k \in \mathbb{N}$, we have

$$b_k := \max\{\Phi_\lambda(u) : u \in Y_k,\ \|u\| = r_k\} < 0.$$

By changing k_0 by other more large, if necessary, we have $\rho_k > r_k > 0$, for all $k \geqslant k_0$.

(B_3) : Claim. $d_k = \inf\{\Phi_\lambda(u) : u \in Z_k,\ \|u\| \leqslant \rho_k\} \to 0$ as $k \to +\infty$.

Indeed. Notice that $Y_k \cap Z_k \neq \emptyset$ and $0 < r_k < \rho_k$, so, we have

$$d_k = \inf\{\Phi_\lambda(u) : u \in Z_k,\ \|u\| \leqslant \rho_k\} \leqslant b_k = \max\{\Phi_\lambda(u) : u \in Y_k,\ \|u\| = r_k\} < 0.$$

By (f_0), we obtain $|F(x,u)| \leqslant \overline{C}(|u| + |u|^{\nu(x)})$, for all $x \in \Omega$, $t \in \mathbb{R}$. Consider $\Gamma_1 : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ and $\Gamma_2 : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\Gamma_1(u) = \int_\Omega \lambda\overline{C}|u|^{\nu(x)}\,dx \text{ and } \Gamma_2(u) = \int_\Omega \lambda\overline{C}|u|\,dx. \tag{1.27}$$

We have $\Gamma_i(0) = 0$, $i = 1, 2$, and they are weakly-strongly continuous. Let us denote

$$\gamma_k = \sup\{|\Gamma_1(u)| : u \in Z_k,\ \|u\| \leqslant 1\} \text{ and } \theta_k = \sup\{|\Gamma_2(u)| : u \in Z_k,\ \|u\| \leqslant 1\}.$$

As the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\nu(\cdot)}(\Omega)$ is compact and by Lemma 1.6, we get $\lim_{k\to+\infty} \gamma_k = \lim_{k\to+\infty} \theta_k = 0$.

Consider $v \in Z_k$ with $\|v\| = 1$ and $0 < t < \rho_k$. Then, by (a_1) and (1.27), we have

$$\begin{aligned}\Phi_\lambda(tv) &= \int_\Omega \frac{1}{p(x)} A(|\nabla tv|^{p(x)})\,dx - \lambda\int_\Omega F(x,tv)\,dx \\ &\geqslant -\lambda\int_\Omega F(x,tv)\,dx \geqslant -\Gamma_1(tv) - \Gamma_2(tv),\end{aligned}$$

and since

$$\Gamma_1(tv) \leqslant t^{\nu^+}\Gamma_1(v) \text{ and } \Gamma_2(tv) = t\Gamma_2(v),$$

we achieve

$$\Phi_\lambda(tv) \geqslant -\rho_k^{\nu^+}\Gamma_1(v) - \rho_k\Gamma_2(v) \geqslant -\rho_k^{\nu^+}\gamma_k - \rho_k\theta_k,$$

for all $t \in (0, \rho_k)$ and $v \in Z_k$ with $\|v\| = 1$. Hence,

$$d_k \geqslant -\rho_k^{\nu^+}\gamma_k - \rho_k\theta_k$$

and as $d_k < 0$ for all $k \geqslant k_0$, we have $\lim_{k\to+\infty} d_k = 0$.

Thus, by Theorem 4.4, there exists a sequence of negative critical values converging to 0, which concludes the proof of Theorem 1.4. □

The proof of Theorem 1.5 is obtained as a result of the following two lemmas with the hypotheses $(a_0), (a_1), (f_0)$, and (f_1).

Lemma 1.9. *There exist positive constants* S_λ *and* ρ_λ *such that* $\lim_{\lambda\to 0^+}\mathsf{S}_\lambda = +\infty$ *and* $\Phi_\lambda(u) > \mathsf{S}_\lambda > 0$ *whenever* $\|u\| = \rho_\lambda$.

Proof. Let $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| > 1$. From (a_1) and (f_0), we have

$$\begin{aligned}\Phi_\lambda(u) &\geqslant \frac{\alpha}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda c_4\int_\Omega (|u|^\nu + 1)\,dx\\ &\geqslant \frac{\alpha}{p^+}\|u\|^{p^-} - \lambda c_{12}^*\|u\|^{\nu^+} - \lambda c_{13}|\Omega|.\end{aligned}$$

Consider $\gamma \in \left(0, \frac{1}{\nu^- - p^-}\right)$ and $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $\|u\| = \lambda^{-\gamma}$. We define $\rho_\lambda := \lambda^{-\gamma}$ and we remark that $\rho_\lambda > 1$ for λ small enough. Thus we obtain

$$\Phi_\lambda(u) \geqslant \frac{\alpha}{p^+}\lambda^{-\gamma p^-} - c_{12}^*\lambda^{1-\gamma\nu^+} - \lambda c_{13}|\Omega|.$$

Since $\gamma < \frac{1}{\nu^+ - p^-}$ and $-\gamma p^- < 1 - \gamma\nu^+$, we get $\mathsf{S}_\lambda := \frac{\alpha}{p^+}\lambda^{-\gamma p^-} - c_{12}^*\lambda^{1-\gamma\nu^+} - \lambda c_{13}|\Omega| \to +\infty$ as $\lambda \to 0^+$. Hence, there exists $\overline{\lambda} > 0$ small enough such that $\mathsf{S}_\lambda > 0$ for all $\lambda \in (0, \overline{\lambda})$. Therefore, we have

$$\Phi_\lambda(u) \geqslant \mathsf{S}_\lambda > 0 = \Phi_\lambda(0),$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| = \rho_\lambda = \lambda^{-\gamma}$ and $\lambda \in (0, \overline{\lambda})$. Thus, S_λ verifies the assertion of the Lemma. □

Lemma 1.10. *Let* $\omega \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}$. *Then*

$$\lim_{t\to+\infty}\Phi_\lambda(t\omega) = -\infty.$$

Proof. By the hypothesis (f_1) and (a_1), for $t > 1$, we estimate

$$\begin{aligned}\Phi_\lambda(t\omega) &= \int_\Omega \frac{1}{p(x)}A(|\nabla t\omega|^{p(x)})\,dx - \lambda\int_\Omega F(x,t\omega)\,dx\\ &\leqslant \frac{\beta}{p^-}t^{p^+}\|\omega\|^{p^+} - \lambda M c t^{p^+}\|\omega\|^{p^+} - \lambda C_M|\Omega|.\end{aligned}\tag{1.28}$$

It is clear that we can choose $M > 0$ large enough such that

$$\frac{\beta}{p^-}\|\omega\|^{p^+} - \lambda c M\|\omega\|^{p^+} < 0,$$

hence, it follows from (1.28) that

$$\lim_{t\to+\infty} \Phi_\lambda(t\omega) = -\infty.$$

□

1.2.5 Proof of Theorem 1.5

Proof. The hypothesis (a_0), (a_1), (f_0), (f_1), (f_3), $(\mathcal{T})$, and the Lemma 1.2 imply that the functional Φ_λ satisfy the $(C)_c$ condition. As a result of Lemma 1.9, Lemma 1.10, and Mountain Pass Theorem, we find that there is a nontrivial critical point u_λ for Φ_λ such that

$$\Phi_\lambda(u_\lambda) = c \geqslant \mathsf{S}_\lambda.$$

Moreover from $(a_1), (f_0)$, we have

$$\begin{aligned}\Phi_\lambda(u_\lambda) &\leqslant \frac{\beta}{p^-}\int_\Omega |\nabla u_\lambda|^{p(x)}\,dx + \lambda c_4\int_\Omega |u_\lambda|^{\nu(x)}\,dx + \lambda C|\Omega| \\ &\leqslant \frac{\beta}{p^-}\max\{\|u_\lambda\|^{p^+}, \|u_\lambda\|^{p^-}\} + \lambda c_* \max\{\|u_\lambda\|^{\nu^+}, \|u_\lambda\|^{\nu^-}\} + \lambda C|\Omega|.\end{aligned} \tag{1.29}$$

Taking $\lambda \to 0^+$ in (1.29) as $\mathsf{S}_\lambda \to +\infty$, then we obtain

$$\lim_{\lambda\to0^+} \|u_\lambda\| = +\infty.$$

□

1.2.6 Proof of Theorem 1.6

The proof of the Theorem 1.6 is obtained as a consequence of the following lemma.

Lemma 1.11. *Suppose (f_5) and $q^+ < p^-$.*

(i) Φ_λ is bounded from below;

(ii) Φ_λ satisfied the (PS) condition.

Proof.

(i) By (f_5), (a_1), and $\|u\| > 1$, for each $\lambda > 0$, we have

$$\begin{aligned}\Phi_\lambda(u) &\geqslant \frac{\alpha}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{\lambda C_1}{q^-}\int_\Omega |u|^{q(x)}\,dx\\ &\geqslant \frac{\alpha}{p^+}\min\{\|u\|^{p^-}, \|u\|^{p^+}\} - \frac{\lambda C_1}{q^-}\max\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\}\\ &\geqslant \frac{\alpha}{p^+}\|u\|^{p^-} - \frac{C^*}{q^-}\|u\|^{q^+}.\end{aligned}$$

Since $q^+ < p^-$, follows that Φ_λ is coercive. Therefore, Φ_λ is bounded from below.

(ii) Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ such that

$$\Phi_\lambda(u_n) \to c \text{ and } \Phi'_\lambda(u_n) \to 0 \text{ in } (W_0^{1,p(\cdot)}(\Omega))', \text{ as } n \to +\infty.$$

Since Φ_λ is coercive, we have $\{u_n\}$ bounded in $W_0^{1,p(\cdot)}(\Omega)$. Hence, by the *Eberlein – Šmulian* Theorem, up to a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, we have $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)}(\Omega)$. Moreover, we have

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega),\ u_n \to u \text{ in } L^{q(\cdot)}(\Omega),\ \text{and } u_n \to u \text{ a.e. in } \Omega,\ \text{as } n \to +\infty.$$

Since $\Phi'_\lambda(u_n) \to 0$ in $(W_0^{1,p(\cdot)}(\Omega))'$, as $n \to +\infty$, and $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, we obtain

$$\langle \Phi'_\lambda(u_n), u_n - u\rangle \to 0 \text{ as } n \to +\infty,$$

that is,

$$\begin{aligned}\langle \Phi'_\lambda(u_n), u_n - u\rangle &= \int_\Omega \left(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\right)(\nabla u_n - \nabla u)\,dx\\ &\quad -\lambda\int_\Omega f(x,u_n)(u_n - u)\,dx \longrightarrow 0 \text{ as } n \to +\infty.\end{aligned}$$

On the other hand, by (f_4) and Hölder inequality, we get

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| &\leqslant C_1\left|\int_\Omega |u_n|^{q(x)-1}(u_n-u)\,dx\right|\\ &\leqslant C_1\left\||u_n|^{q(x)-1}\right\|_{\frac{q(\cdot)}{q(\cdot)-1}}\|u_n-u\|_{q(\cdot)},\end{aligned}$$

so, taking into account that $u_n \to u$ in $L^{q(\cdot)}(\Omega)$, we achieve

$$\int_\Omega f(x,u_n)(u_n-u)\,dx \to 0 \text{ as } n \to +\infty.$$

Hence,

$$\int_\Omega \left(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\right)(\nabla u_n - \nabla u)\,dx \to 0 \text{ as } n \to +\infty.$$

Since Ψ' is of type (S_+) (see Lemma 4.7), we obtain $u_n \to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$ as $n \to +\infty$, concluding the proof of the Palais Smale condition.

□

1.2.7 Proof of Theorem 1.6

Proof. Since $W_0^{1,p(\cdot)}(\Omega)$ is a reflexive and separable Banach space, for each $k \in \mathbb{N}$, consider a k-dimensional linear subspace $\mathfrak{X}_k \subset C_0^\infty(\Omega)$ of $W_0^{1,p(\cdot)}(\Omega)$. Note that, the embedding $\mathfrak{X}_k \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous (Proposition 4.2). Thus, the norms of $W_0^{1,p(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$ are equivalent in $\mathfrak{X}_k$.

By (a_1) and (f_5), we have

$$\Phi_\lambda(u) \leqslant \frac{\beta}{p^-}\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda C_0 \int_\Omega |u|^{q(x)}\,dx.$$

From Proposition 4.3, if $\|u\| < 1$, we have $\rho_{p(\cdot)}(|\nabla u|) \leqslant \|u\|^{p^-}$ and $\rho_{q(\cdot)}(u) \geqslant \|u\|_{q(\cdot)}^{q^+}$ for every $u \in W_0^{1,p(\cdot)}(\Omega)$. Also, by the equivalence of norms in $\mathfrak{X}_k$, there exist a constant $C(k) > 0$ such that $C(k)\|u\|^{q^+} \leqslant \int_\Omega |u|^{q(x)}\,dx$ for every $u \in W_0^{1,p(\cdot)}(\Omega)$. Therefore, we get

$$\begin{aligned}\Phi_\lambda(u) &\leqslant \frac{\beta}{p^-}\|u\|^{p^-} - \lambda C(k)C_0\|u\|^{q^+}\\ &\leqslant \|u\|^{q^+}\left(\frac{\beta}{p^-}\|u\|^{p^- - q^+} - \lambda C(k)C_0\right),\end{aligned}$$

for every $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| < 1$. Let $\mathcal{R} \in (0,1)$ such that

$$\frac{\beta}{p^-}\mathcal{R}^{p^- - q^+} < \lambda C(k)C_0.$$

Thus, for all $0 < \mathcal{R}_0 < \mathcal{R}$, we achieve

$$\begin{aligned} \Phi_\lambda(u) &\leqslant \mathcal{R}_0^{q^+}\left(\frac{\beta}{p^-}\mathcal{R}_0^{p^- - q^+} - \lambda C(k)C_0\right) \\ &< 0 = \Phi_\lambda(0), \end{aligned}$$

for all $u \in \mathbb{S}^k_{\mathcal{R}_0} = \{u \in \mathcal{X}_k : \|u\| = \mathcal{R}_0\}$. So, we conclude that

$$\sup_{\mathbb{S}^k_{\mathcal{R}_0}} \Phi_\lambda(u) < 0 = \Phi_\lambda(0).$$

Since $\mathfrak{X}_k$ and $\mathbb{R}^k$ are isomorphic and $\mathbb{S}^k_{\mathcal{R}_0}$ is homeomorphic to $(k-1)$ dimensional sphere $\mathbb{S}^{k-1}$ in $\mathbb{R}^k$. Then, we have $\gamma(\mathbb{S}^k_{\mathcal{R}_0}) = k$.

By using the Lemma 1.11, we obtain that the Euler Lagrange functional Φ_λ satisfies the Palais Smale condition and Φ_λ is bounded from below. Also, by (f_4), it follows that Φ_λ is an even functional. Then, due to the Theorem 4.6, Φ_λ possesses at least k pairs of different critical points. Since k is arbitrary, we obtain infinitely many critical points of Φ_λ. □

Chapter

2

Existence and asymptotic behaviour of solution for a quasilinear Kirchhoff type equation with variable critical growth exponent

The aim of this chapter is to prove the existence of nontrivial weak solution for a class of nonlocal problems involving a superlinear nonlinearity with critical growth. We consider the nonlinear elliptic equation:

$$\begin{cases} -M(\mathscr{A}(u))\,\mathrm{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_\lambda)$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a real parameter. We consider the set $\mathscr{C} := \{x \in \Omega : s(x) = \gamma^*(x)\} \neq \emptyset$. Moreover, let $p(x)$, $q(x)$, $r(x)$, and $s(x)$ be continuous functions on $\overline{\Omega}$ verifying the inequalities

$$\begin{gathered} 1 < p^- \leqslant p(x) \leqslant p^+ < q^- \leqslant q(x) \leqslant q^+ < N, \\ \text{and} \\ \gamma^- \leqslant \gamma(x) \leqslant \gamma^+ < r^- \leqslant r(x) \leqslant r^+ < s^- \leqslant s(x) \leqslant \gamma^*(x) < +\infty, \end{gathered} \qquad (2.1)$$

for all $x \in \overline{\Omega}$, where $p^- := \min_{x\in\overline{\Omega}} p(x)$, $p^+ := \max_{x\in\overline{\Omega}} p(x)$, and analogously to r^-, r^-, q^-, q^+, γ^-, γ^+, s^-, and s^+, with $\gamma(x) = (1 - \mathcal{H}(\kappa_3))p(x) + \mathcal{H}(\kappa_3)q(x)$, and the variable critical

Sobolev exponent $\gamma^*(x)$ defined by

$$\gamma^*(x) = \begin{cases} \frac{N\gamma(x)}{N-\gamma(x)}, & \text{se } \gamma(x) < N, \\ +\infty, & \text{se } \gamma(x) \geqslant N, \end{cases}$$

for all $x \in \overline{\Omega}$, where $\mathcal{H} : \mathbb{R}_0^+ \to \{0,1\}$ is given by

$$\mathcal{H}(\kappa) = \begin{cases} 1, & \text{if } \kappa > 0, \\ 0, & \text{if } \kappa = 0. \end{cases}$$

The operator $\mathscr{A} : X \to \mathbb{R}$ is defined by

$$\mathscr{A}(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\, dx,$$

where $A(\cdot)$ is the function $A(t) = \displaystyle\int_0^t a(\kappa) d\kappa$ and the function $a(\cdot)$ is described in the hypothesis (a_0). The Banach space X will be described in Section 2.1.

In this chapter, we consider the function $a : \mathbb{R}^+ \to \mathbb{R}^+$ verifying the hypotheses:

(a_0) $a(\cdot)$ is of the class C^1;

(a_1) There are positive constants, κ_0, κ_1, κ_2, and κ_3 is a nonnegative constant such that

$$\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2 \tau^{\frac{q(x)-p(x)}{p(x)}} \leqslant a(\tau) \leqslant \kappa_1 + \kappa_3 \tau^{\frac{q(x)-p(x)}{p(x)}}, \quad \forall \tau \geqslant 0, \quad \forall x \in \overline{\Omega};$$

(a_2) There is $c > 0$ such that

$$\min\left\{ a(\tau^{p(x)})\tau^{p(x)-2}, a(\tau^{p(x)})\tau^{p(x)-2} + \tau \frac{\partial (a(\tau^{p(x)})\tau^{p(x)-2})}{\partial \tau} \right\} \geqslant c\tau^{p(x)-2},$$

for almost everywhere $x \in \Omega$ and for all $\tau > 0$;

(a_3) There are positive constants α and θ such that

$$A(\tau) \geqslant \frac{1}{\alpha} a(\tau)\tau \quad \text{with} \quad \gamma^+ < \theta < s^- \quad \text{and} \quad \frac{q^+}{p^+} \leqslant \alpha < \frac{\theta}{p^+},$$

for all $\tau \geqslant 0$.

In terms of the Kirchhoff's function, we will assume a general structure over $M : \mathbb{R}_0^+ \to \mathbb{R}$. We suppose that M is a continuous function verifying the conditions:

$(\mathcal{M}_1)$ There is $m_0 > 0$ such that $M(t) \geqslant m_0 = M(0) > 0$ for all $t \in \mathbb{R}_0^+$;

$(\mathcal{M}_2)$ The function M is increasing in $\mathbb{R}_0^+$.

From now on, we will denote $\mathscr{M}(t) := \int_0^t M(\tau)d\tau$ for all $t \in \mathbb{R}_0^+$.

We suppose that the nonlinearity $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the properties:

(f_1)
$$\lim_{|t|\to 0} \frac{f(x,t)}{t^{q^+-1}} = 0 \text{ uniformly in } x \in \Omega;$$

(f_2)
$$\lim_{|t|\to +\infty} \frac{f(x,t)}{t^{r^--1}} = 0 \text{ uniformly in } x \in \Omega;$$

(f_3) for $\gamma^+ < \theta < s^-$ given in (a_3),

$$0 < \theta F(x,t) = \theta \int_0^t f(x,s)ds \leqslant tf(x,t), \quad \forall x \in \Omega, \quad \forall t > 0.$$

Our main result establishes the existence of at least one solution to the problem $(\mathscr{P}_\lambda)$ under the previous hypotheses.

Theorem 2.1. *Assume the hypotheses $(a_0) - (a_3)$, (2.1), $(\mathcal{M}_2) - (\mathcal{M}_2)$, and $(f_1) - (f_3)$. Then, there exists $\overline{\lambda} > 0$, such that the problem $(\mathscr{P}_\lambda)$ has at least one nontrivial weak solution in X for each $\lambda \geqslant \overline{\lambda}$. Moreover, if u_λ is a solution to the problem $(\mathscr{P}_\lambda)$, then*

$$\lim_{\lambda\to+\infty} \|u_\lambda\| = 0.$$

2.1 Variational framework

Let us considerthe variable exponent Sobolev space $W^{1,h(\cdot)}(\Omega)$ is defined by

$$W^{1,h(\cdot)}(\Omega) = \Big\{u \in L^{h(\cdot)}(\Omega) : |\nabla u| \in L^{h(\cdot)}(\Omega)\Big\}.$$

The corresponding norm for this space is

$$\|u\|_{1,h(\cdot)} = \|u\|_{h(\cdot)} + \|\nabla u\|_{h(\cdot)}.$$

We define the space $W_0^{1,h(\cdot)}(\Omega)$ as the closure of $C_0^{+\infty}(\Omega)$ in $W^{1,h(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,h(\cdot)}$.

By Proposition 4.2, the spaces $L^{h(\cdot)}(\Omega)$, $W^{1,h(\cdot)}(\Omega)$ and $W_0^{1,h(\cdot)}(\Omega)$ are separable and reflexive Banach spaces when $h^- > 1$, and $\|\nabla u\|_{h(\cdot)}$ and $\|u\|_{1,h(\cdot)}$ are equivalent norms on $W_0^{1,h(\cdot)}(\Omega)$. We will use $\|\nabla u\|_{h(\cdot)}$ to replace $\|u\|_{1,h(\cdot)}$ in the following discussions.

We define the reflexive Banach space

$$X := W_0^{1,p(\cdot)}(\Omega) \cap W_0^{1,\gamma(\cdot)}(\Omega),$$

endowed with the norm

$$\|u\| := \|\nabla u\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u\|_{q(\cdot)}.$$

The weak solutions of $(\mathscr{P}_\lambda)$ coincide with the critical points of the Euler-Lagrange functional $\Phi_\lambda : X \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = \mathscr{M}(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\,dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}\,dx.$$

Note that the functional Φ_λ is Fréchet differentiable in $u \in X$ and

$$\langle \Phi_\lambda'(u), v\rangle = M(\mathscr{A}(u)) \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v dx - \lambda \int_\Omega f(x,u)v dx - \int_\Omega |u|^{s(x)-2}uv dx$$

for all $v \in X$.

Definition 2.1. *We say that $\{u_n\}_{n\in\mathbb{N}} \subset X$ is a Palais-Smale sequence for the functional Φ_λ at the level c_λ if*

$$\Phi_\lambda(u_n) \to c_\lambda \text{ and } \Phi_\lambda'(u_n) \to 0 \text{ in } X'. \tag{2.2}$$

Moreover, if all Palais-Smale sequence at the level c_λ possesses a convergent subsequence in X, we say that Φ_λ satisfies the Palais-Smale condition at the level c_λ.

2.2 The auxiliary problem $(\mathscr{M}_\lambda)$

We only know, by $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$, that the function M is bounded from below, which makes difficult prove that the functional $\Phi_{\omega,\lambda}$ has the geometry of the

Mountain Pass Theorem and that the sequence of Palais-Smale is bounded in X. Hence, we will introduce the auxiliary problem by defining the auxiliary functional $\Phi_{\omega,\lambda}$.

