**UNIVERSIDADE FEDERAL DE SÃO CARLOS**
CENTRO DE CIÊNCIAS EXATAS E DE TECNOLOGIA
PROGRAMA DE PÓS-GRADUAÇãO EM MATEMÁTICA

# Weingarten Surfaces in Three-Dimensional Product Geometries

Mannaim Gennaro Vitti

São Carlos-SP
Dezembro de 2024

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Mannaim Gennaro Vitti

Orientador: Prof. Dr. Alexandre Paiva Barreto

Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de São Carlos como parte dos requisitos para a obtenção do Título de Doutor.

São Carlos-SP
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## Folha de Aprovação

Defesa de Tese de Doutorado do candidato Mannaim Gennaro Vitti, realizada em 28/02/2025.

### Comissão Julgadora:

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O Relatório de Defesa assinado pelos membros da Comissão Julgadora encontra-se arquivado junto ao Programa de Pós-Graduação em Matemática.

*For Lídia Bilia Camargo*

*Memento homo. Memento mori.*

## Para além da curva da estrada

Para além da curva da estrada
Talvez haja um poço, e talvez um castelo,
E talvez apenas a continuação da estrada.
Não sei nem pergunto.
Enquanto vou na estrada antes da curva
Só olho para a estrada antes da curva,
Porque não posso ver senão a estrada antes da curva.
De nada me serviria estar olhando para outro lado
E para aquilo que não vejo.
Importemo-nos apenas com o lugar onde estamos.
Há beleza bastante em estar aqui e não noutra parte qualquer.
Se há alguém para além da curva da estrada,
Esses que se preocupem com o que há para além da curva da estrada.
Essa é que é a estrada para eles.
Se nós tivermos que chegar lá, quando lá chegarmos saberemos.
Por ora só sabemos que lá não estamos.
Aqui há só a estrada antes da curva, e antes da curva
Há a estrada sem curva nenhuma.

— Alberto Caeiro in Poemas Inconjuntos

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# Abstract

In this work, we seek to obtain information about certain types of Weingarten surfaces, approaching this problem through nonlinear partial differential equations. We study particular cases of translation surfaces in the product spaces $H^2 \times \mathbb{R}$ and $H^3$ and Helical surfaces in $S^2 \times \mathbb{R}$ that verify the property of being a Weingarten surface. In addition, we also investigate translation surfaces in the Euclidean space $\mathbb{E}^3$ that are particular cases of Weingarten Flow Solitons.

**Palavras-chave**: Weingarten Surfaces, Product Geometries, Weingarten Flow, Solitons, Nonlinear Partial Differential Equations, Fourier Method, Separation of Variables, Heat Equation.

# Resumo

Neste trabalho, buscamos obter informações de certos tipos de superfícies de Weingarten, abordando esse problema por meio de equações diferenciais parciais não lineares. Estudamos casos particulares de superfícies de translação nos espaços produto $H^2 \times \mathbb{R}$ e $H^3$ e Helicoidais em $S^2 \times \mathbb{R}$ que verificam a propriedade de ser superfície de Weingarten. Além disso, também investigamos superfícies de translação no espaço Euclidiano $\mathbb{E}^3$ que sejam casos particulares de Sólitons do Fluxo de Weingarten.

**Keywords**: Superfícies de Weingarten, Geometrias Produto, Fluxo de Weingarten, Sólitons, Equações Diferenciais Parciais não Lineares, Método de Fourier, Separação de Variáveis, Equação do Calor.

# Contents

# Introduction

A surface immersed in Euclidean space $\mathbb{E}^3$ is said to be a Weingarten surface when its Gaussian (denoted by $K$) and Mean (denoted by $H$) curvatures verify a smooth relation (function)

$$\Psi(H,K) \equiv 0.$$

The study of these surfaces is a classical problem in Differential Geometry, which began with the works of the German mathematician Julius Weingarten, published in the mid-19th century. Within this field, a problem of great interest is the classification of surfaces. Part of what is currently done in classification problems consists of reducing such problems to Nonlinear Partial Differential Equations (see [5], [6], [7], [16]). For example, if we consider a surface defined by a graph $(x,y,u(x,y))$ in $\mathbb{E}^3$, then its Gaussian curvature is

$$K = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1+u_x^2+u_y^2)^2}.$$

Understanding when this graph has constant Gaussian curvature $c$ leads us to the nonlinear partial differential equation

$$c = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1+u_x^2+u_y^2)^2}.$$

Our goal is to investigate such Weingarten surfaces in other three-dimensional spaces, restricting them to specific families of immersions. This restriction leads us to the study of techniques that allow us to analyze second-order Nonlinear Partial Differential Equations.

The first chapter will be reserved to present results contained in the Theory of Isometric Embeddings and also in the Theory of Plane Algebraic Curves. In the second chapter, we define Three-Dimensional Weingarten Surfaces, Translation Surfaces in $H^2 \times \mathbb{R}$ and $H^3$, and Helical Surfaces in $S^2 \times \mathbb{R}$ in order to determine, in some particular cases, when the latter are part of the Weingarten Surfaces. This is done by transferring the problem to the task of investigating Nonlinear Partial Differential Equations. In the third chapter, we introduce the Weingarten Flow, the surfaces called Solitons, and briefly discuss how such surfaces generalize, in a certain sense, those defined in the previous chapter. In addition, we repeat the approach, contained in the second chapter, of trying to obtain properties of surfaces, through Nonlinear Partial Differential Equations, for Solitons. Finally, the last chapter is dedicated to applying the ideas developed in the previous chapters to obtain particular solutions for Nonlinear Partial Differential Equations, which do not come from Differential Geometry, developing a small connection with the method proposed by Fourier (most known as Method of Separation of Variables for the Heat Equation).

# CHAPTER 1

# Preliminary

In this chapter, we will present basic results that we will use throughout the work.

## 1.1 Isometric Immersions

**Lemma 1.1.1.** *Let $M$ be a Riemannian 3-manifold and $\{E_1, E_2, E_3\}$ a global orthonormal frame over $M$. Defining*

$$\nabla^{\times}_{E_j} E_i = \sum_{k=1}^{3} \Gamma^k_{ji} E_k$$

*the following holds:*

$$\Gamma^k_{ji} = -\frac{1}{2}\left([E_i, E_k] \bullet_{\times} E_j + [E_j, E_k] \bullet_{\times} E_i + [E_i, E_j] \bullet_{\times} E_k\right),$$

*where $\bullet_{\times}$ and $\nabla^{\times}$ are, respectively, the metric and the connection on $M$.*

**Proof:** By the compatibility of the metric

$$\begin{aligned}
E_i(E_j \bullet_{\times} E_k) &= \nabla^{\times}_{E_i} E_j \bullet_{\times} E_k + E_j \bullet_{\times} \nabla^{\times}_{E_i} E_k \\
E_j(E_k \bullet_{\times} E_i) &= \nabla^{\times}_{E_j} E_k \bullet_{\times} E_i + E_k \bullet_{\times} \nabla^{\times}_{E_j} E_i \\
E_k(E_i \bullet_{\times} E_j) &= \nabla^{\times}_{E_k} E_i \bullet_{\times} E_j + E_i \bullet_{\times} \nabla^{\times}_{E_k} E_j.
\end{aligned}$$

Adding the first and second equality and subtracting the third we obtain, by the symmetry,

$$\begin{aligned}
E_i(E_j \bullet_{\times} E_k) + E_j(E_k \bullet_{\times} E_i) - E_k(E_i \bullet_{\times} E_j) =& [E_i, E_k] \bullet_{\times} E_j + [E_j, E_k] \bullet_{\times} E_i + [E_i, E_j] \bullet_{\times} E_k \\
&+ 2E_k \bullet_{\times} \nabla^{\times}_{E_j} E_i,
\end{aligned}$$

and since $E_r \bullet_{\times} E_s = \delta_{rs}$,

$$E_k \bullet_{\times} \nabla^{\times}_{E_j} E_i = -\frac{1}{2}\left([E_i, E_k] \bullet_{\times} E_j + [E_j, E_k] \bullet_{\times} E_i + [E_i, E_j] \bullet_{\times} E_k\right).$$

As $\nabla^{\times}_{E_j} E_i = \sum_k \Gamma^k_{ji} E_k$ we conclude,

$$\Gamma^k_{ji} = -\frac{1}{2}\left([E_i,E_k]\bullet_\times E_j + [E_j,E_k]\bullet_\times E_i + [E_i,E_j]\bullet_\times E_k\right).$$

$\square$

**Lemma 1.1.2.** *Let $M$ be a Riemannian 3-manifold and $\Omega \subset \mathbb{R}^2$ an open set. Let $\psi : \Omega \subset \mathbb{R}^2 \to M$ immersion. Then the coefficients of the Second Fundamental Form associated to $\psi$ are*

$$\begin{aligned}
\mathcal{E}(t,s) &= \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s)\\
\mathcal{F}(t,s) &= \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s)\\
\mathcal{G}(t,s) &= \left(N \bullet_\times \nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}\right)(t,s),
\end{aligned}$$

*where $N$ is the Normal of $\psi$ and $\bullet_\times$ is the metric of $M$.*

**Proof:** Let $\Omega$ be equipped with the metric induced by $\psi$. The Weingarten map of $\psi$ is defined as $\mathcal{S} : \mathfrak{X}(\Omega) \to \mathfrak{X}(\Omega)$ such that $d\psi(\mathcal{S}(X)) = -\nabla_X N$. Observe that $\mathcal{S}$ is well-defined since $N$ has norm 1. Furthermore, we know that such a map is linear and self-adjoint. We have,

$$\begin{aligned}
\mathcal{S}(\overline{e_1}) &= a_{tt}\overline{e_1} + a_{st}\overline{e_2}\\
\mathcal{S}(\overline{e_2}) &= a_{ts}\overline{e_1} + a_{ss}\overline{e_2}
\end{aligned}$$

then,

$$\begin{aligned}
\mathcal{S}(\overline{e_1})\bullet_\times \overline{e_2} &= a_{tt}\overline{e_1}\bullet_\times \overline{e_2} + a_{st}\overline{e_2}\bullet_\times \overline{e_2}\\
\mathcal{S}(\overline{e_2})\bullet_\times \overline{e_2} &= a_{ts}\overline{e_1}\bullet_\times \overline{e_2} + a_{ss}\overline{e_2}\bullet_\times \overline{e_2}\\
\mathcal{S}(\overline{e_1})\bullet_\times \overline{e_1} &= a_{tt}\overline{e_1}\bullet_\times \overline{e_1} + a_{st}\overline{e_2}\bullet_\times \overline{e_1}\\
\mathcal{S}(\overline{e_2})\bullet_\times \overline{e_1} &= a_{ts}\overline{e_1}\bullet_\times \overline{e_1} + a_{ss}\overline{e_2}\bullet_\times \overline{e_1},
\end{aligned}$$

and defining $\mathfrak{e} = \overline{e_1}\bullet_\times\overline{e_1}$, $\mathfrak{f} = \overline{e_1}\bullet_\times\overline{e_2}$, $\mathfrak{g} = \overline{e_2}\bullet_\times\overline{e_2}$, $\mathcal{E} = \mathcal{S}(\overline{e_1})\bullet_\times\overline{e_1}$, $\mathcal{F} = \mathcal{S}(\overline{e_1})\bullet_\times\overline{e_2}$, $\mathcal{G} = \mathcal{S}(\overline{e_2})\bullet_\times\overline{e_2}$,

$$\begin{aligned}
\mathcal{F} &= a_{tt}\mathfrak{f} + a_{st}\mathfrak{g}\\
\mathcal{G} &= a_{ts}\mathfrak{f} + a_{ss}\mathfrak{g}\\
\mathcal{E} &= a_{tt}\mathfrak{e} + a_{st}\mathfrak{f}\\
\mathcal{F} &= a_{ts}\mathfrak{e} + a_{ss}\mathfrak{f},
\end{aligned}$$

that is,

$$\begin{aligned}
\begin{pmatrix}\mathcal{E} & \mathcal{F}\\ \mathcal{F} & \mathcal{G}\end{pmatrix} = \begin{pmatrix}a_{tt} & a_{st}\\ a_{ts} & a_{ss}\end{pmatrix}\begin{pmatrix}\mathfrak{e} & \mathfrak{f}\\ \mathfrak{f} & \mathfrak{g}\end{pmatrix} &\Rightarrow \begin{pmatrix}a_{tt} & a_{st}\\ a_{ts} & a_{ss}\end{pmatrix} = \begin{pmatrix}\mathcal{E} & \mathcal{F}\\ \mathcal{F} & \mathcal{G}\end{pmatrix}\begin{pmatrix}\mathfrak{e} & \mathfrak{f}\\ \mathfrak{f} & \mathfrak{g}\end{pmatrix}^{-1}\\
&\Rightarrow \begin{pmatrix}a_{tt} & a_{st}\\ a_{ts} & a_{ss}\end{pmatrix} = \frac{1}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\begin{pmatrix}\mathcal{E} & \mathcal{F}\\ \mathcal{F} & \mathcal{G}\end{pmatrix}\begin{pmatrix}\mathfrak{g} & -\mathfrak{f}\\ -\mathfrak{f} & \mathfrak{e}\end{pmatrix}
\end{aligned}$$

concluding that,

$$a_{tt} = \frac{\mathcal{E}\mathfrak{g} - \mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2} \;\;,\;\; a_{st} = \frac{\mathcal{F}\mathfrak{e} - \mathcal{E}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}$$
$$a_{ts} = \frac{\mathcal{F}\mathfrak{g} - \mathcal{G}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2} \;\;,\;\; a_{ss} = \frac{\mathcal{G}\mathfrak{e} - \mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}.$$

On the other hand, since our immersion is isometric,

$$\begin{aligned}
\mathcal{F}(t,s) = \left(\mathcal{S}\left(\overline{e_1}\right) \bullet_{\times} \overline{e_2}\right)(t,s) &= \left(-\nabla_{\overline{e_1}} N \bullet_{\times} \frac{\partial \psi}{\partial s}\right)(t,s) \\
&= \overline{e_1}\left(-N \bullet_{\times} \frac{\partial \psi}{\partial s}\right)(t,s) - \left(-N \bullet_{\times} \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s) \\
&= \left(N \bullet_{\times} \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s),
\end{aligned}$$

and similarly we conclude,

$$\begin{aligned}
\mathcal{E}(t,s) = \left(\mathcal{S}\left(\overline{e_1}\right) \bullet_{\times} \overline{e_1}\right)(t,s) &= \left(N \bullet_{\times} \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s) \\
\mathcal{G}(t,s) = \left(\mathcal{S}\left(\overline{e_2}\right) \bullet_{\times} \overline{e_2}\right)(t,s) &= \left(N \bullet_{\times} \nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}\right)(t,s).
\end{aligned}$$

□

## 1.2 Plane Algebraic Curves

This section is devoted to state a result from the theory of Plane Algebraic Curves. In this work, we reduce many of our problems to polynomials in two variables. To exemplify the type of approach we use, consider two polynomials $P(X,Y)$ and $Q(X,Y)$. Suppose that there exists a continuous curve $\alpha : I \to \mathbb{R}^2$ such that

$$P \circ \alpha \equiv 0 \equiv Q \circ \alpha$$

that is, $\alpha$ vanishes both polynomials. If such polynomials have no factors in common then $\alpha$ is constant. More precisely, denoting $V(P) := \{p \in \mathbb{R}^2 | P(p) = 0\}$, we have made use of the following result:

**Proposition 1.2.1.** *Let P and Q be polynomials in $\mathbb{R}[X,Y]$ with no commom factors. Then $V(P,Q) = V(P) \bigcap V(Q)$ is a finite set of points.*

Indeed, by this result, the curve $\alpha$ is contained in $V(P,Q)$, which is a finite set of points, that is, the only possibility for $\alpha$ is to be constant. This result was taken from the reference [8] (Proposition 2). Since this reference is no longer published, only in digital form (free) on the personal page of the author William Fulton, the reference [9] (Theorem 1.4 item *a*) can also be consulted.

## 1.3 Scientific WorkPlace

Throughout this work, we will use the Scientific WorkPlace Software recurrently. This is a software for scientific word processing that includes a symbolic algebra system (SAS). This last capability is the one that interests us. Many of the equations that will be discussed here will be analyzed as polynomials. For example, consider the equation

$$c^2 = \frac{s^4\left((1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)\right)^2}{4\left(s^2(v'(t))^2+s^2(u'(s))^2+1\right)^3}.$$

The type of approach we use repeatedly throughout the text is the following: first we eliminate the variables from the functions $u$ and $v$. We will obtain

$$c^2 = \frac{s^4\left((1+s^2(u')^2)v'' - s((u')^2+(v')^2)u' + (1+s^2(v')^2)u''\right)^2}{4\left(s^2(v')^2+s^2(u')^2+1\right)^3}.$$

We can rewrite it as

$$0 = s^4\left((1+s^2(u')^2)v'' - s((u')^2+(v')^2)u' + (1+s^2(v')^2)u''\right)^2 - 4c^2\left(s^2(v')^2+s^2(u')^2+1\right)^3.$$

At this point, we use the **collect** function, using $v'', v'$ as the variables to be collected, obtaining

$$\begin{aligned}
0 &= s^4(1+s^2(u')^2)^2(v'')^2 - 2s^4(1+s^2(u')^2)(su'-s^2u'')(v')^2v'' + 2s^4(u''-s(u')^3)(1+s^2(u')^2)v'' \\
&\quad -4c^2s^6(v')^6 + (s^4(su'-s^2u'')^2 - 12c^2s^4(1+s^2(u')^2))(v')^4 \\
&\quad + (-2s^4(u''-s(u')^3)(su-s^2u'') - 12c^2s^2(1+s^2(u')^2)^2)(v')^2 + s^4(u''-s(u')^3)^2 - 4c^2(1+s^2(u')^2)^3.
\end{aligned}$$

If we divide the equation by the coefficient of $(v')^6$ we will have

$$\begin{aligned}
0 &= -\frac{s^4(1+s^2(u')^2)^2}{4c^2s^6}(v'')^2 + \frac{2s^4(1+s^2(u')^2)(su'-s^2u'')}{4c^2s^6}(v')^2v'' - \frac{2s^4(u''-s(u')^3)(1+s^2(u')^2)}{4c^2s^6}v'' \\
&\quad + (v')^6 - \frac{(s^4(su'-s^2u'')^2 - 12c^2s^4(1+s^2(u')^2))}{4c^2s^6}(v')^4 \\
&\quad + \frac{2s^4(u''-s(u')^3)(su-s^2u'') + 12c^2s^2(1+s^2(u')^2)^2}{4c^2s^6}(v')^2 + \frac{s^4(u''-s(u')^3)^2 - 4c^2(1+s^2(u')^2)^3}{4c^2s^6}.
\end{aligned}$$

Thus, we can observe that the curve $((v')^2, v'')$ vanishes out the family of polynomials

$$P_s(X,Y) = A_sY^2 + B_sXY + C_xY + X^3 + D_sX^2 + E_sX + F_s$$

where

$$\begin{aligned}
A_s &= -\frac{s^4(1+s^2(u')^2)^2}{4c^2s^6} & D_s &= -\frac{(s^4(su'-s^2u'')^2 - 12c^2s^4(1+s^2(u')^2))}{4c^2s^6} \\
B_s &= \frac{2s^4(1+s^2(u')^2)(su'-s^2u'')}{4c^2s^6} & E_s &= \frac{2s^4(u''-s(u')^3)(su-s^2u'') + 12c^2s^2(1+s^2(u')^2)^2}{4c^2s^6} \\
C_s &= -\frac{2s^4(u''-s(u')^3)(1+s^2(u')^2)}{4c^2s^6} & F_s &= \frac{s^4(u''-s(u')^3)^2 - 4c^2(1+s^2(u')^2)^3}{4c^2s^6}.
\end{aligned}$$

We also use the **factor** function to simplify expressions. For example, the coefficient $A_s$ would become $-\frac{(1+s^2(u')^2)^2}{4c^2s^2}$. For simple expressions, this may seem unnecessary, but for long expressions, simplifications are sometimes useful. We use the **collect** function when we want to write long expressions in the form of polynomials in a given variable, and the **factor** function when we want to simplify expressions.

**Remark:** The division by the factor $-4c^2s^6$ aims to obtain, in the polynomial $P_s$, a monic coefficient. This is to ensure that, using the Theory of Plane Algebraic Curves, such a family of polynomials, if irreducible, has to be multiple by constants (and, in this case, necessarily only by 1) or, then, the curve that vanishes this family of polynomials is constant.

# CHAPTER 2

# Weingarten Surfaces

A typical problem in the field of differential geometry is that of classifying surfaces. Given a surface on a three-dimensional Riemannian manifold, a common question is whether such a surface has constant Mean curvature or constant Gaussian curvature. This problem sometimes reduces us to solving a non-linear partial differential equation. As an example, if we consider a surface defined by a graph $(x, y, u(x, y))$ in $\mathbb{E}^3$ then its Gaussian curvature is

$$K = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}.$$

We say that a surface is Weingarten when its Mean and Gaussian curvatures verify a relation

$$\Psi(H, K) = 0$$

for some function $\Psi \in C^\infty(\mathbb{R}^2, \mathbb{R})$. If we consider $c$ constant and the functions $\Psi(X, Y) = Y - c$, $\Psi(X, Y) = X - c$ and $\Psi(X, Y) = 4X^2 - 2Y - c$, we would have, respectively, surfaces of constant Gaussian curvature, constant Mean curvature, and constant norm of the Second Fundamental Form. Throughout the next sections, we will focus on obtaining information from such surfaces. We will work on the problems

$$K = c$$

$$H = c$$

$$aH + bK = c$$

$$4H^2 - 2K = c.$$

More precisely, we will name it as follows:

**Definition 2.0.1** (Weingarten Surfaces)**.** Let $M$ be a Riemannian 3-manifold and $\Omega \subset \mathbb{R}^2$ an open set. Consider $\psi : \Omega \subset \mathbb{R}^2 \to M$ an isometric immersion. We denote $H : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ and $K : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ as, respectively, the Mean Curvature and Gaussian Curvature of $\psi$. Furthermore:

i) We say that $\psi$ is **Flat** if

$$K \equiv 0.$$

ii) We say that $\psi$ has **Constant Gaussian Curvature** if
$$K \equiv c,$$
for a constant $c \in \mathbb{R}$.

iii) We say that $\psi$ is **Minimal** if
$$H \equiv 0.$$

iv) We say that $\psi$ has **constant Mean Curvature** if
$$H \equiv c,$$
for constant $c \in \mathbb{R}$.

v) We say that $\psi$ is a **Null Linear Weingarten Form** if
$$aH + bK \equiv 0,$$
for non-zero constants $a, b \in \mathbb{R}$.

v) We say that $\psi$ is a **Linear Weingarten Form** if
$$aH + bK \equiv c,$$
for non-zero constants $a, b, c \in \mathbb{R}$.

vi) We say that $\psi$ has **prescribed Length of the Second Fundamental Form** if
$$|A|^2(t,s) = c(s)$$
or equivalently,
$$4H^2(t,s) - 2K(t,s) = c(s)$$
for a $c \in C(\mathbb{R}, (0, +\infty))$.

vi) We say that $\psi$ is of **Weingarten** type if
$$\Psi(H,K) \equiv 0$$
for $\Psi \in C^{\infty}(\mathbb{R}^2, \mathbb{R})$.

## 2.1 Translation Surfaces in $H^2 \times \mathbb{R}$ (Half-plane Model)

Essentially, a Translation Surface is defined as the sum of two curves. In Euclidean space, for example, considering two curves $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$, we define the Translation Surface by $\psi(t,s) = \alpha(t) + \beta(s)$. In this section, we will reduce ourselves to curves that are contained in the orthogonal planes of space and that are determined by real functions. In $H^2 \times \mathbb{R}$ this reduces us to three types, the other possibilities being obtained by symmetries of the curves that determine them. More precisely:

$$\psi_I(t,s) := (t,0,v(t)) + (0,s,u(s)) = (t,s,v(t)+u(s))$$
$$\psi_{II}(t,s) := (v(t),0,t) + (u(s),s,0) = (v(t)+u(s),s,t)$$
$$\psi_{III}(t,s) := (t,v(t),0) + (0,u(s),s) = (t,v(t)+u(s),s).$$

Here, we will focus on the first two cases, **Type I** and **Type II**. First, we will define the space. Then, we will find the curvatures of each family of surfaces.

**Definition 2.1.1** ($H^2 \times \mathbb{R}$)**.** The hyperbolic space is the set

$$H^2 = \{(x,y) \in \mathbb{R}^2 | y > 0\},$$

with the Riemannian metric,

$$(u \bullet v)_{(x,y)} = \frac{\langle (u_1,u_2),(v_1,v_2)\rangle}{y^2},$$

where $(x,y) \in H^2$, $u = (u_1,u_2)$ and $v = (v_1,v_2)$ since $u,v \in T_pH^2 \simeq \mathbb{R}^2$.

We consider in $H^2 \times \mathbb{R}$ the product metric, that is,

$$(u \bullet_\times v)_{(x,y,z)} = \frac{\langle (u_1,u_2),(v_1,v_2)\rangle}{y^2} + u_3v_3,$$

where $(x,y,z) \in H^2 \times \mathbb{R}$, $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3)$ since $T_p\left(H^2 \times \mathbb{R}\right) \simeq \mathbb{R}^3$.

Consider $\{e_1,e_2,e_3\}$ a canonical basis of $\mathbb{R}^3$. Then,

$$E_1(x,y,z) = \varphi(x,y,z)e_1$$
$$E_2(x,y,z) = \varphi(x,y,z)e_2$$
$$E_3(x,y,z) = e_3$$

generates a global orthonormal frame over $H^2 \times \mathbb{R}$, where $\varphi : H^2 \times \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x,y,z) = y$. Then, we will obtain the Connection, in this frame, at $H^2 \times \mathbb{R}$. This is done by obtaining the Christoffel Symbols.

