UNIVERSIDADE FEDERAL DE SÃO CARLOS
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Integral Closure, Whitney Stratification, and Bi-Lipschitz Invariants on Toric Varieties

Amanda Santos Araújo

São Carlos-SP
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Amanda Santos Araújo

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aos meus pais e ao meu irmão,
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Resumo

Neste trabalho investigamos a interação entre estruturas combinatórias, propriedades algébricas, geométricas e aspectos métricos na Teoria das Singularidades, tendo como cenário principal as variedades tóricas afins. O objetivo central é desenvolver uma abordagem que permita descrever invariantes e propriedades de singularidades a partir de dados combinatórios associados a semigrupos e poliedros de Newton. Inicialmente, estudamos o fecho integral de ideais não degenerados em anéis de funções analíticas associados a variedades tóricas afins. Mostramos que, sob condições adequadas de não degenerescência em relação ao poliedro de Newton, o fecho integral desses ideais admite uma descrição combinatória precisa, estendendo resultados clássicos conhecidos no caso regular. Em seguida, investigamos superfícies tóricas em $\mathbb{C}^3$ e construímos uma estratificação de Whitney adaptada à sua estrutura combinatória. Demonstramos que, sob condições apropriadas, é possível obter uma estratificação com número reduzido de estratos, preservando as condições de regularidade de Whitney. Por fim, analisamos a invariância bi-Lipschitz de invariantes locais de singularidades. Estabelecemos condições algébricas que garantem a invariância do expoente de Łojasiewicz e da obstrução de Euler local. Os resultados obtidos evidenciam que variedades tóricas constituem um ambiente particularmente adequado para traduzir fenômenos analíticos, algébricos e métricos em termos combinatórios, fornecendo novas ferramentas para o estudo sistemático de singularidades em contextos mais gerais.

Palavras-chave Variedades tóricas afins; fecho integral; poliedros de Newton; estratificações de Whitney; invariantes bi-Lipschitz.

Abstract

This work, we investigate the interation between combinatorial structures, algebraic and geometric properties, and metric aspects in Singularity Theory, with affine toric varieties as the main setting. The central objective is to develop an approach that allows one to describe invariants and properties of singularities using combinatorial data associated with semigroups and Newton polyhedra. First, we study the integral closure of non-degenerate ideals in rings of analytic functions associated with affine toric varieties. We show that, under suitable non-degeneracy conditions with respect to the Newton polyhedron, the integral closure of such ideals admits a precise combinatorial description, extending classical results known in the regular case. Next, we investigate toric surfaces in $\mathbb{C}^3$ and construct a Whitney stratification adapted to their combinatorial structure. We prove that, under appropriate conditions, it is possible to obtain a stratification with a reduced number of strata while preserving Whitney regularity conditions. Finally, we analyze the bi-Lipschitz invariance of local invariants of singularities. We establish algebraic conditions ensuring the invariance of the Łojasiewicz exponent and the local Euler obstruction. The obtained results show that toric varieties constitute a particularly suitable setting to translate analytic, algebraic, and metric phenomena into combinatorial terms, providing new tools for the systematic study of singularities in more general contexts.

Keywords: Affine toric varieties; integral closure; Newton polyhedra; Whitney stratifications; bi-Lipschitz invariants.

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List of Symbols

$\mathbb{Z}$: Integer numbers;

$\mathbb{C}$: Complex numbers;

$\mathbb{R}$: Real numbers;

$\mathbb{T}$: Algebraic n-dimensional torus;

$\mathbb{C}\{x_1,\dots,x_n\}$: Ring of convergent complex power series in n variables;

$\mathscr{O}_{X,x}$: Ring of germs of analytic functions in a neighborhood of $x \in X$;

$\mathscr{O}_X$: Ring of germs of analytic functions in a neighborhood of $0 \in X$;

$\mathscr{O}_n$: Local ring of germs of holomorphic functions defined in a neighborhood of $0 \in \mathbb{C}^n$;

$\bar{I}$: The integral closure of ideal I;

$\mathscr{L}_J(I)$: Łojasiewicz of I with respect J;

$e(I)$: Hilbert–Samuuel multiplicity of I;

$\ell(\mathscr{O}_X/I^k\mathscr{O}_X)$: the colength of I^k;

$X(S)$: Affine toric variety generated by S;

$cone(S)$: convex polyhedral cone associated with S;

$\widetilde{cone}(S)$: dual cone of $cone(S)$;

$\Gamma_+(I)$: Newton polyhedron of I.

Introduction

Singularity theory aims to understand the behavior of objects at points where regularity breaks down. Such points arise naturally in various mathematical contexts, including the study of algebraic varieties, problems in analysis, and differential systems. Although they correspond to singular points, singularities often concentrate essential information about the local and global geometry of the underlying space.

The study of equisingularities constitutes one of the main approaches to understanding the behavior of singularities in families of varieties, seeking to identify conditions under which the type of singularity remains stable along the parameters. In parallel, the analysis of singularities also involves the decomposition of singular spaces into geometrically more regular pieces.

From this perspective, Whitney stratifications constitute a fundamental tool. Introduced by Whitney [53], such stratifications allow one to decompose a singular space into a finite family of smooth manifolds, arranged so as to satisfy regularity conditions that control the interaction between the strata. This approach enables a more refined analysis of the structure of singularities and plays a central role in problems of equisingularity, stability, and topological invariance.

In addition to geometric approaches based on stratifications, algebraic tools also play a fundamental role. Among them, the theory of integral closure stands out. This tool has been widely used in the formulation and study of equisingularity conditions in families of analytic varieties. In particular, Teissier [49] showed that relations of integral dependence between ideals allow one to characterize equisingularity conditions in families of hypersurface germs. Subsequently, Gaffney [26] developed an interpretation of Whitney conditions in terms of the integral closure of modules. These results reveal a deep connection between algebraic structures and geometric properties, showing that techniques from commutative algebra can be used to control the geometric behavior of families of analytic varieties.

Despite its relevance, the explicit determination of the integral closure of an ideal is, in general, a delicate problem. There are, however, particular situations in which this computation admits a precise combinatorial description. A classical example occurs in the case of monomial ideals, where the integral closure can be characterized in terms of the associated Newton polyhedron. More precisely, the integral closure is generated by the monomials whose exponents belong to the Newton polyhedron of the ideal.

This relationship between integral closure and Newton polyhedra was later extended by Saia [48], who characterized a more general class of ideals in $\mathcal{O}_n$ whose integral closure can be described in terms of the combinatorial structure of the Newton polyhedron. The ideals in this class are known as Newton non-degenerate ideals. The motivation for this result is related to the work of Yoshinaga [54] on analytic functions that are non-degenerate in the sense of Kouchnirenko [36], in which the Newton polyhedron plays a central role in the description of local invariants.

It is worth noting that Newton polyhedra constitute a fundamental tool in singularity theory, establishing a natural bridge between convex geometry and analytic invariants. Works by authors such as Oka [43], Wall [52], Fukui [23] and Bivià-Ausina [6] show that the combinatorial structure of these polyhedra provides deep information about invariants of singularities.

In addition to algebraic and geometric perspectives, singularity theory can also be approached from a metric point of view. In this context, one seeks to understand which properties of a singular space remain invariant under transformations that preserve distances up to multiplicative constants, known as bi-Lipschitz equivalences. These equivalences provide an intermediate notion between topological equivalence and analytic equivalence, allowing singularities to be compared from a more flexible geometric perspective.

A central question in this context is the metric version of Zariski's multiplicity conjecture [55], which asks whether multiplicity is preserved under bi-Lipschitz homeomorphisms. Although this conjecture remains open in general, several partial results have revealed a strong interaction between algebraic invariants and metric properties, as evidenced by works of L. Birbrair, J. F. Bobadilla, C. Bivià-Ausina, A. Fernandes, T. Fukui, Z. Jelonek, M. Pe Pereira and J. E. Sampaio, among others.

Another invariant that has attracted significant interest in this context is the local Euler obstruction, introduced by MacPherson [39] in the study of Chern classes for singular varieties. This invariant provides a refined measure of the topological complexity of a singularity and is closely related to polar multiplicities. Although it is known that the Euler obstruction is an analytic invariant, its invariance under bi-Lipschitz equivalences remains, to a large extent, an open problem.

Motivated by the interplay between the algebraic, geometric, and metric perspectives discussed above, it is natural to seek classes of singular spaces in which these structures can be analyzed more explicitly. In this context, affine toric varieties arise not merely as a convenient class of examples, but as a structural setting in which the relationship between algebra, geometry, and combinatorics can be effectively explored.

In general, an affine toric variety is constructed from a finitely generated semigroup $S \subset \mathbb{Z}^n$, and its algebraic and geometric structure is strongly determined by the convex cone generated by S. Moreover, these varieties are characterized by the existence of an action of the algebraic torus $(\mathbb{C}^*)^n$, which has a dense orbit and whose dynamics directly reflect the combinatorial data associated with the semigroup. This interplay between the torus action and the geometry of the cone provides a particularly rich and explicit description of the variety.

The space $\mathbb{C}^n$ constitutes a particular case of this construction, in which the associated semigroup is generated by the canonical basis of $\mathbb{R}^n$. In this context, many classical results in singularity theory already incorporate, in an implicit way, arguments of a combinatorial nature.

More generally, affine toric varieties form a substantially broader class, including singular examples. Extending results from the regular case to this setting therefore requires new approaches capable of dealing with the difficulties introduced by singularities. In particular, it becomes essential to develop tools that allow combinatorial and algebraic methods to be adapted to the study of these varieties in greater generality.

In this direction, one is led to ask to what extent the tools from the regular case, in particular in the local ring $\mathscr{O}_n$, can be extended to $\mathscr{O}_{X(S)}$. In this context, properties such as integral closure, Newton non-degeneracy conditions, and invariants of singularities no longer admit such explicit descriptions.

The main objective of this thesis is to develop an approach that essentially integrates combinatorial structures, algebraic properties, and metric aspects in singularity theory. More precisely, we prove that, in the context of affine toric varieties, the combinatorics associated with semigroups and Newton polyhedra can describe the integral closure of ideals, construct adapted stratifications, and analyze the bi-Lipschitz invariance of some local invariants in singularity theory.

The main contributions of this thesis can be summarized as follows:

- We study the integral closure of non-degenerate ideals in rings of analytic functions associated with affine toric varieties. We show that, under suitable non-degeneracy conditions with respect to the Newton polyhedron, the integral closure of these ideals admits a precise combinatorial description, extending classical results.

- We construct a Whitney stratification on non-normal toric surfaces in $\mathbb{C}^3$, adapted to their combinatorial structure and with a reduced number of strata.

- We establish algebraic conditions that ensure the bi-Lipschitz invariance of local invariants, such as the Łojasiewicz exponent and the local Euler obstruction.

This thesis is organized as follows.

Chapter 1. In this chapter we present of the preliminary material required throughout the thesis. We recall the basic notions of germs of analytic spaces and introduce the theory of integral closure of ideals, together with its main properties. We also discuss the Łojasiewicz exponent and its role as a measure of the complexity of singularities. Furthermore, we review Whitney stratifications and their fundamental properties, as well as the notion of bi-Lipschitz equivalence and the local Euler obstruction. We conclude the chapter with an introduction to affine toric varieties, emphasizing their combinatorial structure and the role they play in the developments of this work.

Chapter 2. In this chapter, we study the integral closure of ideals in rings of analytic functions defined on affine toric varieties. After introducing Newton polyhedra and the notion of non-degeneracy, we establish preliminary results relating non-degeneracy and toroidal embeddings, highlighting how the combinatorial structure of the underlying semigroup influences algebraic properties of ideals. We then establish one of the main results of this chapter (see Corollary 2.27), which shows that if $I \subset \mathscr{O}_{X(S)}$ is a non-degenerate ideal, then its integral closure is completely determined by its Newton polyhedron. More precisely, the integral closure can be described as the ideal generated by all monomials whose exponents lie in the Newton polyhedron of I, that is,

$$\bar{I} = \langle x^k : k_1 b_1 + \cdots + k_r b_r \in \Gamma_+(I) \rangle.$$

This result extends classical descriptions known in the regular case to the toric setting and provides a precise combinatorial characterization of the integral closure in this context.

Chapter 3. In this chapter we study toric surfaces in $\mathbb{C}^3$ and the construction of Whitney stratifications adapted to their combinatorial structure. Using the semigroup description of these surfaces, we first analyze the stratification induced by the torus action and its relation to the orbit–cone correspondence. Within this framework, we prove that, under suitable conditions on the parameters defining the surface (see Theorem 3.4), the pair of strata $(\mathrm{Reg}(V(I)), \{0\} \times \{0\} \times \mathbb{C})$ satisfies the Whitney conditions. Building on this result, we obtain a simplified description of the stratification. More precisely, Corollary 3.8 shows that if $V(I) \subset \mathbb{C}^3$ is the toric surface defined by $I = \langle x^p z^r - y^q \rangle$, with $\gcd(q,r) = 1$, $r > q > 1$, and $p > q$, then the set

$$\{\mathrm{Reg}(V(I)), \mathbb{C}^* \times \{0\} \times \{0\}, \{0\} \times \{0\} \times \mathbb{C}\}$$

defines a Whitney stratification of $V(I)$.

This result shows that Whitney regularity is compatible with a stratification that is strictly simpler than the one induced by the torus action, reducing the number of strata while preserving the required regularity conditions.

Chapter 4. In this chapter, we investigate the bi-Lipschitz invariance of local invariants of singularities in the setting of germs of affine toric varieties. Based on the work of Bivià-Ausina and Fukui [8], whose arguments also serve as a foundation for our proofs, we provide algebraic conditions that guarantee their preservation. While their results concern ideals in $\mathscr{O}_n$, we extend the setting to ideals in $\mathscr{O}_{X(S)}$, where $(X(S),0)$ is a germ of an affine toric variety associated with a semigroup S. In this framework, we prove that bi-Lipschitz equivalent ideals have the same order and, in the case of finite colength, the same Łojasiewicz exponent (see Theorem 4.1). We also investigate the bi-Lipschitz invariance of the local Euler obstruction. In Theorem 4.8, we show that this invariant is preserved for hypersurfaces with isolated singularities under suitable non-degeneracy conditions. Altogether, these results reveal a strong interplay between algebraic, combinatorial, and metric aspects of singularity theory.

Appendix A. In this appendix, we present computational implementations that support the theoretical results developed in the thesis. Using Oscar.jl, we provide explicit codes for the examples discussed in each chapter, as well as auxiliary routines used in the computations.

CHAPTER 1

Preliminaries

The purpose of this chapter is to present the notations and conventions adopted throughout the text. Preliminary results essential for reading are also provided. We review basic notions of analytic geometry and commutative algebra, including germs of analytic spaces, integral closure of ideals, the Łojasiewicz exponent, Whitney stratifications and Verdier's condition, bi-Lipschitz equivalence, the local Euler obstruction, and affine toric varieties. These concepts play a central role in the subsequent chapters. For more details on these topics see [4], [42], and [32].

1.1 Germs of Analytic Spaces

Following [18], we begin our study with a very important concept, the concept of germ.

Definition 1.1. Let X be a topological space and consider $x \in X$. We say that two subsets $Y_1, Y_2 \subseteq X$ define the same *germ at x* if there exists an open neighborhood U of x in X, such that

$$Y_1 \cap U = Y_2 \cap U.$$

This defines an equivalence relation on the set of subsets of X. The equivalence class we call the *germ at x* and the germ of $Y \subset X$ is denoted by (Y,x). Moreover, we say that a germ (Z,x) is contained in a germ (Y,x) (denoted by $(Z,x) \subset (Y,x)$) if there exist representatives Z of (Z,x) and Y of (Y,x) such that $Z \subset Y$. We define $(Y,x) \cap (Y',x) = (Y \cap Y',x)$ and $(Y,x) \cup (Y',x) = (Y \cup Y',x)$. Note that these operations do not depend on the choice of representatives.

Definition 1.2. Let (X,x) and (Y,y) be two germs of topological spaces. A *germ of map* $f : (X,x) \to (Y,y)$ is defined as an equivalence class of maps $f : X \to Y$, with $f(x) = y$, and where X and Y are representatives of (X,x) and (Y,y) respectively. Two such maps $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ are called *equivalent* if they coincide on an open neighborhood of x contained in $U_1 \cap U_2$.

A germ of a homeomorphism (resp. injective germ, isomorphism germ, etc.) is a germ that admits a representative which is a homeomorphism (resp. injective, isomorphism, etc.).

We denote by $\mathscr{O}_{\mathbb{C}^n,x}$ the local ring of germs of holomorphic functions defined in a neighborhood of $x \in \mathbb{C}^n$. We assume the classical results establishing the equivalence between holomorphic functions of several variables defined in a neighborhood of $0 \in \mathbb{C}^n$ and the elements of the ring of convergent power series $\mathbb{C}\{x_1,\dots,x_n\}$ (see [31, p. 66]). In this setting, we denote by $\mathscr{O}_n := \mathscr{O}_{\mathbb{C}^n,0}$.

Theorem 1.3. *The ring $\mathscr{O}_n$ of germs of holomorphic functions of n variables is isomorphic to the ring of convergent complex power series in n variables $\mathbb{C}\{x_1,\dots,x_n\}$.*

Thus, we will denote a holomorphic function or a convergent series in the same way.

Definition 1.4. A space $X \subset \mathbb{C}^n$ is said to be *analytic* if, for every point $p \in X$, there exists an open neighborhood V_p of p in $\mathbb{C}^n$ and finitely many holomorphic functions $f_1,\dots,f_s$ defined on V_p such that

$$X \cap V_p = \{x \in V_p : f_1(x) = \cdots = f_s(x) = 0\}.$$

We will use the notation $V(f_1,\dots,f_s) := \{x \in V_p : f_1(x) = \cdots = f_s(x) = 0\}$ to denote the zeros of $f_1,\dots,f_s$ in V_p. A germ of an analytic space (X,x) is a germ at x of an analytic subset of $\mathbb{C}^n$. Moreover, let $I = (f_1,...,f_s) \subset \mathscr{O}_{\mathbb{C}^n,x}$ be an ideal. Then we define the germ of the analytic space (X,x) by

$$(X,x) = (V(I),x).$$

Note that $(V(I),x) = \bigcap_{i=1}^{s}(V(f_i),x)$. This definition is independent of the choice of generators f_i of I.

Let (X,x) be a germ of an analytic space. Then we define the ideal $\mathscr{I}(X,x)$ of (X,x) by

$$\mathscr{I}(X,x) = \{f \in \mathscr{O}_{\mathbb{C}^n,x} : (X,x) \subset (V(f),x)\}.$$

Definition 1.5. Let $(X,x) \subset (\mathbb{C}^n,x)$ be a germ of an analytic space. A *germ of an analytic function* $f : (X,x) \longrightarrow (\mathbb{C},y)$ is a germ of a map $f : X \to \mathbb{C}$ such that some representative of f is the restriction to X of an analytic function defined on an open neighborhood of x in $\mathbb{C}^n$.

The germs of analytic functions on (X,x) form a $\mathbb{C}$-algebra, denoted by $\mathscr{O}_{X,x}$, which is called the *ring of germs of analytic functions* on (X,x).

In particular, when $x = 0$, we shall denote $\mathscr{O}_{X,0}$ simply by $\mathscr{O}_X$.

A *germ of an analytic map* (or simply a *map*) $\varphi = (f_1,\dots,f_m) : (X,x) \longrightarrow (Y,y) \subset (\mathbb{C}^m,y)$ is defined analogously.

The map $\varphi : (X,x) \to (Y,y)$ is called an *isomorphism* if there exists an analytic map $\psi : (Y,y) \longrightarrow (X,x)$ such that

$$\psi \circ \varphi = \mathrm{id}_{(X,x)} \quad \text{and} \quad \varphi \circ \psi = \mathrm{id}_{(Y,y)}.$$

In this case, the germs (X,x) and (Y,y) are said to be *isomorphic*.

To better understand the structure of analytic germs and how analytic mappings act on them, we list some elementary but useful facts in the next lemma (see [18]).

Lemma 1.6. *(i) Let $(X,x) \subset (\mathbb{C}^n,x)$ be a germ of an analytic space, and let $\mathscr{I}(X,x)$ be the ideal of X. Then*

$$\mathscr{O}_{X,x} = \mathscr{O}_{\mathbb{C}^n,x}/\mathscr{I}(X,x).$$

(ii) Let $(X,x) \subset (\mathbb{C}^n,x)$ be a germ of a submanifold. Then $\mathscr{O}_{X,x} \cong \mathbb{C}\{x_1,\ldots,x_k\}$ for some k.

(iii) Let $\varphi : (X,x) \longrightarrow (Y,y)$ be a germ of an analytic mapping. Then, by composition, φ induces a homomorphism of $\mathbb{C}$-algebras $\varphi^* : \mathscr{O}_{Y,y} \longrightarrow \mathscr{O}_{X,x}, \quad f \longmapsto f \circ \varphi$. Moreover, for two germs of analytic mappings φ and ψ, $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$.

1.2 The integral closure of ideals

Integral closure has occupied a significant place in number theory and algebraic geometry since the nineteenth century, and its modern formulation for ideals originates in the work of Krull and Zariski in the 1930s. The integral closure of ideals refers to the set of elements that are roots of polynomials with coefficients in the respective ideals, enabling the study of their algebraic relations and extensions. This concept is fundamental in commutative algebra, particularly in the analysis of non-normal rings and algebraic geometry.

The references for this section are [33] and [38].

Definition 1.7. Let I be an ideal in a ring R. An element $h \in R$ is said to be *integral* over I if there exist an integer n and elements $a_i \in I^i$, $i = 1,\ldots,n$, such that

$$h^n + a_1 h^{n-1} + \cdots + a_n = 0.$$

Such an equation is called an equation of integral dependence of h over I (of degree n). The set of all elements that are integral over I is called the *integral closure* of I, and is denoted by $\overline{I}$. If $\overline{I} = I$, then I is called integrally closed. If $I \subseteq J$ are ideals, we say that J is integral over I if $J \subseteq \overline{I}$.

A basic example is the following:

Example 1.8. For arbitrary elements $x,y \in R$, $xy \in \overline{(x^2,y^2)}$. In fact, take $n = 2$, $a_1 = 0 \in (x^2,y^2)$ and $a_2 = -x^2y^2 \in (x^2,y^2)^2$, notice that

$$(xy)^2 + a_1(xy) + a_2 = 0$$

is an equation of integral dependence of xy over (x^2,y^2).

In the following we present basic properties of the integral closure of ideals.

Properties 1.9. *Let R be a ring and $I \subseteq R$ an ideal. Then,*

1. $I \subseteq \overline{I}$;

2. *If $I \subseteq J$ are ideals of R, then $\overline{I} \subseteq \overline{J}$;*

3. $\overline{I} \subset \sqrt{I}$;

4. *The nilradical $\sqrt{0}$ of the ring is contained in $\overline{I}$;*

5. *Intersections of integrally closed ideals are integrally closed.*

6. *The following property is called **persistence**: If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(\overline{I}) \subseteq \overline{\varphi(I)S}$;*

7. *Another important property is **contraction**: If $\varphi : R \to S$ is a ring homomorphism and I is an integrally closed ideal of S, then $\varphi^{-1}(I)$ is integrally closed on R.*

8. *In particular, if R is a subring of S, and I an integrally closed ideal of S, then $I \cap R$ is an integrally closed ideal in R.*

Proof. We prove only items (6) and (7), since the remaining statements are straightforward and follow from the definitions.

(6) If $h \in \overline{I}$, then $h^n + a_1 h^{n-1} + \cdots + a_n = 0$, with $a_i \in I^i$ and $n \in Z^+$. Thus,

$$\varphi(h)^n + \varphi(a_1)\varphi(h)^{n-1} + \cdots + \varphi(a_n) = 0,$$

with $\varphi(a_i) \in (\varphi(I)S)^i$. Therefore, $\varphi(h) \in \overline{\varphi(I)S}$.

(7) If $h \in \overline{I}$, we have the integral dependence equation

$$h^n + a_1 h^{n-1} + \cdots + a_n = 0,$$

where $a_i \in (\varphi^{-1}(I))^i$. It follows that

$$\varphi(h)^n + \varphi(a_1)\varphi(h)^{n-1} + \cdots + \varphi(a_n) = 0,$$

where $\varphi(a_i) \in I^i$. We see that $\varphi(h) \in \overline{I}$. Therefore $h \in \varphi^{-1}(I)$, since I is integrally closed. □

The following equality rephrase integral closure.

Proposition 1.10. *[33, Proposition 1.1.7] Let R be a ring, I an ideal of R and $r \in R$. Then: $r \in \overline{I}$ if and only if there exists an integer n such that $(I + (r))^n = I(I + (r))^{n-1}$.*

One can also use modules to express integral dependence.

Corollary 1.11. *[33, Corollary 1.1.8] Let I be an ideal of R and $r \in R$. Then, the following are equivalent:*

(a) r is integral over I;

(b) There exists a finitely generated R-module M such that $rM \subseteq IM$ *and such that whenever* $aM = 0$ *for some* $a \in R$*, then* $ar \in \sqrt{0}$.

Moreover, if I is finitely generated and contains a non-zerodivisor, r is integral over I if and only if there exists a finitely generated faithful R-module M such that $IM = (I + (r))M$.

Now, we introduce reductions, which are an extremely useful tool for integral closure in general.

Definition 1.12. Let $J \subseteq I$ be ideals in R. We say that J is a reduction of I if there exists a non-negative integer n such that $I^{n+1} = JI^n$.

Proposition 1.10 proved the following:

Corollary 1.13. *[33, Corollary 1.2.2] An element* $r \in R$ *is integral over J if and only if J is a reduction of* $J + (r)$.

Note that if $JI^n = I^{n+1}$, then for all positive integer m, we have $I^{m+n} = JI^{m+n-1} = \cdots = J^m I^n$. In particular, if $J \subseteq I$ is a reduction, there exists an integer n such that for all $m \geqslant 1$, $I^{m+n} \subseteq J^m$.

The reduction property is transitive:

Proposition 1.14. *[33, Proposition 1.2.4] Let* $K \subseteq J \subseteq I$ *be ideals in R. Then,*

(i) If K is a reduction of J and J is a reduction of I, then K is a reduction of I.

(ii) If K is a reduction of I, then J is a reduction of I.

(iii) If I is finitely generated, $J = K + (r_1, \ldots, r_k)$*, and K is a reduction of I, then K is a reduction of J.*

In case the ideals are not finitely generated, the conclusion of Proposition 1.14 (item (iii)) need not hold.

The following corollary establishes a connection between integral closure and reductions.

Corollary 1.15. *[33, Corollary 1.2.5] Let* $K \subseteq I$ *be ideals in R. Assume that I is finitely generated. Then K is a reduction of I if and only if* $I \subseteq \overline{K}$.

Using reductions, one obtains that the integral closure $\overline{I}$ is an ideal, as stated in the following corollary.

Corollary 1.16. *[33, Corollary 1.3.1] The integral closure of an ideal in a ring is an integrally closed ideal (in the same ring), i.e.,* $\overline{\overline{I}} = \overline{I}$.

The following lemma connects the notion of integral dependence on an ideal, with the classical notion of integral dependence on a ring.

Lemma 1.17. *[38, Lemma 1.3] Let $R[T]$ be the polynomial ring with an indeterminate T over R.Let $h \in R$ and I an ideal of R. The element h is an integer over I if and only if hT is an integer over the ring $\mathscr{P}(I) = \bigoplus_{n\in\mathbb{N}} I^nT^n$.*

The next result relates the behaviour of the integral closure with powers of an ideal.

Proposition 1.18. *[38, Corollary 1.8] Let I be an ideal of R. Then,*

(i) $(\overline{I})^n \subset \overline{I^n}$.

(ii) $I \cdot \overline{I^n} \subset \overline{I^{n+1}}$.

As a consequence of Corollary 1.11, we obtain the following result.

Corollary 1.19. *[38, Corollary 1.11] If I and J are ideals in R then $\overline{I} \cdot \overline{J} \subseteq \overline{I \cdot J}$.*

Up to this point, we have taken a purely algebraic approach to the integral closure of an ideal. Now, we will consider the case where R is the local ring of a reduced complex analytic variety, that is, $R = \mathscr{O}_{X,x}$. Recall that a complex analytic space X is said to be reduced if, for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ has no nonzero nilpotent elements. This setting offers additional interesting properties beyond those previously discussed.

The next theorem is one of the most important results in this section, offering multiple perspectives on the integral closure of an ideal within the framework of analytic geometry.