We will do a truncation of the function M and we will study the truncated problem. Let θ be as in (f_3) and $\omega \in \mathbb{R}$ such that $m_0 < \omega < \frac{\theta m_0}{p^+\alpha}$. Moreover, by $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ there exists $t_0 > 0$ such that $M(t_0) = \omega$. Thus, we define

$$M_\omega(t) = \begin{cases} M(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ \omega & \text{if } t \geqslant t_0. \end{cases}$$

So, we introduce the auxiliary problem

$$\begin{cases} -M_\omega(\mathscr{A}(u)) \operatorname{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{$\mathscr{M}_\lambda$}$$

where a, f, and λ defined as in problem $(\mathscr{P}_\lambda)$. Note that

$$m_0 \leqslant M_\omega(t) \leqslant \omega < \frac{\theta m_0}{p^+\alpha}, \ \forall t \geqslant 0. \tag{2.3}$$

Now, the next step is proof that the auxiliary problem $(\mathscr{M}_\lambda)$ admits a nontrivial solution. But for that, we need some technical results.

Theorem 2.2. *Assume (a_0), (a_1), $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, and $(f_0) - (f_3)$. Then there exists $\underline{\lambda} > 0$ such that the problem $(\mathscr{M}_\lambda)$ has a nontrivial weak solution, for each $\lambda \geqslant \underline{\lambda}$.*

2.2.1 Technical lemmas for the problem $(\mathscr{M}_\lambda)$

We observe that the Euler-Lagrange functional, $\Phi_{\omega,\lambda} : X \to \mathbb{R}$, associated to the auxiliary problem $(\mathscr{M}_\lambda)$ is given by

$$\Phi_{\omega,\lambda}(u) = \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda \int_\Omega F(x,t)\,dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}\,dx,$$

where $\mathscr{M}_\omega(t) = \int_0^t M_\omega(s)\,ds$. Moreover, the Fréchet derivative of $\Phi_{\omega,\lambda}$ in $u \in X$ is

$$\begin{aligned} \langle \Phi'_{\omega,\lambda}(u), v\rangle =& M_\omega(\mathscr{A}(u)) \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v\,dx - \lambda \int_\Omega f(x,u)v\,dx \\ &- \int_\Omega |u|^{s(x)-2}uv\,dx, \end{aligned} \tag{2.4}$$

for all $v \in X$.

Now, we will verify that $\Phi_{\omega,\lambda}$ has the geometric features required by the Mountain Pass Theorem.

Lemma 2.1. *Suppose that* $(a_0)-(a_3)$, (f_1), (f_2), $(\mathcal{M}_1)$, *and* $(\mathcal{M}_2)$ *are hold. Then, there are* $\mathcal{R}, \rho > 0$ *such that*

$$\Phi_{\omega,\lambda}(u) \geqslant \mathcal{R} > 0,$$

for any $u \in X$ *with* $\|u\| = \rho$.

Proof. **Claim**. Let us fix $\lambda > 0$. By using (f_1) and (f_2), for any $\varepsilon > 0$, we obtain $C(\varepsilon) > 0$ such that

$$|F(x,t)| \leqslant \frac{\varepsilon}{q^+}|t|^{q^+} + \frac{C(\varepsilon)}{r^-}|t|^{r^-}, \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{2.5}$$

Indeed, by (f_1), given $\varepsilon > 0$, there exist $\delta > 0$ such that

$$|f(x,t)| \leqslant \varepsilon |t|^{q^+-1}, \tag{2.6}$$

for all $0 < |t| \leqslant \delta$. Also, by (f_2), given $\varepsilon > 0$ there exists $\mathscr{R} > 0$ such that

$$|f(x,t)| \leqslant \varepsilon |t|^{r^--1}, \tag{2.7}$$

for all $|t| \geqslant \mathscr{R}$.

Note that $\frac{f(x,t)}{t^{r^--1}}$ is continuous in the compact interval $[\delta, \mathscr{R}]$. Thus, there is a constant $C > 0$ such that

$$|f(x,t)| \leqslant C|t|^{r^--1}, \tag{2.8}$$

always that $\delta \leqslant |t| \leqslant \mathscr{R}$. Then, by (2.6), (2.7), and (2.8), we have

$$\begin{aligned} |f(x,t)| &\leqslant \varepsilon |t|^{q^+-1} + \varepsilon |t|^{r^--1} + C|t|^{r^--1} \\ &\leqslant \varepsilon |t|^{q^+-1} + (\varepsilon + C)|t|^{r^--1}, \end{aligned}$$

that is

$$|f(x,t)| \leqslant \varepsilon |t|^{q^+-1} + C(\varepsilon)|t|^{r^--1},$$

where $C(\varepsilon) = \varepsilon + C$. Therefore, we achieve

$$|F(x,t)| \leqslant \frac{\varepsilon}{q^+}|t|^{q^+} + \frac{C(\varepsilon)}{r^-}|t|^{r^-},$$

so, we conclude the proof of the **Claim**.

From $(\mathcal{M}_1)$, (a_1), and (2.5), we achieve

$$\begin{aligned}\Phi_{\omega,\lambda}(u) \geqslant m_0 \left[\int_\Omega \frac{\kappa_0}{p(x)} |\nabla u|^{p(x)}\, dx + \mathcal{H}(\kappa_3)\kappa_2 \int_\Omega \frac{1}{q(x)} |\nabla u|^{q(x)}\, dx\right] \\ - \frac{\lambda\varepsilon}{q^+}\int_\Omega |u|^{q^+}\, dx - \lambda\frac{C(\varepsilon)}{r^-}\int_\Omega |u|^{r^-}\, dx - \frac{1}{s^-}\int_\Omega |u|^{s(x)}\, dx.\end{aligned}$$

Consider $0 < \|u\| = \|\nabla u\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u\|_{q(\cdot)} = \rho < 1$. By Propositon 4.3, and Proposition 4.2, there are positive constants C_1, C_2, C_3, and C_4 such that

$$\begin{aligned}\Phi_{\omega,\lambda}(u) \geqslant & C_1(\|\nabla u\|_{p(\cdot)}^{q^+} + \mathcal{H}(\kappa_3)\|\nabla u\|_{q(\cdot)}^{q^+}) - \frac{\lambda\varepsilon}{q^+}\int_\Omega |u|^{q^+}\, dx \\ & - \lambda\frac{C(\varepsilon)}{r^-}\int_\Omega |u|^{r^-}\, dx - \frac{1}{s^-}\int_\Omega |u|^{s(x)}\, dx \\ \geqslant & \left(C_1 - \frac{\lambda\varepsilon C_2}{q^+}\right)\|u\|^{q^+} - \frac{\lambda C(\varepsilon)C_3}{r^-}\|u\|^{r^-} - \frac{C_4}{s^-}\|u\|^{s^-},\end{aligned}$$

where $C_1 = \min\{\frac{m_0\kappa_0}{p^+}, \frac{m_0\kappa_2}{q^+}\}$.

Let us fix $\varepsilon > 0$, small enough, such that $C_5 := C_1 - \frac{\lambda\varepsilon C_3}{q^+} > 0$. Then there exist positive constants C_5, C_6, and C_7 such that

$$\Phi_{\omega,\lambda}(u) \geqslant C_5\|u\|^{q^+} - \lambda C_6\|u\|^{r^-} - C_7\|u\|^{s^-}.$$

Let $\mathcal{L}(t) = C_5 t^{q^+} - C_6 t^{r^-} - C_7 t^{s^-}$, since $q^+ < r^- < s^-$, then it is easy to see that we can choose $\rho > 0$ small enough there is $\mathcal{R} > 0$ such that

$$\Phi_{\omega,\lambda}(u) \geq \mathcal{R} > 0 \text{ as } \|u\| = \rho.$$

Indeed, let $\rho > 0$ to be determined later. Let $u \in X$ with $\|u\| = \rho$, we have

$$\Phi_{\omega,\lambda}(u) \geqslant C_5\rho^{q^+} - \lambda C_6\rho^{r^-} - C_7\rho^{s^-}.$$

We showed that there exists $\rho > 0$ such

$$\mathcal{R} = C_5\rho^{q^+} - \lambda C_6\rho^{r^-} - C_7\rho^{s^-} > 0,$$

or equivalently

$$\rho^{s^-}\left(\frac{C_5}{\rho^{s^- - q^+}} - \frac{\lambda C_6}{\rho^{s^- - r^-}} - C_7\right) > 0.$$

We know that $\rho > 0$, then it is sufficient to have

$$\frac{C_5}{\rho^{s^- - q^+}} - \frac{\lambda C_6}{\rho^{s^- - r^-}} - C_7 > 0,$$

or equivalently

$$\frac{C_5}{\rho^{s^- - q^+}} - \frac{\lambda C_6}{\rho^{s^- - r^-}} > C_7,$$

that is

$$\frac{1}{\rho^{s^- - r^-}}\left(\frac{C_5}{\rho^{r^- - q^+}} - \lambda C_6\right) > C_7.$$

Since $q^+ < r^- < s^-$, we have

$$\frac{1}{\rho^{s^- - r^-}}\left(\frac{C_5}{\rho^{r^- - q^+}} - \lambda C_6\right) \to +\infty, \text{ as } \rho \to 0^+.$$

Hence, as $q^+ < r^- < s^-$, follows that there are $0 < \rho < 1$, small enough,t and $\mathcal{R} > 0$ such that

$$\Phi_{\omega,\lambda}(u) \geq \mathcal{R} > 0 \text{ as } \|u\| = \rho.$$

□

Lemma 2.2. *Suppose* $(a_0) - (a_3)$, $(\mathcal{M}_2)$, *and* (f_3) *hold. Then, for each* $\lambda > 0$, *there is a nonnegative function* $e \in X$, *independent of* λ, *such that* $\|e\| > r$ *and* $\Phi_{\omega,\lambda}(e) < 0$.

Proof. Fix $\lambda > 0$. We have, by (f_3), positive constants k_1, k_2 satisfying

$$F(x,t) \geqslant k_1 t^\theta - k_2,$$

for all $t > 0$ and $x \in \Omega$. Note that, from (a_1) and (2.3), we obtain

$$\begin{aligned}
\mathscr{M}_\omega(\mathscr{A}(tu_0)) =& \int_0^{\mathscr{A}(tu_0)} M_\omega(t)\,dt \\
\leqslant& \omega \mathscr{A}(tu_0) \\
\leqslant& \omega\left(\int_\Omega \frac{1}{p(x)} A(|\nabla(tu_0)|^{p(x)})\,dx\right) \\
\leqslant& \omega \int_\Omega \left(\frac{k_1}{p(x)}|\nabla(tu_0)|^{p(x)} + \frac{k_3}{q(x)}|\nabla(tu_0)|^{q(x)}\right) dx \\
\leqslant& \omega\Bigg(\frac{k_1}{p^-}\max\left\{\|\nabla tu_0\|_{p(\cdot)}^{p^+}, \|\nabla tu_0\|_{p(\cdot)}^{p^-}\right\} \\
&+ \frac{k_3}{q^-}\max\left\{\|\nabla tu_0\|_{q(\cdot)}^{q^+}, \|\nabla tu_0\|_{q(\cdot)}^{q^-}\right\}\Bigg).
\end{aligned}$$

Then, choosing a nonnegative function $u_0 \in C_0^{+\infty}(\Omega)\backslash\{0\}$ with $\|u_0\| = 1$ and $u_0 \geqslant 0$ in Ω, we have

$$\begin{aligned}
\Phi_{\omega,\lambda}(tu_0) \;\leqslant\; & \omega\left(\frac{k_1}{p^-}\max\left\{\|\nabla tu_0\|_{p(\cdot)}^{p^+}, \|\nabla tu_0\|_{p(\cdot)}^{p^-}\right\} + \frac{k_3}{q^-}\max\left\{\|\nabla tu_0\|_{q(\cdot)}^{q^+}, \|\nabla tu_0\|_{q(\cdot)}^{q^-}\right\}\right) \\
& -\lambda t^\theta k_1 \int_\Omega u_0^\theta\,dx + \lambda k_2|\Omega| - \frac{1}{s^-}\min\left\{\|tu_0\|_{s(\cdot)}^{s^-}, \|tu_0\|_{s(\cdot)}^{s^+}\right\}.
\end{aligned}$$

Thus, by (2.1), and $\gamma^+ < \theta < s^-$, we get $\Phi_{\omega,\lambda}(tu_0) \to -\infty$ as $t \to +\infty$. Then, the lemma is proved by taking $e = t_* u_0$ with $t_* > 0$ large enough. □

From Lemma 2.1, Lemma 2.2, and of the Mountain Pass Theorem without Palais-Smale condition (see [15], Theorem 2.2), we obtain a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\Phi_{\omega,\lambda}(u_n) \to c_{\omega,\lambda} \text{ and } \Phi'_{\omega,\lambda}(u_n) \to 0,$$

where

$$c_{\omega,\lambda} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi_{\omega,\lambda}(\gamma(t)) > 0$$

and

$$\Gamma := \big\{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \Phi_{\omega,\lambda}(\gamma(1)) < 0\big\}.$$

Now, to overcome the lack of compactness due to the presence of a critical term, we will introduce an asymptotic condition for the level $c_{\omega,\lambda}$ and the boundedness of the sequence $\{u_n\}_{n\in\mathbb{N}}$ in X.

Lemma 2.3. *Suppose* $(a_0), (a_1)$, $(\mathcal{M}_1), (\mathcal{M}_2)$, *and* $(f_1) - (f_3)$. *Then*

$$\lim_{\lambda\to+\infty} c_{\omega,\lambda} = 0.$$

Proof. Let $e \in X$ be as in Lemma 2.2, thus $\lim\limits_{t\to+\infty} \Phi_{\omega,\lambda}(te) = -\infty$, for each $\lambda > 0$. So, there is $t_\lambda > 0$ such that

$$\Phi_{\omega,\lambda}(t_\lambda e) = \max_{t\geqslant 0} \Phi_{\omega,\lambda}(te).$$

Hence, we get

$$\langle \Phi'_{\omega,\lambda}(t_\lambda e), t_\lambda e\rangle = 0.$$

Therefore, from (2.4) follows that

$$M_\omega(\mathscr{A}(t_\lambda e)) \int_\Omega a(|(\nabla(t_\lambda e))|^{p(x)})|\nabla(t_\lambda e)|^{p(x)}dx = \lambda t_\lambda \int_\Omega f(x, t_\lambda e)e dx + \int_\Omega |t_\lambda e|^{s(x)}dx. \tag{2.9}$$

By construction $e \geqslant 0$ a.e. in Ω. Therefore, by (f_3) and (2.9), we get

$$M_\omega\left(\mathscr{A}(t_\lambda e)\right) \int_\Omega a(|\nabla(t_\lambda e)|^{p(x)})|\nabla(t_\lambda e)|^{p(x)}\, dx \geqslant \int_\Omega |(t_\lambda e)|^{s(x)}\, dx. \tag{2.10}$$

On the other hand, by (2.3), (a_1), and Proposition 4.3, we obtain

$$\begin{aligned} M_\omega(\mathscr{A}(t_\lambda e)) \int_\Omega a(|\nabla(t_\lambda e)|^{p(x)})|\nabla(t_\lambda e)|^{p(x)}dx &\leqslant \omega \int_\Omega a(|\nabla(t_\lambda e)|^{p(x)})|\nabla(t_\lambda e)|^{p(x)}dx \\ &\leqslant \omega \left(k_1 \int_\Omega |\nabla(t_\lambda e)|^{p(x)}dx + k_3 \int_\Omega |\nabla(t_\lambda e)|^{q(x)}dx\right) \\ &\leqslant \omega \Bigg(k_1 \max\left\{\|\nabla t_\lambda e\|_{p(\cdot)}^{p^-}, \|\nabla t_\lambda e\|_{p(\cdot)}^{p^+}\right\} \\ &\qquad + k_3 \max\left\{\|\nabla t_\lambda e\|_{q(\cdot)}^{q^-}, \|\nabla t_\lambda e\|_{q(\cdot)}^{q^+}\right\}\Bigg). \end{aligned} \tag{2.11}$$

Thus, by (2.10), (2.11), and Proposition 4.3, we achieve

$$\begin{aligned}\omega\left(k_1\max\left\{\|\nabla t_\lambda e\|_{p(\cdot)}^{p^-},\|\nabla t_\lambda e\|_{p(\cdot)}^{p^+}\right\}\right. &\left.+k_3\max\left\{\|\nabla t_\lambda e\|_{q(\cdot)}^{q^-},\|\nabla t_\lambda e\|_{q(\cdot)}^{q^+}\right\}\right)\\ &\geqslant \int_\Omega |t_\lambda e|^{s(x)}\,dx \\ &\geqslant \min\left\{\|t_\lambda e\|_{s(\cdot)}^{s^-},\|t_\lambda e\|_{s(\cdot)}^{s^+}\right\}.\end{aligned} \tag{2.12}$$

Claim. The sequence $\{t_\lambda\}$ is bounded in $\mathbb{R}$. Indeed, supposing that $\{t_\lambda\}$ is unbounded, there is a subsequence with $t_{\lambda_n}\to+\infty$ as $n\to+\infty$. Then, we obtain by (2.12) that

$$\frac{\omega k_1\|e\|_{1,p(\cdot)}}{t_{\lambda_n}^{q^+-p^+}}+\omega k_3\|e\|_{1,q(\cdot)}\geqslant t_{\lambda_n}^{s^--q^+}\|e\|_{s(\cdot)},$$

so, taking into account (2.1), and taking the limit as $n\to+\infty$, we have a contradiction. Hence, $\{t_\lambda\}$ is bounded in $\mathbb{R}$. The **Claim** is proved.

Consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n\to+\infty$ and let $t_0\geqslant 0$ be such that $t_{\lambda_n}\to\bar{t}$ as $n\to+\infty$. From continuity of M_ω, we have $\{M_\omega(\mathscr{A}(t_{\lambda_n}e))\}_{n\in\mathbb{N}}$ bounded, and so, there is $C>0$ such that

$$M_\omega(\mathscr{A}(t_{\lambda_n}e))\int_\Omega a(|\nabla(t_{\lambda_n}e)|^p)|\nabla(t_{\lambda_n}e)|^p dx\leqslant C \text{ for all } n\in\mathbb{N},$$

which implies, due to (2.9), that

$$\lambda_n t_{\lambda_n}\int_\Omega f(x,t_{\lambda_n}e)e\,dx+\int_\Omega t_{\lambda_n}^{s(x)}|e|^{s(x)}dx\leqslant C \text{ for all } n\in\mathbb{N}. \tag{2.13}$$

We claim that $\bar{t}=0$. Indeed. If $\bar{t}>0$, then, as f is continuous, by (f_1), (f_2), and by the Lebesgue Dominated Convergence Theorem, we get

$$\int_\Omega f(x,t_{\lambda_n}e(x))t_{\lambda_n}e(x)\,dx\to\int_\Omega f(x,\bar{t}ze(x))\bar{t}e(x)\,dx>0,\ n\to+\infty,$$

so, we obtain

$$\lambda_n t_{\lambda_n}\int_\Omega f(x,t_{\lambda_n}e)e\,dx+\int_\Omega t_{\lambda_n}^{s(x)}|e|^{s(x)}dx\to+\infty,\quad \text{as } n\to+\infty,$$

which contradicts (2.13). Thus, we conclude that $\bar{t}=0$.

Consider the path $\gamma_*(t)=te$ for $t\in[0,1]$. Note that $\gamma_*\in\Gamma$, then, by (f_3), we have

$$0 < c_{\lambda_n} \leqslant \max_{t\in[0,1]} \Phi_{\omega,\lambda_n}(\gamma_*(t)) = \Phi_{\omega,\lambda_n}(t_{\lambda_n} e) \leqslant \mathscr{M}_\omega(\mathscr{A}(t_{\lambda_n} e)). \tag{2.14}$$

On the other hand, as M_ω is continuous and $\bar{t} = 0$, we obtain

$$\lim_{n\to+\infty} \mathscr{M}_\omega(\mathscr{A}(t_{\lambda_n} e)) = 0,$$

so, by (2.14), follows that

$$\lim_{n\to+\infty} c_{\omega,\lambda_n} = 0.$$

Moreover, due to (f_3), we get that $\{c_{\omega,\lambda}\}_\lambda$ is a monotone sequence. Hence, we conclude that

$$\lim_{\lambda\to+\infty} c_{\omega,\lambda} = 0.$$

□

Lemma 2.4. *Suppose* $(\mathcal{M}_1), (\mathcal{M}_2)$, (f_1), *and* (f_2). *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence in* X *such that*

$$\Phi_{\omega,\lambda}(u_n) \to c_{\omega,\lambda} \text{ and } \Phi'_{\omega,\lambda}(u_n) \to 0 \text{ in } X'. \tag{2.15}$$

Then $\{u_n\}_{n\in\mathbb{N}}$ *is bounded in* X.

Proof. From (a_3), $(\mathcal{M}_1)$, (f_3), (2.3), and (2.15), we have

$$\begin{aligned}
C\,(1+\|u_n\|) &\geqslant \Phi_{\omega,\lambda}(u_n) - \frac{1}{\theta}\langle \Phi'_{\omega,\lambda}(u_n), u_n\rangle \\
&= \mathscr{M}_\omega(\mathscr{A}(u_n)) - \lambda \int_\Omega F(x,u_n)\,dx - \int_\Omega \frac{1}{s(x)}|u_n|^{s(x)}\,dx \\
&\quad - \frac{1}{\theta}\left(M_\omega(\mathscr{A}(u_n)) - \lambda \int_\Omega f(x,u_n)u_n\,dx - \int_\Omega |u_n|^{s(x)}\,dx \right) \\
&\geqslant \mathscr{M}_\omega(\mathscr{A}(u_n)) - \frac{1}{\theta} M_\omega(\mathscr{A}(u_n)) \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx \\
&\geqslant \frac{m_0}{p^+} \int_\Omega A(|\nabla u_n|^{p(x)})\,dx - \frac{\omega}{\theta} \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx \\
&\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta} \right) \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx.
\end{aligned}$$

Then, by (a_1), there exist positive constants C_8 and C_9 such that

$$C(1+\|u_n\|) \geqslant C_8 \left(\int_\Omega |\nabla u_n|^{p(x)}\,dx \right) + C_9 \mathcal{H}(\kappa_3) \left(\int_\Omega |\nabla u_n|^{q(x)}\,dx \right). \tag{2.16}$$

Suppose, by contradiction, that there is a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, with $\|u_n\| \to +\infty$.

If $\kappa_3 = 0$, we have

$$C(1 + \|u_n\|) \geqslant C_8\|u_n\|^{p^-},$$

which is a contradiction because $p^- > 1$. Thus, we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.

On the other hand, if $\kappa_3 > 0$, we will analyze the cases:

(i) $\|\nabla u_n\|_{p(\cdot)} \to +\infty$ and $\|\nabla u_n\|_{q(\cdot)} \to +\infty$ as $n \to +\infty$;

(ii) $\|\nabla u_n\|_{p(\cdot)} \to +\infty$ and $\|\nabla u_n\|_{q(\cdot)}$ is bounded;

(iii) $\|\nabla u_n\|_{p(\cdot)}$ is bounded and $\|\nabla u_n\|_{q(\cdot)} \to +\infty$.