**Lemma 2.1.1.** $[E_i,E_j] \equiv 0$ *with the exception of:*

$$[E_1,E_2] \equiv -E_1.$$

**Proof:** The cases $[E_1,E_1] \equiv [E_2,E_2] \equiv [E_3,E_3] \equiv 0$ are evident. Let $f : H^2 \times \mathbb{R} \to \mathbb{R}$ be a smooth function. We have,

$$\begin{aligned}
([E_1,E_3]f)(x,y,z) &= E_1(E_3 f)(x,y,z) - E_3(E_1 f)(x,y,z) \\
&= \left(E_1\left(\frac{\partial f}{\partial z}\right)\right)(x,y,z) - \left(E_3\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z) \\
&= \left(\varphi\frac{\partial^2 f}{\partial x\partial z}\right)(x,y,z) - \left(\varphi\frac{\partial^2 f}{\partial z\partial x}\right)(x,y,z) \\
&= 0,
\end{aligned}$$

then $[E_1,E_3] \equiv 0$ and similarly $[E_2,E_3] \equiv 0$. Finally,

$$\begin{aligned}
([E_1,E_2]f)(x,y,z) &= E_1(E_2 f)(x,y,z) - E_2(E_1 f)(x,y,z) \\
&= \left(E_1\left(\varphi\frac{\partial f}{\partial y}\right)\right)(x,y,z) - E_2\left(\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z) \\
&= \left(\varphi\frac{\partial}{\partial x}\left(\varphi\frac{\partial f}{\partial y}\right)\right)(x,y,z) - \left(\varphi\frac{\partial}{\partial y}\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z) \\
&= \varphi(x,y,z)\left(\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial y} + \varphi\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial \varphi}{\partial y}\frac{\partial f}{\partial x} - \varphi\frac{\partial^2 f}{\partial y\partial x}\right)(x,y,z) \\
&= -\left(\varphi\frac{\partial f}{\partial x}\right)(x,y,z) \\
&= -E_1(x,y,z)f,
\end{aligned}$$

as we wanted. □

From the previous lemma, we can find the Christoffel expressions:

**Lemma 2.1.2** (Christoffel)**.** $\Gamma_{ji}^k \equiv 0$ *with the exception of:*

$$\Gamma_{11}^2 \equiv 1, \;\; \Gamma_{12}^1 \equiv -1.$$

**Proof:** By Lemma 2.1.1 and by the expression (Lemma 1.1.1)

$$\Gamma_{ji}^k = -\frac{1}{2}\left([E_i,E_k]\bullet_\times E_j + [E_j,E_k]\bullet_\times E_i + [E_i,E_j]\bullet_\times E_k\right),$$

the null cases are evident. We will calculate $\Gamma_{11}^2$ and the case $\Gamma_{12}^1$ is analogous. We have

$$\begin{aligned}
\Gamma_{11}^2(x,y,z) &= -\frac{1}{2}\left([E_1,E_2]\bullet_\times E_1 + [E_1,E_2]\bullet_\times E_1 + [E_1,E_1]\bullet_\times E_2\right)(x,y,z) \\
&= -([E_1,E_2]\bullet_\times E_1)(x,y,z) \\
&= (E_1\bullet_\times E_1)(x,y,z) \\
&= 1.
\end{aligned}$$

□

Since $\nabla_{E_j}^\times E_i = \sum_k \Gamma_{ji}^k E_k$ we obtain:

**Lemma 2.1.3** (Connection)**.** *The following holds:*

$$\begin{array}{lll} \nabla^{\times}_{E_1}E_1=E_2 & \nabla^{\times}_{E_2}E_1=0 & \nabla^{\times}_{E_3}E_1=0\\ \nabla^{\times}_{E_1}E_2=-E_1 & \nabla^{\times}_{E_2}E_2=0 & \nabla^{\times}_{E_3}E_2=0\\ \nabla^{\times}_{E_1}E_3=0 & \nabla^{\times}_{E_2}E_3=0 & \nabla^{\times}_{E_3}E_3=0. \end{array}$$

### 2.1.1 *Curvatures of Type I Translation Surfaces*

This section will be devoted to determine the curvatures of a Type I Translation Surface in $H^2\times\mathbb{R}$. Define the **Type I Translation Surface** $\psi:\Omega\subset\mathbb{R}^2\to H^2\times\mathbb{R}$ by $\psi(t,s)=(t,s,v(t)+u(s))$, where $u\in C^2(J,\mathbb{R})$, $v\in C^2(I,\mathbb{R})$ and $J\times I\subset(0,+\infty)\times\mathbb{R}$. We have,

$$\begin{aligned}\frac{\partial\psi}{\partial t}(t,s)&=\big(1,0,v'(t)\big)\\ &=\frac{1}{\varphi\circ\psi(t,s)}E_1\circ\psi(t,s)+v'(t)E_3\circ\psi(t,s),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\psi}{\partial s}(t,s)&=\big(0,1,u'(s)\big)\\ &=\frac{1}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)+u'(s)E_3\circ\psi(t,s).\end{aligned}$$

The normal $N$ at $\psi(t,s)$ will be given by

$$N(t,s)=-\frac{v'(t)}{\omega(t,s)}E_1\circ\psi(t,s)-\frac{u'(s)}{\omega(t,s)}E_2\circ\psi(t,s)+\frac{1}{\omega(t,s)\varphi\circ\psi(t,s)}E_3\circ\psi(t,s),$$

where $\omega(t,s)=\sqrt{(v'(t))^2+(u'(s))^2+\frac{1}{(\varphi\circ\psi)^2(t,s)}}$.

By Lemma 1.1.2 we conclude that to know the Weingarten map $\mathcal{S}$ we do not need to differentiate the normal field, we just need to know $\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}$, $\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}$ e $\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}$. We have,

**Lemma 2.1.4.** *The following holds:*

$$\begin{aligned}
&\textbf{\textit{i)}}\ \left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=\frac{1}{(\varphi\circ\psi)^2(t,s)}E_2\circ\psi(t,s)+v''(t)E_3\circ\psi(t,s).\\
&\textbf{\textit{ii)}}\ \left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=-\frac{1}{(\varphi\circ\psi)^2(t,s)}E_1\circ\psi(t,s).\\
&\textbf{\textit{iii)}}\ \left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s)=\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)+u''(s)E_3\circ\psi(t,s).
\end{aligned}$$

**Proof:**

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_1\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\ &+v'(t)\left(\nabla_{\overline{e_1}}E_3\circ\psi\right)(t,s)+\overline{e_1}\big(v'(t)\big)E_3\circ\psi(t,s)\\ =&\frac{1}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1+\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\ &+v'(t)\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3+v''(t)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1 =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_1\right)\circ\psi(t,s)+v'(t)\left(\nabla^{\times}_{E_3}E_1\right)\circ\psi(t,s)\\ =&\frac{1}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)\end{aligned}$$

and $\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3 = 0$, therefore,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s) =\frac{1}{(\varphi\circ\psi)^2(t,s)}E_2\circ\psi(t,s)+v''(t)E_3\circ\psi(t,s).$$

Also,

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s) =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_2\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\ &+u'(s)\left(\nabla_{\overline{e_1}}E_3\circ\psi\right)(t,s)+\overline{e_1}\left(u'(s)\right)E_3\circ\psi(t,s)\\ =&\frac{1}{\varphi\circ\psi(t,\theta)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2+\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\ &+u'(s)\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3+\frac{\partial}{\partial t}\left(u'(s)\right)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2 =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_2\right)\circ\psi(t,s)+v'(t)\left(\nabla^{\times}_{E_3}E_2\right)\circ\psi(t,s)\\ =&-\frac{1}{\varphi\circ\psi(t,s)}E_1\circ\psi(t,s)\end{aligned}$$

consequently,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s) =-\frac{1}{(\varphi\circ\psi)^2(t,s)}E_1\circ\psi(t,s).$$

Finally,

$$\begin{aligned}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s) =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_2}}E_2\circ\psi\right)(t,s)+\overline{e_2}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\ &+u'(s)\left(\nabla_{\overline{e_2}}E_3\circ\psi\right)(t,s)+\overline{e_2}\left(u'(s)\right)E_3\circ\psi(t,s)\\ =&\frac{1}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2+\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\ &+u'(s)\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3+u''(s)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2 =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_2\right)\circ\psi(t,s)+u'(s)\left(\nabla^{\times}_{E_3}E_2\right)\circ\psi(t,s)\\ =&0\end{aligned}$$

and $\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3 = 0$, thus,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s) =\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)+u''(s)E_3\circ\psi(t,s).$$

□

With this we have

**Lemma 2.1.5.** *The following holds:*

$$\textbf{\textit{i)}}\mathcal{E}(t,s) = \frac{sv''(t) - u'(s)}{s^2\omega(t,s)}.$$
$$\textbf{\textit{ii)}}\mathcal{F}(t,s) = \frac{v'(t)}{s^2\omega(t,s)}.$$
$$\textbf{\textit{iii)}}\mathcal{G}(t,s) = \frac{su''(s) + u'(s)}{s^2\omega(t,s)}.$$
$$\textbf{\textit{iv)}}\mathfrak{e}(t,s) = \frac{1}{s^2} + (v'(t))^2.$$
$$\textbf{\textit{v)}}\mathfrak{f}(t,s) = u'(s)v'(t).$$
$$\textbf{\textit{vi)}}\mathfrak{g}(t,s) = \frac{1}{s^2} + (u'(s))^2.$$

**Proof:**

$$\begin{aligned}
\mathcal{E}(t,s) &= \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s) \\
&= -\frac{u'(s)}{\omega(t,s)(\varphi\circ\psi)^2(t,s)} + \frac{v''(t)}{\omega(t,s)\varphi\circ\psi(t,s)} \\
&= \frac{sv''(t) - u'(s)}{s^2\omega(t,s)}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{F}(t,s) &= \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s) \\
&= \frac{v'(t)}{\omega(t,s)(\varphi\circ\psi)^2(t,s)} \\
&= \frac{v'(t)}{s^2\omega(t,s)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{G}(t,s) &= \left(N \bullet_\times \nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}\right)(t,s) \\
&= -\frac{u'(s)}{\omega(t,s)} \frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right) + \frac{u''(s)}{\omega(t,s)\varphi\circ\psi(t,s)} \\
&= \frac{su''(s) + u'(s)}{s^2\omega(t,s)}.
\end{aligned}$$

The metric coefficients are,

$$\begin{aligned}
\mathfrak{e}(t,s) &= (\overline{e_1} \bullet_\times \overline{e_1})(t,s) \\
&= \left(\frac{\partial \psi}{\partial t} \bullet_\times \frac{\partial \psi}{\partial t}\right)(t,s) \\
&= \frac{1}{(\varphi\circ\psi)^2(t,s)} + (v'(t))^2 \\
&= \frac{1}{s^2} + (v'(t))^2.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathfrak{f}(t,s) = (\overline{e_1} \bullet_\times \overline{e_2})(t,s) &= \left(\frac{\partial \psi}{\partial t} \bullet_\times \frac{\partial \psi}{\partial s}\right)(t,s) \\
&= v'(t)u'(s).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathfrak{g}(t,s) &= (\overline{e_2} \bullet_\times \overline{e_2})\,(t,s)\\
&= \left(\frac{\partial \psi}{\partial s} \bullet_\times \frac{\partial \psi}{\partial s}\right)(t,s)\\
&= \frac{1}{(\varphi\circ\psi)^2(t,s)} + (u'(s))^2\\
&= \frac{1}{s^2} + (u'(s))^2.
\end{aligned}$$

□

**Proposition 2.1.1.** *The following holds:*

$$H(t,s) = \frac{(1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)}{2s\omega^3(t,s)},$$

*e*

$$K(t,s) = \frac{s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2}{s^2\omega^4(t,s)},$$

*e*

$$\begin{aligned}
|A|^2(t,s) &= \frac{\left((1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)\right)^2}{s^2\omega^6(t,s)}\\
&\quad - \frac{2\omega^2(t,s)\left(s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2\right)}{s^2\omega^6(t,s)}.
\end{aligned}$$

**Proof:**

$$\begin{aligned}
H(t,s) &= \frac{1}{2}\mathrm{tr}(\mathcal{S})(t,s)\\
&= \frac{1}{2}(a_{tt}+a_{ss})\\
&= \frac{1}{2}\left(\frac{\mathcal{E}\mathfrak{g}+\mathcal{G}\mathfrak{e}-2\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\\
&= \frac{1}{2}\left(\frac{\left(\frac{sv''(t)-u'(s)}{s^2\omega(t,s)}\right)\left(\frac{1}{s^2}+(u'(s))^2\right) + \left(\frac{su''(s)+u'(s)}{s^2\omega(t,s)}\right)\left(\frac{1}{s^2}+(v'(t))^2\right) - \frac{2u'(s)(v'(t))^2}{s^2\omega(t,s)}}{\left(\frac{1}{s^2}+(v'(t))^2\right)\left(\frac{1}{s^2}+(u'(s))^2\right) - (u'(s)v'(t))^2}\right)\\
&= \frac{1}{2}\left(\frac{\frac{sv''(t)-u'(s)}{s^4\omega(t,s)} + \frac{(sv''(t)-u'(s))(u'(s))^2}{s^2\omega(t,s)} + \frac{su''(s)+u'(s)}{s^4\omega(t,s)} + \frac{(su''(s)+u'(s))(v'(t))^2}{s^2\omega(t,s)} - \frac{2u'(s)(v'(t))^2}{s^2\omega(t,s)}}{\frac{1}{s^2}\left(\frac{1}{s^2}+(u'(s))^2+(v'(t))^2\right)}\right)\\
&= \frac{1}{2}\left(\frac{\frac{s(v''(t)+u''(s))}{s^4\omega(t,s)} + \frac{(sv''(t)-u'(s))(u'(s))^2+(su''(s)+u'(s))(v'(t))^2}{s^2\omega(t,s)} - \frac{2u'(s)(v'(t))^2}{s^2\omega(t,s)}}{\frac{1}{s^2}\omega^2(t,s)}\right)\\
&= \frac{1}{2}\left(\frac{v''(t)+u''(s)}{s\omega^3(t,s)} + \frac{(sv''(t)-u'(s))(u'(s))^2+(su''(s)+u'(s))(v'(t))^2}{\omega^3(t,s)} - \frac{2u'(s)(v'(t))^2}{\omega^3(t,s)}\right)\\
&= \frac{1}{2}\left(\frac{v''(t)+u''(s)}{s\omega^3(t,s)} + \frac{s(u'(s))^2v''(t)+s(v'(t))^2u''(s)}{\omega^3(t,s)} - \frac{((u'(s))^2+(v'(t))^2)u'(s)}{\omega^3(t,s)}\right)\\
&= \frac{1}{2s\omega^3(t,s)}\left((1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)\right).
\end{aligned}$$

Also,

$$
\begin{aligned}
K(t,s) &= \det(\mathcal{S}) \\
&= a_{tt}a_{ss} - a_{ts}a_{st} \\
&= \left(\frac{\mathcal{E}\mathfrak{g} - \mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{G}\mathfrak{e} - \mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s) - \left(\frac{\mathcal{F}\mathfrak{g} - \mathcal{G}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{F}\mathfrak{e} - \mathcal{E}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G}\mathfrak{g}\mathfrak{e} - \mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f} - \mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e} + (\mathcal{F}\mathfrak{f})^2 - \left(\mathcal{F}^2\mathfrak{g}\mathfrak{e} - \mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f} - \mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e} + \mathcal{E}\mathcal{F}\mathfrak{f}^2\right)}{(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2)^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G}(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2) - \mathcal{F}^2(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2)}{(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2)^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G} - \mathcal{F}^2}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s) \\
&= \frac{\left(\frac{sv''(t) - u'(s)}{s^2\omega(t,s)}\right)\left(\frac{su''(s) + u'(s)}{s^2\omega(t,s)}\right) - \frac{(v'(t))^2}{s^4\omega^2(t,s)}}{\frac{1}{s^2}\omega^2(t,s)} \\
&= \frac{s^2v''(t)u''(s) + sv''(t)u'(s) - su''(s)u'(s) - (u'(s))^2 - (v'(t))^2}{s^2\omega^4(t,s)} \\
&= \frac{s(su''(s) + u'(s))v''(t) - (su''(s) + u'(s))u'(s) - (v'(t))^2}{s^2\omega^4(t,s)}.
\end{aligned}
$$

Finally, being $\lambda_1(t,s)$ and $\lambda_2(t,s)$ the principal curvatures we have,

$$
\begin{aligned}
|A|^2(t,s) &= \lambda_1^2(t,s) + \lambda_2^2(t,s) \\
&= 4\frac{(\lambda_1 + \lambda_2)^2(t,s)}{4} - 2\lambda_1(t,s)\lambda_2(t,s) \\
&= 4H^2(t,s) - 2K(t,s) \\
&= \left(\frac{(1 + s^2(u'(s))^2)v''(t) - s((u'(s))^2 + (v'(t))^2)u'(s) + (1 + s^2(v'(t))^2)u''(s)}{s\omega^3(t,s)}\right)^2 \\
&\quad - 2\left(\frac{s(su''(s) + u'(s))v''(t) - (su''(s) + u'(s))u'(s) - (v'(t))^2}{s^2\omega^4(t,s)}\right) \\
&= \frac{\left((1 + s^2(u'(s))^2)v''(t) - s((u'(s))^2 + (v'(t))^2)u'(s) + (1 + s^2(v'(t))^2)u''(s)\right)^2}{s^2\omega^6(t,s)} \\
&\quad - \frac{2\omega^2(t,s)\left(s(su''(s) + u'(s))v''(t) - (su''(s) + u'(s))u'(s) - (v'(t))^2\right)}{s^2\omega^6(t,s)}.
\end{aligned}
$$

□

### 2.1.2 *Type I Translation Surfaces - Flat case*

By Proposition 2.1.1 we conclude that a **Type I Translation Surface** is **Flat** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.2.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, is Flat if and only if,*

$$0 = s(su''(s) + u'(s))v''(t) - (su''(s) + u'(s))u'(s) - (v'(t))^2.$$

**Proposition 2.1.3.** *Let* $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ *given by* $\psi(t,s) = (t,s,v(t)+u(s))$, *where* $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ *and* $J \times I \subset (0,+\infty) \times \mathbb{R}$, *Flat. Then* $v(t) = \mu t + \tau$, *for* $\mu, \tau$ *constants, and* $I \subset \mathbb{R}$.

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.1.2, the functions $u, v$ satisfy the equation

$$0 = -(v'(t))^2 + s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s).$$

Defining the family of irreducible polynomials

$$P_s(X,Y) = -X + s(su''(s)+u'(s))Y - (su''(s)+u'(s))u'(s),$$

we observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$. By the theory of plane algebraic curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = -X + s(su''(s)+u'(s))Y - (su''(s)+u'(s))u'(s),$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2$ constants such that,

$$s(su''(s)+u'(s)) = K_1$$

and

$$-(su''(s)+u'(s))u'(s) = K_2.$$

Thus,

$$K_2 = -\frac{K_1}{s}u'(s). \tag{2.1}$$

From (2.1), $K_1 = 0$ if and only $K_2 = 0$ and so

$$0 = -(v'(t))^2 + s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) = -(v'(t))^2,$$

that is, the curve $((v'(t))^2, v''(t))$ is constant. Consider the case $K_1$ and $K_2$ nonzero. From (2.1) we obtain $u(s) = -\frac{K_2}{2K_1}s^2 + K_3$. Then,

$$K_1 = s(su''(s)+u'(s)) = -2s^2\frac{K_2}{K_1},$$

that is, $s$ is constant, a contradiction. Thus, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.1.1.** *Let* $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ *given by* $\psi(t,s) = (t,s,v(t)+u(s))$, *where* $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ *and* $J \times I \subset (0,+\infty) \times \mathbb{R}$, *Flat. Then* $v(t) = \mu t + \tau$, *for* $\mu, \tau$ *constants, and* $I \subset \mathbb{R}$. *Moreover,*

$i)$ *If $\mu = 0$ then*

$$u(s) = D \pm C \ln s,$$

*for constants $C \geqslant 0$ and $D$, and $J \subset (0, +\infty)$.*

$ii)$ *If $\mu \neq 0$ then*

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{C^2}{r^2} - \mu^2} dr,$$

*for $C > 0$ e $D$ constants, and $J \subset \left(0, \frac{C}{\mu}\right)$.*

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.1.3 we have $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Then, by Proposition 2.1.2, the functions $u, v$ satisfy the equation

$$0 = -\mu^2 - (su''(s) + u'(s))u'(s),$$

i.e.,

$$\begin{aligned} -\mu^2 &= su''(s)u'(s) + (u'(s))^2 \\ &= \frac{s}{2}\frac{d}{ds}\left[(u'(s))^2\right] + (u'(s))^2, \end{aligned} \tag{2.2}$$

and then,

$$\begin{aligned} -2\mu^2 s &= s^2 \frac{d}{ds}\left[(u'(s))^2\right] + 2s(u'(s))^2 \\ &= \frac{d}{ds}\left[s^2(u'(s))^2\right]. \end{aligned} \tag{2.3}$$

If $\mu = 0$ we have from $(2.3)$ that there is constant $C \geqslant 0$ such that

$$C^2 = s^2(u'(s))^2.$$

If $C = 0$ then $u(s) = D$ constant. If $C > 0$ then $u'$ does not change sign (otherwise there would exist $s_k \in J$ such that $u'(s_k) = 0$, i.e., $C = 0$) and consequently we have two distinct solutions in $J \subset (0, +\infty)$,

$$u(s) = D \pm C \ln s.$$

If $\mu \neq 0$ we have from $(2.3)$ that there is constant $C > 0$ such that

$$C^2 - \mu^2 s^2 = s^2(u'(s))^2.$$

From $(2.2)$ $u'$ does not change sign (otherwise $\mu = 0$) and consequently we have two distinct solutions in $J \subset \left(0, \frac{C}{\mu}\right)$,

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{C^2}{r^2} - \mu^2} dr.$$

$\square$

### 2.1.3 *Type I Translation Surfaces - Constant Gauss Curvature case*

**Lemma 2.1.6.** *Let $\mu$, $c>0$, $C>0$ and $F\geqslant 0$ be constants. Consider the function $g:J\subset(0,+\infty)\to\mathbb{R}$ given by*

$$g(s)=\frac{1}{\frac{1}{C(F+\mu^2)+1}+2c\ln\left(\frac{s}{C}\right)}-(s^2\mu^2+1).$$

*Then, $g>0$ only in the interval $\left(Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}},\tilde{x}\right)\bigcap J$, where $\tilde{x}$ is the* ***unique*** *solution of the equation*

$$\tilde{x}=Ce^{\frac{-\tilde{x}\mu^2+C\mu^2+CF}{2c(\tilde{x}^2\mu^2+1)(C\mu^2+CF+1)}}.$$

**Proof:** To obtain $g>0$ we necessarily need that

$$\frac{1}{\frac{1}{C(F+\mu^2)+1}+2c\ln\left(\frac{s}{C}\right)}>0. \tag{2.4}$$

Since $c>0$ we obtain from (2.4)

$$\ln\left(\frac{s}{C}\right)>-\frac{1}{2c(C(F+\mu^2)+1)},$$

and applying the exponential

$$s>Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}},$$

i.e., $s\in\left(Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}},+\infty\right)$. So, since we want $g>0$, we have

$$\frac{1}{\frac{1}{C(F+\mu^2)+1}+2c\ln\left(\frac{s}{C}\right)}-(s^2\mu^2+1)>0,$$

that is,

$$\ln\left(\frac{s}{C}\right)<\frac{-s\mu^2+C\mu^2+CF}{2c(s^2\mu^2+1)(C\mu^2+CF+1)}$$

and applying the exponential

$$s<Ce^{\frac{-s\mu^2+C\mu^2+CF}{2c(s^2\mu^2+1)(C\mu^2+CF+1)}}. \tag{2.5}$$

Define, for $x\geqslant 0$, the function $\xi(x)=x-Ce^{\frac{-x\mu^2+C\mu^2+CF}{2c(x^2\mu^2+1)(C\mu^2+CF+1)}}$, and observe that

$$\xi'(x)=1+\frac{x\mu^2C}{c(x^2\mu^2+1)^2}e^{\frac{-x\mu^2+C\mu^2+CF}{2c(x^2\mu^2+1)(C\mu^2+CF+1)}}>0,$$

that is, $\xi$ is strictly increasing. Since $\xi(0)<0$ and $\xi(+\infty)>0$ we conclude that there exists a unique $\tilde{x}$ such that $\xi(\tilde{x})=0$. Furthermore, we conclude that in the interval $(0,\tilde{x})$ we have $\xi(s)<0$ and therefore (2.5) holds. Thus, we conclude that $g>0$ in the intersection of the obtained intervals, that is, $\left(Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}},+\infty\right)\bigcap(0,\tilde{x})$. Finally, observe that

$$\frac{-\tilde{x}\mu^2+C\mu^2+CF}{2c(\tilde{x}^2\mu^2+1)(C\mu^2+CF+1)}+\frac{1}{2c(C(F+\mu^2)+1)}=\frac{1}{2c(\tilde{x}^2\mu^2+1)}>0,$$

i.e.,

$$\frac{-\tilde{x}\mu^2 + C\mu^2 + CF}{2c(\tilde{x}^2\mu^2+1)(C\mu^2+CF+1)} > -\frac{1}{2c(C(F+\mu^2)+1)},$$

and then, applying the exponential and multiplying by $C$,

$$\tilde{x} = Ce^{\frac{-\tilde{x}\mu^2+C\mu^2+CF}{2c(\tilde{x}^2\mu^2+1)(C\mu^2+CF+1)}} > Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}},$$

concluding

$$\left(Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}}, +\infty\right) \bigcap (0,\tilde{x}) = \left(Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}}, \tilde{x}\right).$$

□

By Proposition 2.1.1 we conclude that a **Type I Translation Surface** has **Constant Gauss curvature** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.4.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, has constant Gaussian Curvature $c$ , if, and only if,*

$$c = \frac{s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2}{s^2\left((v'(t))^2 + (u'(s))^2 + \frac{1}{s^2}\right)^2}.$$

**Proposition 2.1.5.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, with constant Gaussian Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.1.4, the functions $u, v$ satisfy the equation

$$c = \frac{s^2\left(s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2\right)}{\left(s^2(v'(t))^2 + s^2(u'(s))^2 + 1\right)^2},$$

or equivalently

$$\begin{aligned} 0 =&(v'(t))^4 - \left(\frac{u'(s)+su''(s)}{cs}\right)v''(t) + \left(\frac{2c+2cs^2(u'(s))^2+1}{cs^2}\right)(v'(t))^2 \\ &+ \frac{c(s^2(u'(s))^2+1)^2 + s^2u'(s)(u'(s)+su''(s))}{cs^4}. \end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = X^2 + A_sY + B_sX + C_s,$$

em que,

$$\begin{aligned} A_s &= -\frac{u'(s)+su''(s)}{cs} \\ B_s &= \frac{2c+2cs^2(u'(s))^2+1}{cs^2} \\ C_s &= \frac{c(s^2(u'(s))^2+1)^2 + s^2u'(s)(u'(s)+su''(s))}{cs^4}, \end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** If there exists $s_0 \in J$ such that $A_{s_0} \neq 0$ then the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$. Indeed, if given $s \in V$ the polynomial $P_s$ is reducible then there exist polynomials of degree 1 such that

$$P_s(X,Y) = (X + fY + g)(X + hY + i),$$

that is,

$$0 = X^2 + A_sY + B_sX + C_s - (X + fY + g)(X + hY + i)$$

and reorganizing

$$0 = (-f-h)XY + (B_s - g - i)X + (-fh)Y^2 + (A_s - fi - gh)Y + C_s - gi$$

that occurs if, and only if,

$$\begin{aligned}
-f-h &= 0\\
B_s - g - i &= 0\\
-fh &= 0\\
A_s - fi - gh &= 0\\
C_s - gi &= 0.
\end{aligned}$$

By the first and third equalities we obtain $f = h = 0$. Consequently, by the fifth equality, $A_s = 0$, a contradiction with the continuity of $A_s$ in $V$ and thus we proof the claim.