Theorem 1.20. *[38, Theorem 2.1] Let X be a reduced complex analytic space, $x \in X$, and let I be an ideal of $\mathscr{O}_{X,x}$ and $h \in \mathscr{O}_{X,x}$. The following are equivalent:*

(i) $h \in \overline{I}$;

(ii) (Growth condition) For each system of generators $g_1, \ldots, g_s$ of I there exists a neighbourhood U of x and a constant $C > 0$ such that

$$|h(y)| \leqslant C \sup\{|g_1(z)|, \ldots, |g_s(y)|\},$$

for all $y \in U$;

(iii) (Valuative criterion) For each analytic curve $\varphi : (\mathbb{C}, 0) \to (X, x)$, the germ $h \circ \varphi$ lies in $(\varphi^*(I))\mathscr{O}_1$, where $(\varphi^*(I))\mathscr{O}_1$ denotes the ideal generated by $\varphi^*(I) := \{i \circ \varphi : i \in I\}$.

1.3 Łojasiewicz exponent of ideals

The Łojasiewicz exponent is a fundamental numerical invariant that quantifies the comparative vanishing order of two ideals near a singular point. As shown in [38], this exponent can be characterized algebraically through inclusions between powers of ideals and their integral closures. In this section, we define the Łojasiewicz exponent $\mathscr{L}_J(I)$ and explore its properties, highlighting its connections with other local invariants such as the Hilbert–Samuel multiplicity and the order of an ideal.

We fix the notational conventions used throughout this work. Let (X,x_0) be a germ of analytic space in $\mathbb{C}^n$. Consider two real-valued function germs $a,b:(X,x_0)\to\mathbb{R}$. The relation $a(x)\lesssim b(x)$ near x_0 means that there exists a constant $C>0$ and an open neighborhood U of x_0 in X such that

$$a(x)\leqslant C\cdot b(x),\quad \text{for all } x\in U.$$

Furthermore, we write $a(x)\sim b(x)$ near x_0 whenever both $a(x)\lesssim b(x)$ and $b(x)\lesssim a(x)$ hold in some neighborhood of x_0.

For any n-tuple $x=(x_1,\ldots,x_n)\in\mathbb{C}^n$, we denote its norm by

$$\|x\|=\sqrt{|x_1|^2+\cdots+|x_n|^2}.$$

Definition 1.21. Let X be a reduced equidimensional complex analytic space. Let $I:=\langle f_1,\ldots,f_p\rangle$ and $J:=\langle g_1,\ldots,g_q\rangle$ be two ideals in $\mathscr{O}_X$. Let us consider the maps

$$f=(f_1,\ldots,f_p):(X,0)\to(\mathbb{C}^p,0)$$

$$g=(g_1,\ldots,g_q):(X,0)\to(\mathbb{C}^q,0).$$

The *Łojasiewicz exponent of I with respect to J*, denoted by $\mathscr{L}_J(I)$, is the infimum of the set

$$\{\alpha\in\mathbb{R}_{\geqslant 0}:\|g(x)\|^{\alpha}\lesssim\|f(x)\| \text{ near } 0\}.$$

If this set is empty, we say that $\mathscr{L}_J(I)=\infty$.

The definition of $\mathscr{L}_J(I)$ is independent of the chosen systems of generators of the ideals I and J. Indeed, any two finite generating systems of the same ideal induce equivalent norm functions in a neighborhood of the origin.

It is worth noting that the Łojasiewicz exponent $\mathscr{L}_J(I)$ is finite if and only if $V(I)\subseteq V(J)$, or equivalently, $J\subseteq\sqrt{I}$. This follows from the work of Lejeune-Jalabert and Teissier [38], who established that the existence of a Łojasiewicz inequality between two ideals is equivalent to an integral dependence condition between their powers. Moreover, when finite, $\mathscr{L}_J(I)$ is a positive rational number.

Consider two ideals $I,J\subseteq\mathscr{O}_X$. By [38], the Łojasiewicz exponent of I with respect to J is given by

$$\mathscr{L}_J(I)=\min\left\{\frac{a}{b}\;\middle|\; a,b\in\mathbb{Z}_{\geqslant 1},\quad J^a\subseteq\overline{I^b}\right\}.$$

Let us assume that the ideal $I\subseteq\mathscr{O}_X$ has finite colength, that is, the quotient $\mathscr{O}_X/I$ is a finite-dimensional $\mathbb{C}$-vector space, since $\mathscr{O}_X$ is a $\mathbb{C}$-algebra with residue field $\mathbb{C}$.

In this context, when $J = \mathfrak{m}$ is the maximal ideal of $\mathscr{O}_X$, we denote the number $\mathscr{L}_J(I)$ by $\mathscr{L}_0(I)$. That is,

$$\mathscr{L}_0(I) = \inf\left\{\frac{a}{b} \in \mathbb{Z}_{\geqslant 1} : \quad \mathfrak{m}^a \subseteq \overline{I^b}\right\}.$$

The quantity $\mathscr{L}_0(I)$ is called the *Łojasiewicz exponent* of the ideal I.

The Łojasiewicz exponent is closely related to both the multiplicity and the order of an ideal. To better understand this connection, we recall the necessary definitions and fundamental properties.

We recall that the *order* of a function $f \in \mathscr{O}_X$, denoted by $\operatorname{ord}(f)$, is defined as the largest integer $r \in \mathbb{Z}_{\geqslant 1}$ such that $f \in \mathfrak{m}^r$. Accordingly, the *order of an ideal* $I \subset \mathscr{O}_X$ is defined as $\max\{r : I \subseteq \mathfrak{m}^r\}$.

Definition 1.22. [41] The *Hilbert–Samuel multiplicity* of a finite–colength ideal I in $\mathscr{O}_X$ is defined as

$$e(I) = \lim_{k\to\infty} \frac{n!}{k^n}\ell(\mathscr{O}_X/I^k),$$

where $n = \dim(X)$ and $\ell(\mathscr{O}_X/I^k)$ denotes the colength of I^k, that is, $\dim_{\mathbb{C}}(\mathscr{O}_X/I^k)$.

Some of the main properties that relate the multiplicity of an ideal and its integral closure are:

Proposition 1.23. [27] *Let X be the germ of a d-dimensional complex analytic set. Suppose I and J are ideals of finite co-length in $\mathscr{O}_{X,x}$. Then,*

(i) $\overline{I} = \overline{J} \implies e(I) = e(J)$.

(ii) $I \subset J \implies e(I) \geqslant e(J)$; *if X,x is equidimensional, then* $e(I) = e(J) \implies \overline{I} = \overline{J}$.

(iii) There exists an ideal $I' \subset I$, where I' is generated by d elements such that $e(I') = e(I)$.

Proposition 1.24. *[33] Let X be the germ of a d-dimensional complex analytic set. Suppose I ideal $\mathfrak{m}$-primary of finite co-length in $\mathscr{O}_{X,x}$. If l is a positive integer, then $e(I^l) = l^d e(I)$.*

Remark 1.25. Let $n = \dim(X)$. There is a relationship between the Łojasiewicz exponent, the multiplicity of an ideal, and its order. This relationship stems from the Proposition 1.23 which guarantees inequality $e(\mathfrak{m}^a) \geqslant e(\overline{I^b}) = e(I^b)$. This implies that $a^n \geqslant b^n e(I)$, and consequently, $\mathscr{L}_0(I)^n \geqslant e(I)$. Furthermore, the inclusion $I \subseteq \mathfrak{m}^{\operatorname{ord}(I)}$ yields $e(I) \geqslant e(\mathfrak{m}^{\operatorname{ord}(I)}) = \operatorname{ord}(I)^n e(\mathfrak{m})$. Therefore, we obtain the following inequalities:

$$\mathscr{L}_0(I)^n \geqslant e(I) \geqslant \operatorname{ord}(I)^n. \tag{1.1}$$

1.4 Whitney stratifications

We introduce the notion of Whitney stratification, a concept originally proposed by Whitney that has played a fundamental role in various areas of geometry and topology. For a detailed exposition on this topic, see for instance [29].

Let X be an analytic space in $\mathbb{C}^n$. The goal of stratification theory is to decompose X into a finite family of pairwise disjoint subsets $X_1, \dots, X_s$, such that each X_i is a smooth analytic subspace, called a *stratum*, in such a way that each stratum consists of points having the same type of local behavior, that is, points that are equally singular in a certain geometric or analytic sense.

To make this notion precise, we introduce the formal definition of a locally finite stratification.

Definition 1.26. Let $X \subset \mathbb{C}^n$ be an analytic space. A *locally finite stratification* $\mathscr{V}$ of X is a partition of X into smooth analytic subsets V_α of $\mathbb{C}^n$ (called strata) such that for every point in X there exists a neighborhood in $\mathbb{C}^n$ that contains only a finite number of strata.

The following example illustrates how such a stratification naturally arises in the analytic setting.

Example 1.27. Let $V \subset \mathbb{C}^n$ be an analytic set. Recall that the set ΣV of singular points of V is itself an analytic subset of strictly lower dimension, and that the complement $V \backslash \Sigma V$ is a smooth complex manifold. By repeating this process, one obtains a decreasing filtration

$$V_d \supset V_{d-1} \supset \cdots \supset V_0,$$

where $V_d = V$ and, for each i,

$$V_{i-1} = \begin{cases} \Sigma V_i, & \text{if } \dim V_i = i, \\ V_i, & \text{if } \dim V_i < i. \end{cases}$$

The differences $V_i \backslash V_{i-1}$ form a finite collection of smooth analytic manifolds, each of pure dimension i (or possibly empty). These manifolds constitute a finite stratification of the analytic space V.

To make this construction more explicit, let us consider a classical example.

Example 1.28. Let $V \subset \mathbb{C}^3$ be the algebraic set defined by the equation $x^2 = y^2 z$, known as the *Whitney umbrella*. Consider the subsets of V defined by

$$V_\beta = \Sigma V = \{(0,0,z) \in \mathbb{C}^3\} \subset V \quad \text{and} \quad V_\alpha = V \backslash V_\beta.$$

Then V_α and V_β form a locally finite stratification of V with a 2-dimensional smooth stratum and a 1-dimensional stratum. Geometrically, this stratification decomposes the Whitney umbrella into a smooth surface (with two connected components) and its singular line. The construction is illustrated in Figure 1.1.

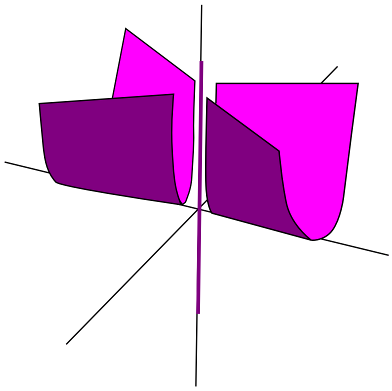

Figure 1.1: Whitney's umbrella stratification

Whitney stratifications

The central notion in the theory of stratifications is the regularity condition between the strata that compose them. In general terms, this condition provides a criterion that controls how the different strata of a stratification meet and relate to each other. In the following, we introduce two fundamental concepts in this theory: the frontier condition and Whitney's regularities conditions.

We say that the stratification $\mathscr{V} = \{V_\alpha\}$ of a subset X of $\mathbb{C}^p$ satisfies the *frontier condition* if for any two strata V_α and V_β such that $V_\beta \cap \overline{V}_\alpha \neq \varnothing$, then $V_\beta \subset \overline{V}_\alpha$, where the $\overline{V}_\alpha$ is the closure of V_α in the topological space X.

We next introduce the Whitney conditions.

Definition 1.29. The stratification $\mathscr{V} = \{V_\alpha\}$ satisfies the *Whitney conditions* if for every pair (V_α, V_β) of strata such that $V_\beta \subset \overline{V}_\alpha$ and for every point y in V_β, we have:

(a) For every sequence of points $\{x_i\}$ in V_α converging to y, such that the limit

$$\lim_{i\to\infty} T_{x_i}(V_\alpha) = T$$

exists in the corresponding Grassmannian, then T contains $T_y(V_\beta)$.

(b) Moreover, if there exists a sequence $\{y_i\}$ of points in V_β converging to y such that the limit of directions

$$\lim_{i\to\infty} \overline{x_i y_i} = \lambda$$

exists in the projective space, then T contains λ.

A stratification $\mathscr{V}$ is called a *Whitney stratification* if it satisfies the frontier condition and the Whitney conditions (a) and (b).

It is important to observe that condition (b) is, in fact, stronger than condition (a). More precisely, it is well known that condition (b) implies condition (a).

Analytic varieties can always be Whitney stratified. More precisely, Whitney proved in [53] that every complex analytic space admits a stratification satisfying conditions (a) and (b).

Example 1.30. The stratification $\{V_\alpha, V_\beta\}$ of the Whitney umbrella, described in Example 1.28, is not a Whitney stratification, since the Whitney condition *(b)* fails at the origin. In fact, along the z–axis the limiting tangent spaces of V_α do not contain the limiting secant lines. A Whitney stratification is obtained by refining the singular stratum according to the sign of z. Then the stratification $\{V_\alpha, V_\beta \backslash \{0\}\}$ satisfies the Whitney conditions and is illustrated in Figure 1.2.

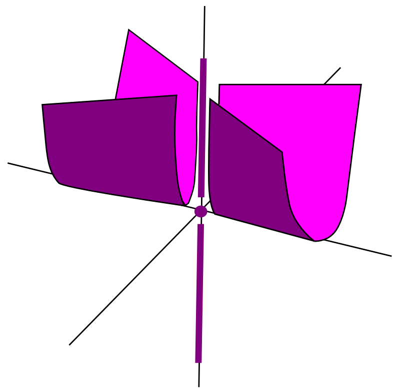

Figure 1.2: Whitney's umbrella Whitney stratification

An algebraic reformulation: the Verdier condition (w)

In [50], Teissier prove that the Whitney conditions are equivalent to Verdier's condition w, in the complex analytic setting. However, its formulation is particularly convenient from an algebraic point of view.

Condition (w) states that the distance between the tangent space to X at a point $x_i \in X_0$, where X_0 denotes the smooth part of X, and the tangent space to Y at y tends to zero at least as fast as the distance between x_i and Y. To formalize this notion, we first recall the definition of the distance between two linear subspaces.

Let A and B be linear subspaces of $\mathbb{C}^n$. The distance of A and B is defined as

$$dist(A,B) = \sup_{\substack{u \in B^\perp - \{0\} \\ v \in A - \{0\}}} \frac{|\langle u, v \rangle|}{\|u\| \, \|v\|},$$

where $B^\perp$ denotes the orthogonal space of B and $\langle \cdot, \cdot \rangle$ is the usual inner product.

Definition 1.31. Let X and Y be two strata in a stratification of a complex analytic space Z, satisfying $\overline{X} \supset Y$, where $\overline{X}$ denotes the closure of the set X in Z. Then the pair (X,Y) satisfies *Verdier's condition* w at $0 \in Y$, if there exists a positive real number C such that

$$dist(T_0Y, T_xX) \leqslant C \cdot dist(x, Y),$$

for all x close to Y, where T_xX denotes the tangent space of X at the point x and $dist(x,Y) = inf\{\|x-y\| : y \in Y\}$.

Teissier showed the importance of the integral dependence relation in determining the Whitney conditions in the case of a family of analytic hypersurfaces [50]. This result was later generalized by Gaffney for any dimension.

Theorem 1.32. *[26, Theorem 2.5] Let X be a complex analytic, reduced, purely d dimensional space, Y an analytic subspace of X purely of t dimensional, 0 a smooth point of Y, and X_0 the set of smooth points on X. Let $F : \mathbb{C}^r \times \mathbb{C}^t \to \mathbb{C}^p$ be a map such that $X = F^{-1}(0)$ with reduced structure, and $\{0\} \times \mathbb{C}^t = Y$, with coordinates $(x_1, \dots, x_r, y_1, \dots, y_t)$. Then $\frac{\partial F}{\partial y_l} \in \overline{\{x_i \frac{\partial F}{\partial x_j}\}\mathscr{O}_X^p}$ for all $l \in \{1, \dots, t\}$, in which $i, j \in \{1, \dots, r\}$ if and only if (X_0, Y) satisfies Verdier's condition (w) at $0 \in Y$, where $\mathscr{O}_X^p$ is the module of germs of analytic maps from X to $\mathbb{C}^p$.*

Whitney stratifications and their equivalent formulations, such as Verdier's condition (w), will be a key tool in Chapter 3 . There, we study non-normal toric surfaces in $\mathbb{C}^3$ through the combinatorial description of their associated semigroups and Newton polyhedra, which allows us to construct a Whitney stratification adapted to their geometry. The main result shows that this stratification has fewer strata than the one induced by the torus action.

1.5 Bi- Lipschitz equivalences

In this section, we recall the notion of bi-Lipschitz equivalence, which plays a fundamental role in the metric study of analytic spaces and singularities. Bi-Lipschitz equivalences preserve distances up to multiplicative constants, providing a natural framework to compare germs of spaces, maps, and ideals. We fix the notational conventions and recall the basic definitions that will be used throughout this work, particularly in Chapter 4, where we study the bi-Lipschitz invariance of the Łojasiewicz exponent, multiplicity, and the Euler obstruction for hypersurfaces. For more details the reader may consult [22, 21, 47].

We start by recalling the definition of a bi-Lipschitz map.

Definition 1.33. A map germ $f : (X,0) \to (Y,0)$ is *Lipschitz* if

$$\|f(x) - f(x')\| \lesssim \|x - x'\| \text{ near } 0.$$

A bi-Lipschitz homeomorphism between $(X,0)$ and $(Y,0)$ is a Lipschitz homeomorphism $h : (X,0) \to (Y,0)$ whose inverse $h^{-1} : (Y,0) \to (X,0)$ is also Lipschitz. Moreover, two subsets X_1 and X_2 of $(X,0)$ are bi-Lipschitz equivalent if there is a bi-Lipschitz homeomorphism $\varphi : (X,0) \to (X,0)$ so that $\varphi(X_1) = X_2$.

Two function germs $f, g : (X,0) \to (\mathbb{C}^p, 0)$ are called *bi-Lipschitz $\mathscr{R}$-equivalent* if there exists a bi-Lipschitz map germ $\phi : (X,0) \to (X,0)$ such that

$$f = g \circ \phi.$$

Similarly, f and g are *bi-Lipschitz $\mathscr{A}$-equivalent* if there exist bi-Lipschitz homeomorphisms $\varphi : (X,0) \to (X,0)$ and $\phi : (\mathbb{C}^p,0) \to (\mathbb{C}^p,0)$ such that

$$\phi \circ f = g \circ \varphi.$$

We also say that f and g are *bi-Lipschitz $\mathscr{K}^*$-equivalent* if there is a bi-Lipschitz homeomorphism $\varphi : (X,0) \to (X,0)$ and a map $A : (X,0) \to GL(\mathbb{C}^p)$ such that A and A^{-1} are Lipschitz and

$$A(x)f(x) = g(\varphi(x))$$

for all x in some open neighborhood of $0 \in X$, where $GL(\mathbb{C}^p)$ denotes the set of all $p \times p$ complex matrices with nonzero determinant.

In particular, bi-Lipschitz $\mathscr{R}$-equivalence implies both bi-Lipschitz $\mathscr{A}$-equivalence and bi-Lipschitz $\mathscr{K}^*$-equivalence.

The notion of *bi-Lipschitz equivalence of ideals*, introduced by Bivià-Ausina and Fukui in [8], is originally defined for ideals in $\mathscr{O}_n$, by comparing their generators via a bi-Lipschitz homeomorphism $\varphi : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$. Using the ideas of Bivià-Ausina and Fukui, here we present this notion to ideals in $\mathscr{O}_X$, where X denotes a germ of analytic space in $\mathbb{C}^n$, and $\mathscr{O}_X$ denotes the ring of germs of analytic functions $(X,0) \to (\mathbb{C},0)$.

Definition 1.34. Let I and J be ideals of $\mathscr{O}_X$. We say that I and J are *bi-Lipschitz equivalent* if there exist families of functions $f_1,\dots,f_p$ and $g_1,\dots,g_q$ in $\mathscr{O}_X$ such that:

1. $\langle f_1,\dots,f_p\rangle \subseteq I$ and $\overline{\langle f_1,\dots,f_p\rangle} = \bar{I}$;
2. $\langle g_1,\dots,g_q\rangle \subseteq J$ and $\overline{\langle g_1,\dots,g_q\rangle} = \bar{J}$;
3. There exists a bi-Lipschitz homeomorphism $\varphi : (X,0) \to (X,0)$ such that

$$\|(f_1(x),\dots,f_p(x))\| \sim \big\|(g_1(\varphi(x)),\dots,g_q(\varphi(x)))\big\| \quad \text{near } 0,$$

where $\bar{I}$ denotes the closure of the ideal I.

It follows directly from the definition that bi-Lipschitz equivalence between two ideals implies bi-Lipschitz equivalence between their respective zero sets. Moreover, if two germs are bi-Lipschitz $\mathscr{R}$-equivalent, then their zero sets are bi-Lipschitz equivalent.

Example 1.35. Let $f \in \mathscr{O}_{X(S)}$ be a nonzero germ and let $\phi : (X(S),0) \longrightarrow (X(S),0)$ be a bi-Lipschitz homeomorphism. Setting $g = f \circ \phi$, the principal ideals $I = \langle f\rangle$ and $J = \langle g\rangle$ are bi-Lipschitz equivalent. Indeed, this follows immediately from Definition 1.34 by taking $p = q = 1$, $f_1 = f$, $g_1 = g$, and using the same bi-Lipschitz homeomorphism ϕ. The norm condition is automatically satisfied since

$$|f(x)| = |g(\phi(x))|.$$

As a concrete example, consider the toric surface $X(S) = \{(x,y,z) \in \mathbb{C}^3;\ xz = y^2\}$, and the germs $f(x,y,z) = x^2+y^2+z^2$, and $g(x,y,z) = x^2 + (y+xy)^2 + \left(z+2y^2+xy^2\right)^2$.

Observe that $\phi(x,y,z) = \left(x,\ y+xy,\ z+2y^2+xy^2\right)$ is a bi-Lipschitz homeomorphism of $X(S)$ satisfying $g = f\circ\phi$. Moreover, ϕ preserves the defining equation of $X(S)$, since

$$x\left(z+2y^2+xy^2\right) = (y+xy)^2.$$

Therefore, the principal ideals $I = \langle f\rangle$ and $J = \langle g\rangle$ are bi-Lipschitz equivalent.

Example 1.36. Let $I,J \subset \mathscr{O}_{X(S)}$ be ideals satisfying $\overline{I} = \overline{J}$. Then I and J are bi-Lipschitz equivalent. Indeed, if $I = \langle f_1,\dots,f_p\rangle$ and $J = \langle g_1,\dots,g_q\rangle$, one may take the identity map $\phi = \mathrm{id}$. The norm condition in Definition 1.34 follows from the characterization of integral closure by the growth condition given in Theorem 1.20. As an illustration, consider the ideals $I = \langle x^2,y^2\rangle$ and $J = \langle x^2+xy,y^2\rangle \subset \mathscr{O}_{X(S)}$. Since $\overline{I} = \overline{J} = \langle x^2,xy,y^2\rangle$, the ideals I and J are bi-Lipschitz equivalent.

1.6 The local Euler obstruction

The local Euler obstruction was defined by MacPherson in [39] as a tool to prove the conjecture about the existence and unicity of the Chern classes in the singular case. Since then, it has been extensively investigated by many authors such as Brasselet and Schwartz [13], Lê and Teissier [37], Kashiwara [34], and others.

Despite its importance, the local Euler obstruction is generally hard to compute from its definition. This has motivated the search for effective formulas that allow one to handle it in concrete situations.

In [12, Theorem 3.1], Brasselet, Lê, and Seade proved a formula to compute the local Euler obstruction using generic linear forms. For this purpose, we first present the definition of generic linear forms.

Let $(X,0) \subset (\mathbb{C}^n,0)$ be a pure-dimensional complex analytic subset $X \subset U$ of an open set $U \subset \mathbb{C}^n$. We consider a complex analytic Whitney stratification $\mathscr{V} = \{V_i\}$ of U adapted to X (i.e. X is a union of strata) and we assume that $\{0\}$ is a stratum. We choose a representative X small enough of $(X,0)$ such that 0 belongs to the closure of all the strata. We write $X = \bigcup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\mathrm{reg}}$ is the set of regular points of X. We assume that the strata $V_0,\dots,V_{q-1}$ are connected. Note that the closures $\overline{V_0},\dots,\overline{V_{q-1}}$ are complex analytic subsets of U.

Definition 1.37. A generic complex linear form (with respect to X) is a complex linear form $\ell : U \to \mathbb{C}$ such that $0 \in \ell^{-1}(0)$ and $\ker(\ell)$ is transversal to all limits of tangent spaces $\{T_{x_n}V_i\}$, for every stratum V_i and every sequence $\{x_n\} \subset V_i$ converging to 0.

Theorem 1.38. *Let $(X,0)$ and $\mathscr{V}$ be given as before, then for each generic linear form l, there exists ε_0 such that for any ε with $0 < \varepsilon < \varepsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X,0)$ is* *equal to*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\varepsilon \cap l^{-1}(\delta)) \cdot \mathrm{Eu}_X(V_i),$$

where χ is the Euler characteristic, $\mathrm{Eu}_X(V_i)$ is the Euler obstruction of X at a point of V_i, $i = 1, \ldots, q$ and $0 < |\delta| \ll \varepsilon \ll 1$.

The local Euler obstruction is known to be an analytic invariant, meaning it remains unchanged under analytic equivalence. However, it is not a topological invariant, as it can vary under homeomorphisms (see [30]). Whether the local Euler obstruction is a bi-Lipschitz invariant remains an open question.

In this thesis, the results addressing this question are presented in Chapter 4, where we provide a partial answer by showing that, under certain conditions, the local Euler obstruction is preserved by bi-Lipschitz transformations. These results thus offer a positive contribution to the understanding of the behavior of the local Euler obstruction within the bi-Lipschitz category. This naturally leads to the study of polar multiplicities, which are closely related to the local Euler obstruction and play a central role in both its computation and geometric interpretation.

The next theorem, proved by Lê and Teissier, presents a way to compute the local Euler obstruction with the aid of Gonzáles-Sprinberg's purely algebraic interpretation of this invariant.

Theorem 1.39 ([37])**.** *Let $X \subset \mathbb{C}^{n+1}$ be an analytic space of dimension d reduced at 0. Then $\mathrm{Eu}_X(0) = \sum_{k=0}^{d-1}(-1)^{d-k-1} m_k(X)$, where $m_k(X)$ denotes the k-th polar multiplicity of X at 0.*

In particular, the first polar multiplicity $m_0(X)$ coincides with the classical notion of multiplicity, which is the central object in the metric version of Zariski's conjecture. This conjecture asks whether multiplicity is preserved under bi-Lipschitz homeomorphisms.

1.7 Affine toric varieties

Toric varieties form a special and well-understood class of algebraic varieties whose geometry can be described using combinatorial data. In this setting, geometric objects are often translated into combinatorial ones such as lattices, polyhedra, cones, and fans. This correspondence provides a powerful bridge between geometry and combinatorics. Rougly speaking, a toric variety is an algebraic variety having an open and dense subset parametrized by monomials, which allows its properties to be studied in a very explicit algebraic and geometric way.

Although at first glance this may seem to be a rather restricted class of varieties, many familiar examples actually are into this category. For instance, $(\mathbb{C}^*)^n = (\mathbb{C}\backslash\{0\})^n$, the affine space $\mathbb{C}^n$. A simple but interesting example is the cuspidal cubic curve given by the equation $y^2 = x^3$, which can be parametrized by $t \mapsto (t^2, t^3)$.

An alternative way to define this important subclass is that they can be described by binomial ideals, making them accessible through semigroup algebras associated with rational polyhedral cones.

In this section, we introduce the basic ideas and constructions related to Affine toric varieties, following standard references such as [16, 10, 20].

Combinatorial definition of affine toric varieties

We present one of the equivalent constructions of affine toric varieties, based on toric ideals. This approach provides a clear algebraic perspective in which an affine toric variety is realized as the zero set of a binomial ideal associated with a finitely generated semigroup $S \subset \mathbb{Z}^n$, highlighting the connection between combinatorics and geometry.

Let $S \subset \mathbb{Z}^n$ be a semigroup generated by a finite set $S_0 = \{b_1, \ldots, b_r\} \subset \mathbb{Z}^n$ satisfying $\mathbb{Z}S_0 = \mathbb{Z}^n$. The set S_0 induces a group homomorphism $\pi_{S_0} : \mathbb{Z}^r \to \mathbb{Z}^n$ given by

$$\pi_{S_0}(\alpha_1, \ldots, \alpha_r) = \alpha_1 b_1 + \cdots + \alpha_r b_r.$$

For each $\alpha = (\alpha_1, \ldots, \alpha_r) \in \ker(\pi_{S_0})$ we set

$$\alpha_+ = \sum_{\alpha_i > 0} \alpha_i e_i \quad \text{and} \quad \alpha_- = -\sum_{\alpha_i < 0} \alpha_i e_i,$$

where $e_1, \ldots, e_r$ are the elements of the standard basis of $\mathbb{R}^r$. Let us observe that $\alpha = \alpha_+ - \alpha_-$ and that $\alpha_+, \alpha_- \in \mathbb{N}^r$.