In the case (i), for n large enough, $\|\nabla u_n\|_{q(\cdot)}^{q^-} \geqslant \|\nabla u_n\|_{q(\cdot)}^{p^-}$. Hence, by (2.16), we get

$$\begin{aligned} C(1 + \|u_n\|) &\geqslant C_8\|\nabla u_n\|_{p(\cdot)}^{p^-} + C_9\mathcal{H}(\kappa_3)\|\nabla u_n\|_{q(\cdot)}^{q^-} \\ &\geqslant C_8\|\nabla u_n\|_{p(\cdot)}^{p^-} + C_9\mathcal{H}(k_3)\|\nabla u_n\|_{q(\cdot)}^{p^-} \\ &\geqslant C_{10}\left(\|\nabla u_n\|_{p(\cdot)} + \mathcal{H}(k_3)\|\nabla u_n\|_{q(\cdot)}\right)^{p^-} \\ &= C_{10}\|u_n\|^{p^-}, \end{aligned}$$

which is on absurd.

In the case (ii), by (2.16), we have

$$\begin{aligned} C(1 + \|\nabla u_n\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u_n\|_{q(\cdot)}) &= C(1 + \|u_n\|) \\ &\geqslant C_{11}\|\nabla u_n\|_{p(\cdot)}^{p^-}, \end{aligned}$$

then, we get

$$C\left(\frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^-}} + \frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^- -1}} + \frac{\|\nabla u_n\|_{q(\cdot)}}{\|\nabla u_n\|_{p(\cdot)}^{p^-}}\right) \geqslant C_{11} > 0.$$

Hence, as $p^- - 1 > 0$, taking the limit as $n \to +\infty$, we obtain a contradiction.

The case (iii) is similar to case (ii).

Hence, we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. □

2.2.2 Proof of Theorem 2.2

Proof. From Lemma 2.1 and Lemma 2.2, the Euler-Lagrange functional $\Phi_{\omega,\lambda}$ associated to the problem $(\mathscr{M}_\lambda)$ verifies the geometric structure of the Mountain Pass Theorem. Then, there is a Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ at the level $c_{\omega,\lambda}$. Moreover, the Lemma 2.3 implies that there exists $\underline{\lambda} > 0$ such that

$$0 < c_{\omega,\lambda} < \left(\frac{1}{\theta} - \frac{1}{s^-}\right) (\overline{m_0\kappa} S)^N, \tag{2.17}$$

for all $\lambda \geqslant \underline{\lambda}$, where $\overline{m_0\kappa} = \min\left\{(m_0(\kappa_0(1-\mathcal{H}(\kappa_3)) + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^-}}, (m_0(\kappa_0(1-\mathcal{H}(\kappa_3)) + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^+}}\right\}$, and S the best positive constant of the Gagliardo-Nirenberg-Sobolev embedding (see 4.1).

Lemma 2.4 implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. Thus, by the *Eberlein-Šmulian* Theorem ([16, Theorem 3.19]), there is a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and $u \in X$, such that

$$\begin{gathered} u_n \rightharpoonup u \text{ weakly in } X, \\ u_n \to u \text{ in } L^{t(\cdot)}(\Omega), \text{ for all } 1 < t^- \leqslant t(x) \leqslant t^+ < \gamma^*(x), \\ u_n \to u \text{ a.e. in } \Omega. \end{gathered} \tag{2.18}$$

Claim 1. We affirm that $u_n \to u$ strongly in $L^{s(\cdot)}(\Omega)$ as $n \to +\infty$. Indeed. We have $\{u_n\}_{n\in\mathbb{N}}$ bounded in X, then, there are two nonnegative measures $\mu, \nu \in \mathscr{M}(\Omega)$ such that

$$\begin{gathered} |\nabla u_j|^{\gamma(x)} \rightharpoonup \mu \quad \text{(weak*-sense of measures) in } \mathscr{M}(\Omega), \\ |u_j|^{s(x)} \rightharpoonup \nu \quad \text{(weak*-sense of measures) in } \mathscr{M}(\Omega). \end{gathered}$$

By the concentration compactness lemma of Lions for variable exponents (see Theorem 4.5), we have there is a countable set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}} \subset \mathscr{C} \subset \Omega$, and sequences

$\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset [0,+\infty)$, satisfying

$$\begin{aligned} \nu &= |u|^{s(x)} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}, \quad \nu_j > 0, \\ \mu &\geqslant |\nabla u|^{\gamma(x)} + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j}, \quad \mu_j > 0, \\ S\nu_j^{\frac{1}{\gamma^*(x_j)}} &\leqslant \mu_j^{\frac{1}{\gamma(x_j)}} \ \forall j \in \mathcal{J}, \end{aligned} \tag{2.19}$$

where δ_{x_j} is the Dirac's delta measure supported on $x_j \in \Omega$ and S the best positive constant of the Gagliardo-Nirenberg-Sobolev embedding (see 4.1). Note that, if $\mathcal{J}$ is empty, then $u_n \to u$ strongly in $L^{s(\cdot)}(\Omega)$.

Suppose, by contradiction, that $\mathcal{J}$ is nonempty and fix $j \in \mathcal{J}$. We consider $\psi \in C_0^{+\infty}(\mathbb{R}^N, [0,1])$ with $|\nabla\psi|_\infty \leqslant 2$ and

$$\psi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geqslant 2. \end{cases}$$

Let us define, for any $\varepsilon > 0$, the cutoff function

$$\psi_{\varepsilon,j}(x) := \psi\left(\frac{x - x_j}{\varepsilon}\right), \ \forall x \in \mathbb{R}^N.$$

We have $\psi_{\varepsilon,j} \in C_0^{+\infty}(\mathbb{R}^N, [0,1])$, $|\nabla\psi_{\varepsilon,j}|_\infty \leqslant \frac{2}{\varepsilon}$, and

$$\psi_{\varepsilon,j}(x) = \begin{cases} 1, & x \in B(x_j, \varepsilon), \\ 0, & \mathbb{R}^N \backslash B(x_j, 2\varepsilon). \end{cases}$$

Since that $\Phi'_{\omega,\lambda}(u_n) \to 0$ in X' and $\{\psi_{\varepsilon,j} u_n\}_{n\in\mathbb{N}}$ is bounded in X, we obtain

$$\langle \Phi'_{\omega,\lambda}(u_n), \psi_{\varepsilon,j} u_n \rangle \to 0 \text{ as } n \to +\infty,$$

that is,

$$
\begin{aligned}
&M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}dx\\
=&-M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\psi_{\varepsilon,j}dx+\lambda\int_\Omega f(x,u_n)u_n\psi_{\varepsilon,j}dx\\
&+\int_\Omega |u_n|^{s(x)}\psi_{\varepsilon,j}dx+o_n(1).
\end{aligned}
\tag{2.20}
$$

Now, we will prove that

$$
\lim_{\varepsilon\to 0}\left\{\limsup_{n\to+\infty} M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}\,dx\right\}=0. \tag{2.21}
$$

By hypotheses (a_1), $(\mathcal{M}_1)$, and (2.3), it is sufficient that

$$
\lim_{\varepsilon\to 0}\left\{\limsup_{n\to+\infty}\left|\int_\Omega |\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}\,dx\right|\right\}=0 \tag{2.22}
$$

and

$$
\lim_{\varepsilon\to 0}\left\{\limsup_{n\to+\infty}\left|\int_\Omega |\nabla u_n|^{q(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}\,dx\right|\right\}=0. \tag{2.23}
$$

We get, by Hölder's inequality, that

$$
\left|\int_\Omega |\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}\,dx\right|\leq C\||\nabla u_n|^{p(x)-1}\|_{\frac{p(\cdot)}{p(\cdot)-1}}\|u_n\nabla\psi_{\varepsilon,j}\|_{p(\cdot)}.
$$

Thence, since that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, there exists a positive constant, C, such that

$$
\left|\int_\Omega |\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}\,dx\right|\leqslant C\|u_n\nabla\psi_{\varepsilon,j}\|_{p(\cdot)}.
$$

By Proposition 4.3, we have

$$
\begin{aligned}
\left|\int_\Omega |\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}dx\right|\leqslant C\max\Bigg\{&\left(\int_\Omega |u_n\nabla\psi_{\varepsilon,j}|^{p(x)}dx\right)^{\frac{1}{p^-}},\\
&\left(\int_\Omega |u_n\nabla\psi_{\varepsilon,j}|^{p(x)}dx\right)^{\frac{1}{p^+}}\Bigg\}.
\end{aligned}
\tag{2.24}
$$

Moreover, since that $u_n \to u$ strongly in $L^{p(x)}(\Omega)$, there is $h \in L^1(\Omega)$ such that

$$u_n(x) \to u(x) \text{ and } |u_n(x)| \leqslant h(x), \text{ a.e. } x \in \Omega.$$

Thus, as $\psi_{\varepsilon,j} \in C_0^{+\infty}(\Omega)$, we have

$$|u_n \nabla \psi_{\varepsilon,j}|^{p(x)} \to |u \nabla \psi_{\varepsilon,j}|^{p(x)} \text{ a.e. } x \in \Omega, \text{ as } n \to +\infty,$$

and there is $C > 0$ such that

$$|u_n \nabla \psi_{\varepsilon,j}|^{p(x)} \leqslant C|h(x)|^{p(x)} \text{ a.e. } x \in \Omega.$$

Therefore, by using the Lebesgue Dominated Convergence Theorem, we achieve

$$\int_\Omega |u_n(x) \nabla \psi_{\varepsilon,j}(x)|^{p(x)}\, dx \to \int_\Omega |u(x) \nabla \psi_{\varepsilon,j}(x)|^{p(x)}\, dx.$$

Thus, by (2.24), we get

$$\limsup_{n \to +\infty} \left| \int_\Omega |\nabla u_n|^{p(x)} u_n \nabla u_n \nabla \psi_{\varepsilon,j}\, dx \right| \leqslant C \max \left\{ \left(\int_\Omega |u \nabla \psi_{\varepsilon,j}|^{p(x)} \right)^{\frac{1}{p^-}}, \left(\int_\Omega |u \nabla \psi_{\varepsilon,j}|^{p(x)} \right)^{\frac{1}{p^-}} \right\}. \tag{2.25}$$

From Hölder's inequality (Lemma 4.1) and the Proposition 4.2, we have

$$\begin{aligned} \int_\Omega |u \nabla \psi_{\varepsilon,j}|^{p(x)}\, dx &\leqslant \overline{C} \| |u|^{p(x)} \|_{L^{\frac{p^*(\cdot)}{p(\cdot)}}(B(x_j,2\varepsilon))} \| |\nabla \psi_{\varepsilon,j}|^{p(x)} \|_{L^{\frac{N}{p(\cdot)}}(B(x_j,2\varepsilon))} \\ &\leqslant C \| |u|^{p(x)} \|_{L^{\frac{p^*(\cdot)}{p(\cdot)}}(B(x_j,2\varepsilon))} \\ &\leqslant C \max \left\{ \|u\|^{p^-}_{L^{p^*(\cdot)}(B(x_j,2\varepsilon))}, \|u\|^{p^+}_{L^{p^*(\cdot)}(B(x_j,2\varepsilon))} \right\}, \end{aligned} \tag{2.26}$$

and, by Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{\varepsilon \to 0} \int_{B(x_j,2\varepsilon)} |u|^{p^*(x)}\, dx = 0. \tag{2.27}$$

Then, from (2.25), (2.26), and (2.27), we get

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n\to+\infty}\left|\int_\Omega |\nabla u_n|^{p(x)-2}u_n\nabla u_n\nabla\psi_{\varepsilon,j}\,dx\right|\right\}=0,$$

which proves (2.22). Analogously, we verify (2.23). Hence, we conclude the prove of (2.21).

We obtain, by (f_1), (f_2), (2.18), and Lebesgue Dominated Convergence Theorem, that

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)u_n\psi_{\varepsilon,j}dx=\int_\Omega f(x,u)u\psi_{\varepsilon,j}\,dx \tag{2.28}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega f(x,u)u\psi_{\varepsilon,j}dx=0. \tag{2.29}$$

Thence, taking the limit in (2.20), as $n\to+\infty$, and by using (2.19) and (2.28), we get

$$\begin{aligned}&\limsup_{n\to+\infty}\left(M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}u_n\nabla u_n\nabla\psi_{\varepsilon,j}dx\right)\\&=-\limsup_{n\to+\infty}\left(M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\psi_{\varepsilon,j}dx\right)+\lambda\int_\Omega f(x,u)u\psi_{\varepsilon,j}dx\\&\quad+\int_\Omega |u|^{s(x)}\psi_{\varepsilon,j}dx+\int_\Omega\psi_{\varepsilon,j}d\nu.\end{aligned} \tag{2.30}$$

Note that, by Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega |u|^{s(x)}\psi_{\varepsilon,j}dx=0.$$

Then, taking $\varepsilon\to 0$ in (2.30) and by using (a_1), $(\mathcal{M}_1)$, (2.21), (2.29), we achieve

$$\begin{aligned}0&=-\lim_{\varepsilon\to 0}\left[\limsup_{n\to+\infty}\left(M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\psi_{\varepsilon,j}\,dx\right)\right]+\nu_j\\&\leqslant -m_0\lim_{\varepsilon\to 0}\left[\limsup_{n\to+\infty}\left(\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\psi_{\varepsilon,j}\,dx\right)\right]+\nu_j\\&\leqslant -m_0\lim_{\varepsilon\to 0}\left[\limsup_{n\to+\infty}\left(\int_\Omega(\kappa_0|\nabla u_n|^{p(x)}+\mathcal{H}(\kappa_3)\kappa_2|\nabla u_n|^{q(x)})\psi_{\varepsilon,j}\,dx\right)\right]+\nu_j.\end{aligned} \tag{2.31}$$

When $\kappa_3=0$, we have $\gamma(x)=p(x)$. Thus, follows from (2.19) and (2.31) that

$$
\begin{aligned}
0 \leqslant & \nu_j - m_0\kappa_0 \lim_{\varepsilon\to 0}\int_\Omega \psi_{\varepsilon,j}d\mu \\
\leqslant & \nu_j - m_0\kappa_0\mu_j - m_0\kappa_0 \lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^{p(x)}\psi_{\varepsilon,j}d\mu \\
= & \nu_j - m_0\kappa_0\mu_j,
\end{aligned} \tag{2.32}
$$

because, by Lebesgue Dominated Convergence Theorem, we have

$$
\lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^{p(x)}\psi_{\varepsilon,j}d\mu = 0.
$$

On the other hand, when $\kappa_3 > 0$, we have $\gamma(x) = q(x)$. Then, it follows from (2.19) and (2.31) that

$$
\begin{aligned}
0 \leqslant & \nu_j - m_0 \lim_{\varepsilon\to 0}\left[\limsup_{n\to+\infty}\left(\int_\Omega \mathcal{H}(\kappa_3)\kappa_2|\nabla u_n|^{q(x)}\psi_{\varepsilon,j}\,dx\right)\right] \\
\leqslant & \nu_j - m_0\mathcal{H}(\kappa_3)\kappa_2 \lim_{\varepsilon\to 0}\int_\Omega \psi_{\varepsilon,j}d\mu \\
\leqslant & \nu_j - m_0\mathcal{H}(\kappa_3)\kappa_2\mu_j - m_0\kappa_0 \lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^{p(x)}\psi_{\varepsilon,j}d\mu \\
= & \nu_j - m_0\mathcal{H}(\kappa_3)\kappa_2\mu_j,
\end{aligned} \tag{2.33}
$$

because, by Lebesgue Dominated Convergence Theorem, we have

$$
\lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^{q(x)}\psi_{\varepsilon,j}d\mu = 0.
$$

Hence, by combining (2.32) and (2.33), we get

$$
m_0((1-\mathcal{H}(\kappa_3))\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2)\mu_j \leqslant \nu_j.
$$

Thence, by (2.19), we conclude that

$$
(\overline{m_0\kappa}S)^N \leqslant \nu_j, \tag{2.34}
$$

where

$$\overline{m_0\kappa} = \min\left\{ (m_0((1-\mathcal{H}(\kappa_3))\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^-}}, (m_0((1-\mathcal{H}(\kappa_3))\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^+}} \right\}.$$

Now, we will prove that the inequality (2.34) cannot occur and, consequently, that $\mathcal{J}$ is empty. From (a_3), $(\mathcal{M}_2)$, and (2.3), we see that

$$\begin{aligned}
&\mathscr{M}(\mathscr{A}(u_n)) - \frac{1}{\theta}M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx \\
&= \int_0^{\mathscr{A}(u_n)} M_\omega(t)\,dt - \frac{1}{\theta}M_\omega \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx \\
&\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right)\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx \\
&\geqslant 0.
\end{aligned} \tag{2.35}$$

Then, by combining (f_3), (2.19), (2.34), and (2.35), we obtain

$$\begin{aligned}
c_{\omega,\lambda} &= \lim_{n\to+\infty}\left[\Phi_{\omega,\lambda}(u_n) - \frac{1}{\theta}\langle \Phi'_\lambda(u_n), u_n\rangle\right] \\
&\geqslant \lim_{n\to+\infty}\int_\Omega \left(\frac{1}{\theta} - \frac{1}{s(x)}\right)|u_n|^{s(x)}\,dx \\
&\geqslant \lim_{n\to+\infty}\int_\Omega \left(\frac{1}{\theta} - \frac{1}{s^-}\right)|u_n|^{s(x)}\,dx \\
&= \left(\frac{1}{\theta} - \frac{1}{s^-}\right)\left(\int_\Omega |u|^{s(x)}\,dx + \sum_{j\in\mathcal{J}}\nu_j\right) \\
&\geqslant \left(\frac{1}{\theta} - \frac{1}{s^-}\right)\nu_j \\
&\geqslant \left(\frac{1}{\theta} - \frac{1}{s^-}\right)(\overline{m_0\kappa_0}S)^N,
\end{aligned}$$

which contradicts (2.17). Thence $\mathcal{J}$ is empty, which implies that $\rho_{s(\cdot)}(u_n) \to \rho_{s(\cdot)}(u)$ as $n \to +\infty$. Hence, by using the Lemma 4.3 and the Proposition 4.3, we conclude that $u_n \to u$ strongly in $L^{s(\cdot)}(\Omega)$ as $n \to +\infty$. This concludes the proof of the **Claim 1**.

Claim 2. We affirm that $u_n \to u$ strongly in X as $n \to +\infty$.

Indeed. From (f_1), (f_2), (2.18), the **Claim 1**, and Lebesgue Dominated

Convergence Theorem, we have

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)(u_n-u)dx=0 \text{ and } \lim_{n\to+\infty}\int_\Omega |u_n|^{s(x)-2}u_n(u_n-u)dx=0. \tag{2.36}$$

Then, as $\{u_n\}_{n\in\mathbb{N}}\subset X$ is a bounded Palais-Smale sequence, we get

$$\begin{aligned}0&=\lim_{n\to+\infty}\langle\Phi'_{\omega,\lambda}(u_n),u_n-u\rangle\\&=\lim_{n\to+\infty}\left[M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx\right].\end{aligned}$$

Thence, by (2.3), we obtain

$$\lim_{n\to+\infty}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx=0,$$

and, consequently, by Lemma 4.9, it follows that $u_n\to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$ as $n\to+\infty$, that is,

$$\lim_{n\to+\infty}\|\nabla u_n-\nabla u\|_{p(\cdot)}=0. \tag{2.37}$$

We point out that by the Hölder's inequality (Lemma 4.1) and (2.37) follows that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx=0. \tag{2.38}$$

If $\kappa_3=0$, we get from (2.37) that $u_n\to u$ strongly in X as $n\to+\infty$. On the other hand, let us consider the case $\kappa_3>0$. Since that $\{u_n\}_{n\in\mathbb{N}}\subset X$ is a bounded Palais-Smale sequence and by using (a_1), (2.3), (2.36), (2.37), and (2.38), we obtain

$$\begin{aligned}o_n(1)&=\langle\Phi'_{\omega,\lambda}(u_n),u_n-u\rangle\\&=M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\&\geqslant C\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\&\geqslant C\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)+|\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\&=C\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1),\end{aligned}$$

so, we have

$$\limsup_{n\to+\infty}\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx\leqslant 0,$$

and, therefeore, by Theorem 4.1, we obtain $u_n \to u$ strongly in $W_0^{1,q(\cdot)}(\Omega)$ as $n \to +\infty$, that is,

$$\lim_{n\to+\infty} \|\nabla u_n - \nabla u\|_{q(\cdot)} = 0. \tag{2.39}$$

Thus, we conclude, by (2.37) and (2.39), that $u_n \to u$ strongly in X as $n \to +\infty$. This concludes the proof of the **Claim 2**.

Hence, by the **Claim 2**, we get that $\Phi_{\omega,\lambda}(u) = c_{\omega\,\lambda} > 0$ and $\Phi'_{\omega,\lambda}(u) = 0$ in X', that is, $u \in X$ is a weak solution of the problem $(\mathscr{M}_\lambda)$. Moreover, as $\Phi_{\omega,\lambda}(u) = c_{\omega\,\lambda} > 0 = \Phi_{\omega,\lambda}(0)$, we obtain that $u \not\equiv 0$. Thus, we conclude the proof of the Theorem 2.2. □

2.3 Proof of Theorem 2.1

Proof. Let $\underline{\lambda}$ be as in the Theorem 2.2. From Theorem 2.2, we have a nontrivial weak solution, $u_\lambda \in X$, of the problem $(\mathscr{M}_\lambda)$, for each $\lambda \geqslant \underline{\lambda}$.

Claim 3. We affirm that exists $\lambda^* \geqslant \underline{\lambda}$ such that

$$\mathscr{A}(u_\lambda) \leqslant t_0, \ \forall \lambda \geqslant \lambda^*,$$

where t_0 is defined at the beginning of Section 2.2.

Indeed. Suppose, by contradiction, that there exists $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_n \to +\infty$ and $\mathscr{A}(u_{\lambda_n}) \geqslant t_0$, as $n \to +\infty$.

From $\mathscr{A}(u_{\lambda_n}) \geqslant t_0$ and (a_1), we get

$$\int_\Omega \left(\kappa_1 |\nabla u_{\lambda_n}|^{p(x)} + \kappa_3 |\nabla u_{\lambda_n}|^{q(x)}\right) \, dx \geqslant t_0. \tag{2.40}$$

Since that u_{λ_n} is a critical point of the functional Φ_{ω,λ_n}, we obtain, by using $(\mathcal{M}_1)$, (2.3),

(a_1), (a_3), and (f_3), that

$$\begin{aligned} c_{\omega,\lambda_n} &= \Phi_{\omega,\lambda_n}(u_{\lambda_n}) - \frac{1}{\theta}\langle \Phi'_{\omega,\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle \\ &\geqslant \mathscr{M}_\omega(\mathscr{A}(u_{\lambda_n})) - \frac{1}{\theta}M_\omega(\mathscr{A}(u_{\lambda_n}))\int_\Omega a(|\nabla u_{\lambda_n}|^{p(x)})|\nabla u_{\lambda_n}|^{p(x)}\,dx \\ &\geqslant \int_0^{\mathscr{A}(u_{\lambda_n})} M_\omega(t)\,dt - \frac{1}{\theta}M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_{\lambda_n}|^{p(x)})|\nabla u_{\lambda_n}|^{p(x)}\,dx \\ &\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right)\int_\Omega a(|\nabla u_{\lambda_n}|^{p(x)})|\nabla u_{\lambda_n}|^{p(x)}\,dx \\ &\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right)\int_\Omega (\kappa_0|\nabla u_{\lambda_n}|^{p(x)} + \mathcal{H}(\kappa_3)\kappa_2|\nabla u_{\lambda_n}|^{q(x)})\,dx. \end{aligned} \tag{2.41}$$

If $\kappa_3 = 0$, by (2.40), we obtain

$$\int_\Omega |\nabla u_{\lambda_n}|^{p(x)}\,dx \geqslant \frac{t_0}{\kappa_1},$$

then, by (2.41), follows that

$$c_{\omega,\lambda_n} \geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right)\left(\frac{\kappa_0}{\kappa_1}\right)t_0 > 0,$$

which is a contradiction, because $\lim_{n\to+\infty} c_{\omega,\lambda_n} = 0$ (see Lemma 2.3). On the other hand, if $\kappa_3 > 0$, multiplying (2.41) by $\kappa_1 \cdot \kappa_3 > 0$ and by using (2.40), we obtain

$$\begin{aligned} \kappa_1\kappa_3 c_{\omega,\lambda_n} &\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right)K\int_\Omega (\kappa_1|\nabla u_{\lambda_n}|^{p(x)} + \kappa_3|\nabla u_{\lambda_n}|^{q(x)})\,dx \\ &\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right)Kt_0 > 0, \end{aligned}$$

where $K = \min\left\{\kappa_0 \cdot \kappa_3, \kappa_1 \cdot \kappa_2\right\} > 0$, which is contradiction, because $\lim_{n\to+\infty} c_{\omega,\lambda_n} = 0$ (see Lemma 2.3). Hence, in both cases, there exists $\lambda^* \geqslant \underline{\lambda}$ such that $\mathscr{A}(u_\lambda) \leqslant t_0$, for all $\lambda \geqslant \lambda^*$. Thus, the **Claim 3** is proved.