If $A_{s_0} \neq 0$ for some $s_0 \in J$, by the claim, the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$, that is, for $s \in V \subset J$. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of plane algebraic curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = X^2 + A_sY + B_sX + C_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2, K_3$ constants such that,

$$\begin{aligned}
A_s &= K_1\\
B_s &= K_2\\
C_s &= K_3,
\end{aligned}$$

that is,

$$\begin{aligned}
K_1 &= -\frac{u'(s) + su''(s)}{cs}\\
K_2 &= \frac{2c + 2cs^2(u'(s))^2 + 1}{cs^2}\\
K_3 &= \frac{c(s^2(u'(s))^2 + 1)^2 + s^2u'(s)(u'(s) + su''(s))}{cs^4}.
\end{aligned}$$

Defining $w(s) = u'(s)$ we obtain

$$\begin{aligned} w'(s) &= -\frac{w(s) + csK_1}{s} \\ s^2 w^2(s) &= \frac{cs^2K_2 - 2c - 1}{2c} \\ cs^4K_3 &= c(s^2w^2(s)+1)^2 + s^2w(s)(w(s) + sw'(s)). \end{aligned}$$

Substituting the first two equalities into the third we have

$$cs^4K_3 = c\left(\frac{cs^2K_2 - 2c - 1}{2c} + 1\right)^2 - cs^3w(s)K_1,$$

thus,

$$s^3w(s)K_1 = \left(\frac{cs^2K_2 - 2c - 1}{2c} + 1\right)^2 - s^4K_3$$

that is,

$$s^3w(s)K_1 = \left(\frac{cs^2K_2 - 2c - 1}{2c}\right)^2 + \frac{cs^2K_2 - 2c - 1}{c} + 1 - s^4K_3$$

and then,

$$sw(s)K_1 = \left(\left(\frac{cs^2K_2 - 2c - 1}{2cs}\right)^2 + \frac{cs^2K_2 - 2c - 1}{cs^2} + 1 - s^2K_3\right).$$

Consequently,

$$\begin{aligned} &\left(\frac{cs^2K_2 - 2c - 1}{2c}\right)K_1^2 \\ &\qquad = s^2w^2(s)K_1^2 \\ &\qquad = (sw(s)K_1)^2 \\ &\qquad = \left(\left(\frac{cs^2K_2 - 2c - 1}{2cs}\right)^2 + \frac{cs^2K_2 - 2c - 1}{cs^2} + 1 - s^2K_3\right)^2, \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &= \left(\left(\frac{cs^2K_2 - 2c - 1}{2cs}\right)^2 + \frac{cs^2K_2 - 2c - 1}{cs^2} + 1 - s^2K_3\right)^2 - \left(\frac{cs^2K_2 - 2c - 1}{2c}\right)K_1^2 \\ &= \left(\frac{c^2s^4K_2^2 - 2c^2s^2K_2 - 2cs^2K_2 + 4c^2 + 4c + 1}{4c^2s^2} + \frac{cs^2K_2 - 2c - 1}{cs^2} + 1 - s^2K_3\right)^2 \\ &\quad - \left(\frac{cs^2K_2 - 2c - 1}{2c}\right)K_1^2 \\ &= \frac{(c^2s^4K_2^2 - 2c^2s^2K_2 - 2cs^2K_2 + 4c^2 + 4c + 1 + 4c^2s^2K_2 - 8c^2 - 4c + 4c^2s^2 - 4c^2s^4K_3)^2}{16c^4s^4} \\ &\quad - \frac{8c^3s^4(cs^2K_2 - 2c - 1)K_1^2}{16c^4s^4} \end{aligned}$$

thus

$$0 = (c^2s^4K_2^2 - 2c^2s^2K_2 - 2cs^2K_2 + 4c^2 + 4c + 1 + 4c^2s^2K_2 - 8c^2 - 4c + 4c^2s^2 - 4c^2s^4K_3)^2 \\ - 8c^3s^4(cs^2K_2 - 2c - 1)K_1^2.$$

Observe that $s^2$ is contained in the set of roots of the non-zero polynomial

$$P(Z) = (c^2Z^2K_2^2 - 2c^2ZK_2 - 2cZK_2 + 4c^2 + 4c + 1 + 4c^2ZK_2 - 8c^2 - 4c + 4c^2Z - 4c^2Z^2K_3)^2 \\ - 8c^3Z^2(cZK_2 - 2c - 1)K_1^2,$$

that is, $s^2$ is constant, a contradiction. Therefore, the curve $((v'(t))^2, v''(t))$ is constant and thus there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$.

If $A_s \equiv 0$ then $P_s(X,Y) = X^2 + B_sX + C_s$, that is, given $s \in J$ we obtain that $v'(t)$ is a root of a polynomial in one variable. Thus, $v'(t)$ is constant and therefore $v(t)$ is a straight line. □

**Proposition 2.1.6.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, with constant Gaussian Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover,*

*i) If $\mu = 0$ then $u'(s) \neq 0$ in $J$.*

*ii) If $\mu \neq 0$ and $c \in \left(-\infty, -\frac{1}{4}\right) \bigcup (0,+\infty)$ then $u'(s) \neq 0$ in $J$.*

*iii) If $\mu \neq 0$, $c \in \left[-\frac{1}{4}, 0\right)$ and $u'(s) = 0$ then*

$$s = \sqrt{\frac{-(1+2c) \pm \sqrt{4c+1}}{2c\mu^2}}.$$

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.1.5 we have $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Then, by Proposition 2.1.4, the functions $u, v$ satisfy the equation

$$c\left(s^2\mu^2 + s^2(u'(s))^2 + 1\right)^2 = -s^3u''(s)u'(s) - s^2(u'(s))^2 - s^2\mu^2$$

and then if there exists $x \in J \subset (0,+\infty)$ such that $u'(x) = 0$,

$$0 = c(x^2\mu^2 + 1)^2 + x^2\mu^2 = c\left(\mu^2x^2\right)^2 + (1+2c)\mu^2x^2 + c. \tag{2.6}$$

**-** *i)***:** If $\mu = 0$ we obtain from (2.6) that $c = 0$, a contradiction, and therefore $u'(s) \neq 0$ in $J$.

**-** *ii)* **e** *iii)***:** Let $\mu \neq 0$. If $c \in (0,+\infty)$ the right-hand side of the equality in (2.6) is strictly positive, therefore it does not hold, that is, it has no roots. Also, the discriminant of the equation in (2.6) is, considering $\mu^2x^2$ as the variable,

$$\Delta = 4c + 1,$$

therefore, if $c \in \left(-\infty, -\frac{1}{4}\right)$ we have no real roots. Finally, if $c \in \left[-\frac{1}{4}, 0\right)$ we have at most two distinct roots. Thus, we obtain from (2.6),

$$\mu^2 x^2 = \frac{-(1+2c) \pm \sqrt{4c+1}}{2c}$$

that is,

$$x = \sqrt{\frac{-(1+2c) \pm \sqrt{4c+1}}{2c\mu^2}}.$$

□

**Theorem 2.1.2.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$, and $J \times I \subset (0, +\infty) \times \mathbb{R}$ with constant Gaussian curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for constant $\mu, \tau$, and $I \subset \mathbb{R}$. Moreover, $u$ satisfies the differential equation*

$$u''(s) = -\frac{c\left(s^2\mu^2 + s^2(u'(s))^2 + 1\right)^2 + s^2(u'(s))^2 + s^2\mu^2}{u'(s)} \tag{2.7}$$

*in the interval*

$$J \subset (0, +\infty) - \left\{ \sqrt{\frac{-(1+2c) + \sqrt{4c+1}}{2c\mu^2}}, \sqrt{\frac{-(1+2c) - \sqrt{4c+1}}{2c\mu^2}} \right\},$$

*if $\mu \neq 0$ and $c \in \left[-\frac{1}{4}, 0\right)$, and in the interval $J \subset (0, +\infty)$ otherwise.*

**Proof:** Suppose $\psi$ has constant Gaussian curvature $c \neq 0$. By Proposition 2.1.5 we have $v(t) = \mu t + \tau$, for constant $\mu, \tau$, and $I \subset \mathbb{R}$. Then, by Proposition 2.1.4, the functions $u, v$ satisfy the equation

$$c\left(s^2\mu^2 + s^2(u'(s))^2 + 1\right)^2 = -s^3 u''(s)u'(s) - s^2(u'(s))^2 - s^2\mu^2$$

and then, by the Proposition 2.1.6,

$$u''(s) = -\frac{c\left(s^2\mu^2 + s^2(u'(s))^2 + 1\right)^2 + s^2(u'(s))^2 + s^2\mu^2}{u'(s)}$$

in the interval

$$J \subset (0, +\infty) - \left\{ \sqrt{\frac{-(1+2c) \pm \sqrt{4c+1}}{2c\mu^2}} \right\},$$

if $\mu \neq 0$ and $c \in \left[-\frac{1}{4}, 0\right)$, and in the interval $J \subset (0, +\infty)$ otherwise. □

**Corollary 2.1.1.** *Assume the conditions of Theorem 2.1.2. If $u$ is a solution of (2.7) then*

$$(u'(s))^2 = \frac{1}{s^2}\left(\frac{1}{\frac{1}{C(F+\mu^2)+1} + 2c\ln\left(\frac{s}{C}\right)} - (s^2\mu^2 + 1)\right),$$

*for constants $C > 0$ and $F \geqslant 0$. Furthermore, if $c > 0$, we have*

$$u(s) = D \pm \int_{s_0}^{s} \frac{1}{r}\sqrt{\frac{1}{\frac{1}{C(F+\mu^2)+1} + 2c\ln\left(\frac{r}{C}\right)} - (r^2\mu^2+1)}dr, \tag{2.8}$$

*in the interval* $\left(Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}}, \tilde{x}\right) \bigcap J$*, where $\tilde{x}$ is the **unique** solution to the equation*

$$\tilde{x} = Ce^{\frac{-\tilde{x}\mu^2 + C\mu^2 + CF}{2c(\tilde{x}^2\mu^2+1)(C\mu^2+CF+1)}},$$

*and $D$ is a constant.*

**Proof:** Suppose $u$ is a solution of (2.7). We have,

$$c\left(s^2\mu^2 + s^2(u'(s))^2 + 1\right)^2 = -s^3u''(s)u'(s) - s^2(u'(s))^2 - s^2\mu^2$$

and then

$$\begin{aligned} cs^4\mu^4 + 2cs^4\mu^2(u'(s))^2 + 2cs^2\mu^2 + cs^4(u'(s))^4 + 2cs^2(u'(s))^2 + c + s^2\mu^2 = &-\frac{s^3}{2}\frac{d}{ds}\left[(u'(s))^2\right] \\ &- s^2(u'(s))^2 \end{aligned}$$

that is,

$$-\frac{2}{s}\left((cs^2\mu^2+2c+1)s^2\mu^2 + (2cs^2\mu^2+2c)s^2(u'(s))^2 + cs^4(u'(s))^4 + c\right) = s^2\frac{d}{ds}\left[(u'(s))^2\right] + 2s(u'(s))^2$$

thus

$$-\frac{2}{s}\left((cs^2\mu^2+2c+1)s^2\mu^2 + (2cs^2\mu^2+2c)s^2(u'(s))^2 + cs^4(u'(s))^4 + c\right) = \frac{d}{ds}\left[s^2(u'(s))^2\right]$$

and defining $g(s) = s^2(u'(s))^2$

$$g'(s) = -\frac{2}{s}\left((cs^2\mu^2+2c+1)s^2\mu^2 + (2cs^2\mu^2+2c)g(s) + cg^2(s) + c\right). \tag{2.9}$$

In (2.9) we have a Riccati differential equation. To solve it, first observe that $\tilde{g}(s) = -(s^2\mu^2+1)$ is a particular solution. Define $G(s) = g(s) - \tilde{g}(s)$, that is, $g(s) = G(s) + \tilde{g}(s)$ which, substituting into (2.9) we obtain

$$G'(s) = -\frac{2c}{s}G^2(s), \tag{2.10}$$

a separable differential equation. Observe that $G \equiv 0$ is a solution of (2.10) and therefore, by the Existence and Uniqueness Theorem, any other solution does not reach zero. If $G \not\equiv 0$ we obtain from (2.10)

$$\int_{s_0}^{s} \frac{G'(r)}{G^2(r)}dr = -2c\int_{s_0}^{s}\frac{dr}{r},$$

and by the change of variables theorem,

$$\int_{G(s_0)}^{G(s)} \frac{dl}{l^2} = -2c\int_{s_0}^{s}\frac{dr}{r}.$$

Consequently,

$$-\frac{1}{G(s)}+\frac{1}{G(s_0)}=-2c\left(\ln(s)-\ln(s_0)\right),$$

thus,

$$\frac{1}{G(s)}=\frac{1}{G(s_0)}+2c\left(\ln(s)-\ln(s_0)\right),$$

and then,

$$G(s)=\frac{1}{\frac{1}{G(s_0)}+2c\ln(s)-2c\ln(s_0)},$$

that is,

$$G(s)=\frac{1}{\frac{1}{s_0^2((u'(s_0))^2+\mu^2)+1}+2c\ln\left(\frac{s}{s_0}\right)},$$

i.e., there exist constants $C$ and $F$, where $C>0$ and $F\geqslant 0$, such that

$$G(s)=\frac{1}{\frac{1}{C(F+\mu^2)+1}+2c\ln\left(\frac{s}{C}\right)}.$$

In short, $G$ is as above or $G\equiv 0$ (as noticed earlier). By the definition of $G$ we obtain

$$g(s)=\frac{1}{\frac{1}{C(F+\mu^2)+1}+2c\ln\left(\frac{s}{C}\right)}-(s^2\mu^2+1),$$

or $g(s)=-(s^2\mu^2+1)$. Since we want $g\geqslant 0$ we obtain that the second case does not occur. Thus,

$$(u'(s))^2=\frac{1}{s^2}\left(\frac{1}{\frac{1}{C(F+\mu^2)+1}+2c\ln\left(\frac{s}{C}\right)}-(s^2\mu^2+1)\right).$$

In the case where $c>0$ we know by Lemma 2.1.6 that $g$ is positive in $\left(Ce^{-\frac{1}{2c(C(F+\mu^2)+1)}},\tilde{x}\right)\bigcap J$ and then, in this interval,

$$u'(s)=\pm\frac{1}{s}\sqrt{\frac{1}{\frac{1}{C(F+\mu^2)+1}+2c\ln\left(\frac{s}{C}\right)}-(s^2\mu^2+1)},$$

which integrating gives us the result. $\square$

**Remark:** In principle, we could write the expression in (2.8) also in the case $c<0$. Our only limitation is the Lemma 2.1.6, in which we precisely estimate the maximal interval in which $g$ is positive. In the case of $c<0$, the argument to explicitly determine the maximal interval is not as clear as that in the positive case.

### 2.1.4 *Type I Translation Surfaces - Minimal case*

By Proposition 2.1.1 we conclude that a **Type I Translation Surface** is **Minimal** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.7.** *The surface* $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ *given by* $\psi(t,s) = (t,s,v(t)+u(s))$*, where* $u \in C^2(J,\mathbb{R})$*,* $v \in C^2(I,\mathbb{R})$ *and* $J \times I \subset (0,+\infty) \times \mathbb{R}$*, is Minimal if and only if ,*

$$0 = (1+s^2(u'(s))^2)v''(t) - s((u'(s))^2 + (v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s).$$

**Proposition 2.1.8.** *Let* $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ *given by* $\psi(t,s) = (t,s,v(t)+u(s))$*, where* $u \in C^2(J,\mathbb{R})$*,* $v \in C^2(I,\mathbb{R})$ *and* $J \times I \subset (0,+\infty) \times \mathbb{R}$*, Minimal. Then* $v(t) = \mu t + \tau$*, for* $\mu, \tau$ *constants, and* $I \subset \mathbb{R}$*.*

**Proof:** Suppose $\psi$ is Minimal. By Proposition 2.1.7, the functions $u, v$ satisfy the equation

$$0 = (1+s^2(u'(s))^2)v''(t) - s((u'(s))^2 + (v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s),$$

or equivalently

$$0 = \frac{s^2u''(s) - su'(s)}{(1+s^2(u'(s))^2)}(v'(t))^2 + v''(t) + \frac{u''(s) - s(u'(s))^3}{(1+s^2(u'(s))^2)}.$$

Defining the family of irreducible polynomials

$$P_s(X,Y) = \frac{s^2u''(s) - su'(s)}{(1+s^2(u'(s))^2)}X + Y + \frac{u''(s) - s(u'(s))^3}{(1+s^2(u'(s))^2)},$$

we observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$. By the theory of plane algebraic curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = \frac{s^2u''(s) - su'(s)}{(1+s^2(u'(s))^2)}X + Y + \frac{u''(s) - s(u'(s))^3}{(1+s^2(u'(s))^2)},$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Then, their coefficients are constant, that is, there are $K_1, K_2$ constants such that,

$$\frac{s^2u''(s) - su'(s)}{1+s^2(u'(s))^2} = K_1$$

and

$$\frac{u''(s) - s(u'(s))^3}{1+s^2(u'(s))^2} = K_2.$$

Setting $u'(s) = w(s)$ we obtains

$$\frac{s^2w'(s) - sw(s)}{1+s^2w^2(s)} = K_1$$

and

$$\frac{w'(s) - sw^3(s)}{1+s^2w^2(s)} = K_2.$$

**Claim 1:** $K_1 = 0$ if and only if $K_2 = 0$. Indeed, if $K_1 = 0$ we have

$$sw'(s) - w(s) = 0,$$

that is,

$$w'(s) = \frac{w(s)}{s},$$

whose solution is $w(s) = \alpha s$, for constant $\alpha$. Therefore,

$$K_2 = \frac{\alpha - \alpha^3 s^4}{1 + \alpha^2 s^4},$$

an then,

$$K_2 + K_2 \alpha^2 s^4 + \alpha^3 s^4 - \alpha = 0.$$

Rewriting we obtain the null polynomial,

$$(K_2 \alpha^2 + \alpha^3) s^4 + K_2 - \alpha = 0,$$

concluding that its coefficients are zero, that is, $K_2 = \alpha$ and consequently $2K_2^3 = 0$. On the other hand, if $K_2 = 0$

$$w'(s) - s w^3(s) = 0,$$

that is,

$$w'(s) = s w^3(s).$$

Thus,

$$K_1 = \frac{s^3 w^3(s) - s w(s)}{1 + s^2 w^2(s)}$$

and then,

$$K_1 + K_1 s^2 w^2(s) - s^3 w^3(s) + s w(s) = 0.$$

Observe that $sw(s)$ is contained in the set of roots of the polynomial

$$P(Z) = K_1 + K_1 Z^2 - Z^3 + Z,$$

and then $sw(s) = \lambda$, that is, $w(s) = \frac{\lambda}{s}$ for some constant $\lambda$. Since $w'(s) = sw^3(s)$ we obtain

$$-\frac{\lambda}{s^2} = w'(s) = s w^3(s) = \frac{\lambda^3}{s^2},$$

consequently $-\lambda = \lambda^3$, whose equality only occurs if $\lambda = 0$. We have $w(s) = 0$ and therefore $K_1 = 0$ since

$$K_1 = K_1 + K_1 s^2 w^2(s) - s^3 w^3(s) + s w(s) = 0,$$

and thus we proof claim 1.

From claim 1 we conclude that if $K_1, K_2$ are zero then,

$$0 = K_1 (v'(t))^2 + v''(t) + K_2 = v''(t),$$

and consequently $(v'(t))^2$ is constant. Thus the curve $((v'(t))^2, v''(t))$ is constant. Let us then consider $K_1, K_2$ both non-zero. We have

$$\frac{s^2w'(s) - sw(s)}{1 + s^2w^2(s)} = K_1$$

and

$$\frac{s^2w'(s) - s^3w^3(s)}{1 + s^2w^2(s)} = s^2K_2,$$

equivalently,

$$K_1 - s^2K_2 = \frac{s^3w^3(s) - sw(s)}{1 + s^2w^2(s)} \tag{2.11}$$

and then

$$0 = K_1 + s^2w^2(s)K_1 - s^2K_2 - s^4w^2(s)K_2 + sw(s) - s^3w^3(s). \tag{2.12}$$

Define $F : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,y) = K_1 + x^2y^2K_1 - x^2K_2 - x^4y^2K_2 + xy - x^3y^3.$$

Observe that $F(s, w(s)) = 0$. Then

$$\frac{\partial F}{\partial x}(x,y) = 2xy^2K_1 - 2xK_2 - 4x^3y^2K_2 + y - 3x^2y^3$$

and

$$\frac{\partial F}{\partial y}(x,y) = 2x^2yK_1 - 2x^4yK_2 + x - 3x^3y^2.$$

**Claim 2:** There exists $s_0 \in I$ such that $\frac{\partial F}{\partial y}(s_0, w(s_0)) \neq 0$. Indeed, if $\frac{\partial F}{\partial y}(s, w(s)) = 0$ for all $s \in I$ we have

$$0 = \frac{\partial F}{\partial y}(s, w(s)) = 2s^2w(s)K_1 - 2s^4w(s)K_2 + s - 3s^3w^2(s),$$

i.e.,

$$0 = 2s^2w(s)\left(K_1 - s^2K_2\right) + s - 3s^3w^2(s),$$

and then

$$0 = 2sw(s)\left(K_1 - s^2K_2\right) + 1 - 3s^2w^2(s).$$

By the equation in $(2.11)$ we obtain

$$0 = 2sw(s)\left(\frac{s^3w^3(s) - sw(s)}{1 + s^2w^2(s)}\right) + 1 - 3s^2w^2(s)$$

then

$$3s^2w^2(s) = \frac{2s^4w^4(s) - 2s^2w^2(s) + 1 + s^2w^2(s)}{1 + s^2w^2(s)},$$

that is,

$$3s^2w^2(s) + 3s^4w^4(s) = 2s^4w^4(s) - s^2w^2(s) + 1$$

concluding

$$0 = s^4w^4(s) + 4s^2w^2(s) - 1.$$

Observe that $s^2w^2(s)$ is contained in the set of roots of the polynomial

$$Q(Z) = Z^2 + 4Z - 1,$$

and then $s^2w^2(s) = \lambda^2$, that is, $w(s) = \dfrac{\pm\lambda}{s}$ for some constant $\lambda > 0$. As

$$\frac{w'(s) - sw^3(s)}{1 + s^2w^2(s)} = K_2$$

we get

$$\frac{\frac{\mp\lambda}{s^2} - s\left(\frac{\pm\lambda}{s}\right)^3}{1+\lambda^2} = K_2,$$

concluding

$$\frac{\mp\lambda \mp \lambda^3}{(1+\lambda^2)K_2} = s^2,$$

that is, $s$ is constant, a contradiction and thus we conclude claim 2.

By claim 2 we can apply the Implicit Function Theorem obtaining

$$w'(s) = \frac{\frac{\partial F}{\partial x}(s,w(s))}{\frac{\partial F}{\partial y}(s,w(s))} = \frac{2sw^2(s)K_1 - 2sK_2 - 4s^3w^2(s)K_2 + w(s) - 3s^2w^3(s)}{2s^2w(s)K_1 - 2s^4w(s)K_2 + s - 3s^3w^2(s)},$$

for some neighborhood of $s_0 \in I$. Since $\frac{s^2w'(s) - sw(s)}{1+s^2w^2(s)} = K_1$, that is,

$$w'(s) = \frac{\left(1+s^2w^2(s)\right)K_1 + sw(s)}{s^2}$$

we have

$$\frac{\left(1+s^2w^2(s)\right)K_1 + sw(s)}{s^2} = \frac{2sw^2(s)K_1 - 2sK_2 - 4s^3w^2(s)K_2 + w(s) - 3s^2w^3(s)}{2s^2w(s)K_1 - 2s^4w(s)K_2 + s - 3s^3w^2(s)},$$

i.e.,

$$\begin{aligned} 0 =& s^2\left(2sw^2(s)K_1 - 2sK_2 - 4s^3w^2(s)K_2 + w(s) - 3s^2w^3(s)\right) \\ & - \left(\left(1+s^2w^2(s)\right)K_1 + sw(s)\right)\left(2s^2w(s)K_1 - 2s^4w(s)K_2 + s - 3s^3w^2(s)\right) \end{aligned}$$

then

$$\begin{aligned} 0 =& 2s^3w^2(s)K_1 - 2s^3K_2 - 4s^5w^2(s)K_2 + s^2w(s) - 3s^4w^3(s) \\ & - \Big(2s^3w^2(s)K_1 - 2s^5w^2(s)K_2 + s^2w(s) - 3s^4w^3(s)\Big) \\ & - \left(1+s^2w^2(s)\right)\left(2s^2w(s)K_1^2 - 2s^4w(s)K_1K_2 + sK_1 - 3s^3w^2(s)K_1\right) \end{aligned}$$

thus

$$\begin{aligned} 0 =& -2s^3K_2 - 2s^5w^2(s)K_2 - s\left(1+s^2w^2(s)\right)\left(2sw(s)K_1^2 - 2s^3w(s)K_1K_2 + K_1 - 3s^2w^2(s)K_1\right) \\ =& -s\left(1+s^2w^2(s)\right)2s^2K_2 - s\left(1+s^2w^2(s)\right)\left(2sw(s)K_1^2 - 2s^3w(s)K_1K_2 + K_1 - 3s^2w^2(s)K_1\right). \end{aligned}$$

and therefore

$$0=-2s^2K_2-2sw(s)K_1^2+2s^3w(s)K_1K_2-K_1+3s^2w^2(s)K_1. \tag{2.13}$$

Define $G:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$G(x,y)=-2xK_2-2yK_1^2+2xyK_1K_2-K_1+3y^2K_1$$

and

$$\overline{G}(x,y)=K_1+y^2K_1-xK_2-xy^2K_2+y-y^3.$$

Observe that, from (2.13) e (2.12), $G(s^2,sw(s))=0=\overline{G}(s^2,sw(s))$. Since the curve $(s^2,sw(s))$ is non-constant we conclude from the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A.0.9$. Therefore, the curve $((v'(t))^2,v''(t))$ is constant and then there exist $\mu,\eta$ such that $v'(t)=\mu$ and $v''(t)=\eta$ (in this case necessarily $\eta=0$) and we conclude that $v(t)=\mu t+\tau$ in $I\subset\mathbb{R}$. □