Consider the ideal

$$I_S = \langle x^{\alpha_+} - x^{\alpha_-};\ \ \alpha \in \ker(\pi_{S_0}) \rangle \subset \mathbb{C}[x_1, \ldots, x_r], \tag{1.2}$$

where $x^\beta = x_1^{\beta_1} \ldots x_r^{\beta_r}$ with $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$. We work with a minimal system of generators of $\ker(\pi_{S_0})$, so that the associated binomials provide a minimal generating set for the ideal I_S.

Definition 1.40. The n-dimensional *affine toric variety* $X(S)$ generated by S is the irreducible variety $\mathscr{V}(I)$ defined by the zero set of the ideal I as in (1.2).

The ideal I_S defined in (1.2) is generated by binomials corresponding to elements of $\ker(\pi_{S_0})$. These binomials determine the algebraic structure of the affine toric variety $X(S)$. The ideal I_S is prime. Therefore, I_S is a *toric ideal* (see [16, Proposition 1.1.9]).

Example 1.41. Consider the affine semigroup $S \subseteq \mathbb{Z}$ generated $S_0 = \{2, 3\}$. The homomorphism induced by S is $\pi_S(\alpha_1, \alpha_2) = 2\alpha_1 + 3\alpha_2$. Its kernel is generated by $\alpha = (3, -2)$, and thus $\alpha_+ = (3, 0)$ and $\alpha_- = (0, 2)$. Consequently, the corresponding toric ideal is

$$I_S = \langle x^3 y^0 - x^0 y^2 \rangle = \langle x^3 - y^2 \rangle \subset \mathbb{C}[x, y].$$

Hence, the affine toric variety $X(S)$ is the cuspidal cubic, i.e. the curve defined by $x^3 = y^2$ in $\mathbb{C}[x, y]$.

Example 1.42. Let $S_0 = \{(1,0),(1,1),(1,2)\} \subset \mathbb{Z}^2$. This set defines the group homomorphism

$$\pi_S(\alpha_1,\alpha_2,\alpha_3) = (\alpha_1+\alpha_2+\alpha_3,\ \alpha_2+2\alpha_3),$$

whose kernel is given by $\ker(\pi_S) = \mathbb{Z}\cdot(1,-2,1)$.

For the element $\alpha = (1,-2,1)$, we have $\alpha_+ = (1,0,1)$ and $\alpha_- = (0,2,0)$. These yield the binomial $x^{\alpha_+} - x^{\alpha_-} = xz - y^2$, and thus the corresponding ideal is

$$I_S = \langle xz - y^2 \rangle \subset \mathbb{C}[x,y,z].$$

Consequently, the affine toric variety associated with this semigroup is $X(S) = V(xz-y^2) \subset \mathbb{C}^3$.

The `Oscar.jl` codes for the computation of the toric ideals in the above examples are collected in the Appendix A.

The torus action

The set $\mathbb{T} = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : z_i \neq 0, \quad i = 1,\ldots,n\} = (\mathbb{C}^*)^n$ is called the *algebraic n-dimensional torus*. Recall that $(\mathbb{C}^*)^n$ is a group under component-wise multiplication: for $t,t' \in (\mathbb{C}^*)^n$,

$$t\cdot t' = (t_1,\ldots,t_n)\cdot(t_1',\ldots,t_n') = (t_1t_1',\ldots,t_nt_n') \in (\mathbb{C}^*)^n.$$

There is a relationship between the toric variety $X(S)$ associated with S and the algebraic n-dimensional torus $\mathbb{T} = (\mathbb{C}^*)^n$. More precisely, the *action of the torus* $\mathbb{T}$ on $X(S)$ is given by

$$\begin{array}{rccc} \phi: & (\mathbb{C}^*)^n \times X(S) & \longrightarrow & X(S) \\ & (t,x) & \longmapsto & (t^{b_1}x_1,\ldots,t^{b_r}x_r) \end{array} \tag{1.3}$$

where $t = (t_1,\ldots,t_n)$, $x = (x_1,\ldots,x_r)$, and $b_1,\ldots,b_r$ are generators of the semigroup S, with $b_i = (b_i^1,\ldots,b_i^n)$.

Moreover, ϕ has an open and dense orbit $\mathscr{O}$ in $X(S)$ that is diffeomorphic to $(\mathbb{C}^*)^n$. Thus the n-dimensional toric variety $X(S)$ contains the torus $\mathbb{T}$ as a Zariski open dense subset. (See Definition 1.1.3. and Theorem 1.1.17 in [16]).

This action is central in the definition of toric varieties: Since the class of an affine toric varieties is precisely the class of affine varieties endowed with an action of the algebraic torus $\mathbb{T}$ having a dense open orbit. The existence of such an action encodes the combinatorial data of the semigroup S, and conversely, the geometry of $X(S)$ reflects the combinatorics of S.

Example 1.43. In Example 1.41, the corresponding torus action is given by $(t,(x,y)) \mapsto (t^2x,,t^3y)$, whereas in Example 1.42 it is described by $((t_1,t_2),(x,y,z)) \mapsto (t_1x,,t_1t_2y,,t_1t_2^2z)$.

Cones and Affine Toric Varieties

In the following, we introduce the concepts of cone and dual cone, which are studied in convex geometry. Those elements will be necessary to the definition of Newton polyhedron which we will use in this work.

Definition 1.44. Let $\{b_1,\dots,b_r\} \subset \mathbb{Z}^n$ be a finite generator set of the semigroup S. The *convex polyhedral cone* associated with S in $\mathbb{R}^n$ is the set

$$cone(S) = \left\{ \sum_{i=1}^{r} \lambda_i b_i \mid \lambda_i \in \mathbb{R} \ , \ \lambda_i \geqslant 0 \right\}.$$

The vectors $b_1,\dots,b_r$ are called generators of the $cone(S)$. Also, we set $cone(\varnothing) = \{0\}$.

A convex polyhedral cone is indeed convex, meaning that if $x, y \in \text{cone}(S)$, then $\lambda x + (1-\lambda)y \in \text{cone}(S)$ for all $0 \leqslant \lambda \leqslant 1$. Moreover, it is a cone in the sense that if $x \in \text{cone}(S)$, then $\lambda x \in \text{cone}(S)$ for all $\lambda \geqslant 0$.

Examples of polyhedral cones include the first quadrant in $\mathbb{R}^2$ and the first octant in $\mathbb{R}^3$ and corresponds to $cone(S_1)$ and $cone(S_2)$, where $S_1 = \langle e_1, e_2 \rangle$ and $S_2 = \langle e_1, e_2, e_3 \rangle$, respectively. Another example is the cone $\text{Cone}(e_1, e_2, e_1+e_3, e_2+e_3) \subset \mathbb{R}^3$, illustrated in Figure 1.3.

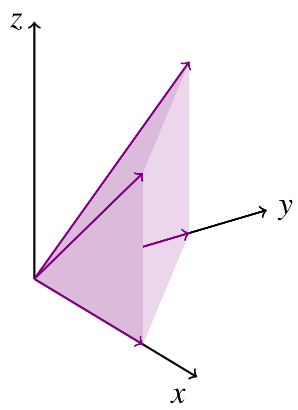

Figure 1.3: $\text{Cone}(e_1, e_2, e_1+e_3, e_2+e_3) \subset \mathbb{R}^3$

The *dimension* of $cone(S)$ is the dimension of the smallest linear subspace of $\mathbb{R}^n$ containing $cone(S)$.

Let $(\mathbb{R}^n)^*$ be the dual space of $\mathbb{R}^n$. To each cone, we associate the *dual cone*, defined by

$$\widecheck{cone}(S) = \{u \in (\mathbb{R}^n)^* : \langle u, v \rangle \geqslant 0, \forall v \in cone(S)\}.$$

The dual of a polyhedral cone is also a polyhedral cone, and the dualization is an involutive operation, that is, the dual cone of $\widecheck{cone}(S)$ is $\text{cone}(S)$, as we can see [16, Proposition 1.2.4].

Moreover, since $\text{cone}(S)$ is generated by vectors in $\mathbb{Z}^n$, its dual cone $\widecheck{cone}(S)$ is also generated by vectors in $\mathbb{Z}^n$ (see [11, p. 5] and [16, p. 29]).

Example 1.45. Let $S = \langle(1,0),(1,1),(1,2)\rangle \subset \mathbb{Z}^2$. The polyhedral cone associated with S and its dual cone are shown in the figure below.

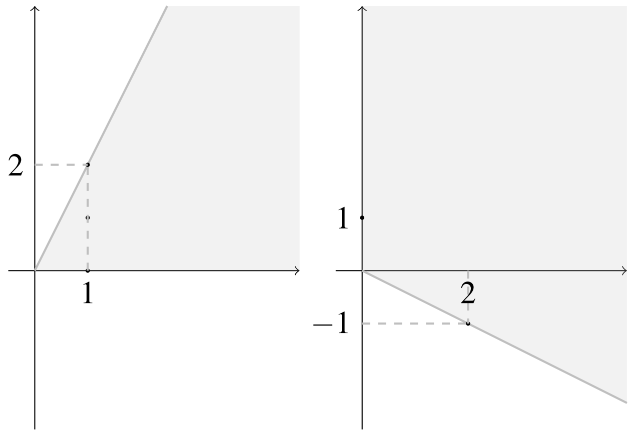

Figure 1.4: Polyhedral cone associated with $S = \langle(1,0),(1,1),(1,2)\rangle$ and its dual cone.

For a point $u \in (\mathbb{R}^n)^*$, we set

$$H_u = \{v \in \mathbb{R}^n : \langle u,v\rangle = 0\} \quad \text{and } H_u^+ = \{v \in \mathbb{R}^n : \langle u,v\rangle \geqslant 0\}.$$

When $u_1,\dots,u_l$ generate $\widetilde{cone}(S)$, the $cone(S)$ can be expressed as

$$cone(S) = H_{u_1}^+ \cap \cdots \cap H_{u_l}^+,$$

which shows that every polyhedral cone arises as the intersection of finitely many closed half-spaces.

Example 1.46. Let $S = \{(1,0),(1,1),(1,2)\} \subset \mathbb{Z}^2$, as in Example 1.45. Recall that the polyhedral cone $cone(S) \subset \mathbb{R}^2$ and its dual cone are illustrated in Figure 1.4. The cones $cone(S)$ and $\widetilde{cone}(S)$ are both of dimension 2.

The $\widetilde{cone}(S)$ is generated by the vectors $u_1 = (1,0)$, $u_2 = (0,1)$ and $u_3 = (2,-1)$. For each generator u_i, consider the closed half-space $H_{u_i}^+ = \{v \in \mathbb{R}^2 \mid \langle u_i,v\rangle \geqslant 0\}$. Explicitly, we have

$$\begin{aligned} H_{(1,0)}^+ &= \{(x,y) \in \mathbb{R}^2 \mid x \geqslant 0\},\\ H_{(0,1)}^+ &= \{(x,y) \in \mathbb{R}^2 \mid y \geqslant 0\},\\ H_{(2,-1)}^+ &= \{(x,y) \in \mathbb{R}^2 \mid 2x-y \geqslant 0\}. \end{aligned}$$

Therefore, $cone(S) = H_{(1,0)}^+ \cap H_{(0,1)}^+ \cap H_{(2,-1)}^+$. This shows that $cone(S)$ is given by $y \geqslant 0$ and $2x-y \geqslant 0$.

The `Oscar.jl` code, presented in Appendix A, computes the dimension of $cone(S)$ and $\widetilde{cone}(S)$, as well as the generators of the dual cone $\widetilde{cone}(S)$.

Definition 1.47. A *face* τ of cone(S) is a subset of the form $\tau = cone(S) \cap H_u$ for $u \in \widetilde{cone}(S)$.

Note that $cone(S)$ is considered to be face of itself, as $0 \in \check{cone}(S)$. If τ is a face of $cone(S)$, we write $\tau \preceq cone(S)$ and $\tau \prec cone(S)$ if $\tau \neq cone(S)$.

In the following, we summarize a list of properties of cones and their faces.

Lemma 1.48. *Let $cone(S)$ be a polyhedral cone. Then:*

(i) Every face of $cone(S)$ is a polyhedral cone.

(ii) An intersection of two faces of $cone(S)$ is also a face of $cone(S)$.

(iii) A face of a face of $cone(S)$ is also a face of $cone(S)$.

(iv) If $\tau \preceq cone(S)$ and $v, w \in cone(S)$, then $v + w \in \tau$ implies $v \in \tau$ and $w \in \tau$.

A *facet* of $cone(S)$ is a face τ of codimension 1, i.e., $\dim \tau = \dim(cone(S)) - 1$. An *edge* of $cone(S)$ is a face of dimension 1.

Example 1.49. Let $S = \langle (1,0),(1,1),(1,2) \rangle \subset \mathbb{Z}^2$ be as in Example 1.45. We describe the faces of the $cone(S)$ and of its dual cone $\check{cone}(S)$.

The $cone(S) \subset \mathbb{R}^2$ is generated by the extremal rays $\tau_1 = \mathbb{R}_{\geqslant 0}(1,0)$ and $\tau_2 = \mathbb{R}_{\geqslant 0}(1,2)$. Its faces consist of the zero face $\{0\}$, the one-dimensional faces τ_1 and τ_2, and the $cone(S)$ itself.

The dual cone $\check{cone}(S) \subset (\mathbb{R}^2)^*$, as described in Example 1.45, has extremal rays $\rho_1 = \mathbb{R}_{\geqslant 0}(0,1)$ and $\rho_2 = \mathbb{R}_{\geqslant 0}(2,-1)$. Accordingly, its faces are the zero face $\{0\}$, the one-dimensional faces ρ_1 and ρ_2, and the $\check{cone}(S)$ itself.

The `Oscar.jl` code used to compute the vectors generating the extremal rays of $cone(S)$ and $\check{cone}(S)$ is presented in the Appendix A.

Here we present a property of cones that we will need.

Definition 1.50. The cone $\mathrm{cone}(S)$ is called *strongly convex* if $\mathrm{cone}(S) \cap \big(-\mathrm{cone}(S)\big) = 0$.

Proposition 1.51. *[16, Proposition 1.2.12] Let* $\mathrm{cone}(S)$ *be a polyhedral cone. The following conditions are equivalent:*

(i) $\mathrm{cone}(S)$ *is strongly convex;*

(ii) $\{0\}$ *is a face of* $\mathrm{cone}(S)$*;*

(iii) $\mathrm{cone}(S)$ *contains no positive-dimensional linear subspace;*

(iv) $\dim \check{cone}(S) = n$.

Example 1.52. Consider the set $S = \{(1,0),(1,1),(1,2)\} \subset \mathbb{Z}^2$, as in Example 1.45. Note that the associated polyhedral cone $\mathrm{cone}(S)$ is a strongly convex polyhedral cone.

In what follows, we assume that $\dim(\mathrm{cone}(S)) = n$ and that $\mathrm{cone}(S)$ is strongly convex.

The Orbit-Cone Correspondence

For normal affine toric varieties, the orbit-cone correspondence holds (see [16, Section 3.2]). More precisely, for each k with $0 \leqslant k \leqslant n$, there is a one-to-one correspondence

$$\{k\text{-dimensional faces of } cone(S)\} \xleftrightarrow{1:1} \{(n-k)\text{-dimensional torus orbits in } X(S)\}\,.$$

Although this correspondence is usually stated for normal toric varieties, it also applies to $X(S)$ in the possibly non-normal case. Indeed, by [16, Theorem 3.A.3.(c)], the torus orbits of $X(S)$ are in bijection with those of its normalization $\widetilde{X(S)}$. The normalization is the affine toric variety associated with the semigroup $\check{\sigma} \cap \mathbb{Z}^s$, where $\sigma = cone(S)$ (see [16, Proposition 1.3.8]). Since $cone(S)$ is strongly convex, we have $\check{\sigma} = cone(S)$. (see [11, Property 1.2.2]). Therefore, the orbit-cone correspondence also holds for $X(S)$, even though $X(S)$ may be non-normal.

Remark 1.53. Since $cone(S)$ is strongly convex, it contains no non-trivial linear subspaces and therefore admits no proper faces of dimension n, by Proposition 1.51. Hence $cone(S)$ is the unique face of maximal dimension. It follows that the torus action on $X(S)$ has a unique orbit of dimension zero.

CHAPTER 2

Integral closure of non-degenerate ideals

In this chapter, we study the integral closure of non-degenerate ideals in $\mathscr{O}_{X(S)}$, where $X(S)$ is a toric variety defined by a semigroup $S \subset \mathbb{Z}_+^n$. The main goal is to understand how the integral closure of such ideals can be described using combinatorial data arising from Newton polyhedra.

This chapter is motivated by results of Saia [48], which relate the integral closure of ideals to Newton polyhedra in the classical setting. Here, we extend this perspective to the toric context, emphasizing the role played by the geometry of the associated polyhedra. We also remark that the results presented in [48] were first extend to Newton non-degenerated modules by Biviá-Ausina in [7].

Our main results are Theorem 2.26 and Corollary 2.27, which provide an explicit description of the integral closure of non-degenerate ideals in terms of the Newton polyhedron. More precisely, we show that the generators of the integral closure are precisely the monomials whose exponents are determined by the Newton polyhedron of the ideal. This provides a concrete description of the integral closure and highlights the close relationship between algebraic and combinatorial structures in the toric setting. These results appear in [2].

Throughout this chapter, we use the OSCAR.jl package to perform the computational calculations associated with the presented examples. All the codes corresponding to these examples, as well as the auxiliary functions created to support the computations, are collected in the Appendix A.

2.1 Newton polyhedra

In this section we present the definition of Newton polyhedron of ideals of the ring $\mathscr{O}_{X(S)}$. For the definition of Newton polyhedron of a function $f : X(S) \to \mathbb{C}$ we can refer to [40] for instance.

Definition 2.1. The *Newton polyhedron* determined by $A \subset S$, denoted by $\Gamma_+(A)$, is the convex hull of the set $\{k + v \ : \ k \in A, v \in cone(S)\}$. If $v \in \widecheck{cone}(S)$, we define

$$\ell(v, \Gamma_+(A)) = \min\{\langle k, v\rangle \ : \ k \in \Gamma_+(A)\}$$

and $\Delta(v,\Gamma_+(A)) = \{k \in \Gamma_+(A) : \langle k,v\rangle = \ell(v,\Gamma_+(A))\}$.

A subset $\Delta \subseteq \Gamma_+(A)$ is called a *face of* $\Gamma_+(A)$ if there exists $v \in \widetilde{cone}(S)$ such that $\Delta = \Delta(v,\Gamma_+(A))$.

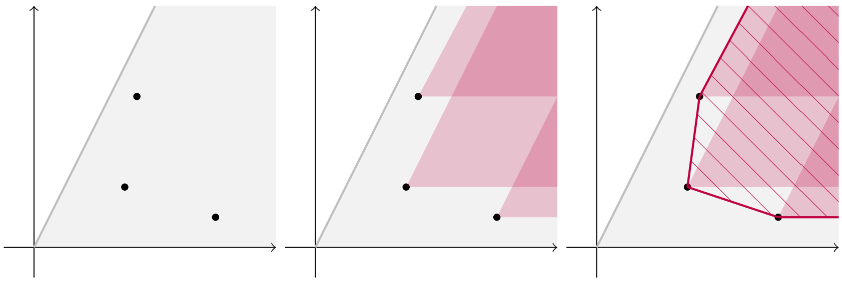

Figure 2.1: Newton polyhedron.

Remark 2.2. Since $\widetilde{cone}(S)$ is generated by vectors in $\mathbb{Z}^n$, for each face Δ of $\Gamma_+(A)$ there exists a vector $v \in \widetilde{cone}(S) \cap \mathbb{Z}^n$ such that

$$\Delta = \Delta(v,\Gamma_+(A)).$$

In particular, $\langle b_i, v\rangle \geqslant 0$ for all i, since $v \in \widetilde{cone}(S)$.

For the sake of simplicity, when it is clear from the context that the polyhedron $\Gamma_+(A)$ is fixed, we will omit it from the notation and write $\Delta(v)$ and $\ell(v)$ instead of $\Delta(v,\Gamma_+(A))$ and $\ell(v,\Gamma_+(A))$.

Since a polyhedron can be described as the intersection of the half-planes defined by its faces, we have an important characterization.

Lemma 2.3. *Let $\Gamma_+(A) \subseteq cone(S)$ be the Newton polyhedron of a set $A \subset S$. Then*

$$\Gamma_+(A) = \{x \in cone(S) : \langle x,v\rangle \geqslant \ell(v,\Gamma_+(A)),\ \textit{for all } v \in \widetilde{cone}(S)\}.$$

Proof. Let $x \in \Gamma_+(A)$. By the definition of $\ell(v,\Gamma_+(A))$, we have

$$\ell(v,\Gamma_+(A)) = \min\{\langle k,v\rangle : k \in \Gamma_+(A)\} \leqslant \langle x,v\rangle, \forall v \in \widetilde{cone}(S)$$

Hence, $x \in \{x \in cone(S) : \langle x,v\rangle \geqslant \ell(v,\Gamma_+(A)),\ \forall v \in \widetilde{cone}(S)\}$.

Conversely, assume that $x \in cone(S)$ and $\langle x,v\rangle \geqslant \ell(v,\Gamma_+(A))$, for all $v \in \widetilde{cone}(S)$. Let $\Delta_1,\dots,\Delta_r$ be the facets (i.e. the faces of codimension 1) of $\Gamma_+(A)$. For each $i \in \{1,\dots,r\}$, there exists $v_i \in \widetilde{cone}(S)$ such that $\Delta_i = \Delta(v_i,\Gamma_+(A))$. In particular, the supporting half-space

$$H_i = \{y \in \mathbb{R}^n : \langle y,v_i\rangle \geqslant \ell(v_i,\Gamma_+(A))\}$$

is exactly the half-space determined by the facet Δ_i.

Since Γ_+ is a polyhedron, we have

$$\Gamma_+(A) = \bigcap_{i=1}^{r} H_i.$$

By the assumption, $\langle x, v_i \rangle \geqslant \ell(v_i, \Gamma_+(A))$, for all $i = 1, \ldots, r$, hence $x \in H_i^+$ for all i. Therefore,

$$x \in \bigcap_{i=1}^{r} H_i = \Gamma_+(A).$$

This completes the proof. $\square$

Example 2.4. Let $cone(S)$ represent the convex polyhedral cone associated with

$$S = \langle (1,3), (1,0), (3,5) \rangle \subset \mathbb{Z}_+^2.$$

Now, consider the set $A = \{(5,0), (2,6), (3,2), (4,4)\}$. The Newton polyhedron determined by A is illustrated in the figure 2.2.

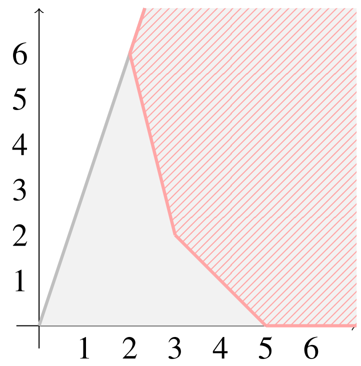

Figure 2.2: Newton polyhedron of $A = \{(5,0), (2,6), (3,2), (4,4)\}$.

After determining the Newton polyhedron $\Gamma_+(A)$, we describe its faces. The zero-dimensional faces of $\Gamma_+(A)$ consist of the vertices $(5,0)$, $(3,2)$, $(2,6)$. The compact one-dimensional faces are the line segments

$$\Delta_1 = \operatorname{conv}(5,0), (3,2), \qquad \Delta_2 = \operatorname{conv}(3,2), (2,6).$$

They are characterized by the vectors $(1,1)$ and $(4,1)$, respectively. More precisely,

$$\Delta_1 = \Delta\big((1,1), \Gamma_+(A)\big), \qquad \ell\big((1,1), \Gamma_+(A)\big) = 5, \text{and}$$

$$\Delta_2 = \Delta\big((4,1), \Gamma_+(A)\big), \qquad \ell\big((4,1), \Gamma_+(A)\big) = 14,$$

with $(1,1), (4,1) \in \breve{cone}(S)$. The non-compact one-dimensional faces of $\Gamma_+(A)$ are

$$\Delta_3 = \Delta\big((3,-1), \Gamma_+(A)\big), \qquad \Delta_4 = \Delta\big((0,1), \Gamma_+(A)\big),$$

with associated vectors $(3,-1)$ and $(0,1)$, respectively. In both cases,

$$\ell\big((3,-1), \Gamma_+(A)\big) = \ell\big((0,1), \Gamma_+(A)\big) = 0,$$

and $(3,-1), (0,1) \in \breve{cone}(S)$.

As shown in Figure 2.3, the Newton polyhedron $\Gamma_+(A)$ is completely determined by the half-spaces associated with the faces Δ_i, $i=1,\dots,4$. More precisely, consider the half-spaces

$$H_1=\{(x,y)\in\mathbb{R}^2 \mid x+y\geqslant 5\},\qquad H_2=\{(x,y)\in\mathbb{R}^2 \mid 4x+y\geqslant 14\},$$

$$H_3=\{(x,y)\in\mathbb{R}^2 \mid 3x-y\geqslant 0\},\qquad H_4=\{(x,y)\in\mathbb{R}^2 \mid y\geqslant 0\}.$$

Then, $\Gamma_+(A)=\bigcap_{i=1}^4 H_i$.

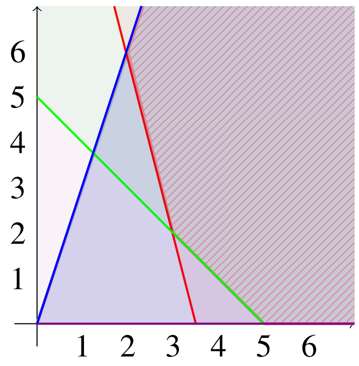

Figure 2.3: Representation of the Newton polyhedron $\Gamma_+(A)$ as the intersection of the half–spaces determined by the faces Δ_i, $i=1,\dots,4$.

Newton Polyhedron of an ideal $I\subset\mathscr{O}_{X(S)}$

In the sequel, we introduce the notion of the Newton polyhedron associated with an ideal of the local ring $\mathscr{O}_{X(S)}$ of germs of analytic functions on $(X(S),0)$.

Let $(X(S),0)\subset(\mathbb{C}^r,0)$ be the germ of the toric variety $X(S)$ at $0\in\mathbb{C}^r$. Recall that a germ of an analytic function $g:(X(S),0)\to(\mathbb{C},0)$ is the restriction to $X(S)$ of the germ at 0 of an analytic function defined in a neighborhood of 0 in $\mathbb{C}^r$ (see Definition 1.5).

Throughout this chapter, whenever we refer to monomials, monomial ideals, or Newton polyhedra, we always consider the canonical coordinate system of the ambient affine space. In particular, monomials are understood with respect to the canonical coordinates $x_1,\dots,x_r$.

We begin by defining the Newton polyhedron associated with a germ in $\mathscr{O}_{X(S)}$, and then extend this construction to ideals of $\mathscr{O}_{X(S)}$.

Definition 2.5. Let $g\in\mathscr{O}_{X(S)}$, i.e., $g=f|_{X(S)}$, where $f:\mathbb{C}^r\to\mathbb{C}$ is an analytic function such that $f=\sum_k a_k x^k$ is its Taylor expansion. The *support* of $g\in\mathscr{O}_{X(S)}$ is defined by the set

$$\operatorname{supp}(g)=\Big\{\nu(k)=k_1b_1+\cdots+k_rb_r\in S:\sum_{\nu(k')=\nu(k)}a_{k'}\neq 0 \text{ and } k=(k_1,\dots,k_r)\Big\},$$

where $b_1,\dots,b_r$ are generators of S. The Newton polyhedron of g is defined by $\Gamma_+(g)=\Gamma_+(\operatorname{supp}(g))$.

If $v\in\widecheck{cone}(S)$, we define $\ell(v,g)=\ell(v,\Gamma_+(g))$ and $\Delta(v,g)=\Delta(v,\Gamma_+(g))$.

Example 2.6. Let $S = \langle (1,0,0),(1,1,0),(1,0,1),(1,2,0),(1,1,1),(1,0,2)\rangle$. Consider

$$g = 80x_4^2 + x_1^2x_3^2 + 5x_1x_2^5x_3 + 60x_4^2x_5 + 9x_6 - x_1^3x_6 - 2x_1^3 \in \mathscr{O}_{X(S)}.$$

The support of g is given by $\operatorname{supp}(g) = \{(7,5,1),(3,5,1),(4,0,2),(1,0,2),(3,0,0),(2,4,0)\}$. Consequently, the Newton polyhedron of I is represented in Figure 2.4.