Thence, we have $M_\omega(\mathscr{A}(u_\lambda)) = M(\mathscr{A}(u_\lambda))$ for all $\lambda \geqslant \lambda^*$, which implies that $\Phi_{\omega,\lambda_n}(u_{\lambda_n}) = \Phi_{\lambda_n}(u_{\lambda_n})$ and $\Phi'_{\omega,\lambda_n}(u_{\lambda_n}) = \Phi'_{\lambda_n}(u_{\lambda_n})$, that is, u_{λ_n} is a nontrivial weak solution of the problem $(\mathscr{P}_\lambda)$, for any $\lambda \geqslant \lambda^*$. Moreover, note that, for each $\lambda \geqslant \lambda^*$, from

$(\mathcal{M}_1)$, (2.3), (a_1), (a_3), (f_3), and Proposition 4.3, we have

$$\begin{aligned}
c_\lambda &= c_{\omega,\lambda} \\
&\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right) \int_\Omega (\kappa_0 |\nabla u_\lambda|^{p(x)} + \mathcal{H}(k_3)\kappa_2 |\nabla u_\lambda|^{q(x)})\, dx \\
&\geqslant \left(\frac{m_0}{p^+\alpha} - \frac{\omega}{\theta}\right) \left[\kappa_0 \min\left\{\|\nabla u_\lambda\|_{p(\cdot)}^{p^+}, \|\nabla u_\lambda\|_{p(\cdot)}^{p^-}\right\} + \mathcal{H}(\kappa_3)\kappa_2 \min\left\{\|\nabla u_\lambda\|_{q(\cdot)}^{q^-}, \|\nabla u_\lambda\|_{q(\cdot)}^{q^+}\right\}\right].
\end{aligned}$$

Hence, by using the Lemma 2.3, we get

$$\lim_{\lambda\to+\infty} \|u_\lambda\| = \lim_{\lambda\to+\infty} (\|\nabla u_\lambda\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u_\lambda\|_{q(\cdot)}) = 0.$$

Thus, we conclude the proof of the Theorem 2.1. □

Chapter

3

Existence of solutions to a class elliptic nonlocal problems with critical growth

The present chapter deals with the existence and multiplicity of weak solutions of the nonlocal elliptic problems involving variable exponents and nonlinearity with critical growth. More precisely, we study the following variable exponent nonlocal elliptic problem:

$$\begin{cases} -M(\mathscr{A}(u))\operatorname{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_\lambda)$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary $\partial\Omega$, and $\lambda > 0$ is a real parameter, $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are continuous function on $\overline{\Omega}$ satisfying the folllowing inequalities

$$\begin{aligned} &1 < r^- \leqslant r(x) \leqslant r^+ < p^- \leqslant p(x) \leqslant p^+ < q^- \leqslant q(x) \leqslant q^+ < N, \\ \text{and} \quad & \\ &1 < r^- \leqslant r(x) \leqslant r^+ < \gamma^- \leqslant \gamma(x) \leqslant \gamma^+ < s^- \leqslant s(x) \leqslant \gamma^*(x) < +\infty, \end{aligned} \qquad (3.1)$$

for all $x \in \overline{\Omega}$, where $p^- := \min_{x\in\overline{\Omega}} p(x)$, $p^+ := \max_{x\in\overline{\Omega}} p(x)$, and analogously to r^-, r^-, q^-, q^+, γ^-, γ^+, s^-, and s^+, with $\gamma(x) = (1-\mathcal{H}(\kappa_3))p(x) + \mathcal{H}(\kappa_3)q(x)$, and the variable critical exponent $\gamma^*(x)$ defined by

$$\gamma^*(x) = \begin{cases} \frac{N\gamma(x)}{N-\gamma(x)}, & \text{se } \gamma(x) < N, \\ +\infty, & \text{se } \gamma(x) \geqslant N, \end{cases}$$

for all $x \in \overline{\Omega}$, where $\mathcal{H} : \mathbb{R}_0^+ \to \{0,1\}$ is given by

$$\mathcal{H}(\kappa) = \begin{cases} 1, & \text{if} \quad \kappa > 0, \\ 0, & \text{if} \quad \kappa = 0. \end{cases}$$

Furthermore, we consider the set $\mathscr{C} = \{x \in \Omega : s(x) = \gamma^*(x)\} \neq \emptyset$. The operator $\mathscr{A} : X \to \mathbb{R}$ is defined by

$$\mathscr{A}(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})\, dx,$$

where $A(\cdot)$ is the function $A(t) = \int_0^t a(\sigma) d\sigma$ and the function $a(\cdot)$ is described in the hypothesis (a_0).

We assume that the function $a : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ verify the following hypotheses:

(a_0) The function $a(\cdot)$ is of class C^1;

(a_1) There exist positive constants, κ_0, κ_1, and κ_2, and a nonnegative constant κ_3, such that

$$\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2 \tau^{\frac{q(x)-p(x)}{p(x)}} \leqslant a(\tau) \leqslant \kappa_1 + \kappa_3 \tau^{\frac{q(x)-p(x)}{p(x)}},$$

for all $\tau \geqslant 0$ and for all $x \in \overline{\Omega}$;

(a_2) There exists $c > 0$ such that

$$\min\left\{ a(\tau^{p(x)})\tau^{p(x)-2}, a(\tau^{p(x)})\tau^{p(x)-2} + \tau \frac{\partial (a(\tau^{p(x)})\tau^{p(x)-2})}{\partial \tau} \right\} \geqslant c\tau^{p(x)-2},$$

for almost every $x \in \Omega$ and for all $\tau > 0$;

(a_3) There is a positive constant α such that

$$A(\tau) \geqslant \frac{1}{\alpha} a(\tau)\tau \text{ with } \frac{r^+}{p^+} < \frac{\gamma^+}{p^+} \leqslant \alpha < \frac{s^-}{p^+},$$

for all $\tau \geqslant 0$.

In terms of Kirchhoff's function, we assume general conditions on $M : \mathbb{R}_0^+ \to \mathbb{R}$. We consider that M is a continuous function that satisfies the conditions:

$(\mathcal{M}_1)$ There exists $m_0 > 0$ such that $M(t) \geqslant m_0 = M(0) > 0$ for all $t \in \mathbb{R}^+$;

($\mathcal{M}_2$) The function M is increasing.

Moreover, we define $\mathscr{M}(t) := \int_0^t M(\tau)d\tau$ for all $t \in \mathbb{R}_0^+$.

We assume that the nonlinearity $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

(f_1) f is odd in the second variable, that is,

$$f(x, -t) = -f(x, t) \text{ for any } (x, t) \in \overline{\Omega} \times \mathbb{R};$$

(f_2) There exist positive constant a_1, a_2 and a function $r \in C^+(\overline{\Omega})$ such that

$$a_1 t^{r(x)-1} \leqslant f(x, t) \leqslant a_2 t^{r(x)-1} \text{ for all } (x, t) \in \overline{\Omega} \times \mathbb{R}_0^+.$$

The main result in this work establishes the existence of infinitely many solutions to problem $(\mathscr{P}_\lambda)$.

Theorem 3.1. *Assume that the functions a, M, and f satisfy the conditions $(a_0) - (a_3)$, $(\mathcal{M}_1) - (\mathcal{M}_2)$, and $(f_1) - (f_2)$. Then, there exists $\overline{\lambda} > 0$, such that problem $(\mathscr{P}_\lambda)$ has infinitely many weak solutions for each $\lambda \in (0, \overline{\lambda})$. Moreover, if u_λ is a solution of problem $(\mathscr{P}_\lambda)$ then*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

We will proof the Theorem (3.1) following the main ideas of [37], [38], [40], and [43]. Since the hypothesis $(\mathcal{M}_1)$ provides only a positive lower bounded for M near zero, this creates serious technical difficulties. We will need to do a truncation on the function M to obtain priori estimates of the boundedness from above. After that, we do another truncation to control the Euler-Lagrange's functional associated with the auxiliary problem. Note that there are serious difficulties to prove the existence of solutions by canonical variational methods. For instance, a difficulty that arises is to prove the Palais-Smale condition due to the lack of compactness of the embedding of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{p^*(\cdot)}(\Omega)$, and, to overcome this difficulty we will use the concentration compactness principle (see [14]). Then, we will obtain infinitely many solutions to the auxiliary problem for any λ sufficiently small by using genus theory as in [10], and, afterwards some estimatives, we concluded that there are infinitely many solutions to problem $(\mathscr{P}_\lambda)$.

3.1 Variational framework

The problem $(\mathscr{P}_\lambda)$ has a variational structure and the natural space to look for solutions are variable exponent Sobolev spaces. Let us consider the Banach space

$$X := W_0^{1,p(\cdot)}(\Omega) \cap W_0^{1,\gamma(\cdot)}(\Omega),$$

endowed with the norm

$$\|u\| = \|\nabla u\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u\|_{q(\cdot)}.$$

We note that the weak solutions of $(\mathscr{P}_\lambda)$ coincides with the critical points of the C^1- functional $\Phi_\lambda : X \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = \mathscr{M}(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\,dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}\,dx,$$

for all $u \in X$. Note that the functional Φ_λ is Fréchet differentiable in $u \in X$ with Fréchet derivative

$$\langle \Phi'_\lambda(u), v\rangle = M(\mathscr{A}(u)) \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u \nabla v dx - \lambda \int_\Omega f(x,u) v dx - \int_\Omega |u|^{s(x)-2} u v dx,$$

for all $v \in X$.

Definition 3.1. *We say that a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ is a Palais-Smale sequence for the functional Φ_λ at level c if*

$$\Phi_\lambda(u_n) \to c \text{ and } \Phi'_\lambda(u_n) \to 0 \text{ in } X', \text{ as } n \to +\infty. \tag{3.2}$$

If (3.2) implies the existence of a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ which converges in X, we say that Φ_λ satisfies the Palais-Smale condition. Moreover, if this subsequence strongly convergent exits only for some c values, we say that Φ_λ satisfies a local Palais-Smale condition.

3.2 The auxiliary problem

Due to presence of the critical term in the nonlinearity of the problem $(\mathscr{P}_\lambda)$, is difficult prove that the functional Φ_λ verifies the Palais-Samle condition. Moreover, the Krasnoselskii's genus theory requires the functional bounded from below. So, we only have the information of that M is an increasing function bounded from below, we will make a truncation in the function M and get a truncated problem $(\mathscr{M}_\lambda)$.

Since $\gamma^+ < s^-$, there exists $\sigma \in \mathbb{R}$ such that $\sigma \in (\gamma^+, s^-)$. Furthermore, by $(\mathcal{M}_2)$, exist $t_0 > 0$ such that $m_0 = M(0) < \omega = M(t_0) < \frac{m_0\sigma}{p^+\alpha}$

$$M_\omega(t) = \begin{cases} M(t) & \text{if } 0 \leqslant t \leqslant t_0, \\ \omega & \text{if } t \geqslant t_0. \end{cases} \tag{3.3}$$

We define the auxiliary truncated problem by

$$\begin{cases} -M_\omega(\mathscr{A}(u)) \operatorname{div}\Big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\Big) = \lambda f(x,u) + |u|^{s(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{$\mathscr{M}_\lambda$}$$

with a, f, and λ defined as in problem $(\mathscr{P}_\lambda)$. Note that, due to (3.3), we have

$$M_\omega(t) \leqslant \omega \text{ for all } t \geqslant 0. \tag{3.4}$$

Theorem 3.2. *Assume* $(a_0)-(a_3)$, $(\mathcal{M}_1)-(\mathcal{M}_2)$, *and* $(f_1)-(f_2)$. *Then, there exists* $\overline{\lambda} > 0$, *such that Problem* $(\mathscr{M}_\lambda)$ *has infinitely many weak solutions in* X *for each* $\lambda \in (0, \overline{\lambda})$.

3.2.1 Variational formulation of the auxiliary problem

We consider the Euler-Lagrange functional $\Phi_{\omega,\lambda} : X \to \mathbb{R}$ associated with problem $(\mathscr{M}_\lambda)$, which is given by

$$\Phi_{\omega,\lambda}(u) = \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\,dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}\,dx$$

where $\mathscr{M}_\omega(t) = \int_0^t M_\omega(s)\,ds$. Moreover, by standard arguments the functional $\Phi_{\omega,\lambda} \in$

$C^1(X,\mathbb{R})$, and

$$\begin{aligned}\langle \Phi'_{\omega,\lambda}(u), v\rangle &= M_\omega(\mathscr{A}(u))\int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\nabla v dx\\ &\quad -\lambda\int_\Omega f(x,u)v dx - \int_\Omega |u|^{s(x)-2}uv dx,\end{aligned}$$

$u\in X$ and for any $v\in X$.

We will study convergence of the Palais-Smale sequence by using the compactness principle of variable exponent Sobolev spaces (see Proposition 4.5). However, we will see that this occurs just below an adequate level, which depends on the best constant in the inequality Gagliardo-Nirenberg-Sobolev for exponents variables, namely,

$$S = S_q(\Omega) = \inf_{v\in C_0^\infty(\Omega)} \frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)}}. \tag{3.5}$$

Lemma 3.1. *Let $\{u_n\}_{n\in\mathbb{N}}\subset X$ be a Palais-Smale sequence for $\Phi_{\omega,\lambda}$, with energy level c_λ, then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.*

Proof. Fix $\lambda>0$. Let $\{u_n\}_{n\in\mathbb{N}}\subset X$ be a Palais-Smale sequence with energy level c_λ (see (3.2)). Hence, there exists $C>0$ such that $\Big|\Phi_{\omega,\lambda}(u_n)\Big| \leqslant C$ for any $n\in\mathbb{N}$. Therefore, by $(\mathcal{M}_1)$, (a_3), (3.4), and remembering that $m_0<\omega<\frac{m_0\sigma}{p^+\alpha}$, we obtain

$$\begin{aligned} C\left(1+\|u_n\|\right) \geqslant& \Phi_{\omega,\lambda}(u_n) - \frac{1}{\sigma}\langle \Phi'_{\omega,\lambda}(u_n), u_n\rangle\\ \geqslant& \mathscr{M}_\omega(\mathscr{A}(u_n)) - \frac{1}{\sigma}M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx\\ &+\lambda\left(\frac{a_1}{\sigma}-\frac{a_2}{r^-}\right)\int_\Omega |u_n|^{r(x)}\,dx\\ \geqslant& \frac{m_0}{p^+}\int_\Omega A(|\nabla u_n|^{p(x)})\,dx - \frac{\omega}{\sigma}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx\\ &+\lambda\left(\frac{a_1}{\sigma}-\frac{a_2}{r^-}\right)\int_\Omega |u_n|^{r(x)}\,dx\\ \geqslant& \left(\frac{m_0}{p^+\alpha}-\frac{\omega}{\sigma}\right)\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx + \lambda\left(\frac{a_1}{\sigma}-\frac{a_2}{r^-}\right)\int_\Omega |u_n|^{r(x)}\,dx.\end{aligned}$$

Then, we have

$$C\left(1+\|u_n\|\right) + \lambda\left(\frac{a_2}{r^-}-\frac{a_1}{\sigma}\right)\int_\Omega |u_n|^{r(x)}\,dx \geqslant \left(\frac{m_0}{p^+\alpha}-\frac{\omega}{\sigma}\right)\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\,dx.$$

Furthermore, by (a_1), there are positive constants C_7, C_8, and C_9 such that

$$\begin{aligned} C(1+\|u_n\|) + C_7 \max\left\{\|u_n\|^{r^+}, \|u_n\|^{r^-}\right\} \geqslant & C_8 \left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right) \\ & + C_9 \mathcal{H}(\kappa_3) \left(\int_\Omega |\nabla u_n|^{q(x)}\,dx\right). \end{aligned} \tag{3.6}$$

Suppose, by contradiction, that up to subsequence $\|u_n\| \to +\infty$.

If $\kappa_3 = 0$, we get from (3.6) that

$$C(1+\|u_n\|) + C_7\|u_n\|^{r^+} \geqslant C_8\|u_n\|^{p^-},$$

which is a contradiction, because $1 < r^+ < p^-$. We conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X.

If $\kappa_3 > 0$. Then, we will analyze the following cases:

(i) $\|\nabla u_n\|_{p(\cdot)} \to +\infty$ and $\|\nabla u_n\|_{q(\cdot)} \to +\infty$ as $n \to +\infty$;

(ii) $\|\nabla u_n\|_{p(\cdot)} \to +\infty$ and $\|\nabla u_n\|_{q(\cdot)}$ is bounded;

(iii) $\|\nabla u_n\|_{p(\cdot)}$ is bounded and $\|\nabla u_n\|_{q(\cdot)} \to +\infty$.

In the case (i), for n large enough, $\|\nabla u_n\|_{q(\cdot)}^{q^- - p^-} \geqslant 1$, this is $\|\nabla u_n\|_{q(\cdot)}^{q^-} \geqslant \|\nabla u_n\|_{q(\cdot)}^{p^-}$. Hence, follows from (3.6) that

$$\begin{aligned} C(1+\|u_n\|) + C_7\|u_n\|^{r^+} &\geqslant C_8\|\nabla u_n\|_{p(\cdot)}^{p^-} + C_9\mathcal{H}(\kappa_3)\|\nabla u_n\|_{q(\cdot)}^{q^-} \\ &\geqslant C_8\|\nabla u_n\|_{p(\cdot)}^{p^-} + C_9\mathcal{H}(k_3)\|\nabla u_n\|_{q(\cdot)}^{p^-} \\ &\geqslant C_{10}\left(\|\nabla u_n\|_{p(\cdot)} + \mathcal{H}(k_3)\|\nabla u_n\|_{q(\cdot)}\right)^{p^-} \\ &= C_{10}\|u_n\|^{p^-}, \end{aligned}$$

therefore, taking the limit, as $n \to +\infty$, we obtain an absurd.

In the case (ii), we achieve by (3.6) that

$$\begin{aligned} C(1+\|u_n\|) + K(\|\nabla u_n\|_{p(\cdot)}^{r^+} + \mathcal{H}(\kappa_3)\|\nabla u_n\|_{q(\cdot)}^{r^+}) &\geqslant C(1+\|u_n\|) + C_7\|u_n\|^{r^+} \\ &\geqslant C_{11}\|\nabla u_n\|_{p(\cdot)}^{p^-}, \end{aligned}$$

so, it follows

$$C\left(\frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^-}}+\frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^--1}}+\frac{\|\nabla u_n\|_{q(\cdot)}}{\|\nabla u_n\|_{p(\cdot)}^{p^-}}\right)+K\left(\frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p^--r^+}}+\frac{\|\nabla u_n\|_{q(\cdot)}^{r^+}}{\|\nabla u_n\|_{p(\cdot)}^{p^-}}\right)\geqslant C_{11}>0.$$

Since $p^- > r^+ > 1$, taking the limit, as $n \to +\infty$, we obtain a contradiction.

The case (iii) is similar to case (ii).

Then, we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. □

Lemma 3.2. *Assume* (a_0), (a_1), (a_2), (a_3), $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, (f_1), *and* (f_2). *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a Palais-Smale sequence at level* c_λ *with*

$$c_\lambda < \left(\frac{1}{\sigma}-\frac{1}{s_{\mathscr{C}}^-}\right)(\overline{m_0\kappa}S)^N - \mathcal{K}\min\left\{\lambda^{\frac{(\frac{s}{r})^-}{(\frac{s}{r})^--1}},\lambda^{\frac{(\frac{s}{r})^+}{(\frac{s}{r})^+-1}}\right\}, \tag{3.7}$$

where

$$\overline{m_0\kappa}=\min\left\{\left[m_0((1-\mathcal{H}(\kappa_3))\kappa_0\mu_j+\mathcal{H}(\kappa_3)\kappa_2\mu_j)\right]^{\frac{1}{\gamma^-}},\ \left[m_0((1-\mathcal{H}(\kappa_3))\kappa_0\mu_j+\mathcal{H}(\kappa_3)\kappa_2\mu_j)\right]^{\frac{1}{\gamma^+}}\right\}$$

and $\mathcal{K}$ *is a positive constant independent of* λ. *Then, up to subsequence,* $\{u_n\}_{n\in\mathbb{N}}$ *is strongly convergent in* X.

Proof. We have by Lemma 3.1 that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. Thus, as X is reflexive, by *Eberlein – Šmulian* Theorem ([16], Theorem 3.19), up to a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, there is $u \in X$, such that

$$u_n \rightharpoonup u \text{ weakly in } X,$$
$$u_n \to u \text{ in } L^{t(\cdot)}(\Omega),\ \text{for all } 1<t^-\leqslant t(x)\leqslant t^+<\gamma^*(x),$$
$$u_n \to u \text{ a.e. in } \Omega.$$

From the concentration compactness principle of Lions for variable exponents (see Proposition 4.5), we have that there exist two nonnegative measures $\mu,\nu \in \mathscr{M}(\Omega)$, a countable set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}} \subset \mathscr{C} \subset \Omega$ and sequences $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset$

$[0,+\infty)$, such that

$$\begin{aligned} |\nabla u_n|^{\gamma(x)} &\rightharpoonup \mu \quad \text{(weak*-sense of measures) in } \mathscr{M}(\Omega), \\ |u_n|^{s(x)} &\rightharpoonup \nu \quad \text{(weak*-sense of measures) in } \mathscr{M}(\Omega), \\ \nu &= |u|^{s(x)} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}, \quad \nu_j > 0, \\ \mu &\geqslant |\nabla u|^{\gamma(x)} + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j}, \quad \mu_j > 0, \end{aligned} \tag{3.8}$$

where δ_{x_j} is the Dirac's delta measure supported on $x_j \in \Omega$ and

$$S\nu_j^{\frac{1}{\gamma^*(x_j)}} \leqslant \mu_j^{\frac{1}{\gamma(x_j)}} \ \forall j \in \mathcal{J}, \tag{3.9}$$

where S is the best positive constant of the Gagliardo-Nirenberg-Sobolev embedding (3.5).

Claim 1. We claim that the set $\mathcal{J}$ is empty. Consequently, $\{u_n\}_{n\in\mathbb{N}}$ converge strongly to u in $L^{s(\cdot)}(\Omega)$.