**Theorem 2.1.3.** *Let $\psi:\Omega\subset\mathbb{R}^2\to H^2\times\mathbb{R}$ given by $\psi(t,s)=(t,s,v(t)+u(s))$, where $u\in C^2(J,\mathbb{R})$, $v\in C^2(I,\mathbb{R})$ and $J\times I\subset(0,+\infty)\times\mathbb{R}$, Minimal. Then $v(t)=\mu t+\tau$, for $\mu,\tau$ constants, and $I\subset\mathbb{R}$. Moreover, $u(s)=D$ for constant $D$ and $J\subset(0,+\infty)$, or*

$$u(s)=D\pm\int_{s_0}^{s}\sqrt{\frac{r^2\mu^2+1}{E^2\mu^2+C^2-r^2}}dr,$$

*for $C>0$, $D$ and $E>0$ constants and $J\subset(0,\sqrt{E^2\mu^2+C^2})$.*

**Proof:** Suppose that $\psi$ is Minimal. By Proposition 2.1.8 we have $v(t)=\mu t+\tau$, for constants $\mu,\tau$, and $I\subset\mathbb{R}$. Then, by Proposition 2.1.7, the functions $u,v$ satisfy the equation

$$0=-s((u'(s))^2+\mu^2)u'(s)+(1+s^2\mu^2)u''(s),$$

and then defining $g(s)=u'(s)$,

$$g'(s)=\frac{s\mu^2g(s)+sg^3(s)}{s^2\mu^2+1}. \tag{2.14}$$

The equation in (2.14) is a Bernoulli differential equation. To solve it we will make the change $h(s)=\frac{1}{g^2(s)}$ obtaining a linear differential equation. Notice that $g\equiv0$ is a solution of (2.14) (so $u$ is constant) and therefore, by the existence and uniqueness theorem, any other solution does not reach zero. If $g\neq0$ we have,

$$h'(s)=-2\left(\frac{s\mu^2h(s)+s}{s^2\mu^2+1}\right),$$

thus,

$$\left(s^2\mu^2+1\right)h'(s)+2s\mu^2h(s)=-2s,$$

that is,

$$\frac{d}{ds}\left[(s^2\mu^2+1)h(s)\right]=-2s$$

which integrating,

$$(s^2\mu^2+1)h(s)-(s_0^2\mu^2+1)h(s_0)=\int_{s_0}^{s}\frac{d}{dr}\left[(r^2\mu^2+1)h(r)\right]dr=\int_{s_0}^{s}-2rdr=-s^2+s_0^2,$$

i.e.,

$$h(s)=\frac{(s_0^2\mu^2+1)h(s_0)+s_0^2-s^2}{s^2\mu^2+1},$$

and since $h>0$,

$$g(s)=\pm\sqrt{\frac{1}{h(s)}}=\pm\sqrt{\frac{s^2\mu^2+1}{(s_0^2\mu^2+1)h(s_0)+s_0^2-s^2}}=\pm\sqrt{\frac{s^2\mu^2+1}{(s_0^2\mu^2+1)\frac{1}{g^2(s_0)}+s_0^2-s^2}}$$

and therefore there exist constants $C>0$, $E>0$ such that

$$g(s)=\pm\sqrt{\frac{s^2\mu^2+1}{E^2\mu^2+C^2-s^2}},$$

concluding by the definition of $g$,

$$u(s)=\int_{s_0}^{s}g(r)dr=D\pm\int_{s_0}^{s}\sqrt{\frac{r^2\mu^2+1}{E^2\mu^2+C^2-r^2}}dr,$$

for $C>0$, $D$ and $E>0$ constants and $J\subset(0,\sqrt{E^2\mu^2+C^2})$. □

### 2.1.5 *Type I Translation Surfaces - Constant Mean Curvature case*

By Proposition 2.1.1 we conclude that a **Type I Translation Surface** has **Constant Mean curvature** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.9.** *The surface $\psi:\Omega\subset\mathbb{R}^2\to H^2\times\mathbb{R}$ given by $\psi(t,s)=(t,s,v(t)+u(s))$, where $u\in C^2(J,\mathbb{R})$, $v\in C^2(I,\mathbb{R})$ and $J\times I\subset(0,+\infty)\times\mathbb{R}$, has constant Mean Curvature $c$ , if, and only if,*

$$c=\frac{(1+s^2(u'(s))^2)v''(t)-s((u'(s))^2+(v'(t))^2)u'(s)+(1+s^2(v'(t))^2)u''(s)}{2s\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)^{\frac{3}{2}}}$$

**Proposition 2.1.10.** *Let $\psi:\Omega\subset\mathbb{R}^2\to H^2\times\mathbb{R}$ given by $\psi(t,s)=(t,s,v(t)+u(s))$, where $u\in C^2(J,\mathbb{R})$, $v\in C^2(I,\mathbb{R})$ and $J\times I\subset(0,+\infty)\times\mathbb{R}$, with constant Mean Curvature $c\neq 0$. Then $v(t)=\mu t+\tau$, for $\mu,\tau$ constants, and $I\subset\mathbb{R}$.*

**Proof:** Suppose $\psi$ has constant Mean Curvature $c\neq 0$. By Proposition 2.1.9 the functions $u,v$ satisfy the equation

$$c^2=\frac{s^4\left((1+s^2(u'(s))^2)v''(t)-s((u'(s))^2+(v'(t))^2)u'(s)+(1+s^2(v'(t))^2)u''(s)\right)^2}{4\left(s^2(v'(t))^2+s^2(u'(s))^2+1\right)^3} \tag{2.15}$$

or equivalently

$$\begin{aligned}
0 =\, & (v'(t))^6 - \frac{(1+s^2(u'(s))^2)^2}{4c^2s^2}(v''(t))^2 - \frac{(1+s^2(u'(s))^2)(su''(s)-u'(s))}{2c^2s}(v'(t))^2v''(t) \\
& + \frac{(s(u'(s))^3-u''(s))(1+s^2(u'(s))^2)}{2c^2s^2}v''(t) \\
& + \frac{(-s^2(u'(s))^2-s^4(u''(s))^2+12c^2+12c^2s^2(u'(s))^2+2s^3u'(s)u''(s))}{4c^2s^2}(v'(t))^4 \\
& + \frac{-s^4(u''(s))^2-s^4(u'(s))^4+6c^2+s^5(u'(s))^3u''(s)+12c^2s^2(u'(s))^2}{2c^2s^4}(v'(t))^2 \\
& + \frac{6c^2s^4(u'(s))^4+s^3u'(s)u''(s)}{2c^2s^4}(v'(t))^2 - \frac{(s^4(u''(s)-s(u'(s))^3)^2-4c^2(1+s^2(u'(s))^2)^3)}{4c^2s^6}.
\end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = X^3 + A_sY^2 + B_sXY + C_sY + D_sX^2 + E_sX + F_s,$$

where,

$$\begin{aligned}
A_s =& -\frac{(1+s^2(u'(s))^2)^2}{4c^2s^2} \\
B_s =& -\frac{(1+s^2(u'(s))^2)(su''(s)-u'(s))}{2c^2s} \\
C_s =& \frac{(s(u'(s))^3-u''(s))(1+s^2(u'(s))^2)}{2c^2s^2} \\
D_s =& \frac{(-s^2(u'(s))^2-s^4(u''(s))^2+12c^2+12c^2s^2(u'(s))^2+2s^3u'(s)u''(s))}{4c^2s^2} \\
E_s =& \frac{-s^4(u''(s))^2-s^4(u'(s))^4+6c^2+s^5(u'(s))^3u''(s)+12c^2s^2(u'(s))^2}{2c^2s^4} \\
& + \frac{6c^2s^4(u'(s))^4+s^3u'(s)u''(s))}{2c^2s^4} \\
F_s =& -\frac{(s^4(u''(s)-s(u'(s))^3)^2-4c^2(1+s^2(u'(s))^2)^3)}{4c^2s^6},
\end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible then there exist polynomials of degree 2 and of degree 1 such that

$$P_s(X,Y) = (X^2 + fXY + gY^2 + hX + iY + j)(X + kY + l)$$

that is,

$$0 = X^3 + A_sY^2 + B_sXY + C_sY + D_sX^2 + E_sX + F_s - (X^2 + fXY + gY^2 + hX + iY + j)(X + kY + l)$$

and reorganizing

$$\begin{aligned}
0 =\, & (-f-k)X^2Y + (D_s-l-h)X^2 + (-g-fk)XY^2 + (B_s-i-fl-hk)XY + (E_s-j-hl)X \\
& + (-gk)Y^3 + (A_s-ki-gl)Y^2 + (C_s-li-jk)Y + F_s - jl
\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{rrr} -f-k=0 & B_s-i-fl-hk=0 & A_s-ki-gl=0 \\ D_s-l-h=0 & E_s-j-hl=0 & C_s-li-jk=0 \\ -g-fk=0 & -gk=0 & F_s-jl=0. \end{array}$$

By the first, third and sixth equality we obtain that $f=g=k=0$. Consequently, by the seventh equality, $A_s=0$, a contradiction since $A_s<0$ and thus we proof the claim.

By the claim the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda=1$ we conclude that

$$P_s(X,Y)=X^3+A_sY^2+B_sXY+C_sY+D_sX^2+E_sX+F_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, then, there are $K_1,K_2,K_3,K_4,K_5,K_6$ constants such that,

$$\begin{aligned} A_s &= K_1 \\ B_s &= K_2 \\ C_s &= K_3 \\ D_s &= K_4 \\ E_s &= K_5 \\ F_s &= K_6, \end{aligned}$$

that is,

$$\begin{aligned} K_1 &= -\frac{(1+s^2(u'(s))^2)^2}{4c^2s^2} \\ K_2 &= -\frac{(1+s^2(u'(s))^2)(su''(s)-u'(s))}{2c^2s} \\ K_3 &= \frac{(s(u'(s))^3-u''(s))(1+s^2(u'(s))^2)}{2c^2s^2} \\ K_4 &= \frac{(-s^2(u'(s))^2-s^4(u''(s))^2+12c^2+12c^2s^2(u'(s))^2+2s^3u'(s)u''(s))}{4c^2s^2} \\ K_5 &= \frac{-s^4(u''(s))^2-s^4(u'(s))^4+6c^2+s^5(u'(s))^3u''(s)+12c^2s^2(u'(s))^2}{2c^2s^4} \\ &\quad +\frac{6c^2s^4(u'(s))^4+s^3u'(s)u''(s))}{2c^2s^4} \\ K_6 &= -\frac{(s^4(u''(s)-s(u'(s))^3)^2-4c^2(1+s^2(u'(s))^2)^3)}{4c^2s^6}. \end{aligned}$$

Setting $w(s) = u'(s)$ we obtain

$$
\begin{aligned}
K_1 &= -\frac{(1+s^2w^2(s))^2}{4c^2s^2}\\
K_2 &= -\frac{(1+s^2w^2(s))(sw'(s)-w(s))}{2c^2s}\\
K_3 &= \frac{(sw^3(s)-w'(s))(1+s^2w^2(s))}{2c^2s^2}\\
K_4 &= \frac{(-s^2w^2(s)-s^4(w'(s))^2+12c^2+12c^2s^2w^2(s)+2s^3w(s)w'(s))}{4c^2s^2}\\
K_5 &= \frac{(-s^4(w'(s))^2-s^4w^4(s)+6c^2+s^5w^3(s)w'(s)+12c^2s^2w^2(s)+6c^2s^4w^4(s)+s^3w(s)w'(s))}{2c^2s^4}\\
K_6 &= -\frac{(s^4(w'(s)-sw^3(s))^2-4c^2(1+s^2(w^2(s))^3)}{4c^2s^6}.
\end{aligned}
$$

By the first equality we have

$$0 = s^4w^4(s)+2s^2w^2(s)+4c^2K_1s^2+1, \tag{2.16}$$

and by the second and third equalities

$$\frac{w(s)}{s}-\frac{2c^2K_2s}{1+s^2w^2(s)} = w'(s) = sw^3-\frac{2c^2K_3s^2}{1+s^2w^2(s)}$$

then,

$$0 = s^4w^5(s)-w(s)-2c^2K_3s^3+2c^2K_2s,$$

that is,

$$0 = s^5w^5(s)-sw(s)-2c^2K_3s^4+2c^2K_2s^2 \tag{2.17}$$

Define $G:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$G(x,y) = y^5-y-2c^2K_3x^2+2c^2K_2x$$

and

$$\overline{G}(x,y) = y^4+2y^2+4c^2K_1x+1.$$

Observe that, from (2.17) e (2.16), $G(s^2,sw(s)) = 0 = \overline{G}(s^2,sw(s))$. Since the curve $(s^2,sw(s))$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A$.0.2. Then, the curve $((v'(t))^2,v''(t))$ is constant and therefore there exist $\mu,\eta$ such that $v'(t)=\mu$ and $v''(t)=\eta$ (in this case necessarily $\eta=0$) and we conclude that $v(t)=\mu t+\tau$ in $I\subset\mathbb{R}$. □

**Theorem 2.1.4.** *Let $\psi:\Omega\subset\mathbb{R}^2\to H^2\times\mathbb{R}$ given by $\psi(t,s)=(t,s,v(t)+u(s))$, where $u\in C^2(J,\mathbb{R})$, $v\in C^2(I,\mathbb{R})$, and $J\times I\subset(0,+\infty)\times\mathbb{R}$, with constant Mean Curvature $c\neq 0$. Then $v(t)=\mu t+\tau$, for constant $\mu,\tau$, and $I\subset\mathbb{R}$. Moreover, the function $u$ satisfies the differential equation*

$$u''(s) = \frac{su'(s)(\mu^2+(u'(s))^2)+2cs\left(\frac{1}{s^2}+\mu^2+(u'(s))^2\right)^{\frac{3}{2}}}{1+\mu^2s^2}. \tag{2.18}$$

**Proof:**Suppose $\psi$ has constant Mean Curvature $c \neq 0$. By Proposition 2.1.9 the functions $u, v$ satisfy the equation

$$c = \frac{(1+s^2(u'(s))^2)v''(t) - s((u'(s))^2 + (v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)}{2s\left((v'(t))^2 + (u'(s))^2 + \frac{1}{s^2}\right)^{\frac{3}{2}}}. \tag{2.19}$$

By Proposition 2.1.10 we have $v(t) = \mu t + \tau$ for constants $\mu, \tau$ and $I \subset \mathbb{R}$. Then, from (2.19),

$$u''(s) = \frac{su'(s)(\mu^2 + (u'(s))^2) + 2cs\left(\frac{1}{s^2} + \mu^2 + (u'(s))^2\right)^{\frac{3}{2}}}{1+\mu^2 s^2}.$$

$\square$

**Corollary 2.1.2.** *Assume the conditions of Theorem* 2.1.4. *Making the change of variable* $g(s) = su'(s)$ *the equation* (2.18) *reduces to*

$$g'(s) = \frac{2c(1+g^2(s)+\mu^2 s^2)^{\frac{3}{2}} + g^3(s) + 2s^2\mu^2 g(s) + g(s)}{s(1+s^2\mu^2)}.$$

*In particular, if* $\mu = 0$*,* $g$ *satisfies the separable differential equation*

$$g'(s) = \frac{(1+g^2(s))\left(2c\sqrt{1+g^2(s)} + g(s)\right)}{s}.$$

**Proof:** Notice that $g'(s) = u'(s) + su''(s)$ and then $sg'(s) = su'(s) + s^2u''(s) = g(s) + s^2u''(s)$. Then,

$$\begin{aligned}
\frac{sg'(s) - g(s)}{s^2} = u''(s) &= \frac{su'(s)(\mu^2 + (u'(s))^2) + 2cs\left(\frac{1}{s^2} + \mu^2 + (u'(s))^2\right)^{\frac{3}{2}}}{1+\mu^2 s^2} \\
&= \frac{su'(s)\left(\mu^2 + \frac{1}{s^2}(su'(s))^2\right) + 2cs\left(\frac{1}{s^2}\right)^{\frac{3}{2}}\left(1+s^2\mu^2+(su'(s))^2\right)^{\frac{3}{2}}}{1+\mu^2 s^2} \\
&= \frac{g(s)\left(\mu^2 + \frac{1}{s^2}g^2(s)\right) + \frac{2c}{s^2}\left(1+s^2\mu^2+g^2(s)\right)^{\frac{3}{2}}}{1+\mu^2 s^2}
\end{aligned}$$

i.e.,

$$g'(s) = \frac{2c(1+g^2(s)+\mu^2 s^2)^{\frac{3}{2}} + g^3(s) + 2s^2\mu^2 g(s) + g(s)}{s(1+s^2\mu^2)}.$$

$\square$

### 2.1.6 *Type I Translation Surfaces of Null Linear Weingarten Form*

By Proposition 2.1.1 we conclude that a **Type I Translation Surface** is **Null Linear Weingarten Form** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.11.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, is Null Linear Weingarten Form, if, and only if,*

$$0 = a\left(\frac{(1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)}{2s\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)^{\frac{3}{2}}}\right)$$
$$+b\left(\frac{s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2}{s^2\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)^2}\right).$$

**Proposition 2.1.12.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, of Null Linear Weingarten Form. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose that $\psi$ is Null Linear Weingarten Form. By Proposition 2.1.11 the functions $u, v$ satisfy the equation (assuming, without loss of generality, $b = 1$),

$$0 = a\left(\frac{s^2\left((1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)\right)}{2\left(s^2(v'(t))^2+s^2(u'(s))^2+1\right)^{\frac{3}{2}}}\right)$$
$$+\frac{s^2\left(s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2\right)}{\left(s^2(v'(t))^2+s^2(u'(s))^2+1\right)^2}.$$

or equivalently

$$\begin{aligned}
0 &= (v''(t))^2(v'(t))^2 + \frac{2s(su''(s)-u'(s))}{s^2(u'(s))^2+1}v''(t)(v'(t))^4 \\
&+ \frac{a^2s^4(s^2(u'(s))^2+1)^3 - 4s^6(u'(s)+su''(s))^2}{a^2s^6(s^2(u'(s))^2+1)^2}(v''(t))^2 + \frac{8s^5(u'(s)+su''(s))}{a^2s^6(s^2(u'(s))^2+1)^2}(v''(t))^2v''(t) \\
&+ \frac{-a^2s^4\left(2(s^2(u'(s))^2+1)^2(su'(s)-s^2u''(s)) - 2s^2(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)\right)}{a^2s^6(s^2(u'(s))^2+1)^2}(v''(t))^2v''(t) \\
&+ \frac{\left(8s^5u'(s)(u'(s)+su''(s))^2 + 2a^2s^4(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)^2\right)}{a^2s^6(s^2(u'(s))^2+1)^2}v''(t) \\
&+ \frac{a^2s^4\left((s^2(u'(s))^2+1)(su'(s)-s^2u''(s))^2 - 2s^2(u''(s)-s(u'(s))^3)(su'(s)-s^2u''(s))\right) - 4s^4}{a^2s^6(s^2(u'(s))^2+1)^2}(v'(t))^4 \\
&+ \frac{a^2s^4\left(s^2(u''(s)-s(u'(s))^3)^2 - 2(u''(s)-(u'(s))^3)(s^2(u'(s))^2+1)(su'(s)-s^2u''(s))\right)}{a^2s^6(s^2(u'(s))^2+1)^2}(v'(t))^2 \\
&- \frac{8s^4u'(s)(u'(s)+su''(s))}{a^2s^6(s^2(u'(s))^2+1)^2}(v'(t))^2 + \frac{s^2(su''(s)-u'(s))^2}{(s^2(u'(s))^2+1)^2}(v'(t))^6 \\
&+ \frac{a^2s^4(u''(s)-s(u'(s))^3)^2(s^2(u'(s))^2+1) - 4s^4(u'(s))^2(u'(s)+su''(s))^2}{a^2s^6(s^2(u'(s))^2+1)^2}.
\end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = XY^2 + A_sY^2 + B_sX^2Y + C_sXY + D_sY + E_sX^3 + F_sX^2 + G_sX + H_s,$$

where,

$$
\begin{aligned}
A_s &= \frac{\left(a^2 s^4 (s^2(u'(s))^2+1)^3 - 4s^6(u'(s)+su''(s))^2\right)}{a^2 s^6 (s^2(u'(s))^2+1)^2}\\
B_s &= \frac{2s(su''(s)-u'(s))}{s^2(u'(s))^2+1}\\
C_s &= \frac{-a^2 s^4\left(2(s^2(u'(s))^2+1)^2(su'(s)-s^2u''(s)) - 2s^2(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)\right)}{a^2 s^6 (s^2(u'(s))^2+1)^2}\\
&\quad + \frac{8s^5(u'(s)+su''(s))}{a^2 s^6 (s^2(u'(s))^2+1)^2}\\
D_s &= \frac{8s^5 u'(s)(u'(s)+su''(s))^2 + 2a^2 s^4 (u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)^2}{a^2 s^6 (s^2(u'(s))^2+1)^2}\\
E_s &= \frac{s^2(su''(s)-u'(s))^2}{(s^2(u'(s))^2+1)^2}\\
F_s &= \frac{a^2 s^4\left((s^2(u'(s))^2+1)(su'(s)-s^2u''(s))^2 - 2s^2(u''(s)-s(u'(s))^3)(su'(s)-s^2u''(s))\right) - 4s^4}{a^2 s^6 (s^2(u'(s))^2+1)^2}\\
G_s &= \frac{a^2 s^4\left(s^2(u''(s)-s(u'(s))^3)^2 - 2(u''(s)-(u'(s))^3)(s^2(u'(s))^2+1)(su'(s)-s^2u''(s))\right)}{a^2 s^6 (s^2(u'(s))^2+1)^2}\\
&\quad - \frac{8s^4 u'(s)(u'(s)+su''(s))}{a^2 s^6 (s^2(u'(s))^2+1)^2}\\
H_s &= \frac{a^2 s^4 (u''(s)-s(u'(s))^3)^2(s^2(u'(s))^2+1) - 4s^4(u'(s))^2(u'(s)+su''(s))^2}{a^2 s^6 (s^2(u'(s))^2+1)^2},
\end{aligned}
$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim 1:** Let $V \subset J$ be an open set. Then there are no constants $M_1, M_2, M_4$ such that

$$
\begin{aligned}
D_s &= M_1 A_s\\
B_s &= M_2\\
H_s &= M_3 A_s\\
M_4 &= \frac{-M_2^2 A_s + 4F_s}{4}\\
M_5 &= \frac{4G_s - M_4 A_s}{4},
\end{aligned}
\tag{2.20}
$$

in the open set $V$. Indeed, if there are such constants $M_1, M_2, M_4, M_5$ we have $M_4 \neq 0$ since from (2.20)

$$
\begin{aligned}
H_s &= M_3 A_s\\
B_s &= M_2\\
0 &= -M_2^2 A_s + 4F_s\\
M_5 &= G_s
\end{aligned}
$$

that is,

$$\begin{aligned}
M_2 &= \frac{2s(su''(s)-u'(s))}{s^2(u'(s))^2+1}\\
M_3\frac{a^2s^4(s^2(u'(s))^2+1)^3-4s^6(u'(s)+su''(s))^2}{a^2s^6(s^2(u'(s))^2+1)^2} &= \frac{a^2s^4(u''(s)-s(u'(s))^3)^2(s^2(u'(s))^2+1)}{a^2s^6(s^2(u'(s))^2+1)^2}\\
&\quad-\frac{4s^4(u'(s))^2(u'(s)+su''(s))^2}{a^2s^6(s^2(u'(s))^2+1)^2}\\
\frac{M_2^2}{4}\frac{\left(a^2s^4(s^2(u'(s))^2+1)^3-4s^6(u'(s)+su''(s))^2\right)}{a^2s^6(s^2(u'(s))^2+1)^2} &= \frac{a^2s^4\left((s^2(u'(s))^2+1)(su'(s)-s^2u''(s))^2\right)-4s^4}{a^2s^6(s^2(u'(s))^2+1)^2}\\
&\quad+\frac{a^2s^4\left(-2s^2(u''(s)-s(u'(s))^3)(su'(s)-s^2u''(s))\right)}{a^2s^6(s^2(u'(s))^2+1)^2}
\end{aligned}$$

$$\begin{aligned}
a^2s^6(s^2(u'(s))^2+1)^2M_5 = {}& a^2s^4\left(s^2(u''(s)-s(u'(s))^3)^2\right)\\
&-2a^2s^4(u''(s)-(u'(s))^3)(s^2(u'(s))^2+1)(su'(s)-s^2u''(s))\\
&-8s^4u'(s)(u'(s)+su''(s)).
\end{aligned}$$

By the first equality above we obtain $u''(s)=\frac{M_2+2su'(s)+s^2(u'(s))^2M_2}{2s^2}$ which substituting into the other three equalities (and multiplying the last one by $s$)

$$\begin{aligned}
0 = {}&-a^2s^8(u'(s))^8+a^2M_2s^7(u'(s))^7+a^2M_3s^{10}(u'(s))^6+\left(a^2-\frac{a^2M_2^2}{4}+M_2^2\right)s^6(u'(s))^6\\
&+(M_2a^2+8M_2)s^5(u'(s))^5+(3a^2M_3-M_2^2M_3)s^8(u'(s))^4+\left(a^2-\frac{3a^2M_2^2}{4}+2M_2^2+16\right)s^4(u'(s))^4\\
&-8M_2M_3s^7(u'(s))^3+(8M_2-a^2M_2)s^3(u'(s))^3+(3M_3a^2-2M_3M_2^2-16M_3)s^6(u'(s))^2\\
&+\left(M_2^2-a^2-\frac{3a^2M_2^2}{4}\right)s^2(u'(s))^2-8M_2M_3s^5u'(s)-a^2M_2su'(s)+(a^2M_3-M_2^2M_3)s^4-\frac{a^2M_2^2}{4}\\
0 = {}&-4a^2M_2(su'(s))^5+(2a^2M_2^2+M_2^4)(su'(s))^48M_2^3(su'(s))^3+(4a^2M_2^2+2M_2^4+16M_2^2)(su'(s))^2\\
&+(4a^2M_2+8M_2^3)su'(s)+2a^2M_2^2+M_2^4-16\\
0 = {}&4a^2M_2s^7(u'(s))^7-(2a^2M_2^2+4a^2)s^7(u'(s))^6+8a^2M_2s^5(u'(s))^5+4a^2M_5s^9(u'(s))^4\\
&+(8a^2-7a^2M_2^2)s^5(u'(s))^4+(16M_2-8a^2M_2)s^4(u'(s))^3+4a^2M_2s^3(u'(s))^3+8a^2M_5s^7(u'(s))^2\\
&+(-8a^2M_2^2-4a^2+64)s^3(u'(s))^2+16M_2-8a^2M_2)s^2u'(s)+4a^2M_5s^5-3a^2M_2^2s.
\end{aligned} \tag{2.21}$$

By the second equality in (2.21) we conclude that $M_2\neq 0$ (otherwise $0=-16$). Then $su'(s)$ vanishes the nonzero polynomial

$$\begin{aligned}
R(Z) = {}&-4a^2M_2Z^5+(2a^2M_2^2+M_2^4)Z^48M_2^3Z^3+(4a^2M_2^2+2M_2^4+16M_2^2)Z^2+(4a^2M_2+8M_2^3)Z\\
&+2a^2M_2^2+M_2^4-16
\end{aligned}$$

and therefore $su'(s)$ is constant. Considering $su'(s)=\lambda$ we obtain by the first and third equalities of

(2.21)

$$
\begin{aligned}
0 =&(a^2M_3 - \lambda^2(2M_3M_2^2 - 3a^2M_3 + 16M_3) - M_2^2M_3 - \lambda^4(M_2^2M_3 - 3a^2M_3) - 8\lambda^3M_2M_3 + a^2\lambda^6M_3)s^4 \\
&- 8\lambda M_2M_3s^4 - a^2\lambda^8 + a^2M_2\lambda^7 + \left(a^2 - \frac{a^2M_2^2}{4} + M_2^2\right)\lambda^6 + (a^2M_2 + 8M_2)\lambda^5 \\
&+ \left(a^2 - \frac{3a^2M_2^2}{4} + 2M_2^2 + 16\right)\lambda^4 + (8M_2 - a^2M_2)\lambda^3 + \left(M_2^2 - a^2 - \frac{3a^2M_2^2}{4}\right)\lambda^2 - a^2M_2(\lambda + \frac{1}{4}) \\
0 =&4a^2M_5(\lambda^2+1)^2s^5 + (\lambda^3(16M_2 - 8a^2M_2) - 3a^2M_2 + \lambda(16M_2 - 8a^2M_2) - \lambda^6(2a^2M_2^2 + 4a^2))s \\
&+ (\lambda^4(8a^2 - 7a^2M_2^2) - \lambda^2(8a^2M_2^2 + 4a^2 - 64))s + 4a^2\lambda^3M_2(\lambda^2+1)^2.
\end{aligned}
\tag{2.22}
$$