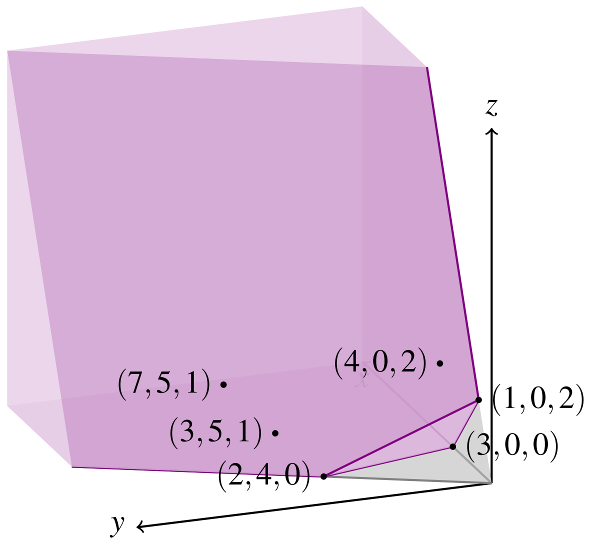

Figure 2.4: Newton polyhedron of g

Consider $G \subseteq \mathscr{O}_{X(S)}$. We define the support of G as

$$\operatorname{supp}(G) = \bigcup_{g\in G} \operatorname{supp}(g), \tag{2.1}$$

and the Newton polyhedron of G as $\Gamma_+(G) = \Gamma_+(\operatorname{supp}(G))$.

Proposition 2.7. *If $I \subseteq \mathscr{O}_{X(S)}$ is an ideal generated by G, then $\Gamma_+(I) = \Gamma_+(G)$.*

Proof. By definition, $\Gamma_+(I) = \Gamma_+(\operatorname{supp}(I))$ and $\Gamma_+(G) = \Gamma_+(\operatorname{supp}(G))$, where $\operatorname{supp}(G)$ denotes the union of the supports of the elements of G. Since $G \subset I$, it follows immediately that $\operatorname{supp}(G) \subset \operatorname{supp}(I)$. Consequently,

$$\operatorname{supp}(G) + cone(S) \subset \operatorname{supp}(I) + cone(S) \subset \Gamma_+(I).$$

By the Definition of the convex hull, this implies that $\Gamma_+(G) \subset \Gamma_+(I)$.

To prove the reverse inclusion, let $f \in I$. Since $\mathscr{O}_{X(S)}$ is Noetherian, the ideal I is finitely generated. Let $G = \{g_1,\ldots,g_m\} \subset I$ be a finite generating set of I. Then there exist $h_1,\ldots,h_m \in \mathscr{O}_{X(S)}$ such that

$$f = \sum_{j=1}^{m} h_j g_j.$$

For each j, we have $\mathrm{supp}(h_j g_j) \subset \mathrm{supp}(h_j) + \mathrm{supp}(g_j)$, for all j. Since $\mathrm{supp}(h_j) \subset cone(S)$, it follows that every exponent vector in $\mathrm{supp}(f)$ belongs to $\mathrm{supp}(g_j) + cone(S)$, for some j. Hence,

$$\mathrm{supp}(f) \subset \Big(\bigcup_{j=1}^{m} \mathrm{supp}(g_j)\Big) + cone(S) = \mathrm{supp}(G) + cone(S),$$

which implies $\mathrm{supp}(f) + cone(S) \subset \mathrm{supp}(G) + cone(S)$. Since this holds for every $f \in I$, we obtain

$$\mathrm{supp}(I) + cone(S) \subset \mathrm{supp}(G) + cone(S).$$

Again by the definition of the convex hull, this yields that $\Gamma_+(I) \subset \Gamma_+(G)$. Combining both inclusions yields $\Gamma_+(I) = \Gamma_+(G)$, which completes the proof. □

Therefore, Proposition 2.7 ensures that the Newton polyhedron $\Gamma_+(I)$ is well defined, that is, it does not depend on the choice of generators of I. If $I \subseteq \mathscr{O}_{X(S)}$ is the ideal generated by G, then $\Gamma_+(I) = \Gamma_+(G)$. Moreover, if $G = \{g_1, \dots, g_m\}$ is a generating system of I, then $\Gamma_+(I)$ is equal to the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_m)$, by basic properties of the convex hull.

Example 2.8. Let $S = \langle (1,0), (1,1), (1,2) \rangle$. Consider the ideal $I \subset \mathscr{O}_{X(S)}$ generated by

$$g_1 = x_1^2 x_2^2 + x_3^4 + x_1^4 x_3 \text{ and } g_2 = x_1^5 x_3 + x_3^3 + x_1^3 x_2^2.$$

The support of I is given by $\mathrm{supp}(I) = \{(4,2), (4,8), (5,2), (6,2), (3,6)\}$. Consequently, the Newton polyhedron of I is represented in Figure 2.5.

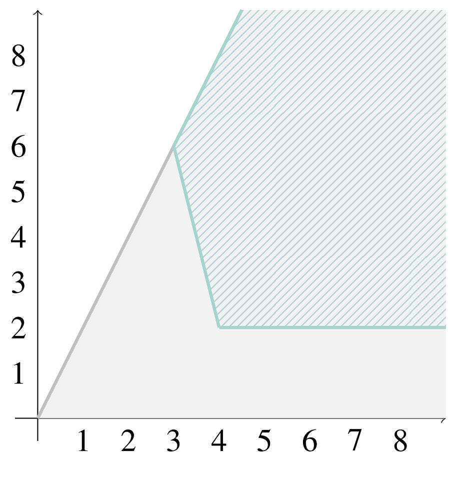

Figure 2.5: Newton polyhedron of $I = \langle x_1^2 x_2^2 + x_3^4 + x_1^4 x_3, x_1^5 x_3 + x_3^3 + x_1^3 x_2^2 \rangle$.

2.2 The integral closure of ideals in $\mathscr{O}_{X(S)}$

In this section, we investigate the relationship between the integral closure of an ideal I and the geometry of its Newton polyhedron $\Gamma_+(I)$. Motivated by the approach of Tessier [51], we first restrict our attention to monomial ideals in $\mathscr{O}_{X(S)}$. In this setting, we show that if $h \in \bar{I}$, then every monomial appearing in h corresponds to a point of $cone(S)$ lying in the Newton polyhedron $\Gamma_+(I)$.

This observation indicates that the integral closure of an ideal I is largely controlled by the combinatorial information encoded in its Newton polyhedron. In particular, it naturally leads to the introduction of a canonical monomial ideal determined solely by $\Gamma_+(I)$, which collects exactly the exponent vectors compatible with this polyhedral data.

Following [7], we denote by I° the ideal of $\mathscr{O}_{X(S)}$ generated by the monomials x^k such that

$$k_1b_1 + \cdots + k_rb_r \in \Gamma_+(I),$$

where $x_1, \ldots, x_r$ is a coordinate system on $X(S)$.

The ideal I° can be regarded as the largest monomial ideal whose Newton polyhedron coincides with $\Gamma_+(I)$, and therefore provides a natural link between I and its associated polyhedral data.

This construction yields a natural framework for describing the integral closure of I in terms of Newton polyhedra. Indeed, the inclusion $I \subset I^\circ$ always holds, and equality characterizes important classes of ideals, most notably the class of non-degenerate ideals. In this sense, I° serves as a bridge between the analytic structure of ideals in $\mathscr{O}_{X(S)}$ and the convex geometry of their associated Newton polyhedra.

The next theorem makes precise the relationship between the integral closure of a monomial ideal and its associated Newton polyhedron.

Theorem 2.9. *Let $I \subset \mathscr{O}_{X(S)}$ be a monomial ideal. If $h \in \bar{I}$ then* $\operatorname{supp}(h) \subset \Gamma_+(I)$.

Proof. Consider $h \in \bar{I} \subseteq \mathscr{O}_{X(S)}$. Note that $h = f|_{X(S)}$, where $f : \mathbb{C}^r \to \mathbb{C}$ is an analytic function with Taylor expansion $f = \sum_k a_k x^k$, $k \in \mathbb{Z}^r_+$. Also observe that I is a monomial ideal in the semigroup ring generated by $S \subseteq \mathbb{Z}^r_+$. According to [14, Theorem 4.45], $\bar{I}$ is also a monomial ideal. Consequently, $x^k \in \bar{I}$, for all x^k that appears in $f = \sum_k a_k x^k$.

Let us denote by $d = (d_1, \ldots, d_n)$ the element of the form $d = \tilde{k}_1 b_1 + \cdots + \tilde{k}_r b_r \in \operatorname{supp}(h)$ for some $\tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_r) \in \mathbb{Z}^r_+$ such that $a_{\tilde{k}} \neq 0$. Let Δ be a face of $\Gamma_+(I)$. By Remark 2.2, there exist $\gamma = (\gamma_1, \ldots, \gamma_n) \in \widecheck{cone}(S) \cap \mathbb{Z}^n$ and $\eta \geqslant 0$ such that

$$\Delta = \{c = (c_1, \ldots, c_n) \in \Gamma_+(I) : \langle c, \gamma \rangle = \eta\}.$$

Let $\varphi : \mathbb{C} \to X(S)$ be the analytic curve defined by

$$\varphi(t) = (t^{\langle b_1, \gamma \rangle}, \ldots, t^{\langle b_r, \gamma \rangle}).$$

Since $X(S) = V(I_S)$ is a toric variety, then I_S is a binomial ideal (see [16]). Therefore, $(1, \ldots, 1) \in X(S)$. Note that, for $t \neq 0$, we have $(t^{\gamma_1}, \ldots, t^{\gamma_n}) \in (\mathbb{C}^*)^n$. Considering the toric action defined in (1.3), we obtain

$$\varphi(t) = \phi((t^{\gamma_1}, \ldots, t^{\gamma_n}), (1, \ldots, 1)) = (t^{\langle b_1, \gamma \rangle}, \ldots, t^{\langle b_r, \gamma \rangle}) \in X(S).$$

So, we have

$$x^{\tilde{k}} \circ \varphi(t) = t^{\langle b_1,\gamma\rangle \tilde{k}_1} \cdots t^{\langle b_r,\gamma\rangle \tilde{k}_r} = t^N,$$

where $N = \langle b_1,\gamma\rangle \tilde{k}_1 + \cdots + \langle b_r,\gamma\rangle \tilde{k}_r$. Let $\upsilon : \mathscr{O}_1 \to \mathbb{Z} \cup \{\infty\}$ be the usual valuation on $\mathscr{O}_1$. Thus, we obtain

$$\upsilon(x^{\tilde{k}} \circ \varphi) = \langle b_1,\gamma\rangle \tilde{k}_1 + \cdots + \langle b_r,\gamma\rangle \tilde{k}_r = N.$$

Since $x^{\tilde{k}} \in \bar{I}$, then $x^{\tilde{k}} \circ \varphi \in \varphi^*(I)\mathscr{O}_1$, i.e., $x^{\tilde{k}} \circ \varphi = f_1(q_1 \circ \varphi) + \cdots + f_l(q_l \circ \varphi)$, where $q_1, \ldots, q_l \in I$ and $f_1, \ldots, f_l \in \mathscr{O}_1$. We obtain

$$\begin{aligned} \upsilon(x^{\tilde{k}} \circ \varphi) \geqslant \min_i \{\upsilon(f_i(q_i \circ \varphi))\} &= \upsilon(f_j(q_j \circ \varphi)) \\ &\geqslant \upsilon(f_j) + \upsilon(q_j \circ \varphi) \\ &\geqslant \upsilon(q_j \circ \psi) \\ &\geqslant \upsilon(g \circ \psi), \end{aligned} \tag{2.2}$$

where g is a generator of I. Taking $g = x^w$ such that $w_1 b_1 + \cdots + w_r b_r \in \Delta$, we have $x^w \circ \varphi(t) = t^{\langle b_1,\gamma\rangle w_1 + \cdots + \langle b_r,\gamma\rangle w_r}$. Then

$$\upsilon(x^w \circ \varphi) = w_1\langle b_1,\gamma\rangle + \cdots + w_r\langle b_r,\gamma\rangle = \langle w_1 b_1 + \cdots + w_r b_r, \gamma\rangle = \eta.$$

Moreover, based on the inequality (2.2), we have

$$\langle d,\gamma\rangle = N = \upsilon(x^{\tilde{k}} \circ \varphi) \geqslant \upsilon(x^w \circ \varphi) = \eta.$$

Then d is above Δ. Therefore, $\operatorname{supp}(h) \subset \Gamma_+(I)$. □

Using this result, we can prove the following corollary.

Corollary 2.10. *Let I be an ideal of $\mathscr{O}_{X(S)}$. Then we have:*

(i) I° *is integrally closed.*

(ii) $\Gamma_+(I) = \Gamma_+(\bar{I})$

Proof. (i) Let $h \in \overline{I^\circ}$. By Theorem 2.9, $\operatorname{supp}(h) \subset \Gamma_+(I^\circ)$. As $\Gamma_+(I^\circ) = \Gamma_+(I)$, the supports of the monomials that appear in $h = \sum_k a_k x^k$ are in $\Gamma_+(I)$, i.e., $k_1 b_1 + \cdots + k_r br \in \Gamma_+(I)$. By definition, these monomials are in I°. It follows that $h \in I^\circ$.

(ii) By definition, $I \subset I^\circ$. So $\bar{I} \subset \overline{I^\circ}$. It follows from (i) that $\bar{I} \subset \overline{I^\circ} = I^\circ$. Thus, we have $I \subset \bar{I} \subset I^\circ$ and, then, $\Gamma_+(I) \subset \Gamma_+(\bar{I}) \subset \Gamma_+(I^\circ) = \Gamma_+(I)$. □

In [5, Prop. 3.8] the authors also study the integral closure of monomial ideals in $\mathscr{O}_{X(S)}$. We remark that in our work the monomial hypothesis is not necessary to obtain the related results.

According to item (ii) of Corollary 2.10, we can affirm that $\bar{I} \subset I^\circ$. However, it is not always true that $I^\circ \subset \bar{I}$, as we can see in the following example.

Example 2.11. Let S be the semigroup generated by the set $\{(1,0),(1,1),(1,2)\} \subset \mathbb{Z}_+^2$. Then, the toric surface $X(S) = V(I_S) \subset \mathbb{C}^3$, where I_S is the ideal generated by the binomial $xz - y^2$. Consider $I \subset \mathscr{O}_{X(S)}$ the ideal generated by $g_1 = x_1^2x_2 + 3x_1^2x_3$ and $g_2 = x_2^3 - x_1x_2 + 2x_3^2$. Then,

$$\mathrm{supp}(I) = \{(3,1),(3,2),(3,3),(2,1),(2,4)\},$$

and the Newton polyhedron of I is represented in the Figure 2.6. Note that $x_2^2 \in I^\circ$, since $(2,2) \in \Gamma_+(I)$. But $x_2^2 \notin \bar{I}$, because there is an analytic curve $\psi : (\mathbb{C},0) \to (X(S),0)$ defined by $\psi(t) = (t,t,t)$ such that $x_2^2 \circ \psi \notin \psi^*(I)\mathscr{O}_1$.

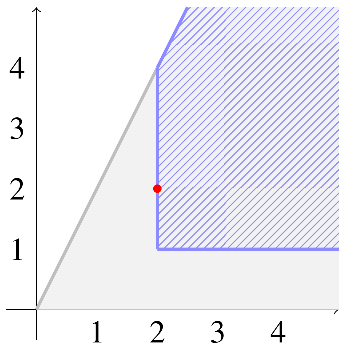

Figure 2.6: Newton polyhedron of $I = \langle x_1^2x_2 + 3x_1^2x_3, x_2^3 - x_1x_2 + x_3^2 \rangle$.

2.3 Non-degenerate ideals

In this section, we introduce the notion of non-degenerate ideals in the local ring $\mathscr{O}_{X(S)}$, motivated by the classical theory of non-degeneracy via Newton polyhedra. Our aim is to relate the algebraic structure of an ideal to the geometry of its Newton polyhedron.

In particular, we establish a relationship between the Newton polyhedron of an ideal $I \subset \mathscr{O}_{X(S)}$ and that of the associated monomial ideal K_I, generated by all monomials belonging to the integral closure of I. This provides a combinatorial framework for studying non-degeneracy in the toric setting.

We now fix some notation associated to a germ in the local ring $\mathscr{O}_{X(S)}$.

Let $g \in \mathscr{O}_{X(S)}$. As we said before, this means that $g = f|_{X(S)}$, where $f : \mathbb{C}^r \to \mathbb{C}$ is an analytic function, and then can be written as $f = \sum_k a_k x^k$. Now, given a compact subset $A \subseteq cone(S)$, we denote by

$$g_A := \sum_{v(k) \in A \cap \mathrm{supp}(g)} a_k x^k.$$

We set $g_A = 0$ whenever $A \cap \mathrm{supp}(g) = \varnothing$.

Let g be a representative of the germ $g \in \mathscr{O}_{X(S)}$ such that $0 \notin \mathrm{supp}(g)$. Then, we associate to g a polynomial $L(g) \in \mathscr{O}_n$ given by

$$L(g) = \sum_{v \in \mathrm{supp}(g)} a_v z^v,$$

where $a_v = \sum_{\nu(k')=\nu(k)} a_{k'} \neq 0$ and $v = \nu(k)$.

Definition 2.12. Let $G = \{g_1, \dots, g_l\} \subseteq \mathscr{O}_{X(S)}$. We say that G is *non-degenerate* if, for each compact face Δ of $\Gamma_+(G)$, the equations

$$L\big((g_1)_\Delta\big) = \cdots = L\big((g_l)_\Delta\big) = 0$$

have no common solution in $(\mathbb{C}^*)^n$.

An ideal $I \subseteq \mathscr{O}_{X(S)}$ is *non-degenerate* if I admits a non-degenerate system of generators.

Remark 2.13. Upon considering $S = \langle e_1, \dots, e_n \rangle$, the toric variety $X(S)$ corresponds to $\mathbb{C}^n$. In this case, we have $cone(S) = \mathbb{R}^n_+$. Thus, the previous definition can be seen as a generalization of the definition of Newton non-degenerate in $\mathscr{O}_n$.

Note that $\operatorname{supp}(L(g)) = \operatorname{supp}(g)$ for all $g \in \mathscr{O}_{X(S)}$. Consequently, the compact faces of $\Gamma_+(L(g))$ are also compact faces of $\Gamma_+(g)$. However, $\Gamma_+(g)$ may have other compact faces. Therefore, if $g \in \mathscr{O}_{X(S)}$ is non-degenerate, then $L(g) \in \mathscr{O}_n$ is Newton non-degenerate. The converse, however, does not hold.

Example 2.14. Let $S = \langle (1,0), (1,1), (1,2) \rangle$. Consider the ideal $I \subset \mathscr{O}_{X(S)}$ given in Example 2.11, that is, the ideal generated by

$$g_1 = x_1^2 x_2 + 3x_1^2 x_3 \text{ and } g_2 = x_2^3 - x_1 x_2 + 2x_3^2.$$

For the face Δ formed by the line segment connecting the vertices $(2,4)$ and $(2,1)$, we have

$$(g_1)_\Delta = 0 \quad \text{and} \quad (g_2)_\Delta = -x_1 x_2 + 2x_3^2.$$

We associate $(g_1)_\Delta$ and $(g_2)_\Delta$ with the following polynomials

$$L((g_1)_\Delta) = 0 \quad \text{and} \quad L((g_2)_\Delta) = -z_1^2 z_2 + 2z_1^2 z_2^4.$$

Note that $L((g_1)_\Delta)$ and $L((g_2)_\Delta)$ cancel simultaneously in $(\frac{1}{2}, \frac{1}{\sqrt[3]{2}}) \in (\mathbb{C}^*)^2$. Therefore, I is degenerate. However, the ideal

$$L(I) = \langle L((g_1)), L((g_2)) \rangle = \langle z_1^3 z_2 + 3z_1^3 z_2^2, z_1^3 z_2^3 - z_1^2 z_2^1 + 2z_1^2 z_2^4 \rangle \subset \mathscr{O}_2$$

is Newton non-degenerate, since $\tilde{\Delta} = \{(2,1)\}$ is unique compact face of the Newton polyhedron of $L(I)$ (Figure 2.7). In this case, we have $L((g_1)_{\tilde{\Delta}}) = 0$ and $L((g_2)_{\tilde{\Delta}}) = -z_1^2 z_2$. Thus,

$$\{(z_1, z_2) \in \mathbb{C}^2 : L((g_1)_{\tilde{\Delta}})(z_1, z_2) = L((g_2)_{\tilde{\Delta}})(z_1, z_2) = 0\} = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}.$$

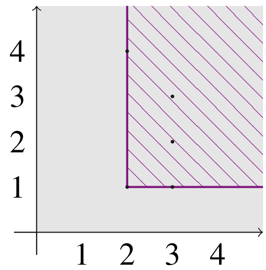

Figure 2.7: Newton polyhedron of $L(I) = \langle z_1^3 z_2 + 3z_1^3 z_2^2, z_1^3 z_2^3 - z_1^2 z_2^1 + 2z_1^2 z_2^4 \rangle \subset \mathscr{O}_2$.

In the above definition we ask only for algebraic conditions. This will be enough for our purposes. However, in the literature, we can find other definitions that also require geometrical conditions. For instance, in [40] Matsui and Takeuchi define the concept of Newton non-degenerate functions and complete intersection to study Milnor fiber over singular toric varieties. For this purpose, they also used geometrical conditions.

Yoshinaga proved in [54] that a function $f \in \mathscr{O}_n$ is Newton non-degenerate if and only if the integral closure of the ideal

$$I(f) = \left\langle x_1 \frac{\partial f}{\partial x_1}, \ldots, x_n \frac{\partial f}{\partial x_n} \right\rangle$$

is generated by the monomials x^k such that $k \in \Gamma_+(f)$. Saia extended this result in [48], proving that an ideal $I \subseteq \mathscr{O}_n$ is Newton non-degenerate if and only if $\overline{I}$ is generated by the monomials x^k such that $k \in \Gamma_+(I)$. Our goal is to extend this result to ideals in $\mathscr{O}_{X(S)}$. For this purpose, we begin by establishing some preliminary results.

Definition 2.15. Let I be an ideal in $\mathscr{O}_{X(S)}$. We define the set $C(\overline{I})$ as the convex hull in $cone(S)$ of the set

$$\bigcup \left\{ \lambda_1 b_1 + \cdots + \lambda_r br : x^\lambda \in \overline{I} \right\},$$

where $\lambda = (\lambda_1, \cdots, \lambda_r)$.

Observe that $C(\overline{I})$ is the Newton polyhedron of the ideal K_I generated by all monomials belonging to the integral closure of I, i.e., $K_I = \langle \{x^m : x^m \in \overline{I}\} \rangle$.

Proposition 2.16. *Let I be an ideal in $\mathscr{O}_{X(S)}$. Then, $C(\overline{I}) = \Gamma_+(K_I)$, where $K_I = \langle \{x^m : x^m \in \overline{I}\} \rangle$.*

Proof. Observe that $\bigcup \{\lambda_1 b_1 + \cdots + \lambda_r br : x^\lambda \in \overline{I}\} = \mathrm{supp}(K_I)$. Thus, $\bigcup \{\lambda_1 b_1 + \cdots + \lambda_r br : x^\lambda \in \overline{I}\} \subset \Gamma_+(K_I)$. It follows that $C(\overline{I}) \subset \Gamma_+(K_I)$, since $C(\overline{I})$ is the smallest convex set containing $\bigcup \{\lambda_1 b_1 + \cdots + \lambda_r br : x^\lambda \in \overline{I}\}$. On the other hand, if $a \in \mathrm{supp}(K_I) + cone(S)$, then

$$a = (m_1 b_1 + \cdots + m_r b_r) + (s_1 b_1 + \cdots + s_r b_r),$$

where $m_1 b_1 + \cdots + m_r b_r \in \mathrm{supp}(K_I)$ and $s_1 b_1 + \cdots + s_r b_r \in cone(S)$.

Since $\overline{I}$ is a ideal of $\mathscr{O}_{X(S)}$, $x^m \in \overline{I}$. Furthermore, $x^m x^s \in \overline{I}$. With this, we have

$$a \in \bigcup \{\lambda_1 b_1 + \cdots + \lambda_r br : x^\lambda \in \overline{I}\}.$$

We conclude $\Gamma_+(K_I) \subseteq C(\bar{I})$, since $\Gamma_+(K_I)$ is the smallest convex set that contains $\mathrm{supp}(K_I) + cone(S)$, $\square$

Lemma 2.17. *Let $I = \langle g_1, \dots, g_l \rangle \subseteq \mathscr{O}_{X(S)}$ be the ideal generated by the set $\{g_1, \dots, g_l\} \subset \mathscr{O}_{X(S)}$. Then $C(\bar{I}) \subseteq \Gamma_+(I)$.*

Proof. If $d = \lambda_1 b_1 + \cdots + \lambda_r b_r \notin \Gamma_+(I)$, there exists some $v \in \breve{cone}(S) \cap \mathbb{Z}^n$ such that

$$\langle d, v \rangle < \ell(v, \Gamma_+(I)).$$

By the definition of $\ell(v, \Gamma_+(I))$, we have

$$\ell(v, \Gamma_+(I)) \leqslant \langle \beta_{j1} b_1 + \cdots + \beta_{jr} b_r, v \rangle,$$

where $\beta_{j1} b_1 + \cdots + \beta_{jr} b_r \in \mathrm{supp}(g) \subset \Gamma_+(I)$, for all $g \in I$. Then

$$\lambda_1 \langle b_1, v \rangle + \cdots + \lambda_r \langle b_r, v \rangle = \langle d, v \rangle < \beta_{j1} \langle b_1, v \rangle + \cdots + \beta_{jr} \langle b_r, v \rangle. \tag{2.3}$$

Consider $\varphi : \mathbb{C} \to X(S)$ defined by

$$\varphi(t) = (t^{\langle b_1, v \rangle}, \dots, t^{\langle b_r, v \rangle}).$$

Since $X(S) = V(I_S)$ is a toric variety, I_S is a binomial ideal. So, $(1, \dots, 1) \in X(S)$. Note that $(t^{v_1}, \dots, t^{v_n}) \in (\mathbb{C}^*)^n$, for $t \neq 0$. Thus, considering the toric action defined in (1.3), we obtain

$$\varphi(t) = \phi((t^{v_1}, \dots, t^{v_n}), (1, \dots, 1)) = (t^{\langle b_1, v \rangle}, \dots, t^{\langle b_r, v \rangle}) \in X(S).$$

Furthermore, $\varphi(0) = (0, \dots, 0) \in X(S)$, since $\dim(cone(S)) = n$ and $cone(S)$ is strongly convex. We have

$$x^\lambda \circ \varphi = t^{\lambda_1 \langle b_1, v \rangle + \cdots + \lambda_r \langle b_r, v \rangle},$$

and for all $g \in I$, we can write

$$g = \sum_{\beta_{j1} b_1 + \cdots + \beta_{jr} b_r \in \mathrm{supp}(g)} a_j x^{\beta_j}.$$

Thus,

$$g \circ \varphi = \sum_{\beta_{j1} b_1 + \cdots + \beta_{jr} b_r \in \mathrm{supp}(g)} a_j t^{\beta_{j1} \langle b_1, v \rangle + \cdots + \beta_{jr} \langle b_r, v \rangle}.$$

By inequality (2.3), $x^\lambda \circ \varphi \notin \varphi^*(I) \mathscr{O}_1$. Therefore, $x^\lambda \notin \bar{I}$. Consequently

$$d \notin \bigcup \left\{ \lambda_1 b_1 + \cdots + \lambda_r br : x^\lambda \in \bar{I} \right\}.$$

We proved that $\bigcup \left\{ \lambda_1 b_1 + \cdots + \lambda_r br : x^\lambda \in \bar{I} \right\} \subset \Gamma_+(I)$. Hence, $C(\bar{I}) \subset \Gamma_+(I)$. $\square$

2.4 Toroidal Embedding

Our main goal is to prove that an ideal $I \subseteq \mathscr{O}_{X(S)}$ is non-degenerate if and only if $C(\bar{I}) = \Gamma_+(I)$. To do this, we will explore the construction of the toroidal embedding associated with the Newton polyhedron $\Gamma_+(I)$ in this chapter.

In [48], Saia generalized Yoshinaga's result [54] using the construction of a toroidal embedding associated with the Newton polyhedron of a finite codimensional ideal I in $\mathscr{O}_n$ to show that I is Newton non-degenerate if and only if the Newton polyhedron of I is the convex hull of the set of n-tuples $m = (m_1, \ldots, m_n)$ such that $x^m \in \bar{I}$. The procedure for constructing the toroidal embeddings associated with a given Newton polyhedron is a local modification of Khovanskii's method of assigning a compact complex non-singular toroidal manifold to an integer-valued compact convex polyhedron in $\mathbb{C}^n$. Here, we apply this procedure to the Newton polyhedron in the $cone(S)$. This construction is the essential tool used to prove our main result.