Indeed. Arguing by contradiction, suppose $\mathcal{J} \neq \emptyset$. Fix $j \in \mathcal{J}$. First, we will prove that

$$\nu_j \geqslant (\overline{m_0 \kappa} S)^N.$$

We consider a cutoff function $\psi \in C_0^\infty(\Omega,[0,1])$ such that $\psi \equiv 1$ in $B(0,1)$ and $\psi \equiv 0$ in $\mathbb{R}^N \setminus B(0,2)$, and $|\nabla \psi|_\infty \leqslant 2$. We define, for all $\varepsilon > 0$, the function

$$\psi_{\varepsilon,j}(x) := \psi\left(\frac{x-x_j}{\varepsilon}\right),$$

thus $\psi_{\varepsilon,j} \in C_0^{+\infty}(\mathbb{R}^N)$, $0 \leqslant \psi_{\varepsilon,j}(x) \leqslant 1$, for all $x \in \mathbb{R}^N$, $|\nabla \psi_{\varepsilon,j}|_\infty \leqslant \frac{2}{\varepsilon}$, and

$$\psi_{\varepsilon,j}(x) = \begin{cases} 1 & \text{if } x \in B(x_j,\varepsilon), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B(x_j, 2\varepsilon). \end{cases}$$

Clearly by Lemma 4.1, Proposition 4.3, and Lemma 4.2 the sequence $\{\psi_{\varepsilon,j} u_n\}_{n\in\mathbb{N}}$ is bounded in X. Thus, as $\{u_n\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence, it follows that $\langle \Phi'_{\omega,\lambda}(u_n), \psi_{\varepsilon,j} u_n\rangle \to 0$ as $n \to +\infty$, that is,

$$\begin{aligned} M_\omega(\mathscr{A}(u_n)) &\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\psi_{\varepsilon,j}\, dx \\ &= - M_\omega(\mathscr{A}(u_n)) \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\psi_{\varepsilon,j}\, dx \\ &\quad + \lambda \int_\Omega f(x,u_n)u_n\psi_{\varepsilon,j}\, dx \\ &\quad + \int_\Omega \psi_{\varepsilon,j}|u_n|^{s(x)}\, dx + o_n(1), \end{aligned} \tag{3.10}$$

where $o_n(1) \to 0$ as $n \to +\infty$.

We obtain by (a_1), $(\mathcal{M}_1)$, and (3.10) that

$$\begin{aligned} M_\omega(\mathcal{A}(u_n)) &\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\psi_{\varepsilon,j}\, dx \\ &\leqslant - m_0 \left(\int_\Omega \kappa_0 |\nabla u_n|^{p(x)}\psi_{\varepsilon,j}\, dx + \mathcal{H}(\kappa_3)\kappa_2 \int_\Omega |\nabla u_n|^{q(x)}\psi_{\varepsilon,j}\, dx \right) \\ &\quad + \lambda \int_\Omega f(x,u_n)u_n\psi_{\varepsilon,j}\, dx + \int_\Omega \psi_{\varepsilon,j}|u_n|^{s(\cdot)}\, dx + o_n(1). \end{aligned} \tag{3.11}$$

Since that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, we have, up to subsequence, that $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $L^{r(\cdot)}(\Omega)$ and that there exists $h \in L^{r(\cdot)}(\Omega)$ such that $|u_n(x)| \leqslant h(x)$ and $u_n(x) \to u(x)$, for a.e. in Ω (see [30], Proposition 2.67). From (f_2) we get

$$|f(x,u_n)u_n\psi_{\varepsilon,j}| \leqslant a_2|h(x)|^{r(x)} \in L^1(\Omega).$$

Hence, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n\to+\infty} \lambda \int_\Omega f(x,u_n(x))u_n(x)\psi_{\varepsilon,j}(x)\, dx = \lambda \int_\Omega f(x,u(x))u(x)\psi_{\varepsilon,j}(x)\, dx. \tag{3.12}$$

Using the Lebesgue Dominated Convergence Theorem, again, we get

$$\begin{gathered} \lim_{\varepsilon\to0^+} \int_\Omega |\nabla u(x)|^{p(x)}\psi_{\varepsilon,j}(x)\, dx = 0,\ \lim_{\varepsilon\to0^+} \int_\Omega f(x,u(x))u(x)\psi_{\varepsilon,j}(x)\, dx = 0, \\ \lim_{\varepsilon\to0^+} \int_\Omega \psi_{i,\varepsilon} d\mu = \mu_i\psi(0), \text{ and } \lim_{\varepsilon\to0^+} \int_\Omega \psi_{i,\varepsilon} d\nu = \nu_i\psi(0). \end{gathered} \tag{3.13}$$

Futhermore, we achieve by (3.8) that

$$\lim_{n\to+\infty}\int_\Omega |u_n(x)|^{s(x)}\psi_{\varepsilon,j}(x)\,dx=\int_\Omega \psi_{\varepsilon,j}(x)d\nu. \tag{3.14}$$

From (a_1), and since $M_\omega(\cdot)$ is bounded, arguing as in [10] to obtain that

$$\lim_{\varepsilon\to0^+}\left[\limsup_{n\to+\infty}\left(M_\omega(\mathscr{A}(u_n)\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n u_n\nabla\psi_{\varepsilon,j}\,dx\right)\right]=0. \tag{3.15}$$

Now, we will use the expressions (3.11), (3.12), (3.13), (3.14), and (3.15) to analyze the cases $\kappa_3=0$ and $\kappa_3>0$.

(I) When $\kappa_3=0$, we have

$$\begin{aligned}0&\leqslant\lim_{\varepsilon\to0^+}\left[\int_\Omega\psi_{\varepsilon,j}d\nu-m_0\kappa_0\int_\Omega\psi_{\varepsilon,j}d\mu\right]\\&\leqslant\lim_{\varepsilon\to0^+}\left[\int_{B(x_j,2\varepsilon)}\psi_{\varepsilon,j}d\nu-m_0\kappa_0\int_{B(x_j,2\varepsilon)}\psi_{\varepsilon,j}d\mu\right]\\&\leqslant\nu_j-m_0\kappa_0\mu_j,\end{aligned}$$

then, we achieve

$$m_0\kappa_0\mu_j\leqslant\nu_j. \tag{3.16}$$

(II) When $\kappa_3>0$, we have

$$\begin{aligned}0&\leqslant\lim_{\varepsilon\to0^+}\left[\int_\Omega\psi_{\varepsilon,j}d\nu-m_0\mathcal{H}(\kappa_3)\kappa_2\int_\Omega\psi_{\varepsilon,j}d\mu)\right]\\&\leqslant\lim_{\varepsilon\to0^+}\left[\int_{B(x_j,2\varepsilon)}\psi_{\varepsilon,j}d\nu-m_0\kappa_0\int_{B(x_j,2\varepsilon)}\psi_{\varepsilon,j}d\mu\right]\\&\leqslant\nu_j-m_0\mathcal{H}(\kappa_3)\kappa_2\mu_j,\end{aligned}$$

thus, we obtain

$$m_0\mathcal{H}(\kappa_3)\kappa_2\mu_j\leqslant\nu_j. \tag{3.17}$$

Consequently, by combining (3.16) and (3.17), we get

$$m_0((1-\mathcal{H}(\kappa_3))\kappa_0\mu_j+\mathcal{H}(\kappa_3)\kappa_2\mu_j)\leqslant\nu_j. \tag{3.18}$$

Therefore, by using (3.9) and (3.18), we obtain

$$(\overline{m_0\kappa}S)^N \leqslant \nu_j, \tag{3.19}$$

where

$$\overline{m_0\kappa} = \min\left\{ (m_0((1-\mathcal{H}(\kappa_3))\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^-}}, (m_0((1-\mathcal{H}(\kappa_3))\kappa_0 + \mathcal{H}(\kappa_3)\kappa_2))^{\frac{1}{\gamma^+}} \right\}.$$

Now, we claim that the inequality (3.19) cannot hold, and hence the set $\mathcal{J}$ is empty. Indeed, remembering that $m_0 \leqslant M_\omega(t) < \frac{\sigma m_0}{p^+\alpha}$, for all $t \in \mathbb{R}$, we get

$$\begin{aligned} c_\lambda =& \Phi_{\omega,\lambda}(u_n) - \frac{1}{\sigma}\langle \Phi'_{\omega,\lambda}(u_n), u_n\rangle + o_n(1) \\ \geqslant& m_0 \int_\Omega \frac{1}{p(x)} A(|\nabla u_n|^{p(x)})\, dx - \lambda \int_\Omega F(x,u_n)\, dx - \int_\Omega \frac{1}{s(x)}|u_n|^{s(x)}\, dx \\ &- \frac{m_0}{p^+\alpha}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\, dx + \frac{\lambda}{\sigma}\int_\Omega f(x,u_n)u_n\, dx + \frac{1}{\sigma}\int_\Omega |u_n|^{s(x)}\, dx + o_n(1). \end{aligned}$$

However, by (a_3), we have

$$\int_\Omega \frac{1}{p(x)} A(|\nabla u_n|^{p(x)})\, dx - \frac{1}{p^+\alpha}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}\, dx \geqslant 0,$$

for all $n \in \mathbb{N}$. Hence, since $0 \leqslant \psi_{\varepsilon,j} \leqslant 1$ and by using (f_1) and (f_2), we obtain

$$\begin{aligned} c_\lambda &\geqslant -\lambda a_2\left(\frac{1}{\sigma} + \frac{1}{r^-}\right)\int_\Omega |u_n|^{r(x)}\, dx + \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)\int_\Omega |u_n|^{s(x)}\, dx + o_n(1) \\ &\geqslant -\lambda a_2\left(\frac{1}{\sigma} + \frac{1}{r^-}\right)\int_\Omega |u_n|^{r(x)}\, dx + \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)\int_\Omega \psi_{\varepsilon,j}|u_n|^{s(x)}\, dx + o_n(1). \end{aligned} \tag{3.20}$$

Taking limit in (3.20) as $n \to +\infty$ and by using (3.8) and Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned} c_\lambda \geqslant& \lim_{n\to+\infty}\left(-\lambda a_2\left(\frac{1}{\sigma} + \frac{1}{r^-}\right)\int_\Omega |u_n|^{r(x)}\, dx + \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)\int_\Omega \psi_{\varepsilon,j}|u_n|^{s(x)}\, dx\right) \\ \geqslant& -\lambda a_2\left(\frac{1}{\sigma} + \frac{1}{r^-}\right)\int_\Omega |u|^{r(x)}\, dx + \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)\int_\Omega \psi_{\varepsilon,j}|u|^{s(x)}\, dx + \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)\nu_j. \end{aligned}$$

On the other hand, observe that

$$|u(x)|^{s(x)}\psi_{\varepsilon,j}(x)\chi_{B(x_j,2\varepsilon)} \to |u(x)|^{s(x)} \text{ as } \varepsilon \to +\infty, \text{ a.e. in } \Omega,$$

and

$$|u(x)|^{s(x)}\psi_{\varepsilon,j}(x)\chi_{B(x_j,2\varepsilon)} \leqslant |u(x)|^{s(x)} \text{ a.e. in } \Omega,$$

thus, by applying the Lebesgue Dominated Convergence Theorem, as $\varepsilon \to 0^+$, we obtain

$$c_\lambda \geqslant -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u|^{r(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega |u|^{s(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\nu_j.$$

Then, by using (3.19) and Hölder inequality (Lemma 4.1), we achieve

$$\begin{aligned} c_\lambda \geqslant & -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\int_\Omega |u|^{r(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega |u|^{s(x)}\,dx + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\left(\overline{m_0\kappa}S\right)^N \\ \geqslant & -\lambda a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\left(\frac{1}{(\frac{s}{r})^-}+\frac{1}{(\frac{s}{s-r})^-}\right)\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}\|1\|_{\frac{s(\cdot)}{s(\cdot)-r(\cdot)}} + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\int_\Omega |u|^{s(x)}\,dx \\ & + \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\left(\overline{m_0\kappa}S\right)^N. \end{aligned} \tag{3.21}$$

Now, we will examine the possible cases:

(i) If $\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}} \geqslant 1$, by Proposition 4.3, we have

$$\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{\left(\frac{s}{r}\right)^-} \leqslant \int_\Omega |u|^{r(x)\cdot\frac{s(x)}{r(x)}}\,dx \leqslant \||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{\left(\frac{s}{r}\right)^+}. \tag{3.22}$$

Hence, by (3.21) and (3.22), we get

$$c_\lambda \geq c_1\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{(\frac{s}{r})^-} - \lambda c_2\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}} + c_3,$$

where $c_1 = \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)$, $c_2 = a_2\left(\frac{1}{\sigma}+\frac{1}{r^-}\right)\left(\frac{1}{(\frac{s}{r})^-}+\frac{1}{(\frac{s}{s-r})^-}\right)\|1\|_{\frac{s(\cdot)}{s(\cdot)-r(\cdot)}}$, and $c_3 = \left(\frac{1}{\sigma}-\frac{1}{s^-}\right)\left(\overline{m_0\kappa}S\right)^N$.

Defining the function $\mathcal{E}_1 : \mathbb{R}^+ \to \mathbb{R}$ given by $\mathcal{E}_1(t) = c_1 t^{(\frac{s}{r})^-} - \lambda c_2 t$ (observe that

$(\frac{s}{r})^- > 1)$, attains its absolute minimum at the point

$$t^* = \left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{1}{(s/r)^- - 1}} > 0.$$

Moreover, note that $r^- \leqslant r(x) \leqslant r^+ < s^-$ and we have

$$\begin{aligned}\mathcal{E}_1(t^*) &= c_1\left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{(s/r)^-}{(s/r)^- - 1}} - \lambda c_2\left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{1}{(s/r)^- - 1}} \\ &= c_1\left(\frac{\lambda c_2}{c_1(s/r)^-}\right)^{\frac{(s/r)^-}{(s/r)^- - 1}}\left(1 - \left(\frac{s}{r}\right)^-\right) \\ &= -\lambda^{\frac{(s/r)^-}{(s/r)^- - 1}}\mathcal{K},\end{aligned}$$

where $\mathcal{K}$ is a positive constant independent of λ.

(ii) If $\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}} < 1$, by Proposition 4.3, we obtain

$$\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{(\frac{s}{r})^+} \leqslant \int_\Omega |u|^{r(x)\cdot\frac{s(x)}{r(x)}}\,dx \leqslant \||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{(\frac{s}{r})^-}. \tag{3.23}$$

Then, by combining (3.21) and (3.23), we achieve

$$c_\lambda \geq c_1\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}}^{(\frac{s}{r})^+} - \lambda c_2\||u|^{r(x)}\|_{\frac{s(\cdot)}{r(\cdot)}} + c_3,$$

where $c_1 = \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)$, $c_2 = a_2\left(\frac{1}{\sigma} + \frac{1}{r^-}\right)\left(\frac{1}{(\frac{s}{r})^-} + \frac{1}{(\frac{s}{s-r})^-}\right)\|1\|_{\frac{s(\cdot)}{s(\cdot)-r(\cdot)}}$, and $c_3 = \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)(\overline{m_0\kappa S})^N$. Therefore, defining the function $\mathcal{E}_2 : \mathbb{R}^+ \to \mathbb{R}$ given by $\mathcal{E}_2(t) = c_1 t^{(\frac{s}{r})^+} - \lambda c_2 t$ (observe that $(\frac{s}{r})^+ > 1$), admits absolute minimum at the point

$$t_* = \left(\frac{\lambda c_2}{(\frac{s}{r})^+ c_1}\right)^{\frac{1}{(s/r)^+ - 1}} > 0.$$

Furthermore, analogously to the previous case (i), we get

$$\mathcal{E}_2(t_*) = -\lambda^{\frac{(s/r)^+}{(s/r)^+ - 1}}\mathcal{K},$$

where $\mathcal{K}$ is a positive constant independent of λ.

Therefore, by using (i) and (ii), we get

$$c_\lambda \geqslant \left(\frac{1}{\sigma} - \frac{1}{s^-}\right)(\overline{m_0\kappa}S)^N - \mathcal{K}\min\left\{\lambda^{\frac{(\frac{s}{r})^-}{(\frac{s}{r})^- - 1}}, \lambda^{\frac{(\frac{s}{r})^+}{(\frac{s}{r})^+ - 1}}\right\}.$$

Then, due to (3.7), we obtain that $\mathcal{J}$ is empty, and consequently $\rho_{s(\cdot)}(u_n) \to \rho_{s(\cdot)}(u)$ as $n \to +\infty$. Thus, from Lemma 4.3 and Proposition 4.3, we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $L^{s(\cdot)}(\Omega)$ as $n \to +\infty$. Thus, we conclude the proof of **Claim 1**.

Claim 2. We affirm that $u_n \to u$ strongly in X as $n \to +\infty$.

In fact. We have that $\Phi'_{\omega,\lambda}(u_n) \to 0$ in X', as $n \to +\infty$, and the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, then

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \langle \Phi'_{\omega,\lambda}(u_n), u_n - u\rangle \\
&= \lim_{n\to+\infty} \Bigg\{ M_\omega(\mathscr{A}(u_n)) \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \\
&\quad - \lambda \int_\Omega f(x,u_n)(u_n - u)\,dx - \int_\Omega |u_n|^{s(x)-2}u_n(u_n-u)\,dx \Bigg\}.
\end{aligned} \tag{3.24}$$

But, as $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, by (a_1), we have that $\{\mathscr{A}(u_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Thus, applying the Bolzano-Weierstrass Theorem, there exists $\widehat{t} \in \mathbb{R}^+$ such that $\mathscr{A}(u_n) \to \widehat{t}$ as $n \to +\infty$, and, as M is continuous,

$$\lim_{n\to+\infty} M_\omega(\mathscr{A}(u_n)) = M_\omega(\widehat{t}) \geq m_0.$$

By (f_2) and the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to+\infty}\left|\lambda \int_\Omega f(x,u_n(x))(u_n(x) - u(x))\,dx\right| = 0.$$

Also, using the Hölder's inequality and since $u_n \to u$ in $L^{s(\cdot)}(\Omega)$ as $n \to +\infty$, we achieve

$$\lim_{n\to+\infty}\left|\int_\Omega |u_n(x)|^{s(x)-2}u_n(x)(u_n(x)-u(x))\,dx\right| = 0.$$

Therefore, follows from (3.24) that

$$\lim_{n\to+\infty}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx=0.$$

Consequently, it follows from Lemma 4.9 that $u_n\to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$ as $n\to+\infty$, that is,

$$\lim_{n\to+\infty}\|\nabla u_n-\nabla u\|_{p(\cdot)}=0. \tag{3.25}$$

In particular, we point out that by the Hölder's inequality and (3.25) follows that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx=0. \tag{3.26}$$

Note that, if $\kappa_3=0$, we get from (3.25) that $u_n\to u$ strongly in X as $n\to+\infty$. Now, considering $\kappa_3>0$. Then, since that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence, by using (a_1), $(\mathcal{M}_1)$, (3.24), (3.25), and (3.26), we obtain

$$\begin{aligned}
o_n(1) &= \langle\Phi'_{\omega,\lambda}(u_n),u_n-u\rangle\\
&= M_\omega(\mathscr{A}(u_n))\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\
&\geqslant C\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\
&\geqslant C\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)+|\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1)\\
&= C\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx+o_n(1),
\end{aligned}$$

so, we get

$$\limsup_{n\to+\infty}\int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n\nabla(u_n-u)\,dx\leqslant 0,$$

Therefeore, by Theorem 4.1, we obtain $u_n\to u$ strongly in $W_0^{1,q(\cdot)}(\Omega)$ as $n\to+\infty$, that is,

$$\lim_{n\to+\infty}\|\nabla u_n-\nabla u\|_{q(\cdot)}=0. \tag{3.27}$$

Then, we conclude, by (3.25) and (3.27), that $u_n\to u$ strongly in X as $n\to+\infty$.

This concludes the proof of the **Claim 2**. □

3.2.2 A second auxiliary truncation argument

We will prove in the next lemma that the Euler-Lagrange functional $\Phi_{\omega,\lambda}$ is unbounded from below in X.

Lemma 3.3. *The Euler-Lagrange functional $\Phi_{\omega,\lambda}$ associated with $(\mathscr{P}_\lambda)$ is not bounded from below in X.*

Proof. Let us take $\upsilon \in X \setminus \{0\}$. Then, by using (a_1), (f_2) and (3.4), we have for each $t > 1$ that

$$\begin{aligned}
\Phi_{\omega,\lambda}(t\upsilon) \leqslant & \int_0^{\mathscr{A}(t\upsilon)} M_\omega(t)\,dt - \lambda \int_\Omega F(x, t\upsilon)\,dx - \int_\Omega \frac{1}{s(x)}|t\upsilon|^{s(x)}\,dx \\
\leqslant & \frac{\omega\kappa_1}{p^-}\int_\Omega |\nabla(t\upsilon)|^{p(x)}\,dx + \frac{\omega\kappa_3}{q^-}\int_\Omega |\nabla(t\upsilon)|^{q(x)}\,dx \\
& + \frac{C}{r^+}\int_\Omega |t\upsilon|^{r(x)}\,dx - \frac{1}{s^+}\int_\Omega |t\upsilon|^{s(x)}\,dx \\
\leqslant & \frac{\omega\kappa_1}{p^-}t^{p^+}\int_\Omega |\nabla\upsilon|^{p(x)}\,dx + \frac{\kappa_3}{q^-}t^{q^+}\int_\Omega |\nabla\upsilon|^{q(x)}\,dx \\
& + \frac{C}{r^+}t^{r^+}\int_\Omega |\upsilon|^{r(x)}\,dx - \frac{1}{s^+}t^{s^-}\int_\Omega |\upsilon|^{s(x)}\,dx.
\end{aligned}$$

Therefore, since $1 < r^- \leqslant r(x) \leqslant r^+ < \gamma^- \leqslant \gamma(x) \leqslant \gamma^+ < s^-$ for all $x \in \overline{\Omega}$, we get

$$\lim_{t\to+\infty} \Phi_{\omega,\lambda}(t\upsilon) = -\infty. \qquad \square$$

Now, following the ideas of Alonso and Azorero (see [10]), we will obtain a second auxiliary functional that will be bounded from below.