In the first equality we have a zero polynomial (2.22) of the form $js^4 + k$. Then both coefficients are zero and, in particular, the second coefficient must be zero, that is,

$$
\begin{aligned}
0 =& - a^2\lambda^8 + a^2M_2\lambda^7 + \left(a^2 - \frac{a^2M_2^2}{4} + M_2^2\right)\lambda^6 + (a^2M_2 + 8M_2)\lambda^5 + \left(a^2 - \frac{3a^2M_2^2}{4} + 2M_2^2 + 16\right)\lambda^4 \\
&+ (8M_2 - a^2M_2)\lambda^3 + \left(M_2^2 - a^2 - \frac{3a^2M_2^2}{4}\right)\lambda^2 - a^2M_2\lambda - \frac{a^2M_2^2}{4}
\end{aligned}
$$

and then $\lambda \neq 0$ (otherwise $M_2 = 0$). Using this information in the second equality of (2.22) we conclude that $s$ vanishes the nonzero polynomial

$$
\begin{aligned}
S(Z) =&4a^2M_5(\lambda^2+1)^2Z^5 + (\lambda^3(16M_2 - 8a^2M_2) - 3a^2M_2 + \lambda(16M_2 - 8a^2M_2) - \lambda^6(2a^2M_2^2 + 4a^2))Z \\
&+ (\lambda^4(8a^2 - 7a^2M_2^2) - \lambda^2(8a^2M_2^2 + 4a^2 - 64))Z + 4a^2\lambda^3M_2(\lambda^2+1)^2,
\end{aligned}
$$

that is, $s$ is constant, a contradiction. Thus, we conclude that $M_4 \neq 0$. From (2.20)

$$
\begin{aligned}
D_s =&M_1A_s \\
B_s =&M_2 \\
M_4 =&\frac{-M_2^2A_s + 4F_s}{4}.
\end{aligned}
$$

Isolating $u''(s)$ in the second equality and substituting it in the others we obtain

$$
\begin{aligned}
0 =& \ 2a^2s^9(u'(s))^7 + a^2M_1s^{10}(u'(s))^6 - a^2M_2s^8(u'(s))^6 + (2a^2 - 2M_2^2)s^7(u'(s))^5 \\
&- M_1(M_2^2 - 3a^2)s^8(u'(s))^4 - (3M_2a^2 + 16M_2)s^6(u'(s))^4 - 8M_1M_2s^7(u'(s))^3 \\
&- (2a^2 + 4M_2^2 + 32)s^5(u'(s))^3 - M_1(2M_2^2 - 3a^2 + 16)s^6(u'(s))^2 - (3M_2a^2 + 16M_2)s^4(u'(s))^2 \\
&- 8M_1M_2s^5u'(s) - (2a^2 + 2M_2^2)s^3u'(s) - M_1(M_2^2 - a^2)s^4 - a^2M_2s^2 \\
0 =& \ 4a^2M_2s^5(u'(s))^5 + 4a^2M_4s^6(u'(s))^4 - (2a^2M_2^2 + M_2^4)s^4(u'(s))^4 - 8M_2^3s^3(u'(s))^3 \\
&+ 8a^2M_4s^4(u'(s))^2 - (4a^2M_2^2 + 2M_2^4 + 16M_2^2)s^2(u'(s))^2 - (4a^2M_2 + 8M_2^3)su'(s) \\
&+ 4a^2M_4s^2 - 2a^2M_2^2 - M_2^4 + 16
\end{aligned}
$$

and dividing the first by $s^2$

$$\begin{aligned}
0 &= 2a^2s^7(u'(s))^7 + a^2M_1s^8(u'(s))^6 - a^2M_2s^6(u'(s))^6 + (2a^2-2M_2^2)s^5(u'(s))^5\\
&- M_1(M_2^2-3a^2)s^6(u'(s))^4 - (3M_2a^2+16M_2)s^4(u'(s))^4 - 8M_1M_2s^5(u'(s))^3\\
&- (2a^2+4M_2^2+32)s^3(u'(s))^3 - M_1(2M_2^2-3a^2+16)s^4(u'(s))^2 - (3M_2a^2+16M_2)s^2(u'(s))^2\\
&- 8M_1M_2s^5u'(s) - (2a^2+2M_2^2)s^3u'(s) - M_1(M_2^2-a^2)s^4 - a^2M_2s^2\\
0 &= 4a^2M_2s^5(u'(s))^5 + 4a^2M_4s^6(u'(s))^4 - (2a^2M_2^2+M_2^4)s^4(u'(s))^4 - 8M_2^3s^3(u'(s))^3\\
&+ 8a^2M_4s^4(u'(s))^2 - (4a^2M_2^2+2M_2^4+16M_2^2)s^2(u'(s))^2 - (4a^2M_2+8M_2^3)su'(s)\\
&+ 4a^2M_4s^2 - 2a^2M_2^2 - M_2^4 + 16.
\end{aligned} \tag{2.23}$$

Define $T:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{T}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}
T(x,y) =& 4a^2M_2y^5 + 4a^2M_4xy^4 - (2a^2M_2^2+M_2^4)y^4 - 8M_2^3y^3 + 8a^2M_4xy^2 - (4a^2M_2^2+2M_2^4+16M_2^2)y^2\\
&- (4a^2M_2+8M_2^3)y + 4a^2M_4x - 2a^2M_2^2 - M_2^4 + 16
\end{aligned}$$

and

$$\begin{aligned}
\overline{T}(x,y) =& 2a^2y^7 + a^2M_1xy^6 - a^2M_2y^6 + (2a^2-2M_2^2)y^5 - M_1(M_2^2-3a^2)xy^4 - (3M_2a^2+16M_2)y^4\\
&- 8M_1M_2xy^3 - (2a^2+4M_2^2+32)y^3 - M_1(2M_2^2-3a^2+16)xy^2 - (3M_2a^2+16M_2)y^2\\
&- 8M_1M_2xy - (2a^2+2M_2^2)y - M_1(M_2^2-a^2)x - a^2M_2.
\end{aligned}$$

Observe that, from (2.23), $T(s^2,su'(s)) = 0 = \overline{T}(s^2,su'(s))$. Since the curve $(s^2,su'(s))$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $T$ and $\overline{T}$ must have at least one factor in common and since $M_4\neq 0$ we obtain a contradiction with Lemma $A$.0.11 and thus proof the claim.

**Claim 2:** $4F_s + B_s^2A_s - 2B_sC_s < 0$ for all $s\in J$. Indeed, define:

$$\begin{aligned}
\eta &= u'(s) + su''(s)\\
\delta &= u'(s) - su''(s)\\
\gamma &= s^2(u'(s))^2 + 1\\
\rho &= u''(s) - s(u'(s))^3.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
A_s &= \frac{a^2\gamma^3 - 4s^2\eta^2}{a^2s^2\gamma^2}\\
B_s &= -\frac{2s\delta}{\gamma}\\
C_s &= \frac{8\eta - 2a^2\left(\gamma^2\delta - s\rho\gamma\right)}{a^2s\gamma^2}\\
F_s &= \frac{a^2s^2\gamma\delta^2 - 2s^3a^2\rho\delta - 4}{a^2s^2\gamma^2}
\end{aligned}$$

thus,

$$\begin{aligned}
4F_s + B_s^2 A_s - 2B_s C_s &= -16\frac{(s^2\delta\eta - \gamma)^2}{a^2 s^2 \gamma^2} \\
&= -16\frac{\left(s^2\delta\eta - s^2\left(\frac{\eta+\delta}{2}\right)^2 - 1\right)^2}{a^2 s^2\gamma^2} \\
&= -\frac{\left(s^2\delta^2 + s^2\eta^2 - 2s^2\delta\eta + 4\right)^2}{a^2 s^2 \gamma^2} \\
&= -\frac{\left(s^2(\delta-\eta)^2 + 4\right)^2}{a^2 s^2\gamma^2} < 0,
\end{aligned}$$

and thus we conclude the proof of the claim.

**Claim 3:** Let $V \subset J$ be open set. If $A_s$ and $B_s$ are constants in $V$ then $A_s, B_s$ are nonzero in $V$. Indeed, if there exist constants $K_1, K_2$ such that

$$\begin{aligned}
A_s =& K_1 \\
B_s =& K_2,
\end{aligned}$$

that is,

$$\begin{aligned}
K_1 &= \frac{\left(a^2 s^4 (s^2(u'(s))^2+1)^3 - 4s^6(u'(s)+su''(s))^2\right)}{a^2 s^6 (s^2(u'(s))^2+1)^2} \\
K_2 &= \frac{2s(su''(s) - u'(s))}{s^2(u'(s))^2+1},
\end{aligned}$$

then, setting $w(s) = u'(s)$, we obtain

$$\begin{aligned}
K_1 &= \frac{\left(a^2 s^4 (s^2 w^2(s)+1)^3 - 4s^6(w(s)+sw'(s))^2\right)}{a^2 s^6 (s^2 w^2(s)+1)^2} \\
K_2 &= \frac{2s(sw'(s) - w(s))}{s^2 w^2(s)+1}.
\end{aligned} \tag{2.24}$$

By the second equality we have,

$$w'(s) = \frac{K_2 w^2(s)}{2} + \frac{w(s)}{s} + \frac{K_2}{2s^2} \tag{2.25}$$

and by the first equality,

$$\begin{aligned}
0 = &-4s^4(w'(s))^2 - 8s^3 w'(s)w(s) + a^2 s^6 w^6(s) + (3a^2 s^4 - K_1 a^2 s^6)w^4(s) \\
&+ (3a^2 s^2 - 2K_1 a^2 s^4 - 4s^2)w^2(s) + a^2 - K_1 a^2 s^2
\end{aligned} \tag{2.26}$$

that substituting one into the other we obtain

$$\begin{aligned}
0 =& s^6 w^6(s) - K_1 s^6 w^4(s) + \left(\frac{3a^2 - K_2^2}{a^2}\right) s^4 w^4(s) - \frac{8K_2}{a^2} s^3 w^3(s) - 2K_1 s^4 w^2(s) \\
&- \left(\frac{2K_2^2 - 3a^2 + 16}{a^2}\right) s^2 w^2(s) - \frac{8K_2}{a^2} sw(s) - K_1 s^2 - \left(\frac{K_2^2 - a^2}{a^2}\right).
\end{aligned} \tag{2.27}$$

If $K_1 = 0$ then

$$0 = s^6 w^6(s) + \left(\frac{3a^2 - K_2^2}{a^2}\right) s^4 w^4(s) - \frac{8K_2}{a^2} s^3 w^3(s) - \left(\frac{2K_2^2 - 3a^2 + 16}{a^2}\right) s^2 w^2(s)$$
$$- \frac{8K_2}{a^2} sw(s) - \left(\frac{K_2^2 - a^2}{a^2}\right)$$

i.e., $sw(s)$ is constant because it is contained in the set of roots of the non-zero polynomial

$$P(Z) = Z^6 + \left(\frac{3a^2 - K_2^2}{a^2}\right) Z^4 - \frac{8K_2}{a^2} Z^3 - \left(\frac{2K_2^2 - 3a^2 + 16}{a^2}\right) Z^2 - \frac{8K_2}{a^2} Z - \left(\frac{K_2^2 - a^2}{a^2}\right)$$

thus $w(s) = \frac{\lambda}{s}$. From (2.24) we have

$$0 = \frac{\left(a^2 s^4 (s^2 \left(\frac{\lambda}{s}\right)^2 + 1)^3 - 4s^6 \left(\left(\frac{\lambda}{s}\right) + s\left(\frac{-\lambda}{s^2}\right)\right)^2\right)}{a^2 s^6 (s^2 \left(\frac{\lambda}{s}\right)^2 + 1)^2} = \frac{\lambda^2 + 1}{s^2} > 0,$$

a contradiction, therefore $K_1 \neq 0$.
If $K_2 = 0$ from (2.25) we obtain that $w'(s) = \frac{w(s)}{s}$, whose solutions are $w(s) = \alpha s$. We obtain from (2.24)

$$K_1 = \frac{\left(a^2 s^4 (s^4 \alpha^2 + 1)^3 - 16 s^8 \alpha^2\right)}{a^2 s^6 (s^4 \alpha^2 + 1)^2}$$

then

$$0 = a^2 \alpha^6 s^{16} - a^2 \alpha^4 K_1 s^{14} + 3a^2 \alpha^4 s^{12} - 2a^2 \alpha^2 K_1 s^{10} + (3a^2 \alpha^2 - 16\alpha^2) s^8 - a^2 K_1 s^6 + a^2 s^4$$

i.e., $s^2$ is constant because it is contained in the set of roots of the non-zero polynomial

$$Q(Z) = a^2 \alpha^6 Z^8 - a^2 \alpha^4 K_1 Z^7 + 3a^2 \alpha^4 Z^6 - 2a^2 \alpha^2 K_1 Z^5 + (3a^2 \alpha^2 - 16\alpha^2) Z^4 - a^2 K_1 Z^3 + a^2 Z^2,$$

a contradiction. Then $K_2 \neq 0$ and thus we conclude the proof of the claim.

**Claim 4:** Let $V \subset J$ be an open set. Then $A_s, B_s, C_s$ are not mutually constant in $V$. Indeed, if there exist constants $K_1, K_2, K_3$ such that

$$A_s = K_1$$
$$B_s = K_2$$
$$C_s = K_3,$$

that is,

$$K_1 = \frac{\left(a^2 s^4 (s^2 (u'(s))^2 + 1)^3 - 4s^6 (u'(s) + su''(s))^2\right)}{a^2 s^6 (s^2 (u'(s))^2 + 1)^2}$$
$$K_2 = \frac{2s(su''(s) - u'(s))}{s^2 (u'(s))^2 + 1}$$
$$K_3 = \frac{-a^2 s^4 \left(2(s^2 (u'(s))^2 + 1)^2 (su'(s) - s^2 u''(s)) - 2s^2 (u''(s) - s(u'(s))^3)(s^2 (u'(s))^2 + 1)\right)}{a^2 s^6 (s^2 (u'(s))^2 + 1)^2}$$
$$+ \frac{8s^5 (u'(s) + su''(s))}{a^2 s^6 (s^2 (u'(s))^2 + 1)^2},$$

then, setting $w(s) = u'(s)$, we obtain

$$
\begin{aligned}
K_1 &= \frac{\left(a^2s^4(s^2w^2(s)+1)^3 - 4s^6(w(s)+sw'(s))^2\right)}{a^2s^6(s^2w^2(s)+1)^2} \\
K_2 &= \frac{2s(sw'(s)-w(s))}{s^2w^2(s)+1} \\
K_3 &= \frac{-a^2s^4\left(2(s^2w^2(s)+1)^2(sw(s)-s^2w'(s)) - 2s^2(w'(s)-sw^3(s))(s^2w^2(s)+1)\right)}{a^2s^6(s^2w^2(s)+1)^2} \\
&\quad + \frac{8s^5(w(s)+sw'(s))}{a^2s^6(s^2w^2(s)+1)^2}.
\end{aligned}
$$

By the second equality we have,

$$
w'(s) = \frac{K_2w^2(s)}{2} + \frac{w(s)}{s} + \frac{K_2}{2s^2}
$$

and by the first equality,

$$
\begin{aligned}
0 = &-4s^4(w'(s))^2 - 8s^3w'(s)w(s) + a^2s^6w^6(s) + (3a^2s^4 - K_1a^2s^6)w^4(s) \\
&+ (3a^2s^2 - 2K_1a^2s^4 - 4s^2)w^2(s) + (a^2 - K_1a^2s^2)
\end{aligned}
$$

that substituting one into the other we obtain

$$
\begin{aligned}
0 = &s^6w^6(s) - K_1s^6w^4(s) + \left(\frac{3a^2-K_2^2}{a^2}\right)s^4w^4(s) - \frac{8K_2}{a^2}s^3w^3(s) - 2K_1s^4w^2(s) \\
&- \left(\frac{2K_2^2-3a^2+16}{a^2}\right)s^2w^2(s) - \frac{8K_2}{a^2}sw(s) - K_1s^2 - \left(\frac{K_2^2-a^2}{a^2}\right).
\end{aligned}
\tag{2.28}
$$

Similarly, by the second and third equalities we obtain

$$
\begin{aligned}
0 = &\left(2a^2K_2+4K_2\right)s^4 - 2a^2s^9w^5(s) + \left(5a^2K_2+4K_2\right)s^6w^2(s) - a^2K_3s^6 + \left(2a^2+16\right)s^5w(s) \\
&+ 4a^2K_2s^8w^4(s) - 2a^2K_3s^8w^2(s) + a^2K_2s^{10}w^6(s) - a^2K_3s^{10}w^4(s).
\end{aligned}
\tag{2.29}
$$

Define $G : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$, $\overline{G} : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$ by

$$
\begin{aligned}
G(x,y) = &y^6 - K_1xy^4 + \left(\frac{3a^2-K_2^2}{a^2}\right)y^4 - \frac{8K_2}{a^2}y^3 - 2K_1xy^2 - \left(\frac{2K_2^2-3a^2+16}{a^2}\right)y^2 \\
&- \frac{8K_2}{a^2}y - K_1x - \left(\frac{K_2^2-a^2}{a^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{G}(x,y) = &\left(2a^2K_2+4K_2\right)x^2 - 2a^2x^2y^5 + \left(5a^2K_2+4K_2\right)x^2y^2 - a^2K_3x^3 + \left(2a^2+16\right)x^2y + 4a^2K_2x^2y^4 \\
&- 2a^2K_3x^3y^2 + a^2K_2x^2y^6 - a^2K_3x^3y^4.
\end{aligned}
$$

Observe that, from (2.28) e (2.29), $G(s^2,sw(s)) = 0 = \overline{G}(s^2,sw(s))$. Since the curve $(s^2,sw(s))$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common and since, by the claim 2, $K_1$ and $K_2$ are non-zero we obtain a contradiction with Lemma $A.0.10$ and thus proof the claim.

**Claim 5:** If $D_{s_0}+A_{s_0}^2B_{s_0}-A_{s_0}C_{s_0}\neq 0$ or $4A_{s_0}^2F_{s_0}+4H_{s_0}-A_{s_0}^3B_{s_0}^2-4A_{s_0}G_{s_0}\neq 0$, for some $s_0\in J$, then the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$. Indeed, if given $s\in V$ the polynomial $P_s$ is reducible then there exist polynomials of degree 2 and of degree 1 such that

$$P_s(X,Y)=(fX^2+gXY+hY^2+iX+jY+k)(lX+mY+n)$$

that is,

$$\begin{aligned}0=&XY^2+A_sY^2+B_sX^2Y+C_sXY+D_sY+E_sX^3+F_sX^2+G_sX+H_s\\&-(fX^2+gXY+hY^2+iX+jY+k)(lX+mY+n)\end{aligned}$$

and reorganizing

$$\begin{aligned}0=&(E_s-fl)X^3+(B_s-fm-gl)X^2Y+(F_s-li-fn)X^2+(1-hl-gm)XY^2+(C_s-mi-gn-jl)XY\\&+(G_s-in-kl)X+(-hm)Y^3+(A_s-hn-jm)Y^2+(D_s-jn-km)Y+(H_s-kn)\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0=&E_s-fl\\0=&B_s-fm-gl\\0=&F_s-li-fn\\0=&1-hl-gm\end{aligned}\qquad\begin{aligned}0=&C_s-mi-gn-jl\\0=&G_s-in-kl\\0=&-hm\end{aligned}\qquad\begin{aligned}0=&A_s-hn-jm\\0=&D_s-jn-km\\0=&H_s-kn.\end{aligned}$$

By the third equality of the second column we obtain that $h=0$ or $m=0$, since by the fourth equality we cannot have $h=0=m$. If $h=0$ and $m\neq 0$ we have

$$\begin{aligned}0=&E_s-fl\\0=&B_s-fm-gl\\0=&F_s-li-fn\end{aligned}\qquad\begin{aligned}g=&\frac{1}{m}\\0=&C_s-mi-gn-jl\\0=&G_s-in-kl\end{aligned}\qquad\begin{aligned}j=&\frac{A_s}{m}\\0=&D_s-jn-km\\0=&H_s-kn,\end{aligned}$$

that is,

$$\begin{aligned}0=&E_s-fl\\0=&B_s-fm-\frac{l}{m}\\0=&F_s-li-fn\end{aligned}\qquad\begin{aligned}0=&C_s-mi-\frac{n}{m}-\frac{lA_s}{m}\\0=&G_s-in-kl\end{aligned}\qquad\begin{aligned}0=&D_s-\frac{nA_s}{m}-km\\0=&H_s-kn.\end{aligned}$$

We have

$$\begin{aligned}0=&E_s-fl\\f=&\frac{mB_s-l}{m^2}\\0=&F_s-li-fn\end{aligned}\qquad\begin{aligned}i=&\frac{mC_s-n-lA_s}{m^2}\\0=&G_s-in-kl\end{aligned}\qquad\begin{aligned}k=&\frac{mD_s-nA_s}{m^2}\\0=&H_s-kn,\end{aligned}$$

that is,

$$\begin{aligned}0=&E_s-\frac{l(mB_s-l)}{m^2}\\0=&F_s-\frac{l(mC_s-n-lA_s)}{m^2}-\frac{n(mB_s-l)}{m^2}\end{aligned}\qquad\begin{aligned}0=&G_s-\frac{n(mC_s-n-lA_s)}{m^2}-\frac{l(mD_s-nA_s)}{m^2}\\0=&H_s-\frac{n(mD_s-nA_s)}{m^2}.\end{aligned}$$

One can see, by the definition of the coefficients of $P_s$, that $E_s = \frac{B_s^2}{4}$. Thus, by the first equality, we have $0 = \frac{B_s^2}{4} - \frac{l(mB_s - l)}{m^2}$ and therefore $B_s = \frac{2l}{m}$. Substituting into the second equality we obtain

$$0 = \frac{4F_s + B_s^2 A_s - 2B_s C_s}{4},$$

a contradiction with the claim 1. Then, $h \neq 0$ and $m = 0$. We have,

$$\begin{aligned} 0 &= E_s - fl & l &= \frac{1}{h} & n &= \frac{A_s}{h} \\ 0 &= B_s - gl & 0 &= C_s - gn - jl & 0 &= D_s - jn \\ 0 &= F_s - li - fn & 0 &= G_s - in - kl & 0 &= H_s - kn, \end{aligned}$$

that is,

$$\begin{aligned} 0 &= E_s - \frac{f}{h} & & & & \\ 0 &= B_s - \frac{g}{h} & 0 &= C_s - \frac{gA_s}{h} - \frac{j}{h} & 0 &= D_s - \frac{jA_s}{h} \\ 0 &= F_s - \frac{i}{h} - \frac{fA_s}{h} & 0 &= G_s - \frac{iA_s}{h} - \frac{k}{h} & 0 &= H_s - \frac{kA_s}{h}. \end{aligned}$$

We have

$$\begin{aligned} f &= hE_s & & & & \\ g &= hB_s & 0 &= C_s - \frac{gA_s}{h} - \frac{j}{h} & 0 &= D_s - \frac{jA_s}{h} \\ 0 &= F_s - \frac{i}{h} - \frac{fA_s}{h} & 0 &= G_s - \frac{iA_s}{h} - \frac{k}{h} & 0 &= H_s - \frac{kA_s}{h}, \end{aligned}$$

that is,

$$\begin{aligned} 0 &= F_s - \frac{i}{h} - E_s A_s & 0 &= G_s - \frac{iA_s}{h} - \frac{k}{h} & & \\ 0 &= C_s - B_s A_s - \frac{j}{h} & 0 &= D_s - \frac{jA_s}{h} & 0 &= H_s - \frac{kA_s}{h}. \end{aligned}$$

We have

$$\begin{aligned} i &= h(F_s - E_s A_s) & 0 &= G_s - \frac{iA_s}{h} - \frac{k}{h} & & \\ j &= h(C_s - B_s A_s) & 0 &= D_s - \frac{jA_s}{h} & 0 &= H_s - \frac{kA_s}{h}, \end{aligned}$$

that is,

$$0 = G_s - (F_s - E_s A_s)A_s - \frac{k}{h} \qquad 0 = D_s - (C_s - B_s A_s)A_s \qquad 0 = H_s - \frac{kA_s}{h}.$$

We have

$$k = h(G_s - (F_s - E_s A_s)A_s) \qquad 0 = D_s - (C_s - B_s A_s)A_s \qquad 0 = H_s - \frac{kA_s}{h},$$

that is,

$$\begin{aligned} 0 &= D_s - (C_s - B_s A_s)A_s = D_s - C_s A_s + B_s A_s^2 \\ 0 &= H_s - (G_s - (F_s - E_s A_s)A_s)A_s = H_s - G_s A_s + F_s A_s^2 - E_s A_s^3, \end{aligned}$$

and remembering that $E_s = \frac{B_s^2}{4}$, we conclude

$$\begin{aligned} 0 &= D_s - C_s A_s + B_s A_s^2 \\ 0 &= \frac{4H_s - 4G_s A_s + 4F_s A_s^2 - B_s^2 A_s^3}{4} \end{aligned}$$

for all $s \in V$, a contradiction with the hypothesis, and thus we proof the claim.