For any ideal $I \subseteq \mathscr{O}_{X(S)}$ we describe a partition of $\widetilde{cone}(S)$ into convex cones with respect to $\Gamma_+(I)$.

Definition 2.18. The vector $a = (a_1, \ldots, a_n) \in \widetilde{cone}(S)$ is called a *primitive integer* vector if a is the vector with minimum length in $C(a) \cap (\mathbb{Z}^n - \{0\})$, where $C(a)$ is the half ray emanating from 0 passing thorough a.

We define an equivalence relation on $\widetilde{cone}(S)$ by

$$a \sim a' \text{ if only if } \Delta(a) = \Delta(a').$$

Considering this equivalence, any equivalence class is naturally identified with a convex cone with its vertex at zero, specified by finitely many linear equations and strictly linear inequalities with rational coefficients.

The closures of equivalence classes specify a partition Σ_0 of $\widetilde{cone}(S)$ into closed convex cones that have the properties:

- If σ_1 is a face of a cone σ in Σ_0, then $\sigma_1 \in \Sigma_0$.
- For any cones σ_1 and σ_2 in Σ_0, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

In this way, based on Σ_0 and using the algorithm described in the proof of Theorem 11 from [35], we construct a partition Σ of $\widetilde{cone}(S)$ into finitely many closed convex cones with their vertices at zero such that:

1. Any cone belonging to Σ lies in one of the cones in Σ_0 and is specified by finitely many linear equations and linear inequalities with rational coefficients.
2. If σ_1 is a face of a cone σ in Σ, then $\sigma_1 \in \Sigma$.
3. For any cones σ_1 and σ_2 in Σ, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

4. Any cone q-dimensional σ in Σ is simplicial and unimodular, i.e., there exists a set of primitive integer vectors $a^1(\sigma),\dots,a^q(\sigma)$ which are linearly independent over $\mathbb{R}$ and $n-q$ primitive integer vectors $a^{q+1}(\sigma),\dots,a^n(\sigma)$ such that

$$\mathbb{Z}a^1(\sigma)+\cdots+\mathbb{Z}a^n(\sigma)=\mathbb{Z}^n.$$

Based on these properties, we can make the following observations:

For a 1-dimensional cone σ in Σ, where $a^i(\sigma)$ represents the corresponding primitive integer of σ, the set $\Delta(a^i(\sigma))$ is a closed face of dimension $(n-1)$ of $\Gamma_+(I)$.

For each compact face Δ of $\Gamma_+(I)$, there exist a cone $\sigma\in\Sigma$ of dimension n and a subset J of $\{1,\dots,n\}$ such that Δ can be expressed as the intersection $\bigcap_{j\in J}\Delta(a^j(\sigma))$, where $a^1(\sigma),\dots,a^n(\sigma)$ is the corresponding set of primitive integer vectors of σ. We shall denote this face Δ by Δ_J.

Proposition 2.19. *If Δ is a compact face of $\Gamma_+(I)$ and Δ_J is defined as above, then $\Delta_J=\Delta\big(\sum_{i\in J}a^i(\sigma)\big)$.*

Proof. Suppose that $k\notin\Delta_J$. Then, there exists at least one $j\in J$ such that $k\notin\Delta(a^j(\sigma))$, that is,

$$\langle k,a^j(\sigma)\rangle>\ell(a^j(\sigma)).$$

Note that

$$\begin{aligned}\Big\langle k,\sum_{i\in J}a^i(\sigma)\Big\rangle &= \sum_{i\in J}\langle k,a^i(\sigma)\rangle\\ &= \langle k,a^j(\sigma)\rangle+\sum_{i\in J,i\neq j}\langle k,a^i(\sigma)\rangle\\ &\geqslant \langle k,a^j(\sigma)\rangle+\sum_{i\in J,i\neq j}\ell(a^i(\sigma))\\ &> \sum_{i\in J}\ell(a^i(\sigma))=\ell\Big(\sum_{i\in J}a^i(\sigma)\Big).\end{aligned}$$

Thus, $k\notin\Delta\Big(\sum_{i\in J}a^i(\sigma)\Big)$.

On the other hand, if $k\in\Delta_J$, then $k\in\Delta(a^i(\sigma))$ for all $i\in J$. Hence, we have

$$\Big\langle k,\sum_{i\in J}a^i(\sigma)\Big\rangle=\sum_{i\in J}\langle k,a^i(\sigma)\rangle=\sum_{i\in J}\ell(a^i(\sigma))=\ell\Big(\sum_{i\in J}a^i(\sigma)\Big).$$

It follows that $k\in\Delta\Big(\sum_{i\in J}a^i(\sigma)\Big)$. Therefore,

$$\Delta_J=\Delta\Big(\sum_{i\in J}a^i(\sigma)\Big).$$

□

For all n-dimensional cones $\sigma \in \Sigma$, $\bigcap_{i=1}^{n} \Delta(a^i(\sigma))$ is a one-point set.

Let σ be a n-dimensional cone in Σ and $a^1(\sigma),\dots,a^n(\sigma)$ the corresponding set of primitive integer vectors of σ that has been ordered. We associate to each such σ a copy of $\mathbb{C}^n$ denoted by $\mathbb{C}^n(\sigma)$.

Let us denote by $\pi_\sigma : \mathbb{C}^n(\sigma) \to X(S)$ the mapping given by

$$\pi_\sigma(y_1,\dots,y_n) = \left(y_1^{\langle a^1(\sigma),b_1\rangle}\cdots y_n^{\langle a^n(\sigma),b_1\rangle},\dots,y_1^{\langle a^1(\sigma),b_r\rangle}\cdots y_n^{\langle a^n(\sigma),b_r\rangle}\right),$$

where $y_1,y_2,\dots,y_n$ are the coordinates in $\mathbb{C}^n(\sigma)$ and $a^i(\sigma) = (a_1^i(\sigma),\dots,a_n^i(\sigma))$.

Note that π_σ is the composition $\tilde{\phi}\circ\Omega_\sigma$ of the following maps

$$\begin{array}{rccc} \Omega_\sigma: & \mathbb{C}^n(\sigma) & \longrightarrow & (\mathbb{C}^*)^n \\ & (y_1,\dots,y_n) & \longmapsto & \left(y_1^{a_1^1(\sigma)}\cdots y_n^{a_1^n(\sigma)},\dots,y_1^{a_n^1(\sigma)}\cdots y_n^{a_n^n(\sigma)}\right) \end{array}$$

and

$$\begin{array}{rccc} \tilde{\phi}: & (\mathbb{C}^*)^n & \longrightarrow & X(S) \\ & z & \longmapsto & (z^{b_1},\dots,z^{b_r}). \end{array}$$

where $b_1,\dots,b_r$ are the generators of S. The map $\tilde{\phi}$ is a restriction of the torus action given in (1.3) to $(\mathbb{C}^*)^n$ and it is a diffeomorphism from $(\mathbb{C}^*)^n$ to the open dense orbit $\mathscr{O}$ of $X(S)$. In this context, π_σ maps the $(\mathbb{C}^*)^n$ obtained from $\mathbb{C}^n(\sigma)$ onto the $(\mathbb{C}^*)^n$ corresponding to $\mathscr{O}$. Consequently, for every point $w\in\mathscr{O}$, there exists $y_\sigma \in \mathbb{C}^n(\sigma)$ such that $w = \pi_\sigma(y_\sigma)$.

Consider the disjoint union $\mathscr{C} = \bigcup_{\sigma\in\Sigma} \mathbb{C}^n(\sigma)$. We define the following equivalence relation: given two points $y_\sigma \in \mathbb{C}^n(\sigma)$ and $y_\tau \in \mathbb{C}^n(\tau)$,

$$y_\sigma \sim y_\tau \text{ if and only if } \pi_\sigma(y_\sigma) = \pi_\tau(y_\tau).$$

We will denote this set thus obtained by $X = X(\Gamma_+(I)) = \mathscr{C}/\sim$.

Since Σ satisfies properties $1-4$, by Theorems 6, 7 and 8 of [35], we have that X is a non-singular n-dimensional algebraic complex manifold and $\pi : X \to X(S)$ defined by $\pi(y) = \pi_\sigma(y_\sigma)$ is a proper analytic mapping onto $X(S)$, where $y_\sigma \in \mathbb{C}^n(\sigma)$ is a representative of the equivalence class $y \in X$.

Let J be a subset of $\{1,\dots,n\}$. Define

$$E_{\sigma,J} = \{y_\sigma \in \mathbb{C}^n(\sigma) : y_{\sigma,j} = 0, \forall j \in J\} \text{ and } E^*_{\sigma,J} = \{y_\sigma \in E_{\sigma,J} : y_{\sigma,j} \neq 0, \forall j \notin J\},$$

where $y_\sigma = (y_{\sigma,1},\dots,y_{\sigma,n})$.

Based in [24, Lemma 3.1], we have the following lemma.

Lemma 2.20. $\pi_\sigma(E_{\sigma,J}) = 0$ *if only if* $\Delta_J = \bigcap_{i\in J} \Delta(a^i(\sigma))$ *is a compact face of* $\Gamma_+(I)$.

Proof. By definition of π_σ, we will prove

$$\pi_\sigma(E_{\sigma,J}) = \{0\} \text{ if and only if } \Big\langle \sum_{i\in J} a^i(\sigma), b_l \Big\rangle > 0, \text{ for all } l \in \{1,2,\dots,r\}.$$

Consider the set $J = \{1,2\} \subset \{1,\dots,n\}$, without loss of generality. If $y_\sigma = (y_{\sigma,1},\dots,y_{\sigma,n}) \in E_{\sigma,J}$, then we have $y_\sigma = (0,0,y_{\sigma,3},\dots,y_{\sigma,n})$. It follows that

$$\pi_\sigma(y_\sigma) = \left(0^{\langle a^1(\sigma),b_1\rangle+\langle a^2(\sigma),b_1\rangle} y_{\sigma,3}^{\langle a^3(\sigma),b_1\rangle}\cdots y_{\sigma,n}^{\langle a^n(\sigma),b_1\rangle},\dots,0^{\langle a^1(\sigma),b_r\rangle+\langle a^2(\sigma),b_r\rangle} y_{\sigma,3}^{\langle a^3(\sigma),b_r\rangle}\cdots y_{\sigma,n}^{\langle a^n(\sigma),b_r\rangle}\right).$$

Since $\langle a^1(\sigma),b_l\rangle + \langle a^2(\sigma),b_l\rangle > 0$, for all $l \in 1,2,\dots,r$, we have that $\pi_\sigma(y_\sigma) = (0,0,\dots,0)$. Therefore, it follows that $\pi_\sigma(E_{\sigma,J}) = 0$.

Now, suppose there exists some index $l \in 1,2,\dots,r$ such that $\langle a^1(\sigma),b_l\rangle + \langle a^2(\sigma),b_l\rangle = 0$. Consider an element $y = (0,0,y_3,\dots,y_n) \in E_{\sigma,J}$ such that $y_j \neq 0$ for all $j \in 3,\dots,n$. Then,

$$\pi_\sigma(y_\sigma) = \left(0^{\langle a^1(\sigma),b_1\rangle+\langle a^2(\sigma),b_1\rangle} y_{\sigma,3}^{\langle a^3(\sigma),b_1\rangle}\cdots y_{\sigma,n}^{\langle a^n(\sigma),b_1\rangle},\dots,0^{\langle a^1(\sigma),b_r\rangle+\langle a^2(\sigma),b_r\rangle} y_{\sigma,3}^{\langle a^3(\sigma),b_r\rangle}\cdots y_{\sigma,n}^{\langle a^n(\sigma),b_r\rangle}\right).$$

By assumption, for some l, we have

$$0^{\langle a^1(\sigma),b_l\rangle+\langle a^2(\sigma),b_l\rangle} = 0^0 = 1.$$

Therefore,

$$y_3^{\langle a^3(\sigma),b_l\rangle}\cdots y_n^{\langle a^n(\sigma),b_l\rangle} \neq 0,$$

which implies that $\pi_\sigma(y) \neq (0,0,\dots,0)$, a contradiction, since we assumed that $\pi_\sigma(E_{\sigma,J}) = \{0\}$.

Note that $\big\langle \sum_{i\in J} a^i(\sigma), b_l \big\rangle = 0$ for some $l \in \{1,\dots,r\}$ if and only if Δ_J is a face of $\Gamma_+(I)$ parallel to some face of $cone(S)$, i.e., Δ_J is not compact.

Therefore, $\big\langle \sum_{i\in J} a^i(\sigma), b_l \big\rangle > 0$, for all $l \in \{1,2,\dots,r\}$ if, and only if, Δ_J is a compact face. $\square$

Consequently, $y_\sigma \in \pi_\sigma^{-1}(0)$ if and only if there exists a compact face Δ_J such that $y_\sigma \in E_{\sigma,J}$.

For each n-dimensional cone $\sigma \in \Sigma$ and any generator g_i of I, let $h_i(y_\sigma)$ be the analytic germ of function given by

$$h_i(y_\sigma) = \sum_{v\in \operatorname{supp}(g_i)} a_v \cdot y_{\sigma,1}^{\langle a^1(\sigma),v\rangle - \ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\langle a^n(\sigma),v\rangle - \ell(a^n(\sigma))}, \tag{2.4}$$

where $g = \sum_k a_k x^k$, $a_v = a_k$ and $v = k_1 b_1 + \cdots + k_r b_r$. The following equality is satisfied by $h_i(y_\sigma)$:

$$g_i \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\ell(a^n(\sigma))} \cdot h_i(y_\sigma). \tag{2.5}$$

Proposition 2.21. *If $y_\sigma \in E_{\sigma,J}$ then $(g_i)_{\Delta_J} \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\ell(a^n(\sigma))} \cdot h_i(y_\sigma)$.*

Proof. Suppose that $J = \{1,2\} \subset \{1,\dots,n\}$, without loss of generality. For each $g = \sum_k a_k x^k$, let h_i be the analytic germ of the function defined by equation (2.4).

Let $y_\sigma \in E_{\sigma,J}$. Then, we have

$$h_i(y_\sigma) = \sum_{v \in \mathrm{supp}(g_i)} a_v \cdot 0^{\langle a^1(\sigma),v\rangle - \ell(a^1(\sigma))} 0^{\langle a^2(\sigma),v\rangle - \ell(a^2(\sigma))} \cdots y_{\sigma,n}^{\langle a^n(\sigma),v\rangle - \ell(a^n(\sigma))}.$$

If $v \in \Delta_J$, then $\langle a^i(\sigma), v\rangle = \ell(a^i(\sigma))$ for all $i \in J$. Otherwise, we have $\langle a^i(\sigma), v\rangle > \ell(a^i(\sigma))$ for $i \in J$. Thus,

$$h_i(y_\sigma) = \sum_{v \in \mathrm{supp}(g_i) \cap \Delta_J} a_v \cdot y_{\sigma,3}^{\langle a^3(\sigma),v\rangle - \ell(a^3(\sigma))} \cdots y_{\sigma,n}^{\langle a^n(\sigma),v\rangle - \ell(a^n(\sigma))},$$

for all $y_\sigma \in E_{\sigma,J}$.

Therefore, $(g_i)_{\Delta_J} \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\ell(a^n(\sigma))} \cdot h_i(y_\sigma)$, for all $y_\sigma \in E_{\sigma,J}$. □

Furthermore, if $y_\sigma \in E_{\sigma,J}$ then

$$(g_i)_{\Delta_J} \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\ell(a^n(\sigma))} \cdot h_i(y_\sigma). \tag{2.6}$$

Therefore, the next proposition gives us a necessary and sufficient condition so that a monomial x^m belongs to the integral closure of the ideal $I \subseteq \mathscr{O}_{X(S)}$.

Proposition 2.22. *Let I be an ideal of $\mathscr{O}_{X(S)}$ and $x^m \in \mathscr{O}_{X(S)}$. Then $x^m \in \overline{I}$ if and only if*

$$|x^m| \circ \pi_\sigma(y_\sigma) \leqslant \mathscr{C} \cdot \sup_i \{|g_i| \circ \pi_\sigma(y_\sigma)\},$$

for all n-dimensional cone $\sigma \in \Sigma$ and all $y_\sigma \in \pi_\sigma^{-1}(U)$, where U is a neighborhood of 0 in $X(S)$.

Proof. If $x^m \in \overline{I}$, then

$$|x^m| \leqslant \mathscr{C} \cdot \sup_i \{|g_i(x)|\},$$

for all x in a neighborhood U of 0. Thus, for all $y_\sigma \in \pi_\sigma^{-1}(U)$, we have

$$|x^m \circ \pi_\sigma(y_\sigma)| \leqslant \mathscr{C} \cdot \sup_i \{|g_i \circ \pi_\sigma(y_\sigma)|\}.$$

Conversely, consider $\varphi : \mathbb{C} \to \mathscr{O}$ an analytic curve, where $\mathscr{O}$ is the dense open set in $X(S)$ obtained from the torus action (1.3). If $t \in U_0$, where U_0 is a neighborhood of 0 in $\mathbb{C}$, then $\varphi(t) \in \mathscr{O}$. Due to the definition of π_σ, there exists $y_\sigma \in \mathbb{C}^n(\sigma)$ such that $\varphi(t) = \pi_\sigma(y_\sigma)$. It follows that $y_\sigma \in \pi_\sigma^{-1}(\mathscr{O})$. In this way, we have

$$|x^m \circ \varphi(t)| \leqslant \mathscr{C} \cdot \sup_i \{|g_i \circ \varphi(t)|\},$$

for all $t \in U_0$. Therefore, $x^m \circ \varphi \in \overline{\varphi^*(I)}$, for every curve φ in $\mathscr{O}$. We conclude, by Lemma (3.3)[1] in [28], that $x^m \in \overline{I}$. □

Lemma 2.23. *An ideal I is non-degenerate if and only if $\sup_i |h_i(y_\sigma)| \neq 0$ for all n-dimensional cone σ of Σ and all $y_\sigma \in \pi_\sigma^{-1}(0)$.*

[1] We are applying Lemma (3.3) in the particular case where $M = I$ is an ideal.

Proof. Let $y_\sigma \in \pi_\sigma^{-1}(0)$. Then we choose J as large as possible such that $y_\sigma \in E^*_{\sigma,J}$. It follows from Lemma 2.20 that $\Delta_J = \bigcap_{i\in J}\Delta(a^i(\sigma))$ is a compact face of $\Gamma_+(I)$. Since I is non-degenerate, there is no common solution in $(\mathbb{C}^*)^n$ for the equations

$$L\big(g_{1_{\Delta_J}}\big) = \cdots = L\big(g_{s_{\Delta_J}}\big) = 0,$$

where $g_1,\dots,g_s$ are generators of I. Consider $\gamma : \mathbb{C}^n \to X(S)$ defined by $\gamma(z) = (z^{b_1},\dots,z^{b_r})$, where $b_1,\dots,b_r$ are generators of S. We have

$$L(g_{i_{\Delta_J}}) \circ \gamma^{-1} \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma))} \cdots y_{\sigma,n}^{\ell(a^n(\sigma))} \cdot h_i(y_\sigma),$$

for all $y_\sigma \in E_{\sigma,J}$. Then, there is no common solution in $E^*_{\sigma,J}$ for the equations

$$h_1|_{E_{\sigma,J}} = \cdots = h_s|_{E_{\sigma,J}} = 0.$$

Therefore, $\sup|h_i(y_\sigma)| \neq 0$, for all $y_\sigma \in \pi_\sigma^{-1}(0)$.

Conversely, suppose that I is degenerate. Then, there exist a compact face Δ of $\Gamma_+(I)$ and a point z^0 in $(\mathbb{C}^*)^n$ such that

$$L(g_{1_\Delta})(z^0) = \cdots = L(g_{s_\Delta})(z^0) = 0.$$

Let σ be an n-dimensional cone in Σ and $a^1(\sigma),\dots,a^n(\sigma)$ corresponding primitive integers such that

$$\Delta = \Delta_J = \bigcap_{i\in J}\Delta(a^i(\sigma)),$$

for some $J \subset \{1,\dots,n\}$. Due to the definition of π_σ, there is $y^0 \in \mathbb{C}^n(\sigma)$ such that $\tilde{\phi}^{-1} \circ \pi_\sigma(y^0) = z^0$. Take $\overline{y}^0 \in E_{\sigma,J}$, where

$$\overline{y_j}^0 = \begin{cases} y_j^0, & \text{if } j \notin J \\ 0, & \text{if } j \in J. \end{cases}$$

By Lemma 2.20 one has $\overline{y}^0 \in \pi_\sigma^{-1}(0)$, because $\Delta = \Delta_J$ is a compact face of $\Gamma_+(I)$. Therefore, $h_i(\overline{y}^0) = 0$, for all $1 \leqslant i \leqslant s$. We conclude that $\sup\limits_i\{|h_i(\overline{y}^0)|\} = 0$. $\square$

Remark 2.24. One can see $y_\sigma = 0 \in E_{\sigma,J}$, where $J = \{1,\cdots,n\}$. Thus,

$$h_i(0) = \sum_{v\in \operatorname{supp}(g_i)\cap\Delta_J} a_v \cdot 0^{\langle a^1(\sigma),v\rangle - \ell(a^1(\sigma))} \cdots 0^{\langle a^n(\sigma),v\rangle - \ell(a^n(\sigma))}.$$

Since Δ_J is a set of a single point, say q, then at least for some generator g_i, we have $h_i(0) = a_q \neq 0$. Therefore, $\sup_i|h_i(0)| \neq 0$.

Lemma 2.25. *Let $I \subseteq \mathscr{O}_{X(S)}$ be a non-degenerate ideal. Then, for each $m = (m_1,\dots,m_n) \in \mathbb{Z}^n_+$, $x^m \in \overline{I}$ if and only if the inequality*

$$\ell(a^i(\sigma)) \leqslant \langle m_1b_1 + \cdots + m_rb_r, a^i(\sigma)\rangle$$

holds for all $1 \leqslant i \leqslant n$ and all n-dimensional cone $\sigma \in \Sigma$.

Proof. If $x^m \in \bar{I}$ then $m_1b_1+\cdots+m_rb_r \in \Gamma_+(\bar{I})$. Since $\Gamma_+(\bar{I}) = \Gamma_+(I)$, it follows that $\ell(v) \leqslant \langle v, m_1b_1+\cdots+m_rb_r\rangle$, for all $v \in \breve{cone}(S)$. In particular,

$$\ell(a^i(\sigma)) \leqslant \langle a^i(\sigma), m_1b_1+\cdots+m_rb_r\rangle,$$

for all $1 \leqslant i \leqslant n$ and all cones $\sigma \in \Sigma$ of dimension n. For any n-dimensional cone $\sigma \in \Sigma$, we have

$$\sup_i\{|g_i|\circ\pi_\sigma(y_\sigma)\} = |y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}|\cdot\sup_i\{|h_i(y_\sigma)|\}$$

and

$$|x^m|\circ\pi_\sigma(y_\sigma) = |y_{\sigma,1}^{\langle m_1b_1+\cdots+m_rb_r,a^1(\sigma)\rangle}\cdots y_{\sigma,n}^{\langle m_1b_1+\cdots+m_rb_r,a^n(\sigma)\rangle}| = |y_{\sigma,1}^{M_1}\cdots y_{\sigma,n}^{M_n}|,$$

where $M_i := \langle m_1b_1+\cdots+m_rb_r, a^i(\sigma)\rangle$.

Since I is a non-degenerate ideal, by Lemma 2.23 one has $\sup_i\{|h_i(y_\sigma)|\} > 0$, for all $y_\sigma \in \pi_\sigma^{-1}(0)$. Then, there exists $\varepsilon > 0$ such that

$$|y_{\sigma,1}^{M_1}\cdots y_{\sigma,n}^{M_n}| < \varepsilon\cdot\frac{\sup_i\{|g_i|\circ\pi_\sigma(y_\sigma)\}}{\sup_i\{|h_i(y_\sigma)|\}}.$$

Hence, there exists $W \subset \mathbb{C}^n(\sigma)$ a neighborhood of 0 such that $\sup_i\{|h_i(y_\sigma)|\} > 0$, for all $y_\sigma \in W$.

Take $K \subset W$ a compact set with $0 \in int(K)$. By continuity, there exists $\tilde{y} \in K$ such that $\sup_i\{|h_i(y_\sigma)|\} \geqslant \sup_i\{|h_i(\tilde{y})|\}$, for all $y_\sigma \in K$. Define $d := \sup_i\{|h_i(\tilde{y})|\} > 0$ and $U_0 := int(K)$. We have that U_0 is a neighborhood of 0 and $\sup_i\{|h_i(y_\sigma)|\} \geqslant d$, for all $y_\sigma \in U_0$.

Now, we take $U := \pi_\sigma(U_0)$. This implies $\sup_i\{|h_i(y_\sigma)|\} \geqslant d$, for all $y_\sigma \in \pi_\sigma^{-1}(U)$. Thus,

$$|y_{\sigma,1}^{M_1}\cdots y_{\sigma,n}^{M_n}| \leqslant \frac{\varepsilon}{d}\cdot\sup_i\{|g_i|\circ\pi_\sigma(y_\sigma)\}.$$

Therefore, $x^m \in \bar{I}$, by Proposition 1.20. □

2.5 Integral closure of non-degenerate ideals

Now we present one of our main results, characterizing the ideals in $\mathscr{O}_{X(S)}$ whose integral closure is generated by monomials with exponents that belong to the Newton polyhedron defined in cone$(S) \subseteq \mathbb{R}^n_+$, where S is a finitely generated semigroup in $\mathbb{Z}^n_+$.

Theorem 2.26. *Let I be an ideal in $\mathscr{O}_{X(S)}$. Then I is non-degenerate if and only if $\Gamma_+(I) = C(\bar{I})$.*

Proof. In Lemma 2.17, we proved the inclusion $C(\bar{I}) \subset \Gamma_+(I)$. So, it is necessary to prove that I is non-degenerate if and only if $\Gamma_+(I) \subset C(\bar{I})$. Observe that $m_1b_1+\cdots+m_rb_r \in \Gamma_+(I)$ if, and only if, the inequality

$$\ell(v^s) \leqslant \langle m_1b_1+\cdots+m_rb_r, v^s\rangle \tag{2.7}$$

holds for all primitive integers v^s such that $\Delta(v^s)$ is a $(n-1)$-dimensional face of $\Gamma_+(I)$. Thus, it remains to prove the inequality holds if and only if the inequality

$$\ell(a^i(\sigma)) \leqslant \langle m_1b_1+\cdots+m_rb_r, a^i(\sigma)\rangle \tag{2.8}$$

is satisfied for all $1 \leqslant i \leqslant n$ and for every cone $\sigma \in \Sigma$ of dimension n.

By the construction of Σ, the set of primitive vectors v^s such that $\Delta(v^s)$ is an $(n-1)$-dimensional face of $\Gamma_+(I)$ is the set $\bigcup_{\sigma\in\Sigma}\left\{a^i(\sigma)\right\}$. Thus, the inequality (2.7) is valid for all v^s if the inequality (2.8) is valid for all $a^i(\sigma)$.

For any n-dimensional cone $\sigma \in \Sigma$, let σ_0 be the n-dimensional cone of Σ_0 such that $\sigma \subset \sigma_0$. Then, there exists a set of primitive integer vectors $\{v^1,\dots,v^p\}$ such that σ_0 is a linear combination of v^s with coefficients in $\mathbb{R}_+$. If the vectors v^s are numbered appropriately, we can assume that $\sigma_0 = \sum_{s=1}^{p} \mathbb{R}_+ v^s$.

It follows that

$$\bigcap_{s=1}^{p} \Delta(v^s) = \bigcap_{i=1}^{n} \Delta(a^i(\sigma))$$

is an one-point set in $\mathbb{R}^n_+$, say $q = (q_1,\dots,q_n)$. Thus, there are non-negative integers α_{is}, $1 \leqslant i \leqslant n$, such that $a^i(\sigma) = \sum_{s=1}^{p} \alpha_{is} v^s$.

Since q is a vertex of $\Gamma_+(I)$, we have $\langle q, v^s\rangle = \ell(v^s)$ and $\langle q, a^i(\sigma)\rangle = \ell(a^i(\sigma))$, for $1 \leqslant s \leqslant p$ and $1 \leqslant i \leqslant n$. Hence, $\ell(a^i(\sigma)) = \sum_{s=1}^{p} \alpha_{is} l(v^s)$. If the inequality $\ell(v^s) \leqslant \langle m_1b_1+\cdots+m_rb_r, v^s\rangle$ holds, then

$$\ell(a^i(\sigma)) = \sum_{s=1}^{p} \alpha_{is} l(v^s) \leqslant \sum_{s=1}^{p} \alpha_{is}\langle m_1b_1+\cdots+m_rb_r, v^s\rangle = \langle m_1b_1+\cdots+m_rb_r, a^i(\sigma)\rangle.$$

Hence, by Lemma 2.25, $x^m \in \bar{I}$. Therefore, $m_1b_1+\cdots+m_rb_r \in C(\bar{I})$.