Remark that, by (a_1), $(\mathcal{M}_1)$, (f_2), and Proposition 4.3, we have

$$\begin{aligned}
\Phi_{\omega,\lambda}(u) =& \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda\int_\Omega F(x,u)dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}dx \\
=& \int_0^{\mathscr{A}(u)} M_\omega(t)dt - \lambda\int_\Omega F(x,u)dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}dx \\
\geqslant & m_0\int_\Omega \left(\frac{\kappa_0}{p(x)}|\nabla u|^{p(x)} + \mathcal{H}(\kappa_3)\frac{\kappa_2}{q(x)}|\nabla u|^{q(x)}\right)dx - \frac{a_2}{r^-}\int_\Omega |u|^{r(x)}dx \\
& - \frac{1}{s^-}\int_\Omega |u|^{s(x)}dx.
\end{aligned}$$

Also, note that

$$\min\{\|\nabla u\|_{p(\cdot)}^{p^-},\|\nabla u\|_{p(\cdot)}^{p^+}\}+\mathcal{H}(\kappa_3)\min\{\|\nabla u\|_{q(\cdot)}^{q^-},\|\nabla u\|_{q(\cdot)}^{q^+}\}\geqslant \min\{\|\nabla u\|_{\gamma(\cdot)}^{\gamma^-},\|\nabla u\|_{\gamma(\cdot)}^{\gamma^+}\}.$$

Moreover, by applying the Gagliardo-Nirenberg-Sobolev inequality (3.5), we have

$$\int_\Omega |u|^{r(x)}\,dx \leqslant \|u\|_{r(\cdot)}^{r^-}+\|u\|_{r(\cdot)}^{r^+},\quad \|u\|_{r(\cdot)}S_r\leqslant \|\nabla u\|_{\gamma(\cdot)},$$

and

$$\int_\Omega |u|^{r(x)}\,dx \leqslant \frac{1}{S_r^{r^-}}\|\nabla u\|_{\gamma(\cdot)}^{r^-}+\frac{1}{S_r^{r^+}}\|\nabla u\|_{\gamma(\cdot)}^{r^+}.$$

Considering $\|\nabla u\|_{\gamma(\cdot)}\leqslant 1$, we achieve

$$\begin{aligned}\Phi_{\omega,\lambda}(u) \;&\geqslant\; \tfrac{m_0\mathscr{K}}{q^+}\|\nabla u\|_{\gamma(\cdot)}^{\gamma^+}-\tfrac{\lambda a_2}{S_r^{r^-}r^-}\|\nabla u\|_{\gamma(\cdot)}^{r^-}-\tfrac{\lambda a_2}{S_r^{r^+}r^-}\|\nabla u\|_{\gamma(\cdot)}^{r^+}\\ &\quad -\tfrac{1}{S_s^{s^-}s^-}\|\nabla u\|_{\gamma(\cdot)}^{s^-}-\tfrac{1}{S_s^{s^+}s^-}\|\nabla u\|_{\gamma(\cdot)}^{s^+}.\end{aligned}$$

Moreover, there is a positive constant C_r such that

$$\frac{1}{S_r^{r^-}}\|\nabla u\|_{\gamma(\cdot)}^{r^-}+\frac{1}{S_r^{r^+}}\|\nabla u\|_{\gamma(\cdot)}^{r^+}\leqslant C_r(\|\nabla u\|_{\gamma(\cdot)}^{r^-}+\|\nabla u\|_{\gamma(\cdot)}^{r^+}),$$

and, since $\|\nabla u\|_{\gamma(\cdot)}\leqslant 1$, we have that $\|\nabla u\|_{\gamma(\cdot)}^{r^+}\leqslant \|\nabla u\|_{\gamma(\cdot)}^{r^-}$ and $\|\nabla u\|_{\gamma(\cdot)}^{s^+}\leqslant \|\nabla u\|_{\gamma(\cdot)}^{s^-}$, thus

$$\begin{aligned}\Phi_{\omega,\lambda}(u) &\geqslant \frac{m_0\mathscr{K}}{q^+}\|\nabla u\|_{\gamma(\cdot)}^{\gamma^+}-\lambda\overline{C}_r\|\nabla u\|_{\gamma(\cdot)}^{r^-}-\overline{C}_s\|\nabla u\|_{\gamma(\cdot)}^{s^-}\\ &:=\mathscr{G}_\lambda(\|\nabla u\|_{\gamma(\cdot)}),\end{aligned}\tag{3.28}$$

where $\mathscr{G}_\lambda:[0,+\infty[\to\mathbb{R}$ is given by

$$\mathscr{G}_\lambda(t)=\frac{m_0\mathscr{K}}{q^+}t^{\gamma^+}-\lambda\overline{C}_r t^{r^-}-\overline{C}_s t^{s^-}.$$

Now, we will show that there is $\lambda^*>0$ such that $\mathscr{G}_\lambda$ assumes positive values for each $\lambda\in(0,\lambda^*)$. Indeed, since $\gamma^+<s^-$, we can take $\bar{t}$ small enough such that

$$\frac{m_0\mathscr{K}}{q^+}\bar{t}^{\gamma^+}-\overline{C}_s\bar{t}^{s^-}>0,$$

and, we define

$$\overline{\lambda} = \frac{1}{2\overline{C}_r}\frac{1}{\overline{t}^{r^-}}\left(\frac{m_0\mathscr{K}}{q^+}\overline{t}^{\gamma^+} - \overline{C}_s\overline{t}^{s^-}\right).$$

Thus, for each $0 < \lambda < \overline{\lambda}$, follows that $\mathscr{G}_\lambda(\overline{t}) \geqslant \mathscr{G}_{\lambda^*}(\overline{t}) > 0$, this is, $\mathscr{G}_\lambda$ assumes positive values.

On the other hand, taking $\underline{t} < \left(\frac{\lambda q^+\overline{C}_r}{m_0\mathscr{K}}\right)^{\frac{1}{\gamma^+ - r^-}}$, we obtain $\frac{m_0\mathscr{K}}{q^+}t^{\gamma^+} - \lambda\overline{C}_r t^{r^-} < 0$, for all $0 < t < \underline{t}$. Since $\overline{C}_s t^{s^-} > 0$, we have

$$\mathscr{G}_\lambda(t) < \frac{m_0\mathscr{K}}{q^+}t^{\gamma^+} - \lambda\overline{C}_r t^{r^-} < 0$$

for all $t < \underline{t}$. Then, as $\mathscr{G}_\lambda$ assumes positive and negative values and $\lim_{t\to+\infty}\mathscr{G}_\lambda(t) = -\infty$, we conclude, by the Rolle's Theorem, that the function $\mathscr{G}_\lambda$ has exactly two roots(see Proposition 4.1). We will denote this roots by $0 < \mathscr{R}_0(\lambda) < \mathscr{R}_1(\lambda)$.

The following lemma is fundamental to build our truncated functional.

Lemma 3.4. *Assume* $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, (f_1), (f_2), (a_0), (a_1), *and* (3.1). *Then*

$$\lim_{\lambda\to 0^+}\mathscr{R}_0(\lambda) = 0.$$

Proof. Since $\mathscr{G}_\lambda(\mathscr{R}_0(\lambda)) = 0$ and $\mathscr{G}'_\lambda(\mathscr{R}_0(\lambda)) > 0$, we have

$$\frac{m_0\mathscr{K}}{q^+} = \frac{\lambda}{r^-}\overline{C}_r\mathscr{R}_0^{r^- - \gamma^+} + \frac{1}{s^-}\overline{C}_s\mathscr{R}_0^{s^- - \gamma^+} \tag{3.29}$$

and

$$\frac{\gamma^+ m_0\mathscr{K}}{q^+} > \lambda\overline{C}_r\mathscr{R}_0^{r^- - \gamma^+} + \overline{C}_s\mathscr{R}_0^{s^- - \gamma^+}, \tag{3.30}$$

for all $\lambda \in (0, \lambda^*)$. Hence, combining (3.29) and (3.30), we get

$$0 < \mathscr{R}_0 < \lambda^{\frac{1}{s^- - r^-}}\left[\frac{\left(\frac{\gamma^+\overline{C}_r}{r^-} - \overline{C}_r\right)}{\left(\overline{C}_s - \frac{\gamma^+\overline{C}_s}{s^-}\right)}\right]^{\frac{1}{s^- - r^-}}.$$

Therefore, as $s^- > r^-$, taking limit as $\lambda \to 0^+$, we obtain

$$\lim_{\lambda\to 0^+}\mathscr{R}_0(\lambda) = 0.$$

□

Remark 3.1. *Due to Lemma 3.4, we can choose* $\lambda^* \in (0, \overline{\lambda})$ *such that*

$$\left(\frac{1}{\sigma} - \frac{1}{s^-}\right)(\overline{m_0 \kappa} S)^N - \mathcal{K} \min\left\{\lambda^{\frac{(\frac{s}{r})^-}{(\frac{s}{r})^- - 1}}, \lambda^{\frac{(\frac{s}{r})^+}{(\frac{s}{r})^+ - 1}}\right\} > 0$$

and $\mathscr{R}_0(\lambda) < \min\{1, t_0\}$, *for each* $\lambda \in (0, \lambda^*)$. *In particular* $\mathscr{R}_0(\lambda) < \min\{\mathscr{R}_1(\lambda), t_0, 1\}$.

We define the function $\tau : \mathbb{R}_0^+ \to [0,1]$ given by

$$\tau(t) = \begin{cases} 1 & \text{if } t \leqslant \mathscr{R}_0(\lambda), \\ 0 & \text{if } t \geqslant \min\{\mathscr{R}_1(\lambda), 1\}. \end{cases}$$

Note that $\tau \in C_0^{+\infty}(\mathbb{R}_0^+, [0,1])$ and it is nonincreasing.

We define the second auxiliary truncated functional $\overline{\Phi}_{\omega,\lambda} : X \to \mathbb{R}$ given by

$$\overline{\Phi}_{\omega,\lambda}(u) = \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\, dx - \tau(\|u\|) \int_\Omega \frac{1}{s(x)} |u|^{s(x)}\, dx,$$

for all $\lambda \in (0, \overline{\lambda})$. Observe that if $\|u\| \leqslant \mathscr{R}_0(\lambda)$, then $\|\nabla u\|_{\gamma(\cdot)} \leqslant \mathscr{R}_0(\lambda)$, and, consequently $\overline{\Phi}_{\omega,\lambda}(u) = \Phi_{\omega,\lambda}(u)$. Moreover, if $\|u\| \geqslant \|\nabla u\|_{\gamma(\cdot)} \geqslant \max\{\mathscr{R}_1(\lambda), 1\}$, then

$$\overline{\Phi}_{\omega,\lambda}(u) = \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\, dx.$$

Lemma 3.5. *The functional* $\overline{\Phi}_{\omega,\lambda}$ *is bounded from below in* X.

Proof. Consider $\|u\| \geqslant \|\nabla u\|_{\gamma(\cdot)} \geqslant 1$. Following as in (3.28) and observing that the norms $\|\cdot\|$ and $\|\cdot\|_\gamma$ are equivalent in X, there are positive constants $\mathscr{K}_1$, $\mathscr{K}_2$ such that

$$\begin{aligned} \overline{\Phi}_{\omega,\lambda}(u) &= \mathscr{M}_\omega(\mathscr{A}(u)) - \lambda \int_\Omega F(x,u)\, dx \\ &\geqslant \mathscr{K}_1 \|u\|^{\gamma^-} - \lambda \mathscr{K}_2 \|u\|^{r^+}, \end{aligned}$$

and, consequently, as $1 < r^+ < \gamma^-$, it follows that $\lim_{\|u\| \to +\infty} \overline{\Phi}_{\omega,\lambda}(u) = +\infty$. Then, we have $\overline{\Phi}_{\omega,\lambda}$ bounded from below in X. □

Now, we will prove a local Palais-Smale condition and a topological result for the truncated functional $\overline{\Phi}_{\omega,\lambda}$.

Lemma 3.6. *If $\overline{\Phi}_{\omega,\lambda}(u) < 0$, then $\|\nabla u\|_{\gamma(\cdot)} < \mathscr{R}_0$ and $\Phi_{\omega,\lambda}(v) = \overline{\Phi}_{\omega,\lambda}(v)$ for all v in a small enough neighborhood of u. Moreover, $\overline{\Phi}_{\omega,\lambda}$ verifies a local Palais-Smale condition for $c_\lambda < 0$, for each $\lambda \in (0, \lambda^*)$.*

Proof. Let us consider $\overline{\Phi}_{\omega,\lambda}(u) < 0$. Supposing by contradiction that $\|\nabla u\|_{\gamma(\cdot)} \geqslant \mathscr{R}_0$, we obtain by the construction of the second truncated functional that $0 > \overline{\Phi}_{\omega,\lambda}(u) \geqslant \mathscr{G}_\lambda(\|\nabla u\|_{\gamma(\cdot)}) \geqslant 0$, which is a contradiction. Therefore, we conclude that $\|\nabla u\|_{\gamma(\cdot)} < \mathscr{R}_0$ and $\Phi_{\omega,\lambda}(u) = \overline{\Phi}_{\omega,\lambda}(u)$. Since that the norms $\|\cdot\|_{\gamma(\cdot)}$ and $\|\cdot\|$ are equivalent, we get, for each $u \in B(0, \mathscr{R}_0)$, that there exists $\varepsilon > 0$ such that $B(u, \varepsilon) \subset B(0, \mathscr{R}_0)$ and $\overline{\Phi}_{\omega,\lambda}(v) = \Phi_{\omega,\lambda}(v)$ for all $v \in B(u, \varepsilon)$ once that $\|\nabla v\|_{\gamma(\cdot)} < \mathscr{R}_0$.

Now, we will prove a local Palais-Smale condition for $\overline{\Phi}_{\omega,\lambda}$ at level $c_\lambda < 0$. We consider $\{u_n\}_{n\in\mathbb{N}}$ a Palais-Smale sequence at level $c_\lambda < 0$. Then, we obtain $\Phi_{\omega,\lambda}(u_n) = \overline{\Phi}_{\omega,\lambda}(u_n) \to c_\lambda < 0$ and $\Phi'_{\omega,\lambda}(u_n) = \overline{\Phi'}_{\omega,\lambda}(u_n) \to 0$ in X. Moreover, as $\overline{\Phi}_{\omega,\lambda}$ is coercive, we get $\{u_n\}_{n\in\mathbb{N}}$ bounded in X. Also, by Remark 3.1, we have

$$\left(\frac{1}{\sigma} - \frac{1}{s^-}\right)(\overline{m_0 \kappa} S)^N - \mathcal{K} \min\left\{\lambda^{\frac{(\frac{s}{r})^-}{(\frac{s}{r})^- - 1}}, \lambda^{\frac{(\frac{s}{r})^+}{(\frac{s}{r})^+ - 1}}\right\} > 0 > c_\lambda,$$

for each $\lambda \in (0, \lambda^*)$. Therefore, from Lemma 3.2, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ is strongly convergent in X. □

We will construct an appropriate minimax sequence of negative critical values for the functional $\overline{\Phi}_{\omega,\lambda}$.

Lemma 3.7. *For every $k \in \mathbb{N} \setminus \{0\}$ there exists $\varepsilon(k) > 0$ such that*

$$\gamma(\overline{\Phi}^{-\varepsilon}_{\omega,\lambda}) \geqslant k,$$

where $\lambda \in (0, \lambda^*)$, $\overline{\Phi}^{-\varepsilon}_{\omega,\lambda} = \{u \in X : \overline{\Phi}_{\omega,\lambda}(u) \leqslant -\varepsilon\}$, and γ is Krasnoselskii's genus.

Proof. Fixed $k \in \mathbb{N}$, since $C_0^{+\infty}(\Omega) \subset X$ has infinite dimension, we can consider $\mathfrak{X}_k$ ($\mathfrak{X}_k \subset C_0^{+\infty}(\Omega)$) a k-linear subspace of X. Then, as $\mathscr{R}_0 < \min\{\mathscr{R}_1(\lambda), t_0, 1\}$, for any

$u \in X$ with $\|u\| = 1$ and $0 < t < \mathscr{R}_0$, we get

$$\begin{aligned}\overline{\Phi}_{\omega,\lambda}(tu) \leqslant & \frac{\omega\kappa_1}{p^-} t^{p^-} \int_\Omega |\nabla u|^{p(x)}\, dx + \frac{\omega\kappa_3}{q^-} t^{q^-} \int_\Omega |\nabla u|^{p(x)}\, dx \\ & - \lambda \frac{a_1 t^{r^+}}{r^+} \int_\Omega |u|^{r(x)}\, dx - \frac{t^{s^+}}{s^+} \int_\Omega |u|^{s(x)}\, dx \\ \leqslant & \frac{\omega\kappa_1}{p^-} t^{p^-} + \frac{\omega\kappa_3}{q^-} t^{q^-} - \frac{\lambda a_1}{r^+} t^{r^+} \alpha_k \\ \leqslant & \left(\frac{\omega\kappa_1}{p^-} + \frac{\omega\kappa_3}{q^-} \right) t^{p^-} - \frac{\lambda a_1}{r^+} t^{r^+} \alpha_k,\end{aligned}$$

where

$$\alpha_k := \inf \left\{ \int_\Omega |u|^{r(x)}\, dx : u \in \mathfrak{X}_k, \|u\| = 1 \right\} > 0.$$

Note that the norms $\|\cdot\|$ and $\|\cdot\|_{r(\cdot)}$ are equivalent on $\mathfrak{X}_k$, once that $\mathfrak{X}_k$ is k-dimensional. Then, there is a positive constant ρ_k such that

$$0 < \rho_k < \min \left\{ \mathscr{R}_0, \left[\frac{\frac{\lambda a_1 \alpha_k}{r^+}}{\omega \left(\frac{\kappa_1}{p^-} + \frac{\kappa_3}{q^-} \right)} \right]^{\frac{1}{p^- - r^+}} \right\}. \tag{3.31}$$

We define

$$\mathbb{S}_{\rho_k} = \{ u \in \mathfrak{X}_k : \|u\| = \rho_k \},$$

which is homeomorphic to $\mathbb{S}^{k-1}$. Therefore, by Corollary 4.1, we have $\gamma(\mathbb{S}_{\rho_k}) = k$. Also, as $r^+ < p^-$, for any $u \in \mathbb{S}_{\rho_k,k}$ and by (3.31), we obtain

$$\begin{aligned}\overline{\Phi}(u)_{\omega,\lambda} = & \overline{\Phi}_{\omega,\lambda}(\rho_k \frac{u}{\|u\|}) \\ \leqslant & \left(\frac{\omega\kappa_1}{p^-} + \frac{\omega\kappa_3}{q^-} \right) \rho_k^{p^-} - \lambda \frac{a_1}{r^+} \rho_k^{r^+} \alpha_k \\ \leqslant & \rho_k^{r^+} \left[\left(\frac{\omega\kappa_1}{p^-} + \frac{\omega\kappa_3}{q^-} \right) \rho_k^{p^- - r^+} - \frac{\lambda a_1}{r^+} \alpha_k \right] < 0.\end{aligned}$$

Thus, we conclude that there exists a positive constant ε such that

$$\overline{\Phi}_{\omega,\lambda}(u) < -\varepsilon \text{ for any } u \in \mathbb{S}_{\rho_k}.$$

Hence, we achieve $\mathbb{S}_{\rho_k} \subset \overline{\Phi}_{\omega,\lambda}^{-\varepsilon}$ and $\gamma(\overline{\Phi}_{\omega,\lambda}^{-\varepsilon}) \geqslant \gamma(\mathbb{S}_{\rho_k}) = k$. □

We define, for any $k \in \mathbb{N} \setminus \{0\}$, the set

$$\Gamma_k = \{C \subset X \setminus \{0\} : C \text{ is closed, } C = -C \text{ and } \gamma(C) \geqslant k\},$$

and the number

$$c_k^\lambda = \inf_{C \in \Gamma_k} \sup_{u \in C} \overline{\Phi}_{\omega,\lambda}(u).$$

Lemma 3.8. *For all $k \in \mathbb{N} \setminus \{0\}$ and $\lambda \in (0, \lambda^*)$, the number c_k^λ is negative.*

Proof. Let $\lambda \in (0, \lambda^*)$ and $k \in \mathbb{N}$. Due to Lemma 3.7, for each $k \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $\gamma(\overline{\Phi}_\lambda^{-\varepsilon}) \geqslant k$. Since $\overline{\Phi}_{\omega,\lambda}$ is continuous and even, we have $\overline{\Phi}_{\omega,\lambda}^{-\varepsilon} \in \Gamma_k$ and

$$\sup_{u \in \overline{\Phi}_{\omega,\lambda}^{-\varepsilon}} \overline{\Phi}_{\omega,\lambda}(u) \leqslant -\varepsilon.$$

From Lemma 3.5, $\overline{\Phi}_{\omega,\lambda}$ is bounded from below, consequently

$$-\infty < c_k^\lambda = \inf_{C \in \Gamma_k} \sup_{u \in C} \overline{\Phi}_{\omega,\lambda}(u) \leqslant \sup_{u \in \overline{\Phi}_{\omega,\lambda}^{-\varepsilon}} \overline{\Phi}_{\omega,\lambda}(u) \leqslant -\varepsilon < 0.$$

□

We will proof in the following lemma the existence of critical points for the functional $\overline{\Phi}_{\omega,\lambda}$.

Lemma 3.9. *Let $k \in \mathbb{N} \setminus \{0\}$ and $\lambda \in (0, \lambda^*)$. If $c_\lambda = c_k^\lambda = c_{k+1}^\lambda = \ldots = c_{k+l}^\lambda$, for some $l \in \mathbb{N}$, then*

$$\gamma(K_{c_\lambda}) \geqslant l + 1,$$

where $K_{c_\lambda} := \{u \in X : \overline{\Phi}_{\omega,\lambda}(u) = c,\ \overline{\Phi}'_{\omega,\lambda}(u) = 0\}$. In particular, each c_k^λ is a critical value of $\overline{\Phi}_{\omega,\lambda}$.

Proof. Let $\lambda \in (0, \lambda^*)$ and $k, l \in \mathbb{N}$. We claim that K_{c_λ} is compact. In fact. Let us consider $\{u_n\}_{n\in\mathbb{N}}$ a sequence in K_{c_λ}. From Lemma 3.8, we have $c_\lambda = c_k^\lambda = c_{k+1}^\lambda = \ldots = c_{k+l}^\lambda$ negative, and, by Lemma 3.6, we have the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X and $\overline{\Phi}_{\omega,\lambda}(u_n) = \Phi_{\omega,\lambda}(u_n)$, for all $n \in \mathbb{N}$. By Lemma 3.2, the functional $\overline{\Phi}_{\omega,\lambda} \equiv \Phi_{\omega,\lambda}$ at level $c_\lambda < 0$ satisfies the Palais-Smale condition in K_{c_λ}. Hence, it follows that K_{c_λ} is compact. Furthermore, since $\overline{\Phi}_{\omega,\lambda}$ is even, $K_{c_\lambda} = -K_{c_\lambda}$.

Suppose, by contradiction, that $\gamma(K_{c_\lambda}) \leqslant l$. Since K_{c_λ} is compact, by using the Proposition 4.7, there exists a closed and symmetric set U, with $K_{c_\lambda} \subset U$ such that $\gamma(U) = \gamma(K_{c_\lambda}) \leqslant l$. Note that we can choose $U \subset \overline{\Phi}^0_{\omega,\lambda}$, since $c_\lambda < 0$, $K_{c_\lambda} \subset \overline{\Phi}^0_{\omega,\lambda}$. Since $\overline{\Phi}_{\omega,\lambda} \equiv \Phi_{\omega,\lambda}$ at any level $c_\lambda < 0$, by the Deformation Lemma 4.5, we obtain an odd homeomorphism $\eta : X \to X$ such that

$$\eta(\overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda} - \mathring{U}) \subset \overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda}, \tag{3.32}$$

for some $\delta \in (0, -c_\lambda)$. Note that $\delta + c_\lambda < 0$, which implies $\overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda} \subset \overline{\Phi}^0_{\omega,\lambda}$. By definition of $c_\lambda = c^\lambda_{k+l} = \inf\limits_{C\in\Gamma_{k+l}} \sup\limits_{u\in C} \overline{\Phi}_{\omega,\lambda}(u)$, there exists $A \in \Gamma_{k+l}$ such that $\sup_{u\in A} \overline{\Phi}_{\omega,\lambda}(u) < c_\lambda + \delta$, so $A \subset \overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda}$. Therefore, by (3.32), we achieve

$$\eta(A - \mathring{U}) \subset \eta(\overline{\Phi}^{c_\lambda+\delta}_{\omega,\lambda} - \mathring{U}) \subset \overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda}. \tag{3.33}$$

We have $A \subset \overline{(A-U)} \cup U$. Then, by the Proposition 4.7, we have

$$\gamma(\overline{A-U}) \geqslant \gamma(A) - \gamma(U) \geqslant (k+l) - l = k.$$

Thus, as η is odd, by Proposition 4.7, we obtain

$$\gamma(\eta(\overline{A-U})) \geqslant \gamma(\overline{A-U}) \geqslant k.$$

Note that $\eta(\overline{A-U})$ is closed and symmetrical, then $\eta(\overline{A-U}) \in \Gamma_k$, Hence, we achieve

$$\sup_{u\in\eta(\overline{A-U})} \overline{\Phi}_{\omega,\lambda}(u) \geqslant c^\lambda_k = c_\lambda. \tag{3.34}$$

On the other hand, since $\overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda}$ is closed and η is a homeomorphism, by (3.33), we get

$$\eta(\overline{A-U}) \subset \overline{\Phi}^{c_\lambda-\delta}_{\omega,\lambda},$$

which is a contradiction with (3.34). Hence, we conclude

$$\gamma(K_{c_\lambda}) \geqslant l + 1.$$

In particular, we obtain $K_{c_\lambda} \neq \emptyset$, that is, c_λ is a critical value of $\overline{\Phi}_{\omega,\lambda}$. □

3.2.3 Proof of Theorem 3.2

Proof. Let $\lambda \in (0, \lambda^*)$. Note that, from Lemma 3.8, we have

$$c_1^\lambda \leqslant c_2^\lambda \leqslant c_3^\lambda \leqslant \cdots < \overline{\Phi}_{\omega,\lambda}(0) = 0.$$

We will consider two possible cases.