**Claim 6:** If $D_s + A_s^2 B_s - A_s C_s = 0$ and $4A_s^2 F_s + 4H_s - A_s^3 B_s^2 - 4A_s G_s = 0$, for all $s \in J$, then the polynomials $P_s$ are written, uniquely, as

$$\begin{aligned}P_s(X,Y) &= \left(\frac{B_s^2}{4}X^2 + B_s XY + Y^2 + \left(\frac{-B_s^2 A_s + 4F_s}{4}\right)X + (C_s - B_s A_s)Y\right)(X + A_s) \\ &\quad + \left(\frac{4G_s + B_s^2 A_s^2 - 4F_s A_s}{4}\right)(X + A_s),\end{aligned}$$

in which both factors are irreducible. Indeed, defining the right side as

$$\begin{aligned}\Phi_s(X,Y) &= \left(\frac{B_s^2}{4}X^2 + B_s XY + Y^2 + \left(\frac{-B_s^2 A_s + 4F_s}{4}\right)X + (C_s - B_s A_s)Y\right)(X + A_s) \\ &\quad + \left(\frac{4G_s + B_s^2 A_s^2 - 4F_s A_s}{4}\right)(X + A_s)\end{aligned}$$

then

$$\begin{aligned}P_s(X,Y) - \Phi_s(X,Y) &= \left(E_s - \frac{B_s^2}{4}\right)X^3 + (D_s + A_s^2 B_s - A_s C_s)Y + \frac{4A_s^2 F_s + 4H_s - A_s^3 B_s^2 - 4A_s G_s}{4} \\ &= \left(E_s - \frac{B_s^2}{4}\right)X^3\end{aligned}$$

and since, by the definition of the coefficients of $P_s$, $E_s = \frac{B_s^2}{4}$ holds, we obtain the desired equality. Finally, $X + A_s$ is irreducible and then it remains to show that

$$\frac{B_s^2}{4}X^2 + B_s XY + Y^2 + \left(\frac{-B_s^2 A_s + 4F_s}{4}\right)X + (C_s - B_s A_s)Y + \left(\frac{4G_s + B_s^2 A_s^2 - 4F_s A_s}{4}\right)$$

is irreducible. Indeed, suppose that there are polynomials of degree 1 such that

$$\begin{aligned}(fX + Y + h)(lX + Y + n) &= \frac{B_s^2}{4}X^2 + B_s XY + Y^2 + \left(\frac{-B_s^2 A_s + 4F_s}{4}\right)X + (C_s - B_s A_s)Y \\ &\quad + \left(\frac{4G_s + B_s^2 A_s^2 - 4F_s A_s}{4}\right)\end{aligned}$$

that is,

$$\begin{aligned}0 &= \left(\frac{B_s^2}{4} - fl\right)X^2 + (B_s - f - l)XY + \left(F_s - \frac{A_s B_s^2}{4} - fn - hl\right)X \\ &\quad + (C_s - A_s B_s - n - h)Y + \frac{A_s^2 B_s^2}{4} - F_s A_s + G_s - hn\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0 &= \frac{B_s^2}{4} - fl \\ 0 &= B_s - f - l \\ 0 &= F_s - \frac{A_s B_s^2}{4} - fn - hl \\ 0 &= C_s - A_s B_s - n - h \\ 0 &= \frac{A_s^2 B_s^2}{4} - F_s A_s + G_s - hn.\end{aligned}$$

We have

$$\begin{aligned}0 &= \frac{B_s^2}{4} - fl\\ l &= B_s - f\\ 0 &= F_s - \frac{A_sB_s^2}{4} - fn - hl\\ n &= C_s - A_sB_s - h\\ 0 &= \frac{A_s^2B_s^2}{4} - F_sA_s + G_s - hn,\end{aligned}$$

that is,

$$\begin{aligned}0 &= \frac{B_s^2}{4} - f(B_s - f)\\ 0 &= F_s - \frac{A_sB_s^2}{4} - f(C_s - A_sB_s - h) - h(B_s - f)\\ 0 &= \frac{A_s^2B_s^2}{4} - F_sA_s + G_s - h(C_s - A_sB_s - h).\end{aligned}$$

By the first equality we obtain $B_s = 2f$. Substituting in the second equality we obtain

$$0 = \frac{4F_s + B_s^2A_s - 2B_sC_s}{4},$$

a contradiction with the claim 2 and thus we proof the claim.

- If $D_{s_0} + A_{s_0}^2B_{s_0} - A_{s_0}C_{s_0} \neq 0$ or $4A_{s_0}^2F_{s_0} + 4H_{s_0} - A_{s_0}^3B_{s_0}^2 - 4A_{s_0}G_{s_0} \neq 0$, for some $s_0 \in J$, by claim 5, the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$, that is, for $s \in V \subset J$. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = XY^2 + A_sY^2 + B_sX^2Y + C_sXY + D_sY + E_sX^3 + F_sX^2 + G_sX + H_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Then, their coefficients are constant, a contradiction with claim 4. Thus, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$.

- If $D_s + A_s^2B_s - A_sC_s = 0$ and $4A_s^2F_s + 4H_s - A_s^3B_s^2 - 4A_sG_s = 0$, for all $s \in J$, we obtain, by the claim 6 which is valid

$$\begin{aligned}P_s(X,Y) = &\left(\frac{B_s^2}{4}X^2 + B_sXY + Y^2 + \left(\frac{-B_s^2A_s + 4F_s}{4}\right)X + (C_s - B_sA_s)Y\right)(X + A_s)\\ &+ \left(\frac{4G_s + B_s^2A_s^2 - 4F_sA_s}{4}\right)(X + A_s),\end{aligned}$$

for all $s \in J$. Suppose that $A_s$ and the curve $((v'(t))^2, v''(t))$ are non-constant. Then, there exist $s_0, s_1 \in J$ such that $A_{s_0} \neq A_{s_1}$ and therefore, by continuity, there exists a neighborhood $V$ of $s_1 \in J$ such that $A_s \neq A_{s_0}$ for all $s \in V$. By the theory of Plane Algebraic Curves, $P_s$, for $s \in V$, must have a factor in common with $P_{s_0}$ and since such a factor cannot be $X + A_s$ it is of degree 2. Since the polynomial of degree 2 has a monic coefficient we obtain that

$$\frac{B_s^2}{4}X^2 + B_sXY + Y^2 + \left(\frac{-B_s^2A_s + 4F_s}{4}\right)X + (C_s - B_sA_s)Y + \left(\frac{4G_s + B_s^2A_s^2 - 4F_sA_s}{4}\right)$$

is constant in the variable $s$, that is, the polynomials are all equal in $V$. Therefore, their coefficients are constant in $V$. As, by hypothesis, $D_s + A_s^2B_s - A_sC_s = 0$, that is, $D_s = A_s(C_s - B_sA_s)$, and $4A_s^2F_s + 4H_s - A_s^3B_s^2 - 4A_sG_s = 0$, that is, $H_s = A_s\left(\frac{4G_s+B_s^2A_s^2-4F_sA_s}{4}\right)$, we obtain

$$\begin{aligned}
D_s &= M_1A_s\\
B_s &= M_2\\
H_s &= M_3A_s\\
M_4 &= \frac{-M_2^2A_s + 4F_s}{4}\\
M_5 &= \frac{4G_s - M_4A_s}{4},
\end{aligned}$$

a contradiction with the claim 1. Therefore, $A_s$ is constant in $J$ or the curve $((v'(t))^2, v''(t))$ is constant. If $((v'(t))^2, v''(t))$ is constant then there exist $\mu, \eta$ such that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. If $A_s$ is constant in $J$ we obtain that if $(v'(t))^2 + A_s = 0$ for all $t \in I$ then $v'(t)$ is constant, that is, $v(t)$ is a straight line. Suppose there exists $t_0 \in J$ such that $(v'(t_0))^2 + A_s \neq 0$. Then, there exists maximal interval $(\alpha, \beta) \subset I$ containing $t_0$ such that $(v'(t))^2 + A_s \neq 0$ for all $t \in (\alpha, \beta)$. Since the curve $((v'(t))^2, v''(t))$ vanishes the family of polynomials $P_s$ we conclude that, its restriction to the interval $(\alpha, \beta)$ vanishes

$$\frac{B_s^2}{4}X^2 + B_sXY + Y^2 + \left(\frac{-B_s^2A_s + 4F_s}{4}\right)X + (C_s - B_sA_s)Y + \left(\frac{4G_s + B_s^2A_s^2 - 4F_sA_s}{4}\right)$$

which, by claim 6, are irreducible and have a monic coefficient. Since, by hypothesis,

$$D_s + A_s^2B_s - A_sC_s = 0,$$

that is,

$$D_s = A_s(C_s - B_sA_s)$$

and

$$4A_s^2F_s + 4H_s - A_s^3B_s^2 - 4A_sG_s = 0,$$

hence

$$H_s = A_s\left(\frac{4G_s + B_s^2A_s^2 - 4F_sA_s}{4}\right)$$

we get

$$\begin{aligned} D_s &= M_1 A_s \\ B_s &= M_2 \\ H_s &= M_3 A_s \\ M_4 &= \frac{-M_2^2 A_s + 4F_s}{4} \\ M_5 &= \frac{4G_s - M_4 A_s}{4}, \end{aligned}$$

a contradiction with the claim 1, and then we conclude by the theory of Plane Algebraic Curves that the curve defined as $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta)$. Then, $(v'(t))^2$ is constant and different from $-A_s$ by the definition of $(\alpha, \beta)$. Finally, notice that $\alpha \notin I$ because otherwise we would have $(v'(\alpha))^2 = -A_s$, a contradiction with the continuity of $v'(t)$. In the same way, $\beta \notin I$, and being $(\alpha, \beta)$ maximal we conclude that $(\alpha, \beta) = I$, that is, $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta) = I$ and therefore $v(t)$ is a line in $I$. $\square$

**Theorem 2.1.5.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, of Null Linear Weingarten Form. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover, the function $u$ satisfies the equation*

$$u''(s) = \frac{((u'(s))^2 + \mu^2)\left(2b + asu'(s)\sqrt{s^2\mu^2 + s^2(u'(s))^2 + 1}\right)}{a(s^2\mu^2 + 1)\sqrt{s^2\mu^2 + s^2(u'(s))^2 + 1} - 2bsu'(s)}.$$

**Proof:** Suppose $\psi$ is of Null Linear Weingarten Form. By Proposition 2.1.12 we have $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Therefore, by Proposition 2.1.11 (assuming without loss of generality $b = 1$),

$$0 = \frac{as^2\left(-s((u'(s))^2 + \mu^2)u'(s) + (1 + s^2\mu^2)u''(s)\right)}{2\left(s^2\mu^2 + s^2(u'(s))^2 + 1\right)^{\frac{3}{2}}} - \frac{s^2\left((su''(s) + u'(s))u'(s) + \mu^2\right)}{\left(s^2\mu^2 + s^2(u'(s))^2 + 1\right)^2},$$

that is,

$$\begin{aligned} 0 = {} & as^2\sqrt{s^2\mu^2 + s^2(u'(s))^2 + 1}\left(-s((u'(s))^2 + \mu^2)u'(s) + (1 + s^2\mu^2)u''(s)\right) \\ & - 2s^2\left((su''(s) + u'(s))u'(s) + \mu^2\right) \end{aligned}$$

i.e.,

$$\begin{aligned} 0 = {} & s^2\left(a(s^2\mu^2 + 1)\sqrt{s^2\mu^2 + s^2(u'(s))^2 + 1} - 2su'(s)\right)u''(s) \\ & - s^2((u'(s))^2 + \mu^2)\left(2 + asu'(s)\sqrt{s^2\mu^2 + s^2(u'(s))^2 + 1}\right) \end{aligned}$$

and therefore

$$\begin{aligned} 0 = {} & s^2\left(a(s^2\mu^2 + 1)\sqrt{s^2\mu^2 + s^2(u'(s))^2 + 1} - 2su'(s)\right)u''(s) \\ & - s^2((u'(s))^2 + \mu^2)\left(2 + asu'(s)\sqrt{s^2\mu^2 + s^2(u'(s))^2 + 1}\right). \end{aligned} \tag{2.30}$$

We have two possible situations. If $\mu = 0$ we have from (2.30) that

$$0 = \left(a\sqrt{s^2(u'(s))^2+1} - 2su'(s)\right)u''(s) - (u'(s))^2\left(2 + asu'(s)\sqrt{s^2(u'(s))^2+1}\right), \tag{2.31}$$

and hence $a\sqrt{s^2(u'(s))^2+1} - 2su'(s) \neq 0$ in $J$. Indeed, if $s_0 \in J$ such that $a\sqrt{s_0^2(u'(s_0))^2+1} - 2s_0u'(s_0) = 0$ we obtain, applying $s_0$ to (2.31),

$$\begin{aligned} 0 =& a\sqrt{s_0^2(u'(s_0))^2+1} - 2s_0u'(s_0) \\ 0 =& (u'(s_0))^2\left(2 + as_0u'(s_0)\sqrt{s_0^2(u'(s_0))^2+1}\right). \end{aligned}$$

By the first equality we have that $\sqrt{s_0^2(u'(s_0))^2+1} = \frac{2s_0u'(s_0)}{a}$ and consequently $u'(s_0) \neq 0$. Substituting in the second equality we have $0 = 2(u'(s_0))^2(s_0^2(u'(s_0))^2+1)$ a contradiction. Therefore from (2.31)

$$u''(s) = \frac{(u'(s))^2\left(2 + asu'(s)\sqrt{s^2(u'(s))^2+1}\right)}{a\sqrt{s^2(u'(s))^2+1} - 2su'(s)}.$$

If $\mu \neq 0$ we can proceed in a similar way obtaining

$$u''(s) = \frac{((u'(s))^2+\mu^2)\left(2 + asu'(s)\sqrt{s^2\mu^2+s^2(u'(s))^2+1}\right)}{a(s^2\mu^2+1)\sqrt{s^2\mu^2+s^2(u'(s))^2+1} - 2su'(s)}$$

in the interval $J$ and thus we conclude the proof. □

### 2.1.7 *Type I Translation Surfaces of Linear Weingarten Form*

By Proposition 2.1.1 we conclude that a **Type I Translation Surface** is of **Linear Weingarten Form** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.13.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, is of Linear Weingarten Form, if , and only if,*

$$\begin{aligned} c = a&\left(\frac{(1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)}{2s\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)^{\frac{3}{2}}}\right) \\ +b&\left(\frac{s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2}{s^2\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)^2}\right). \end{aligned}$$

**Proposition 2.1.14.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, of Linear Weingarten Form with constant $c \neq 0$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose that $\psi$ is of Linear Weingarten Form. By Proposition 2.1.13 the functions $u, v$ satisfy the equation (assuming, without loss of generality, $b = 1$),

$$c = a\left(\frac{s^2\left((1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)\right)}{2\,(s^2(v'(t))^2+s^2(u'(s))^2+1)^{\frac{3}{2}}}\right) + \frac{s^2\left(s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2\right)}{(s^2(v'(t))^2+s^2(u'(s))^2+1)^2}.$$

or equivalently

$$\begin{aligned}
0 &= (v''(t))^2(v'(t))^2 + \frac{(a^2s^4(s^2(u'(s))^2+1)^3 - 4a^2s^6(u'(s)+su''(s))^2)}{a^2s^6(s^2(u'(s))^2+1)^2}(v''(t))^2\\
&+\frac{(8acs^7(u'(s)+su''(s)) - 2a^2s^6(s^2(u'(s))^2+1)(su'(s)-s^2u''(s)))}{a^2s^6(s^2(u'(s))^2+1)^2}v''(t)(v'(t))^4\\
&+\frac{(8a+16c(s^2(u'(s))^2+1))(u'(s)+su''(s))}{as(s^2(u'(s))^2+1)^2}v''(t)(v'(t))^2\\
&-\frac{a(2(s^2(u'(s))^2+1)^2(u'(s)-su''(s)) - 2s(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1))}{as(s^2(u'(s))^2+1)^2}v''(t)(v'(t))^2\\
&+\frac{2as(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)^2}{as^3(s^2(u'(s))^2+1)^2}v''(t)\\
&+\frac{(8c(s^2(u'(s))^2+1)^2 + 8as^2u'(s)(u'(s)+su''(s)))(u'(s)+su''(s))}{as^3(s^2(u'(s))^2+1)^2}v''(t)\\
&-\frac{4c^2s^2}{a^2(s^2(u'(s))^2+1)^2}(v'(t))^8 + \frac{(a^2(su'(s)-s^2u''(s))^2 - 4c(2a+4c(s^2(u'(s))^2+1)))}{a^2(s^2(u'(s))^2+1)^2}(v'(t))^6\\
&+\frac{a^2s^4\left((s^2(u'(s))^2+1)(su'(s)-s^2u''(s))^2 - 2s^2(u''(s)-s(u'(s))^3)(su'(s)-s^2u''(s))\right)}{a^2s^6(s^2(u'(s))^2+1)^2}(v'(t))^4\\
&-\frac{4cs^4\left(2c(s^2(u'(s))^2+1)^2 + 2as^2u'(s)(u'(s)+su''(s)) - \left(2as^2+4cs^2(s^2(u'(s))^2+1)\right)\right)^2}{a^2s^6(s^2(u'(s))^2+1)^2}(v'(t))^4\\
&+\frac{a^2s^4\left(s^2(u''(s)-s(u'(s))^3)^2 - 2(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)(su'(s)-s^2u''(s))\right)}{a^2s^6(s^2(u'(s))^2+1)^2}(v'(t))^2\\
&-\frac{2\left(2c(s^2(u'(s))^2+1)^2 + 2as^2u'(s)(u'(s)+su''(s))\right)\left(2as^2+4cs^2(s^2(u'(s))^2+1)\right)}{a^2s^6(s^2(u'(s))^2+1)^2}(v'(t))^2\\
&+\frac{a^2s^4(u''(s)-s(u'(s))^3)^2(s^2(u'(s))^2+1) - (2c(s^2(u'(s))^2+1)^2 + 2as^2u'(s)(u'(s)+su''(s)))^2}{a^2s^6(s^2(u'(s))^2+1)^2}.
\end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = XY^2 + A_sY^2 + B_sX^2Y + C_sXY + D_sY + E_sX^4 + F_sX^3 + G_sX^2 + H_sX + I_s,$$

where,

$$
\begin{aligned}
A_s &= \frac{a^2s^4(s^2(u'(s))^2+1)^3-4a^2s^6(u'(s)+su''(s))^2}{a^2s^6(s^2(u'(s))^2+1)^2}\\
B_s &= \frac{8acs^7(u'(s)+su''(s))-2a^2s^6(s^2(u'(s))^2+1)(su'(s)-s^2u''(s))}{a^2s^6(s^2(u'(s))^2+1)^2}\\
C_s &= \frac{(8a+16c(s^2(u'(s))^2+1))(u'(s)+su''(s))}{as(s^2(u'(s))^2+1)^2}\\
&\quad-\frac{a(2(s^2(u'(s))^2+1)^2(u'(s)-su''(s))-2s(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1))}{as(s^2(u'(s))^2+1)^2}\\
D_s &= \frac{2as(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)^2}{as^3(s^2(u'(s))^2+1)^2}\\
&= \frac{(8c(s^2(u'(s))^2+1)^2+8as^2u'(s)(u'(s)+su''(s)))(u'(s)+su''(s))}{as^3(s^2(u'(s))^2+1)^2}\\
E_s &= -\frac{4c^2s^2}{a^2(s^2(u'(s))^2+1)^2}\\
F_s &= \frac{a^2(su'(s)-s^2u''(s))^2-4c(2a+4c(s^2(u'(s))^2+1))}{a^2(s^2(u'(s))^2+1)^2}\\
G_s &= \frac{a^2s^4\left((s^2(u'(s))^2+1)(su'(s)-s^2u''(s))^2-2s^2(u''(s)-s(u'(s))^3)(su'(s)-s^2u''(s))\right)}{a^2s^6(s^2(u'(s))^2+1)^2}\\
&\quad-\frac{4cs^4\left(2c(s^2(u'(s))^2+1)^2+2as^2u'(s)(u'(s)+su''(s))-\left(2as^2+4cs^2(s^2(u'(s))^2+1)\right)\right)^2}{a^2s^6(s^2(u'(s))^2+1)^2}\\
H_s &= \frac{a^2s^4\left(s^2(u''(s)-s(u'(s))^3)^2-2(u''(s)-s(u'(s))^3)(s^2(u'(s))^2+1)(su'(s)-s^2u''(s))\right)}{a^2s^6(s^2(u'(s))^2+1)^2}\\
&\quad-\frac{2\left(2c(s^2(u'(s))^2+1)^2+2as^2u'(s)(u'(s)+su''(s))\right)\left(2as^2+4cs^2(s^2(u'(s))^2+1)\right)}{a^2s^6(s^2(u'(s))^2+1)^2}\\
I_s &= \frac{a^2s^4(u''(s)-s(u'(s))^3)^2(s^2(u'(s))^2+1)-(2c(s^2(u'(s))^2+1)^2+2as^2u'(s)(u'(s)+su''(s)))^2}{a^2s^6(s^2(u'(s))^2+1)^2},
\end{aligned}
$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots $\{P_s\}_{s\in J}$.

**Claim 1:** Let $V \subset J$ be an open set. Then $B_s, E_s$ are not mutually constant in $V$. Indeed, if there exist constants $K_2, K_5$ such that

$$
\begin{aligned}
B_s &= K_2\\
E_s &= K_5,
\end{aligned}
$$

that is,

$$
\begin{aligned}
K_2 &= \frac{8acs^7(u'(s)+su''(s))-2a^2s^6(s^2(u'(s))^2+1)(su'(s)-s^2u''(s))}{a^2s^6(s^2(u'(s))^2+1)^2}\\
K_5 &= -\frac{4c^2s^2}{a^2(s^2(u'(s))^2+1)^2},
\end{aligned}
$$

then, setting $w(s) = u'(s)$, we obtain

$$\begin{aligned} K_2 &= \frac{8acs^7(w(s)+sw'(s)) - 2a^2s^6(s^2w^2(s)+1)(sw - s^2w'(s))}{a^2s^6(s^2w^2(s)+1)^2} \\ K_5 &= -\frac{4c^2s^2}{a^2(s^2w^2(s)+1)^2}. \end{aligned} \tag{2.32}$$

By the second equality of (2.32) we obtain

$$0 = a^2s^{10}K_5w^4(s) + 2a^2s^8K_5w^2(s) + a^2K_5s^6 + 4c^2s^8$$

which dividing by $s^6$,

$$0 = a^2K_5s^4w^4(s) + 2a^2K_5s^2w^2(s) + a^2K_5 + 4c^2s^2$$

i.e., $w^2(s)$ is a root of the non-zero polynomial

$$Q(Z) = a^2K_5s^4Z^2 + 2a^2K_5s^2Z + a^2K_5 + 4c^2s^2$$

and then $w^2(s) = \frac{-a^2K_5 \pm 2s\sqrt{-a^2c^2K_5}}{a^2s^2K_5}$. Notice that differentiating this equality from both sides we obtain

$$2w(s)w'(s) = \frac{1}{s^3} - \frac{w^2}{s}. \tag{2.33}$$

By the second equality of (2.32) we obtain

$$\begin{aligned} 0 = {} & a^2K_2s^{10}w^4(s) + 2a^2s^9w^3(s) - 2a^2s^{10}w^2(s)w'(s) + 2a^2K_2s^8w^2(s) + (2a^2 - 8ac)s^7w(s) \\ & - (2a^2 + 8ca)s^8w'(s) + a^2K_2s^6 \end{aligned}$$

that dividing by $s^5$ and multiplying by $w(s)$,

$$\begin{aligned} 0 = {} & a^2K_2s^5w^5(s) + 2a^2s^4w^4(s) - 2a^2s^5w^2(s)w(s)w'(s) + 2a^2K_2s^3w^3(s) + (2a^2 - 8ac)s^2w^2(s) \\ & - (2a^2 + 8ca)s^3w(s)w'(s) + a^2K_2sw(s) \end{aligned}$$

and substituting the equality obtained in (2.33) we obtain

$$0 = a^2K_2s^5w^5(s) + 3a^2s^4w^4(s) + 2a^2K_2s^3w^3(s) + 2a(a-2c)s^2w^2(s) + a^2K_2sw(s) - (a^2 + 4ac)$$

and then $sw(s)$ is constant because it is contained in the set of roots of the non-zero polynomial

$$R(Z) = a^2K_2Z^5 + 3a^2Z^4 + 2a^2K_2Z^3 + 2a(a-2c)Z^2 + a^2K_2Z - (a^2 + 4ac)$$

thus $w(s) = \frac{\lambda}{s}$. From the third equality in (2.32) we obtain

$$s^2 = -\frac{K_5a^2(\lambda^2+1)^2}{c^2}$$

that is, $s^2$ is constant, a contradiction and thus we proof the claim.

**Claim 2:** If $D_{s_0} - A_{s_0}(C_{s_0} - A_{s_0}B_{s_0}) \neq 0$ or $I_{s_0} + A_{s_0}^2(G_{s_0} - A_{s_0}(F_{s_0} - A_{s_0}E_{s_0})) \neq 0$, for some $s_0 \in J$, then the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$. Indeed, observe that

$$P_s(X,Y) = (A_s + X)Y^2 + (B_sX^2 + C_sX + D_s)Y + E_sX^4 + F_sX^3 + G_sX^2 + H_sX + I_s$$

then if given $s \in V$ the polynomial $P_s$ is reducible we have two possible situations. In the first case

$$P_s(X,Y) = (R(X)Y + S(X))\,(T(X)Y + U(X))$$

then $R(X)T(X) = X + A_s$ and then we can assume without loss of generality $T(X) = 1$ and $R(X) = X + A_s$, that is,

$$P_s(X,Y) = ((A_s + X)Y + S(X))\,(Y + U(X))$$

then $U(X)(A_s + X) + S(X) = B_sX^2 + C_sX + D_s$ e $S(X)U(X) = E_sX^4 + F_sX^3 + G_sX^2 + H_sX + I_s$. Since $E_s \neq 0$ we have that

$$\deg\left(S(X)U(X)\right) = \deg\left(E_sX^4 + F_sX^3 + G_sX^2 + H_sX + I_s\right) = 4$$

and

$$\deg\left(U(X)(A_s + X) + S(X)\right) = \deg\left(B_sX^2 + C_sX + D_s\right) \leqslant 2$$

and then we have the possibilities

$$\begin{aligned}
&\deg(S(X)) = 0 \ \text{ and } \ \deg(U(X)) = 4 \ \text{ then } \ \deg\left(U(X)(A_s + X) + S(X)\right) = 5 > 2\\
&\deg(S(X)) = 1 \ \text{ and } \ \deg(U(X)) = 3 \ \text{ then } \ \deg\left(U(X)(A_s + X) + S(X)\right) = 4 > 2\\
&\deg(S(X)) = 2 \ \text{ and } \ \deg(U(X)) = 2 \ \text{ then } \ \deg\left(U(X)(A_s + X) + S(X)\right) = 3 > 2\\
&\deg(S(X)) = 3 \ \text{ and } \ \deg(U(X)) = 1 \ \text{ then } \ \deg\left(U(X)(A_s + X) + S(X)\right) = 3 > 2\\
&\deg(S(X)) = 4 \ \text{ and } \ \deg(U(X)) = 1 \ \text{ then } \ \deg\left(U(X)(A_s + X) + S(X)\right) = 4 > 2,
\end{aligned}$$

that is, the first case does not occur. In the second case

$$P_s(X,Y) = U(X)\left(R(X)Y^2 + S(X)Y + T(X)\right)$$

where $U(X)$ is non-constant and $T(X)$ is non-zero. Therefore, $R(X)U(X) = X + A_s$ and then we can assume, without loss of generality, $R(X) = 1$ and $U(X) = X + A_s$ and consequently $\deg(S(X)) \leqslant 1$ and $\deg(T(X)) = 3$ (since $E_s \neq 0$). Therefore,

$$\begin{aligned}
S(X) &= fX + g\\
T(X) &= kX^3 + lX^2 + mX + n
\end{aligned}$$

then

$$P_s(X,Y) = (X + A_s)\left(Y^2 + (fX + g)Y + kX^3 + lX^2 + mX + n\right)$$

that is,

$$0 = P_s(X,Y) - (X + A_s)\left(Y^2 + (fX + g)Y + kX^3 + lX^2 + mX + n\right)$$

that reorganizing

$$\begin{aligned} 0 &= (E_s - k)X^4 + (F_s - l - A_s k)X^3 + (B_s - f)X^2 Y + (G_s - m - A_s l)X^2 \\ &\quad + (C_s - g - A_s f)XY + (H_s - n - A_s m)X + (D_s - A_s g)Y + I_s - A_s n \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = E_s - k & 0 = G_s - m - A_s l & \\ 0 = F_s - l - A_s k & 0 = C_s - g - A_s f & 0 = D_s - A_s g \\ & & 0 = I_s - A_s n. \\ 0 = B_s - f & 0 = H_s - n - A_s m & \end{array}$$

We have,

$$\begin{array}{lll} k = E_s & 0 = G_s - m - A_s l & \\ 0 = F_s - l - A_s k & 0 = C_s - g - A_s f & 0 = D_s - A_s g \\ & & 0 = I_s - A_s n, \\ f = B_s & 0 = H_s - n - A_s m & \end{array}$$

that is,

$$\begin{array}{lll} 0 = F_s - l - A_s E_s & 0 = C_s - g - A_s B_s & 0 = D_s - A_s g \\ 0 = G_s - m - A_s l & 0 = H_s - n - A_s m & 0 = I_s - A_s n. \end{array}$$

We have,

$$\begin{array}{lll} l = F_s - A_s E_s & g = C_s - A_s B_s & 0 = D_s - A_s g \\ 0 = G_s - m - A_s l & n = H_s - A_s m & 0 = I_s - A_s n, \end{array}$$

that is,

$$0 = G_s - m - A_s(F_s - A_s E_s) \qquad 0 = D_s - A_s(C_s - A_s B_s) \qquad 0 = I_s - A_s(H_s - A_s m).$$

We have,

$$m = G_s - A_s(F_s - A_s E_s) \qquad 0 = D_s - A_s(C_s - A_s B_s) \qquad 0 = I_s - A_s(H_s - A_s m),$$

concluding

$$\begin{aligned} 0 &= D_s - A_s(C_s - A_s B_s) \\ 0 &= I_s - A_s(H_s - A_s(G_s - A_s(F_s - A_s E_s))) \end{aligned}$$

for all $s \in V$, a contradiction with the hypothesis, and thus we proof the claim.