On the other hand, consider σ an n-dimensional cone of Σ. Then, there exists a vertex $Q = q_1b_1+\cdots+q_rb_r$ of $\Gamma_+(I)$ such that $Q \in \bigcap_{i=1}^{n} \Delta(a^i(\sigma))$. Since $C(\bar{I}) = \Gamma_+(I)$, each vertex Q of $\Gamma_+(I)$ is a vertex of $C(\bar{I})$. By the definition of $C(\bar{I})$, it follows that $x^q \in \bar{I}$, where $q = (q_1,\dots,q_n)$. Therefore, there exists a constant $\mathscr{C} > 0$ such that

$$|x^q| \leqslant \mathscr{C}\cdot \sup_i\{|g_i(x)|\}$$

in a neighborhood U of the origin in $X(S)$, where $g_1, g_2, \dots, g_s$ are generators of I. Hence, for all $y_\sigma \in \pi_\sigma^{-1}(U)$, we have

$$\begin{aligned} |x^q|\circ\pi_\sigma(y_\sigma) = |y_{\sigma,1}^{M_1}\cdots y_{\sigma,n}^{M_n}| &\leqslant \mathscr{C}\cdot\sup_i\{|g_i|\circ\pi_\sigma(y_\sigma)\} \\ &= \mathscr{C}\cdot|y_{\sigma,1}^{\ell(a^1(\sigma))}\cdots y_{\sigma,n}^{\ell(a^n(\sigma))}|\cdot\sup_i\{|h_i(y_\sigma)|\}, \end{aligned}$$

where $M_i = \langle Q, a^i(\sigma)\rangle = \ell(a^i(\sigma))$. Then, $\sup_i\{|h_i(y_\sigma)|\} > 0$ in a neighborhood of $\pi_\sigma^{-1}(0)$. Therefore, by Lemma 2.23, the ideal I is non-degenerate. □

As a direct consequence of Theorem 2.26, Corollary 2.27 allows us to characterize the non-degeneracy of an ideal in terms of its integral closure. This characterization is particularly useful in practice, since the integral closure can be determined from the Newton polyhedron associated to the ideal.

Corollary 2.27. *Let I be an ideal in $\mathscr{O}_{X(S)}$. Then I is non-degenerate if and only if $\overline{I} = I^\circ$.*

Proof. If $\Gamma_+(I) = C(\overline{I})$ then $I^\circ = \langle x^k : k_1b_1 + \cdots + k_rb_r \in C(\overline{I})\rangle = \langle x^k : x^k \in \overline{I}\rangle$. However, $\overline{I} = I^\circ$. Conversely, if $\overline{I} = I^\circ$ then $K_I = \langle x^m : x^m \in I^\circ\rangle = I^\circ$. Thus, we have $C(\overline{I}) = \Gamma_+(k_I) = \Gamma_+(I^\circ) = \Gamma_+(I)$. Therefore, the result follows from Theorem 2.26. $\square$

Example 2.28. Let $S = \langle (1,0),(1,1),(1,2)\rangle$. Consider $I \subset \mathscr{O}_{X(S)}$ the ideal generated by $g_1 = x_1^4x_2 + x_3^2$ and $g_2 = x_1x_2x_3 + x_1^2x_2^3$. Then, $\mathrm{supp}(I) = \{(5,1),(2,4),(3,3),(5,3)\}$. Thus, the Newton polyhedron of I is represented in the Figure 2.8

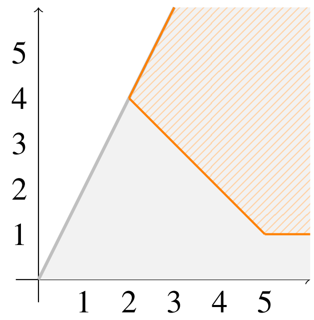

Figure 2.8: Newton polyhedron of $I = \langle x_1^4x_2 + x_3^2, x_1x_2x_3 + x_1^2x_2^3\rangle$.

For the face Δ formed by the line segment connecting the vertices $(5,1)$ and $(2,4)$, we have

$$(g_1)_\Delta = x_1^4x_2 + x_3^2 \quad \text{and} \quad (g_2)_\Delta = x_1x_2x_3.$$

We associate $(g_1)_\Delta$ and $(g_2)_\Delta$ with the following polynomials

$$L((g_1)_\Delta) = z_1^5z_2 + z_1^2z_2^4 \quad \text{and} \quad L((g_2)_\Delta) = z_1^3z_2^3.$$

Thus,

$$\{(z_1,z_2) \in \mathbb{C}^2 : L((g_1)_\Delta)(z_1,z_2) = L((g_2)_\Delta)(z_1,z_2) = 0\} = \{(z_1,z_2) \in \mathbb{C}^2 : z_1z_2 = 0\}.$$

For each compact face Δ of dimension 0, the system $L((g_1)_\Delta) = L((g_2)_\Delta) = 0$ admits no common solution in $(\mathbb{C}^*)^2$, because each $L((g_i)_\Delta)$ is a monomial. Therefore, I is non-degenerate.

Then, we have that the integral closure of I is generated by the monomials x^k such that

$$k_1b_1 + \cdots + k_rb_r \in \Gamma_+(I),$$

by Theorem 2.26.

We now consider the powers of the maximal ideal in the toric setting. In view of the previous characterization of non-degenerate ideals, we obtain the following result.

Proposition 2.29. *Let $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}_{X(S)}$. Then, for every $s \geqslant 1$, the ideal $\mathfrak{m}^s$ is integrally closed.*

Proof. Recall that $\mathfrak{m}^s$ is a monomial ideal. In particular, its support satisfies

$$\operatorname{supp}(\mathfrak{m}^s) = \{\nu(\alpha) \mid |\alpha| \geqslant s\}.$$

Observe that the vertices of $\Gamma_+(\mathfrak{m}^s)$ are contained in $\operatorname{supp}(\mathfrak{m}^s)$. Let Δ be a compact face of $\Gamma_+(\mathfrak{m}^s)$. Then

$$L\big((x^\alpha)_\Delta\big) = \begin{cases} z^{\nu(\alpha)}, & \text{if } \nu(\alpha) \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Since every compact face contains at least one vertex of $\Gamma_+(\mathfrak{m}^s)$, there exists α such that $\nu(\alpha) \in \Delta$, and hence

$$L\big((x^\alpha)_\Delta\big) \neq 0.$$

Moreover, these are monomials, so they do not vanish on $(\mathbb{C}^*)^n$. It follows that $\mathfrak{m}^s$ is non-degenerate. By Corollary 2.27, we conclude that $\overline{\mathfrak{m}^s} = (\mathfrak{m}^s)^\circ$. We prove that $(\mathfrak{m}^s)^\circ = \mathfrak{m}^s$.

We prove that $(\mathfrak{m}^s)^\circ = \mathfrak{m}^s$.

Let $p \in \Gamma_+(\mathfrak{m}^s)$. Then there exist $\nu(k^{(i)}) \in \operatorname{supp}(\mathfrak{m}^s)$ and $c_i \in \operatorname{cone}(S)$ such that

$$p = \sum_{i=1}^{l} \lambda_i\big(\nu(k^{(i)}) + c_i\big),$$

where $\lambda_i \geqslant 0$ and $\sum_{i=1}^{l} \lambda_i = 1$. Since $\nu(k^{(i)}) \in \operatorname{supp}(\mathfrak{m}^s)$, we have $|k^{(i)}| \geqslant s$, and each c_i can be written as

$$c_i = \beta_{i1} b_1 + \cdots + \beta_{ir} b_r, \quad \text{with } \beta_{ij} \geqslant 0.$$

Then

$$\begin{aligned} p &= \sum_{i=1}^{l} \lambda_i\big[(k_1^{(i)} b_1 + \cdots + k_r^{(i)} b_r) + (\beta_{i1} b_1 + \cdots + \beta_{ir} b_r)\big] \\ &= \sum_{i=1}^{l} \lambda_i\big[(k_1^{(i)} + \beta_{i1}) b_1 + \cdots + (k_r^{(i)} + \beta_{ir}) b_r\big] \\ &= \left[\sum_{i=1}^{l} \lambda_i (k_1^{(i)} + \beta_{i1})\right] b_1 + \cdots + \left[\sum_{i=1}^{l} \lambda_i (k_r^{(i)} + \beta_{ir})\right] b_r. \end{aligned}$$

Since $|k^{(i)}| \geqslant s$ and $|\beta_i| := \sum_{j=1}^{r} \beta_{ij} \geqslant 0$, it follows that

$$\sum_{j=1}^{r} \left[\sum_{i=1}^{l} \lambda_i (k_j^{(i)} + \beta_{ij})\right] = \sum_{i=1}^{l} \lambda_i\big(|k^{(i)}| + |\beta_i|\big) \geqslant \left(\sum_{i=1}^{l} \lambda_i\right) s = s.$$

Thus, if $A = (A_1, \ldots, A_r)$ with

$$A_j = \sum_{i=1}^{l} \lambda_i (k_j^{(i)} + \beta_{ij}),$$

then $|A| \geqslant s$, and hence $x^A \in \mathfrak{m}^s$. Therefore,

$$(\mathfrak{m}^s)^\circ \subset \mathfrak{m}^s.$$

The reverse inclusion follows from the definition, and we conclude that $(\mathfrak{m}^s)^\circ = \mathfrak{m}^s$. Therefore, $\overline{\mathfrak{m}^s} = \mathfrak{m}^s$, and hence $\mathfrak{m}^s$ is integrally closed. □

Following this, we present an example addressing non-degenerate ideals in relation to Whitney equisingularity. Using the notation of Gaffney's result and Theorem 2.26 we present the next example.

Example 2.30. Let $F : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$ be defined by $F(x_1,x_2,x_3) = x_1^4x_3^5 - x_2^3$. Note that $F^{-1}(0) = X(S)$, where $S = \langle (1,0),(3,5),(1,3) \rangle$. Consider the ideal I_F generated by $\{x_1\frac{\partial F}{\partial x_1}, x_2\frac{\partial F}{\partial x_1}, x_1\frac{\partial F}{\partial x_2}, x_2\frac{\partial F}{\partial x_2}\}$, i.e.,

$$I_F = \langle 4x_1^4x_3^5, 4x_1^3x_2x_3^5, -3x_1x_2^2, -3x_2^3 \rangle.$$

Thus, we have $\mathrm{supp}(I_F) = \langle (9,15),(11,20),(7,10) \rangle$. Then, the Newton polyhedron of I_F is represented in the Figure 2.9.

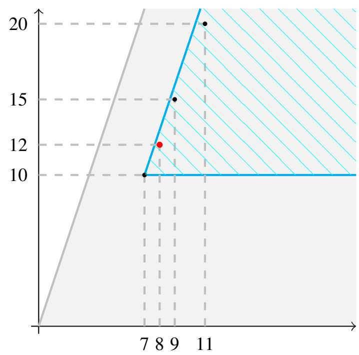

Figure 2.9: Newton polyhedron of $I_F = \langle 4x_1^4x_3^5, 4x_1^3x_2x_3^5, -3x_1x_2^2, -3x_2^3 \rangle$.

Consider the compact face $\Delta = \{(7,10)\}$ and the polynomials

$$L(4x_1^4x_3^5)_\Delta = 0 \qquad L(4x_1^3x_2x_3^5)_\Delta = 0$$
$$L(-3x_1x_2^2)_\Delta = z_1^7z_2^{10} \qquad L(-3x_2^3)_\Delta = 0$$

Note that

$$L(4x_1^4x_3^5)_\Delta = L(4x_1^3x_2x_3^5)_\Delta = L(-3x_1x_2^2)_\Delta = L(-3x_2^3)_\Delta = 0$$

have no common solution in $(\mathbb{C}^*)^2$. Then, I_F is non-degenerate. Thus, since $(8,12) \in \Gamma_+(I_F)$, it follows from the Theorem 2.26 that

$$\frac{\partial F}{\partial x_3} = 5x_1^4x_3^4 \in \overline{I_F}.$$

Therefore, the smooth part of $X(S)$ and $\mathbb{C} \times 0$ satisfies Verdier's condition W, by Theorem 1.32.

This final example highlights the interplay between Newton polyhedra, non-degeneracy, and Whitney conditions, illustrating how explicit face computations lead to Whitney equisingularity in the toric setting. This reasoning motivates the results of Chapter 3, where a canonical Whitney stratification for toric surfaces in $\mathbb{C}^3$ is constructed.

CHAPTER 3

Canonical Whitney stratification of toric surfaces in $\mathbb{C}^3$

After introducing the basic notions of toric varieties, Newton polyhedra, and non-degenerate ideals, together with their connections to integral closure and regularity conditions, we now turn to the setting of affine toric surfaces in $\mathbb{C}^3$. This context offers a natural and concrete framework in which the techniques developed earlier can be applied effectively, allowing for explicit descriptions of both the relevant combinatorial structures and the geometric features of the associated singularities.

In this chapter, we focus on affine toric surfaces embedded in $\mathbb{C}^3$, with particular emphasis on the description of the associated semigroups and on the generators of their defining ideals. Special attention is given to surfaces defined by binomial ideals of the form

$$I = \langle x^p z^r - y^q \rangle,$$

and we show how these surfaces arise naturally as toric varieties associated with suitable finitely generated semigroups. Furthermore, we study a Whitney stratification of these surfaces. By means of Newton polyhedra and the notion of non-degeneracy, we verify the Whitney conditions for natural pairs of strata. The results presented in this chapter are based on and summarize the main findings of [3].

3.1 Toric surfaces in $\mathbb{C}^3$

In this section, we study non-normal toric surfaces $\mathscr{V}(I) \subset \mathbb{C}^3$. The singular set of this class of surfaces is nonempty. Moreover, these singularities are not isolated.

The main focus of this section is Proposition 3.3. In this result, we consider the toric surface defined by the binomial ideal $I = \langle x^p z^r - y^q \rangle$ and, using properties of the numerical semigroup $\langle r, q \rangle$ together with elementary results on Diophantine equations, we characterise the semigroup that defines the surface $\mathscr{V}(I)$.

Related results can be found in [45], [16], [15, Lemma 2.2], and [17, Theorem 4.5], where the authors present generators of the defining ideal for certain classes of toric varieties.

In what follows we consider a toric surface $X(S) \subset \mathbb{C}^3$, where S is generated by $\{\gamma_1, \gamma_2, \gamma_3\} \subset \mathbb{Z}^2_{\geqslant 0}$ and γ_1, γ_2 and γ_3 correspond to the variables x, y and z, respectively.

We begin by briefly recalling a first result which provides, for a toric surface $\mathscr{V}(I)$ associated with a semigroup S generated by $(a_0,0),(a_1,b_1),(a_2,b_2)$, an explicit generator of the defining ideal I in terms of the integers a_i and b_i, for $i \in 0,1,2$.

Proposition 3.1 ([3])**.** *Let $X(S) = \mathscr{V}(I)$ be a toric surface in $\mathbb{C}^3$ defined by the semigroup S minimally generated by the set $\{(a_0,0),(a_1,b_1),(a_2,b_2)\}$, with $a_0 > 0$, and $gcd(b_1,b_2) = 1$.*

(i) If $a_1b_2 - a_2b_1 = 0$, then $I = \langle z^{b_1} - y^{b_2} \rangle$;

(ii) If $a_1b_2 - a_2b_1 > 0$, then

$$I = \begin{cases} \langle y^{a_0b_2} - x^{a_1b_2 - a_2b_1} z^{a_0b_1} \rangle, & \text{if } a_0 \nmid (a_1b_2 - a_2b_1) \\ \langle y^{b_2} - x^{\frac{a_1b_2 - a_2b_1}{a_0}} z^{b_1} \rangle, & \text{if } a_0 | (a_1b_2 - a_2b_1); \end{cases}$$

(iii) If $a_1b_2 - a_2b_1 < 0$, then

$$I = \begin{cases} \langle z^{a_0b_1} - x^{a_2b_1 - a_1b_2} y^{a_0b_2} \rangle, & \text{if } a_0 \nmid (a_1b_2 - a_2b_1) \\ \langle z^{b_1} - x^{\frac{a_2b_1 - a_1b_2}{a_0}} y^{b_2} \rangle, & \text{if } a_0 | (a_1b_2 - a_2b_1). \end{cases}$$

Proposition 3.1 provides an explicit description of the defining ideal I of the toric surface $X(S)$ in terms of the generators of the semigroup S. In particular, the generators of I depend only on the sign of the integer $a_1b_2 - a_2b_1$ and on the divisibility of $a_1b_2 - a_2b_1$ by a_0. This description will be used in the sequel to determine the defining equations of the toric surfaces considered in this chapter.

Remark 3.2. For the proof of the next result, we present some properties of the semigroup $\Gamma = \langle n,m \rangle = \{cn + dm; c,d \in \mathbb{N}\} \subset \mathbb{N}$. For more details see for instance [46]. It is well known that if $gcd(n,m) = 1$, then the set $\mathbb{N} \backslash \Gamma$ is finite and its elements are called gaps of Γ. Moreover, if $1 < n < m$, then the gaps are given by

$$\mathbb{N} \backslash \Gamma = \left\{ am - bn;\ a,b \in \mathbb{N}, 1 \leqslant a \leqslant n-1 \text{ and } 1 \leqslant b \leqslant \left[\frac{am}{n} \right] \right\}, \tag{3.1}$$

where $[x]$ denotes the largest integer that is less than or equal to $x \in \mathbb{R}$, called the integral part of x.

Let $\mathscr{V}(I)$ be the toric surface in $\mathbb{C}^3$ defined by the ideal $I = \langle x^p z^r - y^q \rangle$. It is well known that $\mathscr{V}(I)$ is smooth if $q = 0$ or $q = 1$. Moreover, if $p = r = 1$ and $q > 1$, then $\mathscr{V}(I)$ has an isolated singularity at the origin; in fact, it is an A_q singularity (see, for instance, [16, 25]).

In the next result, we analyze the remaining case.

Proposition 3.3. *Let p,q and r be non-negative integers and $\mathscr{V}(I)$ a toric surface in $\mathbb{C}^3$ defined by the ideal $I = \langle x^p z^r - y^q\rangle$, with $gcd(q,r) = 1$, $q > 1$ and $r \geqslant 1$. Then there exist non-negative integers a_1 and a_2 such that $\mathscr{V}(I) = X(S)$, where S is the semigroup generated by $\{(1,0),(a_1,r),(a_2,q)\}$.*

Proof. We first prove the existence of non-negative integers a_1 and a_2 such that $a_1q - a_2r = p$, that is, (a_1,a_2) is a solution of the Diophantine equation

$$qx - ry = p, \tag{3.2}$$

in the variables x and y. It is well known that the previous equation has solution in $\mathbb{Z}$ if and only if $gcd(q,r) \mid p$. This property is valid in our case, since $gcd(q,r) = 1$. Moreover, if (x_0,y_0) is a particular solution of (3.2), then all integer solutions of this equation can be given by

$$(x_0 - rt,\ y_0 - qt) \tag{3.3}$$

for all $t \in \mathbb{Z}$.

If $r = 1$, then $(a_1,a_2) := (p, p(q-1))$ is a solution of (3.2).

If $r > 1$ consider the semigroup $\Gamma = \langle r,q\rangle$. Since r and q are coprimes, and $q > 1$, then 1 is a gap of Γ. Thus, if $r < q$ by (3.1) there exist $u \in \{1,\dots,r-1\}$ and $v \in \{1,\dots,[\frac{uq}{r}]\}$ such that $uq - vr = 1$. Multiplying this equation by p, we obtain $(a_1,a_2) := (pu, pv)$ as a solution of (3.2).

Similarly, if $r > q$, then there exist $u,w \in \mathbb{N}$ such that $-uq + wr = 1$ for some $w \in \{1,\dots,q-1\}$ and $u \in \{1,\dots,[\frac{wr}{q}]\}$. Consequently $(-pu,-pw)$ is a solution of the previous Diophantine equation. Taking $t = t_0$ an integer in (3.3) satisfying $t_0 < min\left\{\left[\frac{-pu}{r}\right],\left[\frac{-pw}{q}\right]\right\}$, we obtain $(a_1,a_2) := (-pu - rt_0, -pw - qt_0) \in \mathbb{N}^2$ as a solution of (3.2).

In both cases, we obtain the relation

$$p(1,0) + r(a_2,q) = (a_1q - a_2r, 0) + (ra_2, rq) = (a_1q, rq) = q(a_1,r). \tag{3.4}$$

Finally, let $S = \langle(1,0),(a_1,r),(a_2,q)\rangle$, and let $\mathscr{V}(I_S) = X(S)$ be the toric surface associated to S. It follows that $I \subset I_S$. Hence, $X(S) = \mathscr{V}(I_S) \subset \mathscr{V}(I)$. Since $\mathbb{Z}S = \mathbb{Z}^2$, we have that $\dim X(S) = 2$. Moreover $\dim \mathscr{V}(I) = 2$, since it is a hypersurface in $\mathbb{C}^3$. As $X(S)$ and $\mathscr{V}(I)$ are both irreducible varieties of the same dimension, we conclude that $X(S) = \mathscr{V}(I)$. □

Proposition 3.3 allows us to describe the surface $\mathscr{V}(I)$ as an affine toric variety associated with a semigroup generated by $(1,0)$, (a_1,r) and (a_2,q). This description is important because it makes it possible to translate geometric properties of $\mathscr{V}(I)$ into combinatorial properties of the semigroup S, which will be used throughout the remainder of this chapter.

3.2 Canonical Whitney stratification of toric surfaces in $\mathbb{C}^3$

A Whitney stratification of a toric variety $X(S)$ is naturally given by the orbits of the torus action (see [29, p. 21]). By the Orbit–Cone Correspondence, this stratification can be written as

$$\mathscr{V}_S = \{T_\lambda\}_{\lambda \prec \mathrm{cone}(S)} \tag{3.5}$$

where λ denotes a face of cone(S) and T_λ is the orbit of the torus action associated with λ.

Let $\mathscr{V}(I)$ be a toric surface in $\mathbb{C}^3$, where $I=\langle x^pz^r-y^q\rangle$ with $r,q>1$ and $gcd(r,q)=1$. We denote by $Reg(\mathscr{V}(I))$ the smooth part of $\mathscr{V}(I)$ and by $Sing(\mathscr{V}(I))$ its singular set. If $p=1$, then

$$Sing(\mathscr{V}(I))=\mathbb{C}\times\{0\}\times\{0\},$$

whereas for $p>1$ we have

$$Sing(\mathscr{V}(I))=\mathbb{C}\times\{0\}\times\{0\}\cup\{0\}\times\{0\}\times\mathbb{C}.$$

By Proposition 3.3, the surface $\mathscr{V}(I)=X(S)$ can be described as the affine toric variety $X(S)$, where S is the semigroup generated by $\{(1,0),(a_1,r),(a_2,q)\}\subset\mathbb{N}^2$. Consequently, cone$(S)$ is a strongly convex polyhedral cone in $\mathbb{R}^2$. This toric description allows us to translate geometric properties of $\mathscr{V}(I)$ into combinatorial properties of the semigroup S.

As an application of the description of S, we prove in Theorem 3.4 that if $r>q>1$ and $p>q$, then the pair of strata $(Reg(\mathscr{V}(I)),\{0\}\times\{0\}\times\mathbb{C})$ satisfies the Whitney regularity conditions. The proof relies on the theory of integral closure of ideals and its relationship with Whitney regularity, following the approach developed by Teissier [50] and Gaffney [26].

Moreover, when $p>1$, the stratification

$$\{Reg(\mathscr{V}(I)),\mathbb{C}^*\times\{0\}\times\{0\},\{0\}\times\{0\}\times\mathbb{C}^*,\{0\}\times\{0\}\times\{0\}\}$$

is the Whitney stratification of $\mathscr{V}(I)$ determined by the orbits of the torus action (see (3.5)).

In Corollary 3.8, we show that if $r>q>1$ and $p>q$, then the stratification with three strata $\{Reg(\mathscr{V}(I)),\mathbb{C}^*\times\{0\}\times\{0\},\{0\}\times\{0\}\times\mathbb{C}\}$ is a Whitney stratification of $\mathscr{V}(I)$.

Theorem 3.4. *Let $\mathscr{V}(I)\subset\mathbb{C}^3$ be a toric surface defined as the zero set of the ideal $I=\langle x^pz^r-y^q\rangle$, where p,q, and r are non-negative integers such that* $\gcd(q,r)=1$ *with $r>q>1$ and $p>q$. Then the pair $(Reg(\mathscr{V}(I)),\{0\}\times\{0\}\times\mathbb{C})$ satisfies the Whitney conditions.*

Proof. It follows from Proposition 3.3 that there exist integers a_1 and a_2 such that $\mathscr{V}(I)=X(S)$, where S is the semigroup generated by

$$\{(1,0),(a_1,r),(a_2,q)\}.$$

Moreover, (a_1,a_2) is a solution of (3.2). Thus, we have the relation

$$p=a_1q-a_2r. \tag{3.6}$$

Let $F:\mathbb{C}^2\times\mathbb{C}\to\mathbb{C}$ be defined by $F(x,y,z)=x^pz^r-y^q$ which satisfies $F^{-1}(0)=\mathscr{V}(I)$. Consider the ideal $I_F\subset\mathscr{O}_{X(S)}$ generated by the set $\{x\frac{\partial F}{\partial x},y\frac{\partial F}{\partial x},x\frac{\partial F}{\partial y},y\frac{\partial F}{\partial y}\}$ which is given by

$$\{px^pz^r,\ px^{p-1}yz^r,\ -qxy^{q-1},\ -qy^q\}.$$

By (3.4), $supp(x^p z^r) = supp(y^q)$. Thus, $supp(px^p z^r) = supp(-qy^q)$. According to (2.1), $supp(I_F) = \bigcup_{h \in I_F} supp(h)$. Therefore

$$supp(I_F) = \{(p + a_2 r, rq), (p + a_2 r + (a_1 - 1), rq + r), (a_1 q - (a_1 - 1), rq - r)\} + S.$$

Next, we will describe the Newton polyhedron of I_F, which is the convex hull of $supp(I_F) + \text{cone}(S)$ in $\text{cone}(S)$.

As consequence of (3.4), the vectors $v_1 = (1,0)$ and $v_2 = (a_2, q)$ are the rays of $\text{cone}(S)$. Thus, the set $\Gamma_+(-qxy^{q-1}) = (a_1 q - (a_1 - 1), rq - r) + \text{cone}(S)$ is the intersection of the hyperplanes (see Fig 3.1)

$$h_1 : qx - a_2 y \geqslant pq - (p - q) \ \text{ and } \ h_2 : y \geqslant rq - r.$$

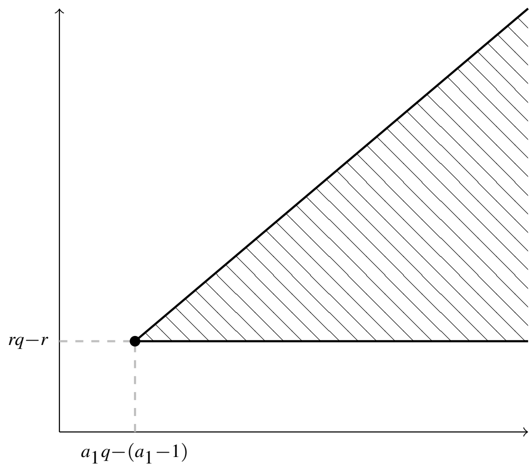

Figure 3.1: Newton polyhedron of $I_F = \langle px^p z^r,\ px^{p-1} yz^r,\ -qxy^{q-1},\ -qy^q \rangle$.

Indeed, by (3.2) the one-dimensional faces of $\Gamma_+(-qxy^{q-1})$ are determined by the vectors $(q, -a_2)$ and $(0,1)$ in $\breve{cone}(S)$. Moreover, by (3.6)

$$\ell((q, -a_2), \Gamma_+(-qxy^{q-1})) = pq - (p - q)$$

and

$$\ell((0,1), \Gamma_+(-qxy^{q-1})) = rq - r.$$

Thus, $\Gamma_+(-qxy^{q-1})$ can be described as the intersection of the closed half-spaces determined by its faces. The point $(p + a_2 r, rq) = (a_1 q, rq)$ lies in the interior of the intersection of these hyperplanes, because

$$q(a_1 q) - a_2(rq) = q(a_1 q - a_2 r) = pq > pq - (p - q) \text{ and } rq > rq - r.$$

Similarly, the point $(p + a_2 r + (a_1 - 1), rq + r) = (a_1 q + (a_1 - 1), rq + r)$ is also in the interior of the intersection of previous hyperplanes, since

$$q(a_1 q + (a_1 - 1)) - a_2(rq + r) = pq + (p - q) > pq - (p - q) \text{ and } rq + r > rq - r.$$

Thus, $(a_1q,rq),(a_1q+(a_1-1),rq+r)\in(a_1q-(a_1-1),rq-r)+\text{cone}(S)$, that is, $(a_1q-(a_1-1),rq-r)$ is the only vertex of $\Gamma_+(I_F)$.