Firstly, if $c_j^\lambda \neq c_{j'}^\lambda$ for all $j, j' \in \mathbb{N}$, $j \neq j'$, that is, $-\infty < c_1^\lambda < c_2^\lambda < \ldots < c_k^\lambda < \ldots < \overline{\Phi}_{\omega,\lambda}(0) = 0$. Moreover, by Lemma 3.9, each c_k^λ is a critical value of $\overline{\Phi}_{\omega,\lambda}$. Consequently, we obtain infinitely many critical points for $\overline{\Phi}_{\omega,\lambda}$. Hence, observing Lemma 3.6, the problem $(\mathscr{M}_\lambda)$ has infinitely many solutions.

Now, suppose that, for some $k \in \mathbb{N} \setminus \{0\}$, there is $l \in \mathbb{N} \setminus \{0\}$ such that $c_k^\lambda = c_{k+l}^\lambda$, so

$$c_\lambda = c_k^\lambda = c_{k+1}^\lambda = \ldots = c_{k+l}^\lambda < 0.$$

Then, from Lemma 3.9, we get

$$\gamma(K_{c_\lambda}) \geqslant l + 1 \geqslant 2.$$

Thus, from Proposition 4.6 the compact set K_{c_λ} has infinitely many points, which are critical points for $\overline{\Phi}_{\omega,\lambda}$. Hence, observing Lemma 3.6, the problem $(\mathscr{M}_\lambda)$ has infinitely many solutions. □

3.3 Proof of Theorem 3.1

Proof. Let $\lambda \in (0, \lambda^*)$ and u_λ a nontrivial solution of Problem $(\mathscr{M}_\lambda)$ (see Theorem 3.2). From Lemma 3.6, we have

$$\|\nabla u_\lambda\|_{\gamma(\cdot)} \leqslant \mathscr{R}_0 < t_0 \text{ and } \overline{\Phi}_{\omega,\lambda}(u_\lambda) = \Phi_{\omega,\lambda}(u_\lambda) < 0.$$

We will analyze the cases $\kappa_3 = 0$ and $\kappa_3 > 0$.

(i) If $\kappa_3 = 0$, by Lemma 3.4, we have $\lim_{\lambda \to 0^+} \|\nabla u_\lambda\|_{p(\cdot)} = 0$. Thus, by (a_1) and changing λ^* by other smaller, if necessary, we obtain

$$\mathscr{A}(u) \leqslant \frac{\kappa_1}{p^-} \|\nabla u_\lambda\|_{p(\cdot)}^{p^-} < t_0, \forall \lambda \in (0, \lambda^*).$$

(ii) If $\kappa_3 > 0$, by Lemma 3.4, we obtain $\lim_{\lambda\to 0^+}\|\nabla u\|_{q(\cdot)} = 0$. Thus, by (a_1), $W_0^{1,q(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$, $p^- < q^-$, and changing λ^* by other smaller, if necessary, there is a positive constant C, such that

$$\begin{aligned}\mathscr{A}(u) &\leqslant \frac{\kappa_1}{p^-}\|\nabla u_\lambda\|_{p(\cdot)}^{p^-} + \frac{\kappa_3}{q^-}\|\nabla u_\lambda\|_{q(\cdot)}^{q^-} \\ &\leqslant \frac{\kappa_1}{p^-}C\|\nabla u_\lambda\|_{q(\cdot)}^{p^-} + \frac{\kappa_3}{q^-}\|\nabla u_\lambda\|_{q(\cdot)}^{q^-} \\ &\leqslant \left(\frac{\kappa_1}{p^-}C + \frac{\kappa_3}{q^-}\right)\|\nabla u_\lambda\|_{q(\cdot)}^{p^-} < t_0, \forall \lambda \in (0,\lambda^*).\end{aligned}$$

Therefore, by (i) and (ii), we conclude that

$$M_\omega(\mathscr{A}(u)) = M(\mathscr{A}(u)), \forall \lambda \in (0,\lambda^*).$$

Consequently, we concluded that u_λ is solution of the problem $(\mathscr{P}_\lambda)$ for each $(0,\lambda^*)$. Moreover, since the problem $(\mathscr{M}_\lambda)$ has infinite many solutions for each $\lambda \in (0,\overline{\lambda})$, it follows that the problem $(\mathscr{P}_\lambda)$ has infinite many solutions for each $\lambda \in (0,\overline{\lambda})$.

Now, we will study the asymptotic behaviour of solutions to problem $(\mathscr{P}_\lambda)$. Remember that

$$\|u_\lambda\| = \|\nabla u_\lambda\|_{p(\cdot)} + \mathcal{H}(\kappa_3)\|\nabla u_\lambda\|_{q(\cdot)}. \tag{3.35}$$

We will analyze the cases $\kappa_3 = 0$ and $\kappa_3 > 0$.

(iii) If $\kappa_3 = 0$, we get from Lemma 3.4 and (3.35) that

$$\lim_{\lambda\to 0^+}\|u_\lambda\| = \lim_{\lambda\to 0^+}\|\nabla u_\lambda\|_{p(\cdot)} = 0.$$

(iv) If $\kappa_3 > 0$, we obtain by the Sobolev embedding $W_0^{1,q(\cdot)}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega)$ and by (3.35) that

$$\begin{aligned}\|u_\lambda\| &= \|\nabla u_\lambda\|_{p(\cdot)} + \|\nabla u_\lambda\|_{q(\cdot)} \\ &\leqslant (C+1)\|\nabla u_\lambda\|_{q(\cdot)}.\end{aligned}$$

Thus, follows from Lemma 3.4 that

$$\lim_{\lambda\to 0^+}\|u_\lambda\| = 0.$$

Hence, by (iii) and (iv), we conclude that

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

□

Chapter 4

Appendix

Proposition 4.1. *Consider* $a_1, a_2, \dots, a_n \in \mathbb{R} \setminus \{0\}$ *and* $\alpha_1, \dots, \alpha_n$ *such that* $\alpha_i \neq \alpha_j$ *for* $i \neq j$. *Then the equation*

$$a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \cdots + a_n x^{\alpha_n} = 0, \ x \in (0, \infty),$$

has a most $n-1$ *roots in* $(0, \infty)$.

The proof follows by induction and by the Rolle's Theorem.

4.1 Variable exponent Lebesgue and Sobolev spaces

Lemma 4.1. *([34, Proposition 2.1])*

(i) The space $\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ *is a separable, uniformly convex Banach space, and its dual space is* $L^{p'(\cdot)}(\Omega)$, *where* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. *For any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\Big| \int_\Omega uv \, dx \Big| \leqslant \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \ \forall u \in L^{p(\cdot)}(\Omega), \, v \in L^{p'(\cdot)}(\Omega).$$

(ii) If $p_1, p_2 \in C^+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ *for all* $x \in \overline{\Omega}$, *then* $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$, *and the embedding is continuous.*

Proposition 4.2. *([34, Proposition 2.5]) Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with smooth boundary* $\partial\Omega$ *and* $p \in C(\overline{\Omega})$ *with* $p(x) < N$ *for all* $x \in \overline{\Omega}$.

(i) $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *are separable and reflexive Banach spaces;*

(ii) if $q \in C^+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$*, then the embedding from* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact;*

(iii) There is a constant $C > 0$*, such that*

$$\|u\|_{p(\cdot)} \leqslant C\|\nabla u\|_{p(\cdot)}, \ \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Lemma 4.2. *([32, Lemma 2.1]) Let* $h \in L^\infty(\Omega)$ *be such that* $1 \leqslant h(x)p(x) \leqslant +\infty$ *for a.e.* $x \in \Omega$*. Let* $u \in L^{p(\cdot)}(\Omega)$*,* $u \not\equiv 0$*. Then*

$$\|u\|_{hp(\cdot)}^{h^-} \leqslant \||u|^{h(\cdot)}\|_{p(\cdot)} \leqslant \|u\|_{hp(\cdot)}^{h^+}, \ \text{if } \|u\|_{hp(\cdot)} \geqslant 1,$$
$$\|u\|_{hp(\cdot)}^{h^+} \leqslant \||u|^{h(\cdot)}\|_{p(\cdot)} \leqslant \|u\|_{hp(\cdot)}^{h^-}, \ \text{if } \|u\|_{hp(\cdot)} \leqslant 1.$$

In particular, if h *is a constant function, then*

$$\||u|^h\|_{p(\cdot)} = \|u\|_{hp(\cdot)}^h.$$

The following lemma is a version of the well-known Brezis-Lieb Lemma for variable exponents, the proof is similar to the constant case, the reader can see for instance [42].

Lemma 4.3. *Suppose* $\{u_n\}_{n\in\mathbb{N}}$ *bounded in* $L^{h(\cdot)}(\Omega)$ *and* $u_n(x) \to u(x)$ *a.e. in* Ω*. Then,* $u \in L^{h(\cdot)}(\Omega)$ *and*

$$\lim_{n\to+\infty}\left(\int_\Omega |u_n|^{h(x)}\,dx - \int_\Omega |u_n - u|^{h(x)}\,dx\right) = \int_\Omega |u|^{h(x)}\,dx.$$

Theorem 4.1. *([34, Theorem 3.1]) Consider the mapping* $L : W_0^{1,q(\cdot)}(\Omega) \to (W_0^{1,q(\cdot)}(\Omega))'$ *defined by*

$$\langle L(u), v\rangle = \int_\Omega |\nabla u|^{q(x)-2}\nabla u\nabla v dx.$$

Then, the mapping L *is of the type* (S_+)*, that is, if* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,q(\cdot)}(\Omega)$ *and* $\limsup\limits_{n\to\infty}\langle L(u_n), u_n - u\rangle \leqslant 0$*, it follow that* $u_n \to u$ *strongly in* $W_0^{1,q(\cdot)}(\Omega)$*.*

An important function in manipulating of the generalized Lebesgue-Sobolev spaces is the $p(\cdot)$-modular function of the $L^{p(\cdot)}(\Omega)$ space, which is the convex function

$\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \ \forall u \in L^{p(\cdot)}(\Omega).$$

Proposition 4.3. *([34, Proposition 2.3]) Let $u \in L^{p(\cdot)}(\Omega)$, then, we have*

(i) $\|u\|_{p(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$;

(ii) If $\|u\|_{p(\cdot)} > 1$ *then* $\|u\|_{p(\cdot)}^{p^-} \leqslant \rho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p^+}$;

(iii) If $\|u\|_{p(\cdot)} < 1$ *then* $\|u\|_{p(\cdot)}^{p^+} \leqslant \rho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p^-}$;

(iv) $\|u\|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u) \to 0$ *and* $\|u\|_{p(\cdot)} \to \infty \Leftrightarrow \rho_{p(\cdot)}(u) \to \infty$.

Proposition 4.4. *([34, Proposition 2.4]) Let $u, u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \dots$, then the following statements are equivalent:*

(i) $\lim_{n\to+\infty} \|u_n - u\|_{p(\cdot)} = 0$;

(ii) $\lim_{n\to+\infty} \rho_{p(\cdot)}(u_n - u) = 0$;

(iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to+\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

In particular, $\rho_{p(\cdot)}$ is continuous in $L^{p(\cdot)}(\Omega)$. Moreover, if $p \in C^+(\overline{\Omega})$, then $\rho_{p(\cdot)}$ is weakly lower semicontinuous.

4.2 Variational theorems

Lemma 4.4. *(Mountain Pass Theorem [23, Theorem I]) Let X be a real Banach space, let $\Phi : X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$ that satisfies the $(C)_c$ condition for any $c > 0$, $\Phi(0) = 0$, and the following conditions hold:*

(i) There exist positive constants ρ and $\mathcal{R}$ such that $\Phi(u) \geqslant \mathcal{R}$ for any $u \in X$ with $\|u\| = \rho$;

(ii) There exists a function $e \in X$ such that $\|e\| > \rho$ and $\Phi_\lambda(e) < 0$.

Then, the functional Φ has a critical value $c \geqslant \mathcal{R}$, that is, there exists $u \in X$ such that $\Phi(u) = c$ and $\Phi'(u) = 0$ in X'.

Theorem 4.2. *([64], Theorem 9.12). Assume that X has infinite dimension and let $\Phi \in C^1(X;\mathbb{R})$ be a functional satisfying the $(C)_c$ condition as well as the following properties:*

(i) $\Phi(0) = 0$ *and there exist two constants* $r, \rho > 0$ *such that* $\Phi_{|\partial B_r} \geqslant \rho$;

(ii) Φ *is even;*

(iii) for all finite dimensional subspace $\widehat{X} \subset X$ *there exists* $\mathcal{R} = \mathcal{R}(\widehat{X}) > 0$ *such that*

$$\Phi(u) \leqslant 0 \text{ for all } u \in \widehat{X} \backslash B_{\mathcal{R}}(\widehat{X}),$$

where $B_{\mathcal{R}}(\widehat{X}) = \{u \in \widehat{X} : \|u\| < \mathcal{R}\}$.

Then Φ possesses an unbounded sequence of critical values characterized by a minimax argument.

Theorem 4.3. *(Fountain Theorem [57, Theorem 2.9]) Assume*

(A_1) *X is a Banach space, $\Phi \in C^1(X,\mathbb{R})$ is an even functional.*
If for every $k \in \mathbb{N}$ there exit $\rho_k > r_k > 0$ such that

(A_2) $b_k := \inf\{\Phi(u) : u \in Z_k, \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$.

(A_3) $a_k := \max\{\Phi(u) : u \in Y_k, \|u\| = \rho_k\} \leqslant 0$.

(A_4) *Φ satisfies the $(C)_c$ condition for every $c > 0$.*

Then Φ has an unbounded sequence of critical points such that $\Phi(u_n) \to +\infty$.

Theorem 4.4. *(Dual Fountain Theorem [73, Theorem 3.18]) Suppose (A_1). If for each $k \geqslant k_0$ there exist $\rho_k > r_k > 0$ such that*

(B_1) $a_k = \inf\{\Phi(u) : u \in Z_k,\ \|u\| = \rho_k\} \geqslant 0$.

(B_2) $b_k = \max\{\Phi(u) : u \in Y_k,\ \|u\| = r_k\} < 0$.

(B_3) $d_k = \inf\{\Phi(u) : u \in Z_k,\ \|u\| \leqslant \rho_k\} \to 0,$ *as* $k \to +\infty$.

(B_4) *Φ satisfies the $(C)^*_c$ condition for every $c \in [d_{k_0}, 0[$.*

Then Φ has a sequence of negative critical values converging to 0.

Lemma 4.5. *(Deformation Lemma [6, Lemma 1.3]) Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and satisfy the (PS) condition. Let $c \in \mathbb{R}$ and N be any neighborhood of $K_c := \{u \in X : \varphi(u) = c, \varphi'(u) = 0\}$. Then there exists $\eta(t, x) \equiv \eta_t(x) \in C([0, 1] \times X, X)$ and positive constants $\overline{\varepsilon} > \varepsilon$ such that:*

(1) $\eta_0(x) = x$ *for all* $x \in X$;

(2) $\eta_t(x) = x$ *for all* $x \notin \varphi^{-1}[c - \overline{\varepsilon}, c - \overline{\varepsilon}]$ *and all* $t \in [0, 1]$;

(3) η_t *is a homeomorphism of* X *onto* X *for all* $t \in [0, 1]$;

(4) $\varphi(\eta_t(x)) \leqslant \varphi(x)$ *for all* $x \in X$, $t \in [0, 1]$;

(5) $\eta_1(A_{c+\varepsilon} - N) \subset A_{c-\varepsilon}$, *where* $A_c = \{x \in X : \varphi(x) \leqslant c\}$ *for any* $c \in \mathbb{R}$;

(6) if $K_c = \emptyset$, $\eta_1(A_{c+\varepsilon}) \subset A_{c-\varepsilon}$;

(7) if φ *is even,* η_t *is odd in* x.

Remark 4.1. *In [23] and [24] the Mountain Pass Theorem is established under the Palais-Smale (PS) condition, and, in [73] the Fountain Theorem and Dual Fountain Theorem are established under Palais-Smale (PS) and $(PS)_c^*$ condition. The Cerami condition (C_c) is more weaker than the known condition of Palais-Smale $(PS)_c$. Since the Deformation Theorem is still valid under the Cerami condition, we see that many critical point theorems like the Mountain Pass Theorem, Fountain Theorem, and Dual Fountain Theorem hold under the Cerami condition.*

Proposition 4.5. *(Concentration compactness lemma of Lions [14]) Let $p(x)$, $q(x)$ be two continuous functions such that*

$$1 < \inf_{x \in \Omega} p(x) \leqslant \sup_{x \in \Omega} p(x) < N \text{ and } 1 \leqslant q(x) \leqslant p^*(x) \text{ in } \Omega.$$

Let $\{u_n\}_{n \in \mathbb{N}}$ be a weakly convergent sequence in $W^{1,p(\cdot)}(\Omega)$ with weak limit u and such that:

$$|\nabla u_n|^{p(x)} \rightharpoonup \mu \text{ weakly-* in the sense of measures.}$$

$$|u_n|^{q(x)} \rightharpoonup \nu \text{ weakly-* in the sense of measures.}$$

In addition we assume that $\mathscr{C} := \{x \in \Omega : q(x) = p^*(x)\}$ *is nomempty. Then, for some countable index set* $\mathcal{J}$, *we have:*

$$\begin{aligned} \nu &= |u|^{q(x)} + \sum_{j \in \mathcal{J}} \nu_j \delta_{x_j}, \ \nu_j > 0, \\ \mu &\geqslant |\nabla u|^{p(x)} + \sum_{j \in \mathcal{J}} \mu_j \delta_{x_j}, \ \mu_j > 0, \\ S\nu_j^{\frac{1}{p^*(x_j)}} &\leqslant \mu_j^{\frac{1}{p(x_j)}}, \ \forall j \in \mathcal{J}. \end{aligned}$$

Where $\{x_j\}_{j \in \mathcal{J}} \subset \mathscr{C}$ *and* S *is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely*

$$S = S_q(\Omega) := \inf_{v \in C_0^{+\infty}(\Omega)} \frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)}}. \tag{4.1}$$

4.3 Krasnoselskii's genus

We will present some basic notions on Krasnoselskii's genus and, for more information about this subject, we refer [7], [6], [18], [24], [29], [53], and [64].

Let E be a real Banach space and let us denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$. Let $A \in \mathfrak{A}$. The Krasnoselskii's genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for every $x \in A$. If such a k does not exist, we set $\gamma(A) = +\infty$. Furthermore, we set $\gamma(\emptyset) = 0$.

Theorem 4.5. *Let* $E = \mathbb{R}^k$ *and* $\partial\Omega$ *be the boundary of an open, symmetric and bounded subset* $\Omega \subset \mathbb{R}^k$ *with* $0 \in \Omega$. *Then* $\gamma(\partial\Omega) = k$.

Corolllary 4.1. $\gamma(\mathbb{S}^{k-1}) = k$ *where* $\mathbb{S}^{k-1}$ *is a unit sphere of* $\mathbb{R}^k$.

Proposition 4.6. *If* $K \in \mathfrak{U}$, $0 \notin K$, *and* $\gamma(K) \geqslant 2$, *then* K *has infinitely many points.*

Theorem 4.6. *(Clark's Theorem) Let* $\Phi \in C^1(X, \mathbb{R})$ *be a functional satisfying the Palais-Smale condition. Suppose that:*

(i) Φ *is bounded from below and even;*

(ii) there is a compact set $K \in \mathfrak{U}$ *such that* $\gamma(K) = k$ *and* $\sup_{x \in K} \Phi(x) < \Phi(0)$.

Then, Φ possesses at least k pairs of distinct critical points and their corresponding critical values are less than $\Phi(0)$.

Proposition 4.7. *([64, Proposition 7.5] and [7, Lemma 10.4]) Let $A, B \in \mathfrak{A}$. Then*

(1) *Mapping property: If there exists $\varphi \in C(A, B)$, odd, then $\gamma(A) \leqslant \gamma(B)$;*

(2) *If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$;*

(3) *Monotonicity property: If $A \subset B$, then $\gamma(A) \leqslant \gamma(B)$;*

(4) *Subadditivity: If $\gamma(A \cup B) \leqslant \gamma(A) + \gamma(B)$;*

(5) *If $\gamma(B) < +\infty$, then $\gamma(\overline{A \setminus B}) \geqslant \gamma(A) - \gamma(B)$;*

(6) *Continuity property: If A is compact, then $\gamma(A) < \infty$, and there exists $\delta > 0$ such that $\gamma(A) = \gamma(N_\delta(A))$ where $N_\delta(A) = \{x \in X : d(x, A) \leqslant \delta\}$;*

(7) *if $\eta \in C(A, E)$ is odd then $\gamma(A) \leqslant \gamma(\eta(A))$.*

4.4 The (S_+) condition

Lemma 4.6. *Suposse that the function a satisfies the hypotheses (a_0), (a_1), and let $\{\varrho_n\}_{n\in\mathbb{N}}, \varrho$ be in $\mathbb{R}^N$ such that*

$$\left(a(|\varrho_n|^{p(x)})|\varrho_n|^{p(x)-2}\varrho_n - a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho\right)(\varrho_n - \varrho) \to 0 \text{ as } n \to +\infty, \tag{4.2}$$

for each $x \in \Omega$. Then, $\{\varrho_n\}_{n\in\mathbb{N}}$ converges to ϱ.