**Claim 3:** If $D_s - A_s(C_s - A_s B_s) = 0$ and $I_s + A_s^2(G_s - A_s(F_s - A_s E_s)) = 0$, for all $s \in J$, then the polynomials $P_s$ are written, uniquely, as

$$\begin{aligned} P_s(X,Y) &= (E_s X^3 + (F_s - A_s E_s)X^2 + B_s XY + (G_s - A_s(F_s - A_s E_s))X + Y^2)(X + A_s) \\ &\quad + ((C_s - A_s B_s)Y - A_s(G_s - A_s(F_s - A_s E_s)))(X + A_s) \end{aligned}$$

in which both factors are irreducible. Indeed, defining the right side as

$$\begin{aligned} \Phi_s(X,Y) &= (E_s X^3 + (F_s - A_s E_s)X^2 + B_s XY + (G_s - A_s(F_s - A_s E_s))X + Y^2)(X + A_s) \\ &\quad + ((C_s - A_s B_s)Y - A_s(G_s - A_s(F_s - A_s E_s)))(X + A_s) \end{aligned}$$

then

$$P_s(X,Y)-\Phi_s(X,Y)=(D_s-A_s(C_s-A_sB_s))Y+I_s+A_s^2(G_s-A_s(F_s-A_sE_s))=0.$$

Finally, $X+A_s$ is irreducible and so it remains to show that

$$E_sX^3+(F_s-A_sE_s)X^2+B_sXY+(G_s-A_s(F_s-A_sE_s))X+Y^2+(C_s-A_sB_s)Y-A_s(G_s-A_s(F_s-A_sE_s))$$

is irreducible. Indeed, suppose that there exist polynomials of degree 2 and degree 1 such that

$$\begin{aligned}&E_sX^3+(F_s-A_sE_s)X^2+B_sXY+(G_s-A_s(F_s-A_sE_s))X+Y^2+(C_s-A_sB_s)Y-A_s(G_s-A_s(F_s-A_sE_s))\\&=(fX^2+gXY+hY^2+jX+kY+l)(pX+qY+r)\end{aligned}$$

that is,

$$\begin{aligned}0=&(E_s-fp)X^3-(fq+gp)X^2Y+(F_s-A_sE_s-fr-jp)X^2-(gq+hp)XY^2\\&+(B_s-gr-jq-kp)XY+(G_s-A_s(F_s-A_sE_s)-jr-lp)X-hqY^3+(1-kq-hr)Y^2\\&+(C_s-A_sB_s-kr-lq)Y-(A_s(G_s-A_s(F_s-A_sE_s))+lr)\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll}0=E_s-fp & & \\ 0=fq+gp & 0=B_s-gr-jq-kp & 0=1-kq-hr\\ 0=F_s-A_sE_s-fr-jp & 0=G_s-A_s(F_s-A_sE_s)-jr-lp & 0=C_s-A_sB_s-kr-lq\\ 0=gq+hp & 0=hq & 0=A_s(G_s-A_s(F_s-A_sE_s))+lr.\end{array}$$

Since $E_s\neq 0$ we have, by the first equality, $f\neq 0$ and $p\neq 0$. We can assume, without loss of generality, $f=E_s$ and $p=1$, obtaining

$$\begin{array}{lll}0=E_sq+g & 0=B_s-gr-jq-k & 0=1-kq-hr\\ 0=F_s-A_sE_s-E_sr-j & 0=G_s-A_s(F_s-A_sE_s)-jr-l & 0=C_s-A_sB_s-kr-lq\\ 0=gq+h & 0=hq & 0=A_s(G_s-A_s(F_s-A_sE_s))+lr.\end{array}$$

We have

$$\begin{array}{lll}g=-E_sq & 0=B_s-gr-jq-k & 0=1-kq-hr\\ j=F_s-A_sE_s-E_sr & 0=G_s-A_s(F_s-A_sE_s)-jr-l & 0=C_s-A_sB_s-kr-lq\\ 0=gq+h & 0=hq & 0=A_s(G_s-A_s(F_s-A_sE_s))+lr,\end{array}$$

that is,

$$\begin{array}{lll}0=-E_sq^2+h & 0=hq & 0=C_s-A_sB_s-kr-lq\\ 0=B_s+E_sqr-(F_s-A_sE_s-E_sr)q-k & 0=1-kq-hr & 0=A_s(G_s-A_s(F_s-A_sE_s))\\ 0=G_s-A_s(F_s-A_sE_s)-(F_s-A_sE_s-E_sr)r-l & & \quad+lr.\end{array}$$

We have,

$$
\begin{aligned}
&h = E_s q^2 && 0 = 1 - kq - hr\\
&0 = B_s + E_s qr - (F_s - A_s E_s - E_s r)q - k && 0 = C_s - A_s B_s - kr - lq\\
&0 = G_s - A_s(F_s - A_s E_s) - (F_s - A_s E_s - E_s r)r - l && 0 = A_s(G_s - A_s(F_s - A_s E_s)) + lr,\\
&0 = hq &&
\end{aligned}
$$

that is,

$$
\begin{aligned}
&0 = B_s + E_s qr - (F_s - A_s E_s - E_s r)q - k && 0 = 1 - kq - E_s q^2 r\\
&0 = G_s - A_s(F_s - A_s E_s) - (F_s - A_s E_s - E_s r)r - l && 0 = C_s - A_s B_s - kr - lq\\
&0 = E_s q^3 && 0 = A_s(G_s - A_s(F_s - A_s E_s)) + lr.
\end{aligned}
$$

By the third equality we have, as $E_s \neq 0$, $q = 0$. By the fourth equality we obtain $0 = 1$, a contradiction, and thus we proof the claim.

- If $D_{s_0} - A_{s_0}(C_{s_0} - A_{s_0}B_{s_0}) \neq 0$ or $I_{s_0} + A_{s_0}^2(G_{s_0} - A_{s_0}(F_{s_0} - A_{s_0}E_{s_0})) \neq 0$, for some $s_0 \in J$, by claim 2, the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$, that is, for $s \in V \subset J$. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = XY^2 + A_s Y^2 + B_s X^2 Y + C_s XY + D_s Y + E_s X^4 + F_s X^3 + G_s X^2 + H_s X + I_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Then, their coefficients are constant, a contradiction with claim 1. Thus, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$.

- If $D_s - A_s(C_s - A_s B_s) = 0$ and $I_s + A_s^2(G_s - A_s(F_s - A_s E_s)) = 0$, for all $s \in J$, we obtain, by the claim 3 that

$$
\begin{aligned}
P_s(X,Y) = {}&(E_s X^3 + (F_s - A_s E_s)X^2 + B_s XY + (G_s - A_s(F_s - A_s E_s))X + Y^2)(X + A_s)\\
&+ ((C_s - A_s B_s)Y - A_s(G_s - A_s(F_s - A_s E_s)))(X + A_s),
\end{aligned}
$$

for all $s \in J$. Suppose that $A_s$ and the curve $((v'(t))^2, v''(t))$ are non-constant. Then, there exist $s_0, s_1 \in J$ such that $A_{s_0} \neq A_{s_1}$ and therefore, by continuity, there exists a neighborhood $V$ of $s_1 \in J$ such that $A_s \neq A_{s_0}$ for all $s \in V$. By the theory of Plane Algebraic Curves, $P_s$, for $s \in V$, must have a factor in common with $P_{s_0}$ and since such a factor cannot be $X + A_s$ it is that one of degree 3. Since the polynomial of degree 3 has a monic coefficient we obtain that

$$E_s X^3 + (F_s - A_s E_s)X^2 + B_s XY + (G_s - A_s(F_s - A_s E_s))X + Y^2 + (C_s - A_s B_s)Y - A_s(G_s - A_s(F_s - A_s E_s))$$

is constant in the variable $s$, that is, the polynomials are all equal in $V$, and therefore, their coefficients are constant in $V$. Therefore, $B_s, E_s$ are constant in $V$,a contradiction with claim 1. Therefore, $A_s$ is constant in $J$ or the curve $((v'(t))^2, v''(t))$ is constant. If $((v'(t))^2, v''(t))$ is constant there exist $\mu, \eta$ such that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. If $A_s$ is constant in $J$ we obtain that if $(v'(t))^2 + A_s = 0$ for all $t \in I$ then $v'(t)$ is constant, that is, $v(t)$ is a straight line. Suppose there exists $t_0 \in J$ such that $(v'(t_0))^2 + A_s \neq 0$. Then there exists a maximal interval $(\alpha, \beta) \subset I$ containing $t_0$ such that $(v'(t))^2 + A_s \neq 0$ for all $t \in (\alpha, \beta)$. Since the curve $((v'(t))^2, v''(t))$ vanishes the family of polynomials $P_s$ we conclude that, its restriction to the interval $(\alpha, \beta)$, vanishes

$$E_s X^3 + (F_s - A_s E_s) X^2 + B_s XY + (G_s - A_s(F_s - A_s E_s))X + Y^2 + (C_s - A_s B_s)Y - A_s(G_s - A_s(F_s - A_s E_s))$$

which, by claim 3, are irreducible and have a monic coefficient. Since $B_s, E_s$ cannot, by claim 1, be mutually constant we conclude by the theory of Plane Algebraic Curves that the curve $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta)$. Therefore, $(v'(t))^2$ is constant and different from $-A_s$ by the definition of $(\alpha, \beta)$. Finally, note that $\alpha \notin I$ because otherwise we would have $(v'(\alpha))^2 = -A_s$, a contradiction with the continuity of $v'(t)$. In the same way, $\beta \notin I$, and being $(\alpha, \beta)$ maximal we conclude that $(\alpha, \beta) = I$, that is, $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta) = I$ and therefore $v(t)$ is a line in $I$. $\square$

**Theorem 2.1.6.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, of Linear Weingarten Form with constant $c \neq 0$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover:*

*i) If $\mu \neq 0$ e $\frac{c}{b} \in \left(-\infty, -\frac{2b^2+a^2}{4b^2}\right] \bigcup \left[-\frac{a^2}{4b^2}, +\infty\right)$ then,*

$$u''(s) = \frac{2bs^2((u'(s))^2 + \mu^2) + 2c(s^2(u'(s))^2 + s^2\mu^2 + 1)^2 + as^2 u'(s)\sqrt{s^2(u'(s))^2 + s^2\mu^2 + 1}}{as^2(s^2\mu^2 + 1)\sqrt{s^2(u'(s))^2 + s^2\mu^2 + 1} - 2bs^3 u'(s)},$$

*in the interval $J$.*

*ii) If $\mu \neq 0$ and $\frac{c}{b} \in \left(-\frac{2b^2+a^2}{4b^2}, -\frac{a^2}{4b^2}\right)$ then,*

$$u''(s) = \frac{2bs^2((u'(s))^2 + \mu^2) + 2c(s^2(u'(s))^2 + s^2\mu^2 + 1)^2 + as^2 u'(s)\sqrt{s^2(u'(s))^2 + s^2\mu^2 + 1}}{as^2(s^2\mu^2 + 1)\sqrt{s^2(u'(s))^2 + s^2\mu^2 + 1} - 2bs^3 u'(s)},$$

*in the interval $J - \left\{\sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2(4bc+a^2)}{\mu^2(4bc+a^2+2b^2)}}}\right\}$.*

*iii) If $\mu = 0$ e $c \neq -\frac{a^2}{4b}$ then,*

$$u''(s) = \frac{2bs^2(u'(s))^2 + 2c(s^2(u'(s))^2 + 1)^2 + as^3(u'(s))^3\sqrt{s^2(u'(s))^2 + 1}}{as^2\sqrt{s^2(u'(s))^2 + 1} - 2bs^3 u'(s)},$$

*in the interval $J$.*

*iv) If $\mu = 0$, $c = -\frac{a^2}{4b}$ and $\frac{a}{b} \in (-\infty, -2] \bigcup [2, +\infty)$ then,*

$$u''(s) = \frac{4b^2s^2(u'(s))^2 - a^2(s^2(u'(s))^2+1)^2 + 2abs^3(u'(s))^3\sqrt{s^2(u'(s))^2+1}}{2abs^2\sqrt{s^2(u'(s))^2+1} - 4b^2s^3u'(s)},$$

*in the interval $J$.*

*v) If $\mu = 0$, $c = -\frac{a^2}{4b}$ and $\frac{a}{b} \in (-2,2)$ then $u(s) = D \pm \frac{a\ln(s)}{\sqrt{4b^2-a^2}}$ in $J$, being $D$ a constant, or*

$$u''(s) = -\frac{2bsu'(s) + a(1+s^2(u'(s))^2)^{\frac{3}{2}}}{2bs^2}$$

*in an open set $V \subset J$.*

**Proof:** Suppose that $\psi$ is of Linear Weingarten Form. By Proposition 2.1.14 we have $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Then, by Proposition 2.1.13 (assuming without loss of generality $b = 1$),

$$c = a\left(\frac{-s((u'(s))^2+\mu^2)u'(s) + (1+s^2\mu^2)u''(s)}{2s\left(\mu^2+(u'(s))^2+\frac{1}{s^2}\right)^{\frac{3}{2}}}\right) + \frac{-(su''(s)+u'(s))u'(s)-\mu^2}{s^2\left(\mu^2+(u'(s))^2+\frac{1}{s^2}\right)^2},$$

that is,

$$c = \frac{as^2(-s((u'(s))^2+\mu^2)u'(s) + (1+s^2\mu^2)u''(s))}{2\left(s^2\mu^2+s^2(u'(s))^2+1\right)^{\frac{3}{2}}} + \frac{2s^2(-(su''(s)+u'(s))u'(s)-\mu^2)}{2\left(s^2\mu^2+s^2(u'(s))^2+1\right)^2},$$

i.e.,

$$\begin{aligned} 0 =& as^2\sqrt{s^2\mu^2+s^2(u'(s))^2+1}(-s((u'(s))^2+\mu^2)u'(s) + (1+s^2\mu^2)u''(s)) \\ &+ 2s^2(-(su''(s)+u'(s))u'(s)-\mu^2) - 2c\left(s^2\mu^2+s^2(u'(s))^2+1\right)^2, \end{aligned}$$

then,

$$\begin{aligned} 0 =& \left(as^2(s^2\mu^2+1)\sqrt{s^2(u'(s))^2+s^2\mu^2+1} - 2s^3u'(s)\right)u''(s) \\ &- \left(2s^2((u'(s))^2+\mu^2) + 2c(s^2(u'(s))^2+s^2\mu^2+1)^2 + as^2u'(s)\sqrt{s^2(u'(s))^2+s^2\mu^2+1}\right). \end{aligned} \tag{2.34}$$

**-** *i)* Let $\mu \neq 0$ and $c \in \left(-\infty, -\frac{2+a^2}{4}\right] \cup \left[-\frac{a^2}{4}, +\infty\right)$. We have from (2.34) that

$$as^2(s^2\mu^2+1)\sqrt{s^2(u'(s))^2+s^2\mu^2+1} - 2s^3u'(s) \neq 0$$

in the interval $J$. Indeed, if there exists $s_0 \in J$ such that $as_0^2(s_0^2\mu^2+1)\sqrt{s_0^2(u'(s_0))^2+s_0^2\mu^2+1} - 2s_0^3u'(s_0) = 0$ we obtain, applying $s_0$ to (2.34)

$$\begin{aligned} 0 =& as_0^2(s_0^2\mu^2+1)\sqrt{s_0^2(u'(s_0))^2+s_0^2\mu^2+1} - 2s_0^3u'(s_0) \\ 0 =& 2s_0^2((u'(s_0))^2+\mu^2) + 2c(s_0^2(u'(s_0))^2+s_0^2\mu^2+1)^2 + as_0^2u'(s_0)\sqrt{s_0^2(u'(s_0))^2+s_0^2\mu^2+1}. \end{aligned}$$

Isolating the root in the first equality and substituting it in the second we obtain

$$\begin{aligned}
&\sqrt{s_0^2(u'(s_0))^2+s_0^2\mu^2+1}=\frac{2s_0u'(s_0)}{a(s_0^2\mu^2+1)}\\
&0=2s_0^2\frac{(a^4\mu^6s_0^8+3a^4\mu^4s_0^6+3a^4\mu^2s_0^4+(a^4+16c)s_0^2)(u'(s_0))^4+a^4(2s_0^2\mu^2+1)(s_0^2\mu^2+1)^3(u'(s_0))^2}{a^4(s_0^2\mu^2+1)^4}\\
&\quad+2s_0^2\frac{a^4\mu^2(s_0^2\mu^2+1)^4}{a^4(s_0^2\mu^2+1)^4}
\end{aligned}$$

that is,

$$\begin{aligned}
&s_0^2(u'(s_0))^2+s_0^2\mu^2+1=\left(\frac{2s_0u'(s_0)}{a(s_0^2\mu^2+1)}\right)^2\\
&0=(a^4\mu^6s_0^8+3a^4\mu^4s_0^6+3a^4\mu^2s_0^4+(a^4+16c)s_0^2)(u'(s_0))^4+a^4(2s_0^2\mu^2+1)(s_0^2\mu^2+1)^3(u'(s_0))^2\\
&\quad+a^4\mu^2(s_0^2\mu^2+1)^4
\end{aligned}$$

i.e.,

$$\begin{aligned}
&0=s_0^2(a+as_0^2\mu^2-2)(a+as_0^2\mu^2+2)(u'(s_0))^2+a^2(s_0^2\mu^2+1)^3\\
&0=(a^4\mu^6s_0^8+3a^4\mu^4s_0^6+3a^4\mu^2s_0^4+(a^4+16c)s_0^2)(u'(s_0))^4+a^4(2s_0^2\mu^2+1)(s_0^2\mu^2+1)^3(u'(s_0))^2\\
&\quad+a^4\mu^2(s_0^2\mu^2+1)^4.
\end{aligned}$$

Observe, in the first equality, that $(a+as_0^2\mu^2-2)(a+as_0^2\mu^2+2)\neq 0$ (otherwise $a^2(s_0^2\mu^2+1)^3=0$, a contradiction) and so we can isolate $(u'(s_0))^2$ in the first equality and substitute it in the second concluding that

$$0=4a^2(s_0^2\mu^2+1)^4\frac{2\mu^2(4c+a^2+2)s_0^4+2\mu^2(4c+a^2+2)s_0^2+a^2+4c}{s_0^2(a+as_0^2\mu^2-2)(a+as_0^2\mu^2+2)}$$

that is,

$$0=2\mu^2(4c+a^2+2)s_0^4+2\mu^2(4c+a^2+2)s_0^2+a^2+4c.$$

i.e.,

$$0=(4c+a^2+2)s_0^4+(4c+a^2+2)s_0^2+\frac{a^2+4c}{2\mu^2}. \tag{2.35}$$

Define $M(c,a)=4c+a^2$. We have,

$$0=(M(c,a)+2)s_0^4+(M(c,a)+2)s_0^2+\frac{M(c,a)}{2\mu^2}.$$

Since $c\in\left(-\infty,-\frac{2+a^2}{4}\right]\cup\left[-\frac{a^2}{4},+\infty\right)$ then $M(c,a)\leqslant -2$ or $M(c,a)\geqslant 0$. If $M(c,a)=-2$ then $M(c,a)=0$, a contradiction. If $M(c,a)<-2$ or $M(c,a)\geqslant 0$ we obtain

$$0=s_0^4+s_0^2+\frac{M(c,a)}{2\mu^2(M(c,a)+2)}>0,$$

a contradiction. Therefore, from (2.34), we conclude

$$u''(s) = \frac{2s^2((u'(s))^2+\mu^2)+2c(s^2(u'(s))^2+s^2\mu^2+1)^2+as^2u'(s)\sqrt{s^2(u'(s))^2+s^2\mu^2+1}}{as^2(s^2\mu^2+1)\sqrt{s^2(u'(s))^2+s^2\mu^2+1}-2s^3u'(s)}$$

in $J$.

**-** *ii*) Let $\mu \neq 0$ and $c \in \left(-\frac{2+a^2}{4}, -\frac{a^2}{4}\right)$. We have from (2.34) that

$$as^2(s^2\mu^2+1)\sqrt{s^2(u'(s))^2+s^2\mu^2+1}-2s^3u'(s) \neq 0$$

in $J - \left\{\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2(4c+a^2)}{\mu^2(4c+a^2+2)}}}\right\}$. Indeed, if there exists

$$s_0 \in J - \left\{\sqrt{-\frac{1}{2}\pm\frac{1}{2}\sqrt{1-\frac{2(4c+a^2)}{\mu^2(4c+a^2+2)}}}\right\}$$

such that

$$as_0^2(s_0^2\mu^2+1)\sqrt{s_0^2(u'(s_0))^2+s_0^2\mu^2+1}-2s_0^3u'(s_0) = 0$$

we obtain, as in (2.35),

$$0 = (4c+a^2+2)s_0^4+(4c+a^2+2)s_0^2+\frac{a^2+4c}{2\mu^2}.$$

Define $M(c,a) = 4c+a^2$. Therefore, since $c \in \left(-\frac{2+a^2}{4}, -\frac{a^2}{4}\right)$, we have $-2 < M(c,a) < 0$. Observe that $s_0^2$ is a root of the polynomial

$$Q(Z) = (M(c,a)+2)Z^2+(M(c,a)+2)Z+\frac{M(c,a)}{2\mu^2}$$

whose $\Delta > 0$ and the roots are $-\dfrac{1}{2}\pm\dfrac{1}{2}\sqrt{1-\dfrac{2M(c,a)}{\mu^2(M(c,a)+2)}}$. Since $s_0 > 0$ we conclude that the only root is the positive one, then

$$s_0 = \sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2M(c,a)}{\mu^2(M(c,a)+2)}}}$$

that is,

$$s_0 = \sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2(4c+a^2)}{\mu^2(4c+a^2+2)}}}$$

a contradiction. Then, from (2.34), we conclude

$$u''(s) = \frac{2s^2((u'(s))^2+\mu^2)+2c(s^2(u'(s))^2+s^2\mu^2+1)^2+as^2u'(s)\sqrt{s^2(u'(s))^2+s^2\mu^2+1}}{as^2(s^2\mu^2+1)\sqrt{s^2(u'(s))^2+s^2\mu^2+1}-2s^3u'(s)}$$

in $J-\left\{\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{2(4c+a^2)}{\mu^2(4c+a^2+2)}}}\right\}$.

**-** *iii*) Let $\mu=0$ and $c\neq -\frac{a^2}{4}$. We have from (2.34) that $as^2\sqrt{s^2(u'(s))^2+1}-2s^3u'(s)\neq 0$ in $J$. Indeed, if there exists $s_0\in J$ such that $as_0^2\sqrt{s_0^2(u'(s_0))^2+1}-2s_0^3u'(s_0)=0$ we obtain, applying $s_0$ to (2.34)

$$\begin{aligned}0 =&as_0^2\sqrt{s_0^2(u'(s_0))^2+1}-2s_0^3u'(s_0)\\ 0 =&2s_0^2(u'(s_0))^2+2c(s_0^2(u'(s_0))^2+1)^2+as_0^3(u'(s_0))^3\sqrt{s_0^2(u'(s_0))^2+1}.\end{aligned}$$

Isolating the square root in the first equality and substituting it in the second

$$\begin{aligned}&as_0^2\sqrt{s_0^2(u'(s_0))^2+1}=2s_0^3u'(s_0)\\ &0=2((u'(s_0))^2s_0^2+1)(c+(1+c)(u'(s_0))^2s_0^2),\end{aligned}$$

and squaring the first equality and rewriting

$$\begin{aligned}&0=s_0^6(a^2-4)(u'(s_0))^2+a^2s_0^4\\ &0=2((u'(s_0))^2s_0^2+1)(c+(1+c)(u'(s_0))^2s_0^2),\end{aligned}$$

that is,

$$\begin{aligned}&0=s_0^2(a^2-4)(u'(s_0))^2+a^2\\ &0=c+(1+c)(u'(s_0))^2s_0^2,\end{aligned}\tag{2.36}$$

then multiplying the first equality by $1+c$ and the second equality by $a^2-4$

$$\begin{aligned}&0=s_0^2(1+c)(a^2-4)(u'(s_0))^2+a^2(1+c)\\ &0=c(a^2-4)+(1+c)(a^2-4)(u'(s_0))^2s_0^2,\end{aligned}$$

and subtracting the second from the first we obtain

$$0=a^2(1+c)-c(a^2-4)=4c+a^2\neq 0$$

a contradiction. Then, from (2.34), we conclude

$$u''(s)=\frac{2s^2(u'(s))^2+2c(s^2(u'(s))^2+1)^2+as^3(u'(s))^3\sqrt{s^2(u'(s))^2+1}}{as^2\sqrt{s^2(u'(s))^2+1}-2s^3u'(s)}$$

in $J$.

**-** *iv*) Let $\mu=0$, $c=-\frac{a^2}{4}$ and $a\in(-\infty,-2]\bigcup[2,+\infty)$. We have from (2.34) that $as^2\sqrt{s^2(u'(s))^2+1}-2s^3u'(s)\neq 0$ in $J$. Indeed, if there exists $s_0\in J$ such that $as_0^2\sqrt{s_0^2(u'(s_0))^2+1}-2s_0^3u'(s_0)=0$ we obtain, as in (2.36),

$$0=s_0^2(a^2-4)(u'(s_0))^2+a^2>0,$$

a contradiction. Then, from (2.34), we conclude

$$u''(s)=\frac{4s^2(u'(s))^2-a^2(s^2(u'(s))^2+1)^2+2as^3(u'(s))^3\sqrt{s^2(u'(s))^2+1}}{2as^2\sqrt{s^2(u'(s))^2+1}-4s^3u'(s)}$$

in $J$.