Since I_F is a monomial ideal, I_F is non-degenerate. By Corollary 2.27, the integral closure of I_F is equal to the ideal I_F°. In this case, to verify whether $\frac{\partial F}{\partial z}=rx^pz^{r-1}$ belongs to $\overline{I_F}$, it suffices to check whether

$$supp\Big(\frac{\partial F}{\partial z}\Big)=\{(a_1q-a_2,rq-q)\}\subset\Gamma_+(I_F).$$

Since $p>q$, the pair $(a_1q-a_2,rq-q)$ satisfies

$$\begin{aligned} q(a_1q-a_2)-a_2(rq-q) &= a_1q^2-a_2q-a_2rq+a_2q \\ &= (a_1q-a_2r)q \\ &= pq>pq-(p-q). \end{aligned}$$

Moreover, as $r>q$, we have $rq-q>rq-r$. Consequently,

$$(a_1q-a_2,rq-q)\in\Gamma_+(I_F).$$

It follows that $\frac{\partial F}{\partial z}=rx^pz^{r-1}\in\overline{I_F}$. Therefore, by Theorem 1.32, the pair $(Reg(\mathscr{V}(I)),\{0\}\times\{0\}\times\mathbb{C})$ satisfies the Whitney conditions. □

Note that this result is obtained by characterizing the integral closure of a non-degenerate ideal by means of a certain monomial ideal. The proof then reduces to showing that the corresponding Newton polyhedron consists of a single vertex and that the other relevant monomials are inside this cone.

The following example makes the proof of Theorem 3.4 explicit in a concrete situation.

Example 3.5. Let $\mathscr{V}(I)=X(S)\subset\mathbb{C}^3$ be the toric surface associated with the semigroup generated by $\{(1,0),\ (14,11),\ (8,7)\}$. By relation (3.6),

$$p=a_1q-a_2r=10.$$

Consider $F:\mathbb{C}^2\times\mathbb{C}\to\mathbb{C}$ defined by

$$F(x,y,z)=x^{10}z^{11}-y^7.$$

Let $I_F\subset\mathscr{O}_{X(S)}$ be the ideal generated by $\{x\frac{\partial F}{\partial x},y\frac{\partial F}{\partial x},x\frac{\partial F}{\partial y},y\frac{\partial F}{\partial y}\}$, that is,

$$\{10x^{10}z^{11},\ 10x^9yz^{11},\ -7xy^6,\ -7y^7\}.$$

Then, supp$(I_F)=\{(98,77),\ (111,88),\ (85,66)\}$. Thus, the Newton polyhedron of I_F is represented in the Figure 3.2.

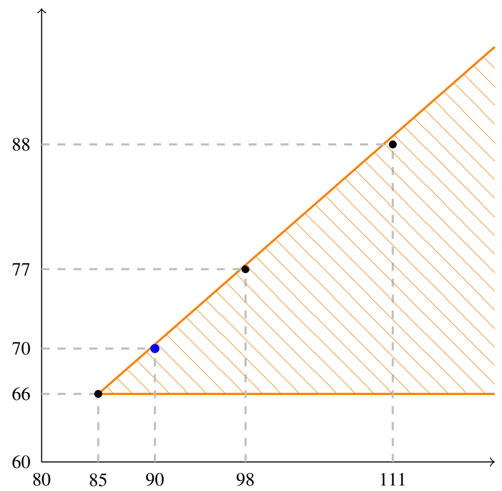

Figure 3.2: Newton polyhedron of $I_F = \langle 10x^{10}z^{11},\ 10x^9yz^{11},\ -7xy^6,\ -7y^7\rangle$.

Since I_F is non-degenerate and $\operatorname{supp}\left(\frac{\partial F}{\partial z}\right) = (90,70) \in \Gamma_+(I_F)$, it follows that $\frac{\partial F}{\partial z} = 11x^{10}z^{10} \in \overline{I_F}$. Therefore, by Theorem 1.32, the pair $(Reg(\mathscr{V}(I)), \{0\} \times \{0\} \times \mathbb{C})$ satisfies the Whitney conditions.

The next example provides an explicit verification of the hypotheses and conclusion of Theorem 3.4.

Example 3.6. Let $\mathscr{V}(I) \subset \mathbb{C}^3$ be the toric surface defined by

$$I = \langle x^5z^3 - y^2\rangle.$$

Here $p = 5$, $q = 2$, and $r = 3$. We verify that the hypotheses of Theorem 3.4 are satisfied:

$$\gcd(2,3) = 1, \quad 3 > 2 > 1, \quad 5 > 2.$$

Hence, by Theorem 3.4, the pair

$$\big(\mathrm{Reg}\mathscr{V}(I)), 0 \times 0 \times \mathbb{C}\big)$$

satisfies the Whitney conditions.

We now present a classical example showing that the hypotheses of Theorem 3.4 are essential.

Example 3.7. (Whitney's umbrella) Let $\mathscr{V}(I) = X(S) \subset \mathbb{C}^3$ be the toric surface associated with the semigroup S generated by $\{(1,0),\ (1,1),\ (0,2)\}$. Then $I = \langle x^2z - y^2\rangle$. It is well known that for this surface the pair $(Reg(\mathscr{V}(I)), \{0\} \times \{0\} \times \mathbb{C})$ does not satisfy the Whitney conditions. In this case, $r = 1 < 2 = q$ and $p = 2 = q$, that is, the hypotheses in Theorem 3.4 are important to guarantee the result.

Whitney stratifications play a crucial role in understanding the local and global structure of singular spaces. By ensuring that the origin lies within a 1-dimensional stratum in $\mathscr{V}(I)$, we guarantee that the origin has the same geometric and topological nature as this stratum.

Corollary 3.8. *Let $\mathscr{V}(I) \subset \mathbb{C}^3$ be the toric surface defined as the zero set of the ideal $I = \langle x^p z^r - y^q \rangle$, where p, q and r are non-negative integers such that $\gcd(q,r) = 1$, with $r > q > 1$ and $p > q$. Then the set*

$$\{Reg(\mathscr{V}(I)), \mathbb{C}^* \times \{0\} \times \{0\}, \{0\} \times \{0\} \times \mathbb{C}\}$$

is a Whitney stratification of $\mathscr{V}(I)$.

Proof. Consider the following sets:

$$T_1 = \{(x,0,0) \in \mathbb{C}^3;\ x \neq 0\} \text{ and } T_3 = \{(0,0,z) \in \mathbb{C}^3;\ z \neq 0\}.$$

From the defining equation of $\mathscr{V}(I)$ we deduce that $T_1, T_3 \subset \mathscr{V}(I)$.

Now, since the vectors $(1,0)$ and (a_2, q) determine the rays of $cone(S)$, the stratification $\mathscr{V}_S = \{T_\lambda\}_{\lambda < \mathrm{cone}(S)}$ contains two 1-dimensional strata.

Recall that the action of the torus $\theta : (\mathbb{C}^*)^2 \times \mathscr{V}(I) \to \mathscr{V}(I)$ is given by

$$\theta((t_1,t_2),(x,y,z)) = (t_1 x, t_1^{a_1} t_2^r y, t_1^{a_2} t_2^q z).$$

It follows that T_1 and T_3 are in fact the 1-dimensional orbits of the action on $\mathscr{V}(I)$. In particular, T_1 and T_3 are strata of the Whitney stratification $\mathscr{V}_{\mathscr{S}}$. Therefore, by Theorem 3.4, the set $\{Reg(\mathscr{V}(I)), \mathbb{C}^* \times \{0\} \times \{0\}, \{0\} \times \{0\} \times \mathbb{C}\}$ is a Whitney stratification of $\mathscr{V}(I)$. □

Corollary 3.8 is conceptually significant because it shows that Whitney regularity is compatible with a simpler stratification than the one induced by the torus action. This simpler Whitney stratification may have further applications in the study of geometric or topological properties of such surfaces.

CHAPTER 4

Bi-Lipschitz Invariants

The results developed in the previous chapters provide algebraic, geometric, and combinatorial tools that are essential for the study carried out in this chapter. In particular, the notions of integral closure of ideals, the Łojasiewicz exponent, Whitney stratifications, and the geometry of affine toric varieties provide a framework for understanding the local structure of singularities. The combinatorial description given by Newton polyhedra also allows us to visualize examples and identify geometric and metric features relevant to the study of bi-Lipschitz invariants.

Within this framework, the study of bi-Lipschitz invariants plays an important role in singularity theory, as these invariants capture geometric properties of analytic varieties beyond their purely algebraic structure. In this chapter, we investigate algebraic conditions that ensure the bi-Lipschitz invariance of two local invariants: the Łojasiewicz exponent, and the Euler obstruction.

We provide a answer, to a particular case, to the problem of the bi-Lipschitz invariance of the local Euler obstruction, proving its invariance for hypersurfaces with isolated singularities under suitable non-degeneracy assumptions. The results presented in this chapter correspond to those obtained in [1].

4.1 The Bi-Lipschitz Invariance of the Łojasiewicz Exponent

This section is based on the work of Bivià-Ausina and Fukui [8]. While their results are established for ideals in $\mathscr{O}_n$, here we develop the corresponding theory for ideals in $\mathscr{O}_{X(S)}$, where $(X(S),0)$ is a germ of an affine toric variety associated with the semigroup S. As a first step, we establish a general result on the bi-Lipschitz equivalence of ideals, which will serve as a fundamental tool for the subsequent analysis of Jacobian ideals and their integral closures.

The proofs of Theorems 4.1, 4.4, and Corollary 4.2 are inspired by the arguments presented in the work of Biviá-Ausina and Fukui [8].

Theorem 4.1. *Let I and J be ideals of $\mathscr{O}_{X(S)}$. If I and J are bi-Lipschitz equivalent, then* $\operatorname{ord}(I) =$

$\mathrm{ord}(J)$. *Moreover, if I and J have finite colength, then* $\mathscr{L}_0(I) = \mathscr{L}_0(J)$.

Proof. Since I and J are bi-Lipschitz equivalent, there exist germs of analytic maps

$$f = (f_1, \dots, f_p) : (X(S), 0) \to (\mathbb{C}^p, 0)$$

and

$$g = (g_1, \dots, g_q) : (X(S), 0) \to (\mathbb{C}^q, 0)$$

such that

$$\overline{I} = \overline{\langle f_1, \dots, f_p \rangle} \quad \text{and} \quad \overline{J} = \overline{\langle g_1, \dots, g_q \rangle},$$

and there exists a bi-Lipschitz homeomorphism $\varphi : (X(S), 0) \to (X(S), 0)$ satisfying

$$\|g(\varphi(x))\| \sim \|f(x)\| \quad \text{near } 0.$$

By symmetry, it suffices to prove that $\mathscr{L}_0(I) \leqslant \mathscr{L}_0(J)$ and $\mathrm{ord}(I) \leqslant \mathrm{ord}(J)$. Let $\mathscr{L}_0(J) = \theta \in \mathbb{R}_{\geqslant 0}$, we have

$$\|x\|^\theta \lesssim \|g(x)\| \quad \text{near } 0.$$

Then

$$\|x\|^\theta \sim \|\varphi(x)\|^\theta \lesssim \|g(\varphi(x))\| \sim \|f(x)\| \quad \text{near } 0,$$

and it follows that $\mathscr{L}_0(I) \leqslant \mathscr{L}_0(J)$. Moreover, by Proposition 2.29, we have

$$\mathrm{ord}(J) = \max\{s : J \subseteq \mathfrak{m}^s\} = \max\{s : J \subseteq \overline{\mathfrak{m}^s}\} = \max\{s : \|g(x)\| \lesssim \|x\|^s \text{ near } 0\},$$

where $\mathfrak{m}$ is the maximal ideal of $\mathscr{O}_{X(S)}$. If $\|f(x)\| \lesssim \|x\|^{ord(I)}$ near 0, then

$$\|g(x)\| \sim \|f(\varphi(x))\| \lesssim \|\varphi(x)\|^{ord(I)} \sim \|x\|^{ord(I)} \text{ near } 0,$$

and we obtain $\mathrm{ord}(I) \leqslant \max\{s : \|g(x)\| \lesssim \|x\|^s \text{ near } 0\} = \mathrm{ord}(J)$. $\square$

The following result shows that, under a strong algebraic rigidity condition, bi-Lipschitz equivalence implies the equality of the integral closures of the corresponding ideals.

Corollary 4.2. *Let I and J be ideals of $\mathscr{O}_{X(S)}$ with finite colength. Let us suppose that $\overline{I} = \mathfrak{m}^{\mathrm{ord}(I)}$. Then, I and J are bi-Lipschitz equivalent if and only if $\overline{I} = \overline{J}$.*

Proof. The converse implication is immediate. If I and J are bi–Lipschitz equivalent, then by Theorem 4.1 we have $\mathrm{ord}(I) = \mathrm{ord}(J)$ and $\mathscr{L}_0(I) = \mathscr{L}_0(J)$. Since $\overline{I} = \mathfrak{m}^{\mathrm{ord}(I)}$, it follows that $\mathscr{L}_0(I) \leqslant \mathrm{ord}(I)$. Moreover, $I \subseteq \mathfrak{m}^{\mathrm{ord}(I)}$ implies $I^q \subseteq \mathfrak{m}^{q\cdot\mathrm{ord}(I)}$ and hence $\overline{I^q} \subseteq \mathfrak{m}^{q\cdot\mathrm{ord}(I)}$ for all $q \in \mathbb{N}$. If $\mathfrak{m}^p \subseteq \overline{I}^q$, then $p \geqslant q \cdot \mathrm{ord}(I)$, so

$$\mathrm{ord}(I) \leqslant \inf\{p/q : \mathfrak{m}^p \subseteq \overline{I}^q\} = \mathscr{L}_0(I).$$

Therefore, $\mathrm{ord}(I) = \mathscr{L}_0(I)$. Since $J \subseteq \mathfrak{m}^{\mathrm{ord}(J)}$, by the inequalities (1.1) we obtain

$$e(\mathfrak{m})\,\mathrm{ord}(I)^n \;\geqslant\; \mathrm{ord}(I)^n = \mathscr{L}_0(I)^n = \mathscr{L}_0(J)^n \;\geqslant\; e(J) \;\geqslant\; e(\mathfrak{m})\,\mathrm{ord}(J)^n = e(\mathfrak{m})\,\mathrm{ord}(I)^n.$$

Hence $e(J) = e(\mathfrak{m}^{\mathrm{ord}(I)})$. By the Rees Multiplicity Theorem (see [33]), $\mathfrak{m}^{\mathrm{ord}(I)} \subseteq \overline{J}$. Therefore, $\overline{J} = \overline{I}$. $\square$

The following example illustrates a situation where the hypothesis $\bar{I} = \mathfrak{m}^{\mathrm{ord}(I)}$ of Corollary 4.2 is satisfied.

Example 4.3. Let $I \subset \mathscr{O}_{X(S)}$ be a non-degenerate ideal, where $X(S)$ is a toric variety with $S = \langle b_1, \ldots, b_r \rangle$. If $\Gamma_+(I) = \Gamma_+(\mathfrak{m}^{\mathrm{ord}(I)})$, then $\bar{I} = \mathfrak{m}^{\mathrm{ord}(I)}$. Indeed, by Corollary 2.27, the integral closure is given by

$$\bar{I} = \langle x^k : k_1 b_1 + \cdots + k_r b_r \in \Gamma_+(I) \rangle.$$

The hypothesis $\Gamma_+(I) = \Gamma_+(\mathfrak{m}^{\mathrm{ord}(I)})$ implies that

$$\bar{I} = \langle x^k : k_1 b_1 + \cdots + k_r b_r \in \Gamma_+(\mathfrak{m}^{\mathrm{ord}(I)}) \rangle = (\mathfrak{m}^{\mathrm{ord}(I)})^\circ = \overline{\mathfrak{m}^{\mathrm{ord}(I)}}.$$

By Proposition 2.29, $\mathfrak{m}^{\mathrm{ord}(I)}$ is non-degenerate and integrally closed. Therefore, $\bar{I} = \mathfrak{m}^{\mathrm{ord}(I)}$.

Before proceeding, we recall that for a germ $f \in \mathscr{O}_{X(S)}$ the *Jacobian ideal* is defined by

$$J(f) = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_r} \right\rangle \subseteq \mathscr{O}_{X(S)}.$$

Throughout this section, whenever the Łojasiewicz exponent $\mathscr{L}_0(J(f))$ is considered, we assume that the Jacobian ideal $J(f) \subset \mathscr{O}_{X(S)}$ has finite colength.

Theorem 4.4. *Let $f, g \in \mathscr{O}_{X(S)}$ be germs with an isolated singularity at the origin. Suppose that f and g are bi-Lipschitz $\mathscr{A}$-equivalent or bi-Lipschitz $\mathscr{K}^*$-equivalent. Then the Jacobian ideals $J(f)$ and $J(g)$ are bi-Lipschitz equivalent. In particular,* $\mathrm{ord}(f) = \mathrm{ord}(g)$ *and* $\mathscr{L}_0(J(f)) = \mathscr{L}_0(J(g))$.

Proof. Suppose that f and g are bi-Lipschitz $\mathscr{A}$-equivalent, that is, there exist bi-Lipschitz homeomorphisms $\varphi : (X(S), 0) \to (X(S), 0)$ and $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $\phi \circ f = g \circ \varphi$.

We identify $(X(S), 0) \subset \mathbb{C}^r$ with an analytic subvariety of $\mathbb{R}^{2r}$; in this setting, the regular part $X(S)_{reg}$ is a smooth real manifold of dimension $2 \dim_{\mathbb{C}} X(S)$.

By Rademacher's theorem, ϕ and φ are differentiable almost everywhere on $X(S)_{reg}$ and on $\mathbb{C}$. Thus, the derivatives $D\phi(x)$ and $D\varphi(x)$ exist almost everywhere on $X(S)_{reg}$. Moreover, since φ and φ^{-1} are bi-Lipschitz, their derivatives (as well as those of ϕ and ϕ^{-1}) exist almost everywhere and are essentially bounded, with bounded inverses as well. For every $x \in X(S)_{reg}$ where the derivatives exist, the chain rule yields

$$\nabla(g \circ \varphi)(x) \;=\; D\varphi(x)^T \nabla g(\varphi(x)), \tag{4.1}$$

where ∇g denotes the real gradient of g restricted to $X(S)_{reg}$. On the other hand,

$$\nabla(\phi \circ f)(x) \;=\; D\phi(f(x)) \nabla f(x). \tag{4.2}$$

Comparing (4.1) and (4.2) and taking norms, we obtain

$$\|D\varphi(x)^T \nabla g(\varphi(x))\| = \|D\phi(f(x)) \nabla f(x)\| \leqslant \|D\phi(f(x))\| \, \|\nabla f(x)\| \leqslant M \|\nabla f(x)\|.$$

Since $D\phi(f(x))$ is invertible with $\|D\phi(f(x))^{-1}\| \leqslant L$, it follows that

$$\|\nabla g(\varphi(x))\| \leqslant \|D\phi(f(x))^{-T}\| \, \|D\phi(f(x))\nabla f(x)\| \leqslant LM\|\nabla f(x)\|.$$

Similarly, using φ^{-1} and ϕ^{-1}, we obtain

$$\|\nabla f(x)\| \leqslant LM\|\nabla g(\varphi(x))\|.$$

Therefore, there exist constants $C_1, C_2 > 0$ such that, for almost every $x \in X(S)_{reg}$,

$$C_1\|\nabla f(x)\| \leqslant \|\nabla g(\varphi(x))\| \leqslant C_2\|\nabla f(x)\|.$$

We now extend this inequality to all points near the origin. Note that the singular locus of $X(S)$, $X(S)_{\text{sing}}$, is closed, of real codimension at least 2, hence of measure zero, and that ∇f and ∇g are continuous on $X(S)$. Thus, an inequality holding almost everywhere on $X(S)_{reg}$ extends, by density and continuity, to all points of $X(S)$.

More precisely, given $x_0 \in X(S)_{reg}$, there exists a sequence $x_n \in X(S)_{reg}$ with $x_n \to x_0$ such that the inequality holds for all x_n. Taking the limit as $n \to \infty$ and using the continuity of ∇f and ∇g, the inequality also holds at x_0. At the origin $0 \in X(S)$, since f and g have isolated singularities, $\nabla f(0) = \nabla g(0) = 0$, which is compatible with the inequality.

We conclude that the inequality holds in a neighborhood $V \subset X(S)$ of the origin. Hence, $J(f)$ and $J(g)$ are bi-Lipschitz equivalent.

Now suppose that f and g are bi-Lipschitz $\mathscr{K}^*$-equivalent. Thus, there exist a bi-Lipschitz homeomorphism $\varphi : (X(S),0) \to (X(S),0)$ and a Lipschitz function $A : (X(S),0) \to \mathbb{C}^*$, with Lipschitz inverse A^{-1}, such that $g \circ \varphi = A \cdot f$ in a neighborhood of the origin in $X(S)$. Then we have that

$$\begin{aligned}
\|(\nabla g)(\varphi(x))\| &\lesssim \|(\nabla g)(\varphi(x))D\varphi(x)\| && \text{(since } \varphi \text{ is bi-Lipschitz)}\\
&= \|\nabla(g\circ\varphi)(x)\| && \text{(by 4.1)}\\
&= \|\nabla A(x)f(x) + A(x)\nabla f(x)\| && \text{(since } g(\varphi(x)) = A(x)f(x))\\
&\leqslant \|\nabla A(x)\|\,|f(x)| + \|A(x)\|\,\|\nabla f(x)\| &&\\
&\lesssim |f(x)| + \|\nabla f(x)\| && \text{(since } A(x) \text{ is Lipschitz)}\\
&\lesssim \|x\|\,\|\nabla f(x)\| + \|\nabla f(x)\| && \text{(since } |f(x)| \lesssim \|x\|\|\nabla f(x)\|)\\
&\lesssim \|\nabla f(x)\|,
\end{aligned}$$

almost everywhere. Similarly, we have

$$\|(\nabla f)(\varphi^{-1}(x))\| \lesssim \|\nabla g(x)\|$$

near 0, and hence we obtain that the ideals $J(f)$ and $J(g)$ are bi-Lipschitz equivalent. □

Corollary 4.5. *Let $f \in \mathscr{O}_{X(S)}$ be such that $\overline{J(f)} = \mathfrak{m}^{\operatorname{ord}(f)-1}$. If $g \in \mathscr{O}_{X(S)}$ is bi-Lipschitz $\mathscr{A}$-equivalent to f or bi-Lipschitz $\mathscr{K}^*$-equivalent to f, then $\overline{J(f)} = \overline{J(g)}$.*

Proof. It follows from Theorem 4.1 and Corollary 4.2. □

Before presenting the example below, we briefly recall the notion of a semi–homogeneous function. A germ $f \in \mathscr{O}_{X(S)}$ is called *semi–homogeneous of order d* if it admits a decomposition of the form

$$f = f_d + (\text{higher order terms}),$$

where f_d is a nonzero homogeneous polynomial of degree d. Equivalently, f is semi–quasi–homogeneous with respect to the weight $(1,\ldots,1)$.

If, in addition, the homogeneous component f_d has an isolated singularity, then $J(f) = \mathfrak{m}^{d-1}$, and hence the hypotheses of Corollary 4.5 are satisfied. The following example illustrates this situation.

Example 4.6. If $f \in \mathscr{O}_n$ is semi–homogeneous of order d and its homogeneous component f_d has an isolated singularity, then

$$\overline{J(f)} = m^{d-1}.$$

This follows from growth estimates comparing $\nabla f = (\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})$ with monomials of degree $d-1$, which ensure that all such monomials belong to the integral closure of the Jacobian ideal.

We can write

$$f = f_d + (\text{terms of order} > d),$$

where f_d is homogeneous of degree d and has an isolated singularity at 0.

Since each partial derivative $\partial f/\partial x_i$ has order $\geqslant d-1$, we obtain $J(f) \subseteq \mathfrak{m}^{d-1}$ and hence $\overline{J(f)} \subseteq \mathfrak{m}^{d-1}$. For the reverse inclusion, let $q(x)$ be any monomial of degree $d-1$. Writing $x = ru$ with $r = \|x\|$ and $u \in S^{2n-1}$, we have

$$\nabla f_d(x) = r^{d-1}\nabla f_d(u) \quad \text{and} \quad q(x) = r^{d-1}q(u).$$

Since f_d has an isolated singularity, $\nabla f_d(u) \neq 0$ for all $u \in S^{2n-1}$; by Weierstrass' Theorem there exist a constant $m_0 > 0$ such that

$$\|\nabla f_d(x)\| \geqslant m_0\|x\|^{d-1}.$$

Taking $M_0 := \max_{u\in S^{2n-1}} |q(u)| < \infty$, we have $|q(x)| \leqslant M_0\|x\|^{d-1}$. Moreover, $\nabla f(x) = \nabla f_d(x) + R(x)$, where $R(x)$ are the derivatives of the terms of order $> d$. Note that $\operatorname{ord} R \geqslant d$, hence $\|R(x)\| \leqslant \frac{m_0}{2}\|x\|^{d-1}$ in a neighborhood of 0, and consequently

$$\|\nabla f(x)\| \geqslant \tfrac{m_0}{2}\|x\|^{d-1}$$

in a neighborhood of 0. Hence,

$$|q(x)| \leqslant \frac{2M_0}{m_0}\|\nabla f(x)\|$$

in a neighborhood of 0. Therefore, in the same neighborhood, we get

$$|q(x)| \leqslant C \max_{1\leqslant i\leqslant n} \left\{ \frac{\partial f}{\partial x_i}(x) \right\}$$

for some constant $C > 0$, which by the Growth Condition implies $q \in \overline{J(f)}$. Since this holds for every monomial of degree $d-1$, we conclude $\mathfrak{m}^{d-1} \subseteq \overline{J(f)}$, and the equality follows.

We now turn to the toric setting. The following example shows that the hypotheses of Corollary 4.5 are not restricted to the classical case of $\mathscr{O}_n$. Indeed, we exhibit a function on an affine toric variety whose Jacobian ideal satisfies these hypotheses, providing an explicit application of the corollary in $\mathscr{O}_{X(S)}$.

Example 4.7. Consider the toric variety $X(S)$ associated with the semigroup $S = \{(1,0),(1,1),(1,2)\}$. Let $g = x^a + y^b + z^a + R(x,y,z) \in \mathscr{O}_{X(S)}$, where $b \geqslant a$ and $R(x,y,z)$ consists of terms of order strictly greater than a. The Newton polyhedron of the Jacobian ideal $J(g)$ is depicted in Figure 4.1.

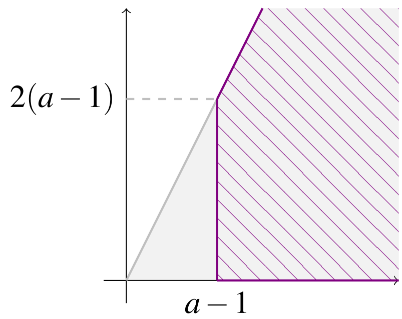

Figure 4.1: Newton polyhedron of $J(g) = \langle ax^{a-1}, by^{b-1}, cz^{c-1} \rangle$.

Note that $J(g)$ is non-degenerate, and by the Corollaries 2.27 and 2.29, we have

$$\begin{aligned} \overline{J(g)} &= \langle x^a y^b z^c \ : \ a(1,0)+b(1,1)+c(1,2) \in \Gamma(J(g)) \rangle \\ &= \langle x^a y^b z^c \ : \ a(1,0)+b(1,1)+c(1,2) \in \Gamma(\mathfrak{m}^{a-1}) \rangle \\ &= \overline{\mathfrak{m}^{a-1}} = \mathfrak{m}^{a-1}. \end{aligned}$$

4.2 The Bi-Lipschitz Invariance of the Euler Obstruction for Hypersurfaces

In this section, we study the bi-Lipschitz invariance of the Euler obstruction for hypersurfaces with an isolated singularity defined in a toric variety $X(S)$. Using the formulas of Matsui and Takeuchi [40] for the Euler characteristic via Newton polyhedra, we show that, under the same hypotheses of Corollary 4.5, this obstruction is preserved under bi-Lipschitz equivalence of ideals.

We emphasize that the results of this section are restricted to hypersurfaces in normal affine toric varieties and rely on non-degeneracy assumptions expressed in terms of the associated Jacobian ideal and Newton polyhedra.

Theorem 4.8. *Let $f, g \in \mathscr{O}_{X(S)}$ be germs with an isolated singularity at the origin such that $\overline{J(f)} = \mathfrak{m}^{\operatorname{ord}(f)-1}$. Suppose that f and g are bi-Lipschitz $\mathscr{A}$-equivalent or bi-Lipschitz $\mathscr{K}^*$-equivalent. Then*

$$Eu_{V(f)}(0) = Eu_{V(g)}(0),$$

where $V(f)$ and $V(g)$ are the hypersurfaces given by the zero sets of f and g, respectively.