Proof. Fixed $x \in \Omega$. We affirm that the sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$. Indeed, suppose, by contradction, that $\{\varrho_n\}_{n\in\mathbb{N}}$ is unbounded, that is, up to a subsequence, still denoted by $\{\varrho_n\}_{n\in\mathbb{N}}$, we have $|\varrho_n|$ converging to $+\infty$. Then, by (a_0), we obtain that Υ' is strictly monotone that is equivalent to say that Υ is strictly convex (see [52, Lemma 15.4]) and due to (4.2), there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$0 \leqslant \left(a(|\varrho_n|^{p(x)})|\varrho_n|^{p(x)-2}\varrho_n - a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho\right)(\varrho_n - \varrho) < C.$$

So, we get

$$a(|\varrho_n|^{p(x)})|\varrho_n|^{p(x)} + a(|\varrho|^{p(x)})|\varrho|^{p(x)} < C + a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho \cdot \varrho_n + a(|\varrho_n|^{p(x)})|t\varrho_n|^{p(x-2)}\varrho \cdot \varrho_n.$$

Then, by (a_1) and Cauchy inequality, we obtain

$$\alpha\left(|\varrho_n|^{p(x)}+|\varrho|^{p(x)}\right)<C+\beta\left(|\varrho|^{p(x)-1}|\varrho_n|+|\varrho_n|^{p(x)-1}|\varrho|\right).$$

Therefore, we have

$$\alpha\left(1+\frac{|\varrho|^{p(x)}}{|\varrho_n|^{p(x)}}\right)<\frac{C}{|\varrho_n|^{p(x)}}+\beta\left(\frac{|\varrho|^{p(x)-1}}{|\varrho_n|^{p(x)-1}}+\frac{|\varrho|}{|\varrho_n|}\right). \tag{4.3}$$

Since we have assumed that $|\varrho_n|\to+\infty$ as $n\to+\infty$, taking limit in (4.3), we obtain $\alpha\leqslant 0$ which is a contradiction with (a_1). Therefore $\{\varrho_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$.

Thus, by Bolzano-Weierstrass Theorem exists a subsequence, still denoted by $\{\varrho_n\}_{n\in\mathbb{N}}$, that converges to $\Sigma\in\mathbb{R}^N$. So, by (4.2) we obtain

$$\begin{aligned}0&=\lim_{n\to+\infty}\left(a(|\varrho_n|^{p(x)})|\varrho_n|^{p(x)-2}\varrho_n-a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho\right)(\varrho_n-\varrho)\\&=\left(a(|\Sigma|^{p(x)})|\Sigma|^{p(x)-2}\Sigma-a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho\right)(\Sigma-\varrho).\end{aligned}$$

From the strict monotonicity Ψ', we have to $\Sigma=\varrho$. □

Lemma 4.7. *Suppose (a_0) and (a_1). Then the functional $\Psi:W_0^{1,p(\cdot)}(\Omega)\to\mathbb{R}$ defined by*

$$\Psi(u):=\int_\Omega\frac{1}{p(x)}A(|\nabla u|^{p(x)})\,dx,\ \forall u\in W_0^{1,p(\cdot)}(\Omega),$$

has a following properties:

(i) The functional Ψ is well defined on $W_0^{1,p(\cdot)}(\Omega)$ and is convex.

(ii) The functional Ψ is of class $C^1(W_0^{1,p(\cdot)}(\Omega),\mathbb{R})$ and weakly lower semicontinuous on $W_0^{1,p(\cdot)}(\Omega)$ with the derivative given by

$$\langle\Psi'(u),v\rangle=\int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,dx,$$

for all $u,v\in W_0^{1,p(\cdot)}(\Omega)$.

Moreover, $\Psi':W_0^{1,p(\cdot)}(\Omega)\to(W_0^{1,p(\cdot)}(\Omega))'$ is strictly monotone and verifies the (S_+) condition, that is, for every sequence $\{u_n\}\subset W_0^{1,p(\cdot)}(\Omega)$ such that $u_n\rightharpoonup u$ weakly in

$W_0^{1,p(\cdot)}(\Omega)$ *and*

$$\limsup_{n\to+\infty}\int_\Omega a(|\nabla u|^{p(x)}|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx\leqslant 0,$$

we have, passing to a subsequence if necessary, $u_n\to u$ *strongly in* $W_0^{1,p(\cdot)}(\Omega)$.

Proof. (i) Let $u\in W_0^{1,p(\cdot)}(\Omega)$, we have

$$\Psi(u)\leqslant\frac{\beta}{p^-}\rho_{p(\cdot)}(u)\leqslant\frac{\beta}{p^-}\max\{\|u\|_{p(\cdot)}^{p^+},\|u\|_{p(\cdot)}^{p^-}\}<+\infty.$$

By condition (a_0) is easy to see that Ψ is convex.

(ii) The proof that the functional Ψ is of class C^1 follows by standards arguments. Also, Ψ is weakly lower semicontinuous in $W_0^{1,p(\cdot)}(\Omega)$ (see [16, Corollary 3.9]). The operator Ψ' is strictly monotone, this follows from the fact that Υ is strictly convex (see [52, Lemma 15.4]). Hence Ψ is strictly convex and, therefore, Ψ' is strictly monotone.

Now we will prove that the operator Ψ' satisfies the condition (S_+).

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ such that $u_n\rightharpoonup u$ weakly in $W_0^{1,p(\cdot)}(\Omega)$ and

$$\limsup_{n\to+\infty}\int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\leqslant 0.$$

Hence, we obtain

$$\limsup_{n\to+\infty}\langle\Psi'(u_n)-\Psi'(u),u_n-u\rangle\leqslant 0. \tag{4.4}$$

On the other hand, by the convexity of Υ, we have

$$(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n-a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u))(\nabla u_n-\nabla u)\geqslant 0,$$

so, we obtain

$$\begin{aligned}
&\liminf_{n\to+\infty}\langle\Psi'(u_n)-\Psi'(u),u_n-u\rangle\\
&=\liminf_{n\to+\infty}\int_\Omega\left(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n-a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n-\nabla u)\,dx\\
&\geqslant 0.
\end{aligned} \tag{4.5}$$

Therefore, by (4.4) and (4.5), we conclude

$$\lim_{n\to+\infty}\int_\Omega\left(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n-a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n-\nabla u)\,dx=0, \tag{4.6}$$

which implies that there is a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, still denoted by itself, such that

$$(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n-a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u))(\nabla u_n-\nabla u)\to 0 \text{ as } n\to+\infty,$$

for a.e. $x\in\Omega$. Hence, by Lemma 4.6, we have $\nabla u_n\to\nabla u$ a.e. $x\in\Omega$, as $n\to+\infty$. Thus, since $a\in C(\mathbb{R}^+,\mathbb{R})$, by using the Lebesgue Dominated Convergence Theorem, it is proved that

$$\int_\Omega(a(|\nabla u_n|^{p(x)})-a(|\nabla u|^{p(x)}))|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\to 0. \tag{4.7}$$

By (4.6), (4.7), and $\alpha\leqslant a(\eta)$, we obtain

$$\begin{aligned}
\langle\Psi'(u_n)-\Psi'(u),u_n-u\rangle=&\int_\Omega(a(|\nabla u_n|^{p(x)})-a(|\nabla u|^{p(x)}))|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\\
&+\int_\Omega a(|\nabla u|^{p(x)})(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)dx\\
\geqslant&\, o_n(1)+\alpha\int_\Omega(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx.
\end{aligned}$$

Therefore, defining $S_n=(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)$, we have

$$\begin{aligned}
\limsup_{n\to+\infty}\int_\Omega S_n(x)\,dx=&\limsup_{n\to+\infty}\int_\Omega(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx\\
\leqslant&\,0.
\end{aligned} \tag{4.8}$$

Let us remember the following inequality, for all $x, y \in \mathbb{R}^N$,

$$|\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle| \geqslant \begin{cases} (p-1)\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if} \quad 1<p<2, \\ \frac{2^{3-p}}{p}|x-y|^p & \text{if} \quad p \geqslant 2. \end{cases}$$

Then, we obtain

$$S_n(x) \geqslant \begin{cases} (p^- -1)\frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p(x)}} & \text{if} \quad 1<p(x)<2, \\ \frac{2^{3-p^+}}{p^+}|\nabla u_n - \nabla u|^{p(x)} & \text{if} \quad p(x) \geqslant 2. \end{cases}$$

Consider the following sets

$$\Omega_+ = \{x \in \Omega : p(x) \geqslant 2\} \text{ and } \Omega_- = \{x \in \Omega : 1 < p(x) < 2\}.$$

(i) If $p(x) \geqslant 2$, we have

$$S_n(x) \geqslant C_p \int_{\Omega_+} |\nabla u_n - \nabla u|^{p(x)}\, dx \geqslant 0,$$

therefore, follows from (4.8) that

$$\lim_{n\to+\infty} \int_{\Omega_+} |\nabla u_n - \nabla u|^{p(x)}\, dx = 0. \tag{4.9}$$

(ii) If $1 < p(x) < 2$, by Hölder inequality, we obtain

$$\begin{aligned} \int_{\Omega_-} |\nabla u_n - \nabla u|^{p(x)}\, dx &= \int_{\Omega_-} \frac{|\nabla u_n - \nabla u|^{p(x)}}{(|\nabla u_n|+|\nabla u|)^{\frac{p(x)(2-p(x))}{2}}} \left((|\nabla u_n| + |\nabla u|)^{\frac{p(x)(2-p(x))}{2}} \right) dx \\ &\leqslant C \|I_1^n\|_{\frac{2}{p(\cdot)}} \|I_2^n\|_{\frac{2}{2-p(\cdot)}}, \end{aligned} \tag{4.10}$$

where C is a positive constant,

$$I_1^n = \frac{|\nabla u_n - \nabla u|^{p(x)}}{(|\nabla u_n| + |\nabla u|)^{\frac{p(x)(2-p(x))}{2}}}, \text{ and } I_2^n = (|\nabla u_n| + |\nabla u|)^{\frac{p(x)(2-p(x))}{2}}.$$

Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)}(\Omega)$, it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$.

Hence, we have

$$0 \leqslant \rho_{\frac{2}{2-p(\cdot)}}(I_2^n) = \int_{\Omega^-} |I_n^2|^{\frac{2}{2-p(\cdot)}}\, dx \leq K$$

for some positive constant K. We also have that

$$0 \leqslant \rho_{\frac{2}{p(\cdot)}}(I_1^n) = \int_{\Omega^-} |I_1^n|^{\frac{2}{p(x)}}\, dx \leqslant C \int_{\Omega} S_n(x) dx,$$

therefore, by using (4.8) and (4.10), we get

$$\lim_{n\to+\infty} \int_{\Omega_-} |\nabla u_n - \nabla u|^{p(x)}\, dx = 0. \tag{4.11}$$

Thus, from (4.9) and (4.11), we achieve that $u_n \to u$ strongly in $W_0^{1,p(\cdot)}(\Omega)$ as $n \to +\infty$. □

4.5 Properties of $\mathscr{A}$

The Lemma 4.8 was motivated by [54] and [66].

Lemma 4.8. *The operator $\mathscr{A}' : X \to X'$ is monotone.*

Proof. First of all, we will prove the following claim.

Claim 1. We have for all $\varrho, \zeta \in \mathbb{R}^N$, with $(\varrho, \zeta) \neq (0,0)$, that

(a)

$$\left(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho - a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta\right)(\varrho - \zeta) \geqslant c\left(\frac{1}{4}\right)^{p^+-2} |\varrho - \zeta|^{p(x)},$$

where $p(x) \geqslant 2$ and $x \in \Omega$.

(b)

$$\left(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho - a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta\right)(\varrho - \zeta) \geqslant c(|\zeta| + |\varrho|)^{p(x)-2}|\varrho - \zeta|^2,$$

where $1 < p(x) < 2$ and $x \in \Omega$.

Indeed. Let $\varrho, \zeta \in \mathbb{R}^N$ with $(\varrho, \zeta) \neq (0,0)$ and suppose, without loss of generality, that $|\varrho| \leqslant |\zeta|$. Then, we obtain

$$\begin{aligned}\big(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho\ &- a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta\big)\,(\varrho-\zeta)\\ &= \sum_{i=1}^{N}(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho_i - a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta_i)(\varrho_i-\zeta_i)\\ &= \sum_{i=1}^{N}(\varphi_i(x,\varrho)-\varphi_i(x,\zeta))(\varrho_i-\zeta_i),\end{aligned} \tag{4.12}$$

where $\varphi_i(x,w) = a(|w|^{p(x)})|w|^{p(x)-2}w_i$ for $i=1,\dots,N$, for all $w\in\mathbb{R}^N$. Moreover, note that

$$\varphi_i(x,\varrho)-\varphi_i(x,\zeta) = \sum_{j=1}^{N}\int_0^1 \frac{\partial\varphi_i(x,z)}{\partial z_j}(\varrho_j-\zeta_j)\,dt, \tag{4.13}$$

where $z = \zeta + t(\varrho-\zeta)$. Therefore, by (4.12) and (4.13), we have

$$\big(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho - a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta\big)\,(\varrho-\zeta) = \sum_{i,j=1}^{N}\int_0^1 \frac{\partial\varphi_i(x,z)}{\partial z_j}(\varrho_i-\zeta_i)(\varrho_j-\zeta_j)\,dt. \tag{4.14}$$

Let us fix $w\in\mathbb{R}^N\setminus\{0\}$ and define $\tau := |z|$. Then, as in [39], we achieve

$$\begin{aligned}\sum_{i,j=1}^{N}\frac{\partial\varphi_i(x,z)}{\partial z_j}w_iw_j &= (p(x)-2)\tau^{p(x)-4}a(\tau^{p(x)})\sum_{i,j=1}^{N}z_iw_iz_jw_j\\ &\quad + a(\tau^{p(x)})\tau^{p(x)-2}\sum_{i=1}^{N}w_i^2 + p(x)a'(\tau^{p(x)})\tau^{2p(x)-4}\sum_{i,j=1}^{N}z_iw_iz_jw_j\\ &= \frac{1}{\tau}\frac{\partial\left(a(\tau^{p(x)})\tau^{p(x)-2}\right)}{\partial\tau}\sum_{i,j=1}^{N}z_iw_iz_jw_j + a(\tau^{p(x)})\tau^{p(x)-2}\sum_{i=1}^{N}w_i^2\\ &= \frac{1}{\tau}\frac{\partial\left(a(\tau^{p(x)})\tau^{p(x)-2}\right)}{\partial\tau}\langle w,z\rangle^2 + a(\tau^{p(x)})\tau^{p(x)-2}|w|^2\\ &= |w|^2\left(\tau\frac{\partial\left(a(\tau^{p(x)})\tau^{p(x)-2}\right)}{\partial\tau}\left\langle\frac{w}{|w|},\frac{z}{\tau}\right\rangle^2 + a(\tau^{p(x)})\tau^{p(x)-2}\right).\end{aligned} \tag{4.15}$$

Setting $\lambda = \left\langle \frac{w}{|w|}, \frac{z}{\tau} \right\rangle^2$ and by using (a_2), we have

$$\begin{aligned}
\lambda\left(\tau\frac{\partial\big(a(\tau^{p(x)})\tau^{p(x)-2}\big)}{\partial\tau}\right) + a(\tau^{p(x)})\tau^{p(x)-2} &= (1-\lambda)a(\tau^{p(x)})\tau^{p(x)-2} \\
&\quad +\lambda\left(\tau\frac{\partial\big(a(\tau^{p(x)})\tau^{p(x)-2}\big)}{\partial\tau} + a(\tau^{p(x)})\tau^{p(x)-2}\right) \\
&\geqslant c\tau^{p(x)-2}.
\end{aligned} \tag{4.16}$$

Thus, by (4.15) and (4.16), we get

$$\sum_{i,j=1}^{N} \frac{\partial\varphi_i(x,z)}{\partial z_j} w_i w_j \geqslant c\tau^{p(x)-2}|w|^2. \tag{4.17}$$

Then, as $z = \zeta + t(\varrho - \zeta)$ and by considering $w = \varrho - \zeta$, we obtain from (4.14) and (4.17) that

$$\big(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho - a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta\big)(\varrho-\zeta) \geqslant c\int_0^1 |\zeta + t(\varrho-\zeta)|^{p(x)-2}|\varrho-\zeta|^2 dt. \tag{4.18}$$

(a) Assume $p(x) \geqslant 2$. Thus, since that $|\varrho| \leqslant |\zeta|$, for all $t \in [0, 1/4]$, we get

$$|\zeta + t(\varrho - \zeta)| \geqslant |\zeta| - \frac{|\zeta - \varrho|}{4} \geqslant \frac{|\zeta - \varrho|}{4}. \tag{4.19}$$

Then, by (4.18) and (4.19), we deduce that

$$\begin{aligned}
\big(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho &- a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta\big)(\varrho-\zeta) \\
&\geqslant c\int_0^{\frac{1}{4}} |\zeta + t(\varrho-\zeta)|^{p(x)-2}|\varrho-\zeta|^2\, dt \\
&\geqslant c\left(\frac{1}{4}\right)^{p^+-2} |\varrho-\zeta|^{p(x)}.
\end{aligned}$$

(b) Assume $1 < p(x) < 2$. Note that $|t\varrho + (1-t))\zeta| \leqslant |\varrho| + |\zeta|$, for all $t \in [0,1]$, it follows that

$$|\zeta + t(\varrho - \zeta)|^{p(x)-2} = |t\varrho + (1-t))\zeta|^{p(x)-2} \geqslant (|\varrho| + |\zeta|)^{p(x)-2}. \tag{4.20}$$

Therefore, by (4.18) and (4.20), we obtain

$$\begin{aligned}\big(a(|\varrho|^{p(x)})|\varrho|^{p(x)-2}\varrho\ -a(|\zeta|^{p(x)})|\zeta|^{p(x)-2}\zeta\big)\,(\varrho-\zeta)&\\ \geqslant c\int_0^1 |\zeta+t(\varrho-\zeta)|^{p(x)-2}|\varrho-\zeta|^2 dt&\\ \geqslant c(|\zeta|+|\varrho|)^{p(x)-2}|\varrho-\zeta|^2.&\end{aligned}$$

Thus, we conclude the prove of **Claim 1.**

Let us consider $\varrho = \nabla u$ and $\zeta = \nabla v$. We obtain of the **Claim 1** that

$$\begin{aligned}\langle \mathscr{A}'(u)-\mathscr{A}'(v), u-v\rangle &= \int_\Omega \big(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)}\nabla u - a(|\nabla v|^{p(x)})|\nabla v|^{p(x)}\nabla v\big)\,(\nabla u-\nabla v)\,dx\\ &\geqslant 0.\end{aligned}$$

□

Lemma 4.9. *Assume* $(a_0)-(a_2)$*. Then, the operator* $\mathscr{A}' : X \to X'$ *verifies the following condition: for every sequence* $\{u_n\}_{n\in\mathbb{N}}$ *in* X *such that* $u_n \rightharpoonup u$ *weakly as* $n\to+\infty$ *in* X *and*

$$\limsup_{n\to+\infty}\langle \mathscr{A}'(u_n), u_n-u\rangle \leqslant 0, \tag{4.21}$$

we have $u_n \to u$ *strongly in* $W_0^{1,p(\cdot)}(\Omega)$ *as* $n\to+\infty$.

Proof. We have that $\mathscr{A}'(u)$ is a continuous linear functional, so, it follows that

$$\lim_{n\to+\infty}\langle \mathscr{A}'(u), u_n-u\rangle = 0.$$

Therefore, by (4.21), we obtain

$$\limsup_{n\to+\infty}\langle \mathscr{A}'(u_n)-\mathscr{A}'(u), u_n-u\rangle \leqslant 0. \tag{4.22}$$

Then, by using the Lemma 4.8 and (4.22), we obtain

$$\lim_{n\to+\infty}\langle \mathscr{A}'(u_n)-\mathscr{A}'(u), u_n-u\rangle = 0,$$

that is,

$$\lim_{n\to+\infty}\int_\Omega \left(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n - a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)dx = 0. \tag{4.23}$$

Claim I. We affirm that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n - \nabla u|^{p(x)}\,dx = 0,$$

which implies that $\{u_n\}_{n\in\mathbb{N}}$ converges stronlgy to u in $W_0^{1,p(\cdot)}(\Omega)$.

In fact. Note that

$$\int_\Omega |\nabla u_n - \nabla u|^{p(x)}\,dx = \int_{\Omega_+^p} |\nabla u_n - \nabla u|^{p(x)}\,dx + \int_{\Omega_-^p} |\nabla u_n - \nabla u|^{p(x)}\,dx,$$

where $\Omega_+^p = \{x\in\Omega : p(x)\geqslant 2\}$ and $\Omega_-^p = \{x\in\Omega : 1 < p(x) < 2\}$.

If $p(x)\geqslant 2$, for all $x\in\Omega$, then, by **Claim 1** (a) of the Lemma 4.8, we have

$$\begin{aligned}\int_\Omega \left(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\right. & \left.-a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx \\ & \geqslant C\int_{\Omega_+^p} |\nabla u_n - \nabla u|^{p(x)}\,dx.\end{aligned}$$

Therefore, by (4.23), we conclude that

$$\lim_{n\to+\infty}\int_{\Omega_+^p} |\nabla u_n - \nabla u|^{p(x)}\,dx = 0. \tag{4.24}$$

On the other hand, if $1 < p(x) < 2$, for all $x\in\Omega$, then, by **Claim 1** (b) of the Lemma 4.8, we get

$$\begin{aligned}\int_\Omega \left(a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)-2}\nabla u_n\right. & \left.-a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx \\ & \geqslant \int_{\Omega_-^p} \sigma^{p(x)-2}|\nabla u_n - \nabla u|^2\,dx,\end{aligned}$$

where $\sigma(x) = C\left(|\nabla u_n(x)| + |\nabla u(x)|\right)$. Hence, by (4.23), we obtain

$$\lim_{n\to+\infty}\int_{\Omega_-^p} \sigma^{p(x)-2}|\nabla u_n - \nabla u|^2\,dx = 0. \tag{4.25}$$

But, by the Hölder's inequality (Lemma 4.1) and by the Lemma 4.2, we get

$$\begin{aligned}
\int_{\Omega^p_-} |\nabla u_n - \nabla u|^{p(x)}\,dx &= \int_{\Omega^p_-} \sigma^{\frac{p(x)(2-p(x))}{2}} \left(\sigma^{\frac{p(x)(p(x)-2)}{2}} |\nabla u_n - \nabla u|^{p(x)}\right)\,dx \\
&\leqslant C \|\sigma^{\frac{p(x)(2-p(x))}{2}}\|_{L^{\frac{2}{2-p(\cdot)}}(\Omega^p_-)} \|\sigma^{\frac{p(x)(p(x)-2)}{2}} |\nabla u_n - \nabla u|^{p(x)}\|_{L^{\frac{2}{p(\cdot)}}(\Omega^p_-)} \\
&\leqslant C \max\left\{ \|\sigma\|_{L^{p(\cdot)}(\Omega^p_-)}^{\left[\frac{p(x)(2-p(x))}{2}\right]^-}, \|\sigma\|_{L^{p(\cdot)}(\Omega^p_-)}^{\left[\frac{p(x)(2-p(x))}{2}\right]^+} \right\} \\
&\quad \max\left\{ \left(\int_{\Omega^p_-} \sigma^{p(x)-2} |\nabla u_n - \nabla u|^2\,dx\right)^{\frac{p^-}{2}}, \right. \\
&\quad \left. \left(\int_{\Omega^p_-} \sigma^{p(x)-2} |\nabla u_n - \nabla u|^2\,dx\right)^{\frac{p^+}{2}} \right\}.
\end{aligned}$$

Therefore, since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ and by using (4.25), we achieve

$$\lim_{n\to+\infty} \int_{\Omega^p_-} |\nabla u_n - \nabla u|^{p(x)}\,dx = 0. \tag{4.26}$$

Thus, by (4.24), (4.26), we get

$$\lim_{n\to+\infty} \int_{\Omega} |\nabla u_n - \nabla u|^{p(x)}\,dx = 0,$$

which concludes the **Claim I**.

Note that, by the Proposition 4.3 and **Claim I**, we have

$$\lim_{n\to+\infty} \|\nabla u_n - \nabla u\|_{p(\cdot)} = 0.$$

□

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