Proof. Since $\overline{J(f)} = \mathfrak{m}^{\operatorname{ord}(f)-1}$, we have $\Gamma_+(J(f)) = \Gamma_+\big(\mathfrak{m}^{\operatorname{ord}(f)-1}\big)$. Hence,

$$\begin{aligned} J(f)^\circ &= \big\langle x^k : k_1\gamma_1 + \cdots + k_r\gamma_r \in \Gamma_+(J(f)) \big\rangle \\ &= \big\langle x^k : k_1\gamma_1 + \cdots + k_r\gamma_r \in \Gamma_+\big(\mathfrak{m}^{\operatorname{ord}(f)-1}\big) \big\rangle \\ &= \big(\mathfrak{m}^{\operatorname{ord}(f)-1}\big)^\circ. \end{aligned}$$

Since $\mathfrak{m}^{\operatorname{ord}(f)-1}$ is non-degenerate, it follows from Corollaries 2.27 and 2.29 that

$$\big(\mathfrak{m}^{\operatorname{ord}(f)-1}\big)^\circ = \overline{\mathfrak{m}^{\operatorname{ord}(f)-1}} = \mathfrak{m}^{\operatorname{ord}(f)-1}.$$

Thus $\overline{J(f)} = J(f)^\circ$. Applying the Corollary 2.27 once more, we conclude that $J(f)$ is a non-degenerate ideal.

By Theorem 4.4, it follows that $J(f)$ and $J(g)$ are bi-Lipschitz equivalent. Since $\overline{J(f)} = \mathfrak{m}^{\operatorname{ord}(f)-1}$, Corollary 4.5 implies that $\overline{J(f)} = \overline{J(g)}$. Hence $\Gamma_+(J(f)) = \Gamma_+(J(g))$. Note that the equality $\Gamma_+(J(f)) = \Gamma_+(\mathfrak{m}^{\operatorname{ord}(f)-1})$ implies that $\Gamma_+(f) = \Gamma_+(\mathfrak{m}^{\operatorname{ord}(f)})$. Applying the same argument to g, we conclude that $\Gamma_+(f) = \Gamma_+(g)$. Moreover,

$$\begin{aligned} J(g)^\circ &= \big\langle x^k : k_1\gamma_1 + \cdots + k_r\gamma_r \in \Gamma_+(J(g)) \big\rangle \\ &= \big\langle x^k : k_1\gamma_1 + \cdots + k_r\gamma_r \in \Gamma_+(J(f)) \big\rangle \\ &= \big(\mathfrak{m}^{\operatorname{ord}(f)-1}\big)^\circ \\ &= J(f)^\circ \\ &= \overline{J(f)} \\ &= \overline{J(g)}, \end{aligned}$$

then, applying the Corollary 2.27 once more, we conclude that $J(g)$ is also an ideal non-degenerate. Thus, f and g are non-degenerate functions in the sense of [40, Definition 3.2].

Now, let $L_f, L_g : \mathbb{C}^r \to \mathbb{C}$ be generic linear forms with respect to $V(f)$ and $V(g)$, respectively. Without loss of generality, we can assume that

$$L_f(x) = a_1x_1 + \cdots + a_rx_r, \quad a_i \neq 0, \quad \text{for all i}$$

$$L_g(x) = b_1x_1 + \cdots + b_rx_r, \quad b_i \neq 0, \quad \text{for all i}$$

so that $\Gamma_+(L_f) = \Gamma_+(L_g)$. Consider the following subvarieties of $X(S)$

$$V(f, L_f) := \{f = L_f = 0\} \subset V(f) := \{f = 0\}, \tag{4.3}$$

$$V(g,L_g) := \{g = L_g = 0\} \subset V(g) := \{g = 0\}. \tag{4.4}$$

Since f and g are non-degenerate functions, for L_f and L_g generic enough, we have (f,L_f) and (g,L_g) as non-degenerate complete intersections, as in [40, Definition 3.9] (see also [43]).

Therefore, by [40, Theorem 3.12], the Euler characteristic of the Milnor fiber $(L_f)_0$ of the function $L_f|_{V(f)} : V(f) \to \mathbb{C}$ at the point $0 \in V(f)$ is given as a sum of volumes of the Newton polyhedron associated to the function $f \cdot L_f$.

Since $\Gamma_+(f) = \Gamma_+(g)$ and $\Gamma_+(L_f) = \Gamma_+(L_g)$, it follows that $\Gamma_+(f \cdot L_f) = \Gamma_+(g \cdot L_g)$. Consequently, we conclude that $\chi((L_f)_0) = \chi((L_g)_0)$.

Moreover, since both $V(f)$ and $V(g)$ have isolated singularities, their local Euler obstructions at the origin coincide. Therefore, the Euler obstructions of $V(f)$ and $V(g)$ are equal. $\square$

In [9] Bobadilla, Fernandes, and Sampaio proved the following theorem.

Theorem 4.9. *Let $X \subset \mathbb{C}^{N+1}$ and $Y \subset \mathbb{C}^{M+1}$ be two complex analytic surfaces. If $(X,0)$ and $(Y,0)$ are bi-Lipschitz homeomorphic, then $m_0(X,0) = m_0(Y,0)$.*

Then, as a corollary of this result and from Theorems 4.8 and 1.39, we obtain.

Corollary 4.10. *Let $f, g \in \mathscr{O}_3$ be germs with an isolated singularity at the origin such that $\overline{J(f)} = \mathfrak{m}^{\mathrm{ord}(f)-1}$. Suppose that f and g are bi-Lipschitz $\mathscr{A}$-equivalent or bi-Lipschitz $\mathscr{K}^*$-equivalent. Then*

$$m_1(X,0) = m_1(Y,0),$$

where $X = f^{-1}(0)$ and $Y = g^{-1}(0)$.

Proof. Let $f, g \in \mathscr{O}_3$ be germs with isolated singularities at the origin such that $J(f) = \mathfrak{m}^{\mathrm{ord}(f)-1}$, and suppose that f and g are bi-Lipschitz $\mathscr{A}$-equivalent or bi-Lipschitz $\mathscr{K}^*$-equivalent. Set $X = f^{-1}(0)$ and $Y = g^{-1}(0)$. Then X and Y are hypersurfaces in $\mathbb{C}^3$, hence complex analytic surfaces. By Theorem 4.8, we have

$$\mathrm{Eu}_X(0) = \mathrm{Eu}_Y(0).$$

On the other hand, by Theorem 4.9, since $(X,0)$ and $(Y,0)$ are bi-Lipschitz homeomorphic, their multiplicities coincide:

$$m_0(X,0) = m_0(Y,0).$$

Now, using the Lê–Teissier formula (Theorem 1.39), for surfaces we have

$$\mathrm{Eu}_X(0) = (-1)^{2-0-1} m_0(X,0) + (-1)^{2-1-1} m_1(X,0) = -m_0(X,0) + m_1(X,0),$$

and similarly,

$$\mathrm{Eu}_Y(0) = -m_0(Y,0) + m_1(Y,0).$$

Since $\mathrm{Eu}_X(0) = \mathrm{Eu}_Y(0)$ and $m_0(X,0) = m_0(Y,0)$, it follows that

$$-m_0(X,0) + m_1(X,0) = -m_0(Y,0) + m_1(Y,0),$$

and therefore $m_1(X,0) = m_1(Y,0)$. $\square$

Example 4.11. Let $f(x,y,z) = x^2 + y^a + z^2 \in \mathscr{O}_3$, where $a \geqslant 2$, and consider the germ

$$g(x,y,z) = x^2 + (y+xy)^a + (z+2y^2+xy^2)^2 + 2(y^2 - xz) \in \mathscr{O}_3.$$

The quadratic part of g is $(x-z)^2 + 2y^2$, whose critical locus is given by $\{x = z,\ y = 0\}$. A direct computation shows that this curve is contained in $\mathrm{Sing}(V(g))$. Hence $\mathrm{Sing}(V(g))$ has dimension at least one, whereas $\mathrm{Sing}(V(f))$ is isolated.

Since the dimension of the singular locus is preserved under bi-Lipschitz homeomorphisms and

$$\dim \mathrm{Sing}(V(f)) \neq \dim \mathrm{Sing}(V(g)),$$

it follows that the germs $(V(f),0)$ and $(V(g),0)$ are not bi-Lipschitz equivalent. In particular, f and g are not bi-Lipschitz $\mathscr{R}$-equivalent.

We now consider the same germs in the toric setting. Let $X(S)$ be the toric variety associated with the semigroup $S = \{(1,0),(1,1),(1,2)\}$. When f and g are regarded as elements of $\mathscr{O}_{X(S),0}$, the situation changes significantly. Indeed, since $y^2 - xz = 0$ on $X(S)$, the term $2(y^2 - xz)$ vanishes identically. Moreover, the map

$$\varphi(x,y,z) = \big(x,\ y+xy,\ z+2y^2+xy^2\big)$$

defines a local bi-Lipschitz homeomorphism of $(X(S),0)$. Therefore,

$$g = f \circ \varphi \quad \text{in } \mathscr{O}_{X(S),0}.$$

It follows that f and g are bi-Lipschitz $\mathscr{R}$-equivalent on $(X(S),0)$, and therefore also bi-Lipschitz $\mathscr{A}$-equivalent. Consequently, by Theorem 4.8,

$$Eu_{V(f)}(0) = Eu_{V(g)}(0).$$

Example 4.12. Consider $f(x,y,z) = x^2 + y^2 + z^2 \in \mathscr{O}_3$. We have $\mathrm{ord}(f) = 2$. The Jacobian ideal is $J(f) = \mathfrak{m}$. Let $\Phi(x,y,z) = (x+y,\ y,\ z)$. Define

$$g(x,y,z) = f(\Phi(x,y,z)) = (x+y)^2 + y^2 + z^2.$$

Since Φ is a linear invertible transformation, it is bi-Lipschitz, and thus f and g are bi-Lipschitz equivalent. Therefore, all the assumptions of Theorem 4.8 are satisfied, and we conclude that

$$Eu_{V(f)}(0) = Eu_{V(g)}(0).$$

APPENDIX A

Computational Calculations in Oscar.jl

This appendix contains the Julia codes used to perform the computations presented throughout this thesis. All symbolic and algebraic calculations were carried out using the OSCAR system [44, 19]. In addition to employing the functionalities already available in OSCAR, we implemented auxiliary functions to support and streamline the examples developed along the text. The scripts reproduce the examples discussed in the main body and may also serve as a reference for verification and further exploration.

A.1 Auxiliary Functions

In this section, we present a collection of auxiliary functions implemented in Oscar.jl, designed to support and automate the computational procedures arising in the examples throughout this thesis. These routines focus primarily on the construction of Newton polyhedra, the identification of their compact faces, and the computation of the associated auxiliary polynomials and their restrictions, thereby providing effective tools for carrying out the required calculations.

Construction of Newton Polyhedron

```
function polyH(C::AbstractMatrix, V::Vector)

cone = positive_hull(C) # C = generating matrix of the cone (columns)
CO   = polyhedron(cone) # V = list of vectors (shifts)
shifts = [CO + v for v in V] # Minkowski sum CO + v_i
P = shifts[1]    # convex hull of all

for i in 2:length(shifts)
P = convex_hull(P, shifts[i])
end

return P
end
```

Algorithm A.1: Function `polyH`

List of all compact faces of Newton Polyhedron

```
function compact_faces(P)
d = dim(P)
compact_faces = Polyhedron[]

for k in 0:d
for F in faces(P, k)
if is_bounded(F)
push!(compact_faces, F)
end
end
end
return  compact_faces
end
```

Algorithm A.2: Function `compact_faces`

Linear combinations with generators of $cone(S)$

```
function nu(k::Vector{Int}, C::Matrix{Int})
#  k=(k_1,...,k_r) and C is matrix of generators {b_1,...,b_r} of cone(S)
return vec(k' * C)  # k_1b_1+\cdots+k_rb_
end
```

Algorithm A.3: Function `nu(k, C)`

Construction of the auxiliary polynomial $L(g)$

```
function L(f, C)

K = base_ring(parent(f))
ef = collect(exponents(f))
cf = collect(coefficients(f))

nu_ef = [nu(k, C) for k in ef]

r = length(nu_ef[1])

names = ["z"*string(i) for i in 1:r]
S, z = polynomial_ring(K, names)

Lf = zero(S)

for (a, v) in zip(cf, nu_ef)
mon = one(S)
for i in 1:r
mon *= z[i]^v[i]
end

Lf += a * mon
end

return Lf
end
```

Algorithm A.4: Function `L(g, C)`

Support of a ideal

```
function supportS(C, gs...)
U = Vector{Vector{Int}}()

for g in gs
append!(U, collect(exponents(L(g, C))))
end

return unique(U)
end
```

Algorithm A.5: Function `supportS(C, gs...)`

Restriction $L(g_\Delta)$

```
function restrict_to_face(Lf, E)

ef = collect(exponents(Lf))
cf = collect(coefficients(Lf))

R = parent(Lf)
r = nvars(R)

LfE = zero(Lf)

for (a, v) in zip(cf, ef)

if v in E

mon = one(R)
for i in 1:r
mon *= gen(R, i)^v[i]
end

LfE += a * mon
end
end

return LfE
end
```

Algorithm A.6: Function `restrict_to_face(Lf, E)`

Restrictions $L((g_i)_\Delta)$ for all compacts faces of $\Gamma(I)$, where g_i are generators of I

```
function L_Delta(P, Ls...)
compact_faces = Polyhedron[]

d = dim(P)

for k in 0:d
for F in faces(P, k)
if is_bounded(F)
push!(compact_faces, F)
end
```

```
end
end

restrictions = []

for E in compact_faces
restricted = [restrict_to_face(L, E) for L in Ls]
push!(restrictions, restricted)
end

return restrictions
end
```

Algorithm A.7: Function `L_Delta(P, Ls...)`

Condition of non-degeneration

```
function is_ND(restr)

for r in restr # r = [f1, f2, ..., fk]
R = parent(r[1])

I = ideal(R, r...)
m = prod(gens(R))
J = ideal(R, m)

sat = saturation(I, J)

if sat != ideal(R, 1)
return "I is degenerate"
end
end

return "I is non degenerate"
end
```

Algorithm A.8: Function `is_ND(L_Delta(P, Ls...))`

A.2 Codes for the examples of Chapter 1

```
julia>A=[2 3]  #Matrix of generators for semigroup S.
1x2 Matrix{Int64}:
2  3
julia>I=toric_ideal(transpose(A))#The toric ideal
Ideal generated by
x1^3 - x2^2
```

Algorithm A.9: Example 1.41

```
julia>B=[1 1 1;0 1 2] #Matrix of generators for semigroup S.
2x3 Matrix{Int64}:
1  1  1
0  1  2
```

```
julia>I=toric_ideal(transpose(B)) # The binomial ideal of the associated toric
  variety.
Ideal generated by
-x1*x3 + x2^2
```

Algorithm A.10: Example 1.42

```
julia>cone=positive_hull([1 0;1 1;1 2]) #Polyhedral cone associated with S
Polyhedral cone in ambient dimension 2
julia>facets_cone=facets(cone) #The faceted inequalities of the cone(S)
2-element SubObjectIterator{LinearHalfspace{QQFieldElem}} over the halfspaces
  of R^2 described by:
-x_2 <= 0
-2*x_1 + x_2 <= 0
julia> coned=polarize(cone) #  The dual cone of cone(S)
Polyhedral cone in ambient dimension 2
julia> H = hilbert_basis(coned) #The Hilbert basis of the dual cone
3-element SubObjectIterator{PointVector{ZZRingElem}}:
[1, 0]
[0, 1]
[2, -1]
julia> dim_cone=dim(cone) #The dimensions of the cone
2
julia> dim_coned=dim(coned) #The dimensions of the dual cone
2
```

Algorithm A.11: Example 1.46

```
julia>cone=positive_hull([1 0;1 1;1 2]) #Polyhedral cone associated with S
Polyhedral cone in ambient dimension 2
julia> rays_cone=rays(cone) #The vectors that determine the rays of the cone(S
  ).
2-element SubObjectIterator{RayVector{QQFieldElem}}:
[1, 0]
[1, 2]
julia> coned=polarize(cone) #  The dual cone of cone(S)
Polyhedral cone in ambient dimension 2
julia> rays_coned=rays(coned)#The vectors that determine the rays of the dual
  cone.
2-element SubObjectIterator{RayVector{QQFieldElem}}:
[0, 1]
[1, -1//2]
```

Algorithm A.12: Example 1.49

A.3 Codes for the examples of Chapter 2

```
julia> C=[1 3;1 0;3 5] #Declare the generating vectors of the cone.
3x2 Matrix{Int64}:
1  3
1  0
3  5
julia> V=[[5,0],[2,6],[3,2],[4,4]]# Set that determines the polyhedron
4-element Vector{Vector{Int64}}:
[5, 0]
[2, 6]
[3, 2]
```

```
[4, 4]
julia> P=polyH(C,V) # Function that determines Newton polyhedron
Polyhedron in ambient dimension 2
julia> n_vertices(P) # Number of vertices of the polyhedron
3
julia> vertices(P) #Vertices of the polyhedron
3-element SubObjectIterator{PointVector{QQFieldElem}}:
[5, 0]
[2, 6]
[3, 2]
julia> n_facets(P) # returns the number of faces of maximum dimension
4
julia> facets(P) # Returns the faces of maximum dimension (the facets) of the
  polyhedron `P` as inequalities.
4-element SubObjectIterator{AffineHalfspace{QQFieldElem}} over the halfspaces
  of R^2 described by:
-3*x_1 + x_2 <= 0
-4*x_1 - x_2 <= -14
-x_1 - x_2 <= -5
-x_2 <= 0
julia> facets(P)[1] # One of the faces of the polyhedron
The half-space of R^2 described by
-3*x_1 + x_2 <= 0
julia> vectors_facets = [-normal_vector(h) for h in facets(P)]
4-element Vector{Vector{QQFieldElem}}:
[3, -1]
[4, 1]
[1, 1]
[0, 1]
```

Algorithm A.13: Example 2.4

```
julia>C=[1 0 0;1 1 0;1 0 1;1 2 0;1 1 1;1 0 2]
6x3 Matrix{Int64}:
1  0  0
1  1  0
1  0  1
1  2  0
1  1  1
1  0  2
julia> R, (x1,x2,x3,x4,x5,x6) = polynomial_ring(QQ, ["x1","x2","x3","x4","x5",
  "x6"])
(Multivariate polynomial ring in 6 variables over QQ, QQMPolyRingElem[x1, x2,
  x3, x4, x5, x6])
julia> f=x1^2*x3^2+5*x1*x2^5*x3+60*x4^2*x5+9*x6-x1^3*x6+80*x4^2-2x1^3
-x1^3*x6 - 2*x1^3 + x1^2*x3^2 + 5*x1*x2^5*x3 + 60*x4^2*x5 + 80*x4^2 + 9*x6
julia> V=supportS(C,f)
5-element Vector{Vector{Int64}}:
[7, 5, 1]
[3, 5, 1]
[2, 4, 0]
[3, 0, 0]
[1, 0, 2]
julia> P=polyH(C,V)
Polyhedron in ambient dimension 3
julia> vertices(P)
3-element SubObjectIterator{PointVector{QQFieldElem}}:
[2, 4, 0]
[3, 0, 0]
```

```
[1, 0, 2]
julia> facets(P)
4-element SubObjectIterator{AffineHalfspace{QQFieldElem}} over the halfspaces
 of R^3 described by:
-2*x_1 + x_2 + x_3 <= 0
-x_3 <= 0
-x_2 <= 0
-4*x_1 - x_2 - 4*x_3 <= -12
julia>  vectors_facets = [-normal_vector(h) for h in facets(P)]
4-element Vector{Vector{QQFieldElem}}:
[2, -1, -1]
[0, 0, 1]
[0, 1, 0]
[4, 1, 4]
```

Algorithm A.14: Example 2.6

```
julia> C=[1 0;1 1;1 2]
3x2 Matrix{Int64}:
1  0
1  1
1  2
julia> R, (x1,x2,x3) = polynomial_ring(QQ, ["x1","x2","x3"])
(Multivariate polynomial ring in 3 variables over QQ, QQMPolyRingElem[x1, x2,
 x3])
julia> g1=x1^2*x2^2+x3^4+x1^4*x3
x1^4*x3 + x1^2*x2^2 + x3^4
julia> g2=x1^5*x3+x3^3+x1^3*x2^2
x1^5*x3 + x1^3*x2^2 + x3^3
julia> V=supportS(C,g1,g2)
5-element Vector{Vector{Int64}}:
[4, 8]
[5, 2]
[4, 2]
[3, 6]
[6, 2]
julia> P=polyH(C,V)
Polyhedron in ambient dimension 2
julia> vertices(P)
2-element SubObjectIterator{PointVector{QQFieldElem}}:
[4, 2]
[3, 6]
julia> facets(P)
3-element SubObjectIterator{AffineHalfspace{QQFieldElem}} over the halfspaces
 of R^2 described by:
-x_2 <= -2
-4*x_1 - x_2 <= -18
-2*x_1 + x_2 <= 0
```

Algorithm A.15: Example 2.8

```
julia> C=[1 0;1 1;1 2]
3x2 Matrix{Int64}:
1  0
1  1
1  2
julia> R, (x1,x2,x3) = polynomial_ring(QQ, ["x1","x2","x3"])
(Multivariate polynomial ring in 3 variables over QQ, QQMPolyRingElem[x1, x2, x3
  ])
```

```
julia> g1=x1^2*x2+3*x1^2*x3
x1^2*x2 + 3*x1^2*x3
julia> g2=x2^3-x1*x2+x3^2
-x1*x2 + x2^3 + x3^2
julia> V=supportS(C,g1,g2)
5-element Vector{Vector{Int64}}:
[3, 2]
[3, 1]
[3, 3]
[2, 4]
[2, 1]
julia> P=polyH(C,V)
Polyhedron in ambient dimension 2
julia> p=[2,2]
2-element Vector{Int64}:
2
2
julia> p in P
true
```

Algorithm A.16: Example 2.11

```
julia> C=[1 0;1 1;1 2]
3x2 Matrix{Int64}:
1  0
1  1
1  2

julia> R, (x1,x2,x3) = polynomial_ring(QQ, ["x1","x2","x3"])
(Multivariate polynomial ring in 3 variables over QQ, QQMPolyRingElem[x1, x2, x3
    ])

julia> g1=x1^2*x2+3*x1^2*x3
x1^2*x2 + 3*x1^2*x3

julia> g2=x2^3-x1*x2+2*x3^2
-x1*x2 + x2^3 + 2*x3^2

julia> V=supportS(C,g1,g2)
5-element Vector{Vector{Int64}}:
[3, 2]
[3, 1]
[3, 3]
[2, 4]
[2, 1]

julia> P=polyH(C,V)
Polyhedron in ambient dimension 2

julia> Lg1=L(g1,C)
3*z1^3*z2^2 + z1^3*z2

julia> Lg2=L(g2,C)
z1^3*z2^3 + 2*z1^2*z2^4 - z1^2*z2

julia> L_Delta(P,Lg1,Lg2)
3-element Vector{Any}:
QQMPolyRingElem[0, 2*z1^2*z2^4]
QQMPolyRingElem[0, -z1^2*z2]
```

```
QQMPolyRingElem[0, 2*z1^2*z2^4 - z1^2*z2]

julia> is_ND(L_Delta(P,Lg1,Lg2))
"I is degenerate"

julia> D=[1 0;0 1]
2x2 Matrix{Int64}:
1  0
0  1

julia> Dv=supportS(D,Lg1,Lg2)
5-element Vector{Vector{Int64}}:
[3, 2]
[3, 1]
[3, 3]
[2, 4]
[2, 1]

julia> PD=polyH(D,Dv)
Polyhedron in ambient dimension 2

julia> L_Delta(PD,Lg1,Lg2)
1-element Vector{Any}:
QQMPolyRingElem[0, -z1^2*z2]

julia> is_ND(L_Delta(PD,Lg1,Lg2))
"I is non degenerate"
```

Algorithm A.17: Example 2.14

```
julia> C=[1 0;1 1;1 2]
3x2 Matrix{Int64}:
1  0
1  1
1  2

julia> R, (x1,x2,x3) = polynomial_ring(QQ, ["x1","x2","x3"])
(Multivariate polynomial ring in 3 variables over QQ, QQMPolyRingElem[x1, x2, x3
   ])

julia> g1=x1^4*x2+x3^2
x1^4*x2 + x3^2

julia> g2=x1*x2*x3+x1^2*x2^3
x1^2*x2^3 + x1*x2*x3

julia> V=supportS(C,g1,g2)
4-element Vector{Vector{Int64}}:
[5, 1]
[2, 4]
[5, 3]
[3, 3]

julia> P=polyH(C,V)
Polyhedron in ambient dimension 2

julia> Lg1=L(g1,C)
z1^5*z2 + z1^2*z2^4

```

```
julia> Lg2=L(g2,C)
z1^5*z2^3 + z1^3*z2^3

julia> L_Delta(P,Lg1,Lg2)
3-element Vector{Any}:
QQMPolyRingElem[z1^5*z2, 0]
QQMPolyRingElem[z1^2*z2^4, 0]
QQMPolyRingElem[z1^5*z2 + z1^2*z2^4, z1^3*z2^3]

julia> is_ND(L_Delta(P,Lg1,Lg2))
"I is non degenerate"
```

Algorithm A.18: Example 2.28

```
julia> C=[1 0;3 5;1 3]
3x2 Matrix{Int64}:
1  0
3  5
1  3
julia> g1=4*x1^4*x3^5
4*x1^4*x3^5
julia> g2=4*x1^3*x2*x3^5
4*x1^3*x2*x3^5
julia> g3=-3*x1*x2^2
-3*x1*x2^2
julia> g4=-3*x2^3
-3*x2^3
julia> V=supportS(C,g1,g2,g3)
3-element Vector{Vector{Int64}}:
[9, 15]
[11, 20]
[7, 10]
julia> P=polyH(C,V)
Polyhedron in ambient dimension 2
julia> Lg1=L(g1,C)
4*z1^9*z2^15
julia> Lg2=L(g2,C)
4*z1^11*z2^20

julia> Lg3=L(g3,C)
-3*z1^7*z2^10

julia> Lg4=L(g4,C)
-3*z1^9*z2^15

julia> L_Delta(P,Lg1,Lg2,Lg3,Lg4)
1-element Vector{Any}:
QQMPolyRingElem[0, 0, -3*z1^7*z2^10, 0]

julia> is_ND(L_Delta(P,Lg1,Lg2,Lg3,Lg4))
"I is non degenerate"

julia> G=5*x1^4*x3^4
5*x1^4*x3^4

julia> S=supportS(C,G)
1-element Vector{Vector{Int64}}:
[8, 12]

```

```
julia> S[1] in P
true
```

Algorithm A.19: Example 2.30

A.4 Codes for the examples of Chapter 3

```
julia> C=[1 0;14 11; 8 7]
3x2 Matrix{Int64}:
1   0
14  11
8   7
julia> R, (x1,x2,x3) = polynomial_ring(QQ, ["x1","x2","x3"])
(Multivariate polynomial ring in 3 variables over QQ, QQMPolyRingElem[x1, x2,
 x3])

julia> g1=10*x1^10*x3^11
10*x1^10*x3^11

julia> g2=10*x1^9*x2*x3^11
10*x1^9*x2*x3^11

julia> g3=-7*x1*x2^6
-7*x1*x2^6

julia> g4=-7*x2^7
-7*x2^7

julia> V=supportS(C,g1,g2,g3,g4)
3-element Vector{Vector{Int64}}:
[98, 77]
[111, 88]
[85, 66]

julia> P=polyH(C,V)
Polyhedron in ambient dimension 2

julia> Lg1=L(g1,C)
10*z1^98*z2^77

julia> Lg2=L(g2,C)
10*z1^111*z2^88

julia> Lg3=L(g3,C)
-7*z1^85*z2^66

julia> Lg4=L(g4,C)
-7*z1^98*z2^77

julia> L_Delta(P,Lg1,Lg2,Lg3,Lg4)
1-element Vector{Any}:
QQMPolyRingElem[0, 0, -7*z1^85*z2^66, 0]

julia> is_ND(L_Delta(P,Lg1,Lg2,Lg3,Lg4))
"I is non degenerate"

```

```
julia> g=11*x1^10*x3^10
11*x1^10*x3^10

julia> v=supportS(C,g)
1-element Vector{Vector{Int64}}:
[90, 70]

julia> v[1]
2-element Vector{Int64}:
90
70

julia> v[1] in P
true
```

Algorithm A.20: Example 3.5

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