**-** $v)$ Let $\mu = 0$, $c = -\frac{a^2}{4}$ and $a \in (-2,2)$. We have from (2.34) that

$$
\begin{aligned}
0 &= s^2(a\sqrt{s^2(u'(s))^2+1} - 2su'(s))u''(s) \\
&\quad - \left(2s^2(u'(s))^2 - \frac{a^2(1+(u'(s))^2)^2}{2} + as^3(u'(s))^3\sqrt{1+s^2(u'(s))^2}\right).
\end{aligned}
\tag{2.37}
$$

It is easy to verify that $U(s) = D \pm \frac{a\ln(s)}{\sqrt{4-a^2}}$, with $D$ constant, is a solution of (2.37) in all $J$. If $u \not\equiv U$ then there exists $s_0 \in J$ such that $a\sqrt{s_0^2(u'(s_0))^2+1} - 2s_0u'(s_0) \neq 0$. Indeed, if $a\sqrt{s^2(u'(s))^2+1} - 2su'(s) = 0$ for all $s \in J$ we have

$$
\left(a\sqrt{s^2(u'(s))^2+1}\right)^2 = (2su'(s))^2
$$

i.e.,

$$
0 = s^2(a^2-4)(u'(s))^2 + a^2,
$$

that is,

$$
(u'(s))^2 = \frac{a^2}{s^2(4-a^2)}
$$

and since $u'(s) \neq 0$ (otherwise $a = 0$) we obtain two distinct differential equations defined by

$$
u'(s) = \pm\frac{a}{s\sqrt{4-a^2}}
$$

thus

$$
u(s) = D \pm \frac{a\ln(s)}{\sqrt{4-a^2}},
$$

a contradiction, since $u \not\equiv U$. Then, since $a\sqrt{s_0^2(u'(s_0))^2+1} - 2s_0u'(s_0) \neq 0$, there exists a neighborhood $V \subset J$ of $s_0$ such that $a\sqrt{s^2(u'(s))^2+1} - 2su'(s) \neq 0$ for all $s \in V \subset J$. Therefore, from (2.37),

$$
u''(s) = \frac{4s^2(u'(s))^2 - a^2(1+(u'(s))^2)^2 + 2as^3(u'(s))^3\sqrt{1+s^2(u'(s))^2}}{2s^2(a\sqrt{s^2(u'(s))^2+1} - 2su'(s))},
$$

in $V \subset J$. Equivalently, multiplying by $2s^2(a\sqrt{s^2(u'(s))^2+1} + 2su'(s))$ both the denominator and the numerator,

$$
u''(s) = -\frac{2su'(s) + a(1+s^2(u'(s))^2)^{\frac{3}{2}}}{2s^2}
$$

in the interval $V \subset J$. □

### 2.1.8 *Type I Translation Surfaces with Prescribed Length of Second Fundamental Form.*

By Proposition 2.1.1 we conclude that a **Type I Translation Surface** has **Prescribed Length of Second Fundamental Form** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.15.** *Let $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, $c \in C(J,(0,+\infty))$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, has Prescribed Length of the Second Fundamental Form $c(s)$, if, and only if,*

$$\begin{aligned} c(s) = & \frac{\left((1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)\right)^2}{s^2\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)^3} \\ & - \frac{2\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)\left(s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2\right)}{s^2\left((v'(t))^2+(u'(s))^2+\frac{1}{s^2}\right)^3}. \end{aligned}$$

**Proposition 2.1.16.** *Let $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, $c \in C(J,(0,+\infty))$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. Consider $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$ with Prescribed Length of Second Fundamental Form $c(s)$. Then $v(t) = \mu t + \tau$, for $\mu,\tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ has Prescribed Length of Second Fundamental Form $c(s) > 0$. By Proposition 2.1.15 the functions $u, v$ satisfy the equation

$$\begin{aligned} c(s) = & \frac{s^4\left((1+s^2(u'(s))^2)v''(t) - s((u'(s))^2+(v'(t))^2)u'(s) + (1+s^2(v'(t))^2)u''(s)\right)^2}{\left(s^2(v'(t))^2+s^2(u'(s))^2+1\right)^3} \\ & - \frac{2s^2\left(s^2(v'(t))^2+s^2(u'(s))^2+1\right)\left(s(su''(s)+u'(s))v''(t) - (su''(s)+u'(s))u'(s) - (v'(t))^2\right)}{\left(s^2(v'(t))^2+s^2(u'(s))^2+1\right)^3} \end{aligned} \tag{2.38}$$

or equivalently

$$\begin{aligned} 0 = & (v'(t))^6 - \frac{(s^2(u'(s))^2+1)^2}{c(s)s^2}(v''(t))^2 + \frac{2u'(s)(s^2(u'(s))^2+2-s^3u'(s)u''(s))}{c(s)s}v''(t)(v'(t))^2 \\ & + \frac{(3c(s)-s^2(u'(s))^2-s^4(u''(s))^2+3c(s)s^2(u'(s))^2+2c(s)s^3u'(s)u''(s)-2)}{c(s)s^2}(v'(t))^4 \\ & + \frac{2u'(s)(s^2(u'(s))^2+1)^2}{c(s)s^3}v''(t) + \frac{3c(s)-4s^2(u'(s))^2-2s^4(u''(s))^2-2s^4(u'(s))^2}{c(s)s^4}(v'(t))^2 \\ & + \frac{6c(s)s^2(u'(s))^2+3c(s)s^4(u'(s))^4+2s^5(u'(s))^3u''(s)-2}{c(s)s^4}(v'(t))^2 \\ & - \frac{(s^2(s^4(u'(s))^6+2s^2(u'(s))^4+s^2(u''(s))^2+2su'(s)u''(s)+2(u'(s))^2)-c(s)(s^2(u'(s))^2+1)^3)}{c(s)s^6}. \end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = X^3 + B_sY^2 + C_sXY + D_sX^2 + E_sY + F_sX + G_s,$$

where,

$$
\begin{aligned}
B_s &= -\frac{(s^2(u'(s))^2+1)^2}{c(s)s^2} \\
C_s &= \frac{2u'(s)(s^2(u'(s))^2+2-s^3u'(s)u''(s))}{c(s)s} \\
D_s &= \frac{(3c(s)-s^2(u'(s))^2-s^4(u''(s))^2+3c(s)s^2(u'(s))^2+2c(s)s^3u'(s)u''(s)-2)}{c(s)s^2} \\
E_s &= \frac{2u'(s)(s^2(u'(s))^2+1)^2}{c(s)s^3} \\
F_s &= \frac{3c(s)-4s^2(u'(s))^2-2s^4(u''(s))^2-2s^4(u'(s))^2}{c(s)s^4} \\
&\quad + \frac{6c(s)s^2(u'(s))^2+3c(s)s^4(u'(s))^4+2s^5(u'(s))^3u''(s)-2}{c(s)s^4} \\
G_s &= -\frac{(s^2(s^4(u'(s))^6+2s^2(u'(s))^4+s^2(u''(s))^2+2su'(s)u''(s)+2(u'(s))^2)-c(s)(s^2(u'(s))^2+1)^3)}{c(s)s^6},
\end{aligned}
$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible then there exist polynomials of degree 2 and degree 1 such that

$$P_s(X,Y) = (X^2 + fXY + gY^2 + hX + iY + j)(X + kY + l)$$

that is,

$$0 = X^3 + B_sY^2 + C_sXY + D_sX^2 + E_sY + F_sX + G_s - (X^2 + fXY + gY^2 + hX + iY + j)(X + kY + l)$$

and reorganizing

$$
\begin{aligned}
0 =& (-f-k)X^2Y + (D_s - l - h)X^2 + (-g - fk)XY^2 + (C_s - i - fl - hk)XY + (F_s - j - hl)X \\
&+ (-gk)Y^3 + (B_s - ki - gl)Y^2 + (E_s - li - jk)Y + G_s - jl
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{array}{rrr}
-f-k=0 & C_s - i - fl - hk = 0 & B_s - ki - gl = 0 \\
D_s - l - h = 0 & F_s - j - hl = 0 & E_s - li - jk = 0 \\
-g - fk = 0 & -gk = 0 & G_s - jl = 0.
\end{array}
$$

By the first, third and sixth equalities we obtain that $f = g = k = 0$. Consequently by the seventh equality, $B_s = 0$, a contradiction since $B_s < 0$ and thus we proof the claim.

By the claim the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = X^3 + B_sY^2 + C_sXY + D_sX^2 + E_sY + F_sX + G_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2, K_3, K_4, K_5, K_6$ constants such that,

$$\begin{aligned} B_s &= K_1 \\ C_s &= K_2 \\ D_s &= K_3 \\ E_s &= K_4 \\ F_s &= K_5 \\ G_s &= K_6, \end{aligned}$$

that is,

$$\begin{aligned}
K_1 &= -\frac{(s^2(u'(s))^2+1)^2}{c(s)s^2} \\
K_2 &= \frac{2u'(s)(s^2(u'(s))^2+2-s^3u'(s)u''(s))}{c(s)s} \\
K_3 &= \frac{(3c(s)-s^2(u'(s))^2-s^4(u''(s))^2+3c(s)s^2(u'(s))^2+2c(s)s^3u'(s)u''(s)-2)}{c(s)s^2} \\
K_4 &= \frac{2u'(s)(s^2(u'(s))^2+1)^2}{c(s)s^3} \\
K_5 &= \frac{3c(s)-4s^2(u'(s))^2-2s^4(u''(s))^2-2s^4(u'(s))^2}{c(s)s^4} \\
&\quad + \frac{6c(s)s^2(u'(s))^2+3c(s)s^4(u'(s))^4+2s^5(u'(s))^3u''(s)-2}{c(s)s^4} \\
K_6 &= -\frac{(s^2(s^4(u'(s))^6+2s^2(u'(s))^4+s^2(u''(s))^2+2su'(s)u''(s)+2(u'(s))^2)-c(s)(s^2(u'(s))^2+1)^3)}{c(s)s^6}.
\end{aligned}$$

Defining $w(s) = u'(s)$ we obtain

$$\begin{aligned}
K_1 &= -\frac{(s^2w^2(s)+1)^2}{c(s)s^2} \\
K_2 &= \frac{2w(s)(s^2w^2(s)+2-s^3w(s)w'(s))}{c(s)s} \\
K_3 &= \frac{(3c(s)-s^2w^2(s)-s^4(w'(s))^2+3c(s)s^2w^2(s)+2c(s)s^3w(s)w'(s)-2)}{c(s)s^2} \\
K_4 &= \frac{2w(s)(s^2w^2(s)+1)^2}{c(s)s^3} \\
K_5 &= \frac{(3c(s)-4s^2w^2(s)-2s^4(w'(s))^2-2s^4w^2(s)+6c(s)s^2w^2(s)+3c(s)s^4w^4(s)+2s^5w^3(s)w'(s)-2)}{c(s)s^4} \\
K_6 &= -\frac{(s^2(s^4w^6(s)+2s^2w^4(s)+s^2(w'(s))^2+2sw(s)w'(s)+2w^2(s))-c(s)(s^2w^2(s)+1)^3)}{c(s)s^6}.
\end{aligned}$$

Since $K_1 \neq 0$ we have, by the first and fourth equalities,

$$w(s) = -\frac{K_4}{2K_1}s,$$

therefore

$$K_1 = -\frac{\left(s^2\left(-\frac{K_4}{2K_1}s\right)^2+1\right)^2}{c(s)s^2} = -\frac{\left(\frac{K_4^4}{16K_1^4}s^8+\frac{K_4^2}{2K_1^2}s^4+1\right)}{c(s)s^2}$$

then

$$c(s) = -\frac{(s^4K_4^2+4K_1^2)^2}{16K_1^5s^2}.$$

that substituting $w(s)$ and $c(s)$ in the second equality we obtain

$$0 = \frac{16K_1^4K_2 - 32s^2K_1^4K_4 + s^8K_2K_4^4 + 8s^4K_1^2K_2K_4^2}{16sK_1^5}$$

i.e.

$$0 = 16K_1^4K_2 - 32s^2K_1^4K_4 + s^8K_2K_4^4 + 8s^4K_1^2K_2K_4^2.$$

Notice that, by the above equality, $s^2$ is contained in the set of roots of the polynomial

$$Q(Z) = 16K_1^4K_2 - 32K_1^4K_4Z + K_2K_4^4Z^4 + 8K_1^2K_2K_4^2Z^2$$

that is, $s^2$ is constant or $Q$ is zero. Since $s$ is not constant we obtain that all the coefficients of $Q$ are zero. Since $K_1 \neq 0$ we conclude that $K_4 = 0$. Then,

$$u'(s) = w(s) = -\frac{K_4}{2K_1}s = 0$$

and

$$c(s) = -\frac{1}{K_1s^2}$$

which substituting in (2.38)

$$0 = s^4\left(-\frac{1}{K_1+2}\right)(v'(t))^3 + s^2\left(-\frac{3}{K_1+2}\right)(v'(t))^2 + -\frac{3}{K_1}v'(t) + s^4v''(t) - \frac{1}{K_1s^2}$$

and multiplying by $s^2$

$$0 = s^6\left(-\frac{1}{K_1+2}\right)(v'(t))^3 + s^4\left(-\frac{3}{K_1+2}\right)(v'(t))^2 + -s^2\frac{3}{K_1}v'(t) + s^6v''(t) - \frac{1}{K_1}$$

and rewriting,

$$0 = \left(v''(t) - (v'(t))^3\left(\frac{1}{K_1}-2\right)\right)s^6 - \left((v'(t))^2\left(\frac{3}{K_1}-2\right)\right)s^4 + \left(\frac{3v'(t)}{K_1}\right)s^2 - \frac{1}{K_1}.$$

Observe that, by the above equality, $s^2$ is contained in the set of roots of the non-zero polynomial

$$R(Z) = \left(v''(t) - (v'(t))^3\left(\frac{1}{K_1}-2\right)\right)Z^3 - \left((v'(t))^2\left(\frac{3}{K_1}-2\right)\right)Z^2 + \left(\frac{3v'(t)}{K_1}\right)Z - \frac{1}{K_1}$$

that is, $s^2$ is constant, a contradiction. Then, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist constants $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. $\square$

**Theorem 2.1.7.** *Let $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, $c \in C(J,(0,+\infty))$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (t,s,v(t)+u(s))$ with Prescribed Length of Second Fundamental Form $c(s)$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover, if $c(s) \geqslant \frac{2s^2\mu^2+1}{(s^2\mu^2+1)^2}$ we have*

$$u''(s) = \frac{su'(s)(s^4\mu^2(u'(s))^2 + s^4\mu^4 - 1) + (1+s^2\mu^2+s^2(u'(s))^2)\sqrt{\tilde{\Delta}(\mu^2,s^2,c(s),(u'(s))^2)}}{s^2(1+s^2\mu^2)^2},$$

*or*

$$u''(s) = \frac{su'(s)(s^4\mu^2(u'(s))^2 + s^4\mu^4 - 1) - (1+s^2\mu^2+s^2(u'(s))^2)\sqrt{\tilde{\Delta}(\mu^2,s^2,c(s),(u'(s))^2)}}{s^2(1+s^2\mu^2)^2},$$

*where*

$$\tilde{\Delta}(x,y,z,w) = (x^2y^3z + 2(z-1)xy^2 + (z-1)y)w + x^3y^3z + (3z-2)x^2y^2 + (3z-2)xy + z.$$

**Proof:** Suppose $\psi$ has Prescribed Length of Second Fundamental Form $c(s)$. By Proposition 2.1.16 we have $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Then, by Proposition 2.1.15

$$\begin{aligned} c(s) = {} & \frac{s^4\left(-s((u'(s))^2+\mu^2)u'(s) + (1+s^2\mu^2)u''(s)\right)^2}{(s^2\mu^2+s^2(u'(s))^2+1)^3} \\ & - \frac{2s^2\left(s^2\mu^2+s^2(u'(s))^2+1\right)\left(-(su''(s)+u'(s))u'(s)-\mu^2\right)}{(s^2\mu^2+s^2(u'(s))^2+1)^3}, \end{aligned}$$

i.e.,

$$\begin{aligned} 0 = {} & s^4(1+s^2\mu^2)^2(u''(s))^2 - 2s^3u'(s)(s^4\mu^2(u'(s))^2+s^4\mu^4-1)u''(s) + s^6(u'(s))^2((u'(s))^2+\mu^2)^2 \\ & - c(s)(s^2(u'(s))^2+s^2\mu^2+1)^3 + 2s^2((u'(s))^2+\mu^2)(s^2(u'(s))^2+s^2\mu^2+1). \end{aligned} \tag{2.39}$$

Define

$$P_s(Z) = A_sZ^2 + B_sZ + C_s$$

where

$$\begin{aligned} A_s &= s^4(1+s^2\mu^2)^2 \\ B_s &= -2s^3u'(s)(s^4\mu^2(u'(s))^2+s^4\mu^4-1) \\ C_s &= s^6(u'(s))^2((u'(s))^2+\mu^2)^2 - c(s)(s^2(u'(s))^2+s^2\mu^2+1)^3 \\ &\quad + 2s^2((u'(s))^2+\mu^2)(s^2(u'(s))^2+s^2\mu^2+1). \end{aligned}$$

Observe that $P_s(u''(s)) = 0$ and furthermore, notice that the discriminant $\Delta(s) > 0$. Indeed, since

$c(s) \geqslant \frac{2s^2\mu^2+1}{(s^2\mu^2+1)^2}$, we have

$$\begin{aligned}
\Delta(s) &= B_s^2 - 4A_sC_s \\
&= 4s^4(1+s^2\mu^2+s^2(u'(s))^2)^2(s^2\mu^2+1)^2(s^2(u'(s))^2+s^2\mu^2+1)c(s) \\
&+ 4s^4(1+s^2\mu^2+s^2(u'(s))^2)^2\Big(-s^2(2s^2\mu^2+1)(u'(s))^2+2s^2\mu^2(s^2\mu^2+1)\Big) \\
&\geqslant 4s^4(1+s^2\mu^2+s^2(u'(s))^2)^2(s^2\mu^2+1)^2(s^2(u'(s))^2+s^2\mu^2+1)\frac{2s^2\mu^2+1}{(s^2\mu^2+1)^2} \\
&+ 4s^4(1+s^2\mu^2+s^2(u'(s))^2)^2\Big(-(s^2(2s^2\mu^2+1)(u'(s))^2+2s^2\mu^2(s^2\mu^2+1)\Big) \\
&= 4s^4(1+s^2\mu^2+s^2(u'(s))^2)^2(s^2\mu^2+1) > 0.
\end{aligned}$$

Then, using the formula for the roots of the second-degree equation we obtain, from (2.39), the differential equations

$$u''(s) = \frac{su'(s)(s^4\mu^2(u'(s))^2+s^4\mu^4-1)+(1+s^2\mu^2+s^2(u'(s))^2)\sqrt{\tilde{\Delta}(\mu^2,s^2,c(s),(u'(s))^2)}}{s^2(1+s^2\mu^2)^2},$$

or

$$u''(s) = \frac{su'(s)(s^4\mu^2(u'(s))^2+s^4\mu^4-1)-(1+s^2\mu^2+s^2(u'(s))^2)\sqrt{\tilde{\Delta}(\mu^2,s^2,c(s),(u'(s))^2)}}{s^2(1+s^2\mu^2)^2},$$

where

$$\tilde{\Delta}(x,y,z,w) = (x^2y^3z+2(z-1)xy^2+(z-1)y)w+x^3y^3z+(3z-2)x^2y^2+(3z-2)xy+z.$$

□

### 2.1.9 *Curvatures of Type II Translation Surfaces*

This section will be devoted to determining the curvatures of a Type II Translation Surface in $H^2 \times \mathbb{R}$. Define the **Type II Translation Surface** $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ by $\psi(t,s) = (v(t)+u(s),s,t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$. We have,

$$\begin{aligned}
\frac{\partial \psi}{\partial t}(t,s) &= \big(v'(t),0,1\big) \\
&= \frac{v'(t)}{\varphi \circ \psi(t,s)} E_1 \circ \psi(t,s) + E_3 \circ \psi(t,s),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \psi}{\partial s}(t,s) &= \big(u'(s),1,0\big) \\
&= \frac{u'(s)}{\varphi \circ \psi(t,s)} E_1 \circ \psi(t,s) + \frac{1}{\varphi \circ \psi(t,s)} E_2 \circ \psi(t,s).
\end{aligned}$$

The normal $N$ at $\psi(t,s)$ will be given by

$$N(t,s) = \frac{1}{\omega(t,s)} E_1 \circ \psi(t,s) - \frac{u'(s)}{\omega(t,s)} E_2 \circ \psi(t,s) - \frac{v'(t)}{\omega(t,s)\varphi \circ \psi(t,s)} E_3 \circ \psi(t,s),$$

where $\omega(t,s)=\sqrt{\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)^2+(u'(s))^2+1}$.

By Lemma 1.1.2 we conclude that to know the Weingarten map $\mathcal{S}$ we do not need to differentiate the normal field, we just need to know $\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}$, $\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}$ and $\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}$. We have,

**Lemma 2.1.7.** *The following holds:*

$$\textit{\textbf{i)}}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=\frac{\partial}{\partial t}\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)+\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)^2E_2\circ\psi(t,s).$$
$$\textit{\textbf{ii)}}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=-\frac{v'(t)}{(\varphi\circ\psi)^2(t,s)}E_1\circ\psi(t,s)+\frac{u'(s)v'(t)}{(\varphi\circ\psi)^2(t,s)}E_2\circ\psi(t,s).$$
$$\begin{aligned}\textit{\textbf{iii)}}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s)=&\left(\frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)-\frac{u'(s)}{(\varphi\circ\psi)^2(t,s)}\right)E_1\circ\psi(t,s)\\&+\left(\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)+\frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)}\right)E_2\circ\psi(t,s).\end{aligned}$$

**Proof:**

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=&\frac{v'(t)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_1\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\&+\left(\nabla_{\overline{e_1}}E_3\circ\psi\right)(t,s)\\=&\frac{v'(t)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1+\frac{\partial}{\partial t}\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\&+\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1=&\frac{v'(t)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_1\right)\circ\psi(t,s)+\left(\nabla^{\times}_{E_3}E_1\right)\circ\psi(t,s)\\=&\frac{v'(t)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)\end{aligned}$$

and $\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3=0$, therefore,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=\frac{\partial}{\partial t}\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)+\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)^2E_2\circ\psi(t,s).$$

Also,

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=&\frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_1\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\&+\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_2\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\=&\frac{u'(s)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1+\frac{\partial}{\partial t}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\&+\frac{1}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2+\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)} E_2 =& \frac{v'(t)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_2\right)\circ\psi(t,s) + \left(\nabla^{\times}_{E_3}E_2\right)\circ\psi(t,s)\\ =& -\frac{v'(t)}{\varphi\circ\psi(t,s)}E_1\circ\psi(t,s)\end{aligned}$$

consequently,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s) = -\frac{v'(t)}{(\varphi\circ\psi)^2(t,s)}E_1\circ\psi(t,s) + \frac{u'(s)v'(t)}{(\varphi\circ\psi)^2(t,s)}E_2\circ\psi(t,s).$$

Finally,

$$\begin{aligned}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s) =& \frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_2}}E_1\circ\psi\right)(t,s) + \overline{e_2}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\ &+\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_2}}E_2\circ\psi\right)(t,s) + \overline{e_2}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\ =&\frac{u'(s)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_1 + \frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\ &+\frac{1}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2 + \frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)} E_1 =& \frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_1\right)\circ\psi(t,s) + \frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_1\right)\circ\psi(t,s)\\ =& \frac{u'(s)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)\end{aligned}$$

and

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)} E_2 =& \frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_2\right)\circ\psi(t,s) + \frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_2\right)\circ\psi(t,s)\\ =& -\frac{u'(s)}{\varphi\circ\psi(t,s)}E_1\circ\psi(t,s)\end{aligned}$$

then,

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s) =& \left(\frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right) - \frac{u'(s)}{(\varphi\circ\psi)^2(t,s)}\right)E_1\circ\psi(t,s)\\ &+\left(\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right) + \frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)}\right)E_2\circ\psi(t,s).\end{aligned}$$

□

With this we have

**Lemma 2.1.8.** *The following holds:*

$$
\begin{aligned}
&\textbf{\textit{i)}}\mathcal{E}(t,s)=\frac{sv''(t)-u'(s)(v'(t))^2}{s^2\omega(t,s)}\\
&\textbf{\textit{ii)}}\mathcal{F}(t,s)=\frac{v'(t)(1+(u'(s))^2)}{s^2\omega(t,s)}.\\
&\textbf{\textit{iii)}}\mathcal{G}(t,s)=\frac{su''(s)-u'(s)-(u'(s))^3}{s^2\omega(t,s)}.\\
&\textbf{\textit{iv)}}\mathfrak{e}(t,s)=\frac{(v'(t))^2+s^2}{s^2}.\\
&\textbf{\textit{v)}}\mathfrak{f}(t,s)=\frac{u'(s)v'(t)}{s^2}.\\
&\textbf{\textit{vi)}}\mathfrak{g}(t,s)=\frac{1+(u'(s))^2}{s^2}.
\end{aligned}
$$

**Proof:**

$$
\begin{aligned}
\mathcal{E}(t,s)=&\left(N\bullet_\times\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)\\
=&\frac{1}{\omega(t,s)}\frac{\partial}{\partial t}\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)-\frac{u'(s)(v'(t))^2}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}\\
=&\frac{v''(t)}{\omega(t,s)\varphi\circ\psi(t,s)}-\frac{u'(s)(v'(t))^2}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}\\
=&\frac{sv''(t)-u'(s)(v'(t))^2}{s^2\omega(t,s)}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathcal{F}(t,s)=&\left(N\bullet_\times\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)\\
=&-\frac{v'(t)}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}-\frac{v'(t)(u'(s))^2}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}\\
=&-\frac{v'(t)(1+(u'(s))^2)}{s^2\omega(t,s)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathcal{G}(t,s)=&\left(N\bullet_\times\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s)\\
=&\frac{1}{\omega(t,s)}\left(\frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)-\frac{u'(s)}{(\varphi\circ\psi)^2(t,s)}\right)-\frac{u'(s)}{\omega(t,s)}\left(\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)+\frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)}\right)\\
=&\frac{su''(s)-u'(s)-(u'(s))^3}{s^2\omega(t,s)}.
\end{aligned}
$$

The metric coefficients are,

$$
\mathfrak{e}(t,s)=\left(\overline{e_1}\bullet_\times\overline{e_1}\right)(t,s)=\left(\frac{\partial\psi}{\partial t}\bullet_\times\frac{\partial\psi}{\partial t}\right)(t,s)=\frac{(v'(t))^2}{(\varphi\circ\psi)^2(t,s)}+1=\frac{(v'(t))^2+s^2}{s^2}.
$$

Also,

$$
\mathfrak{f}(t,s)=\left(\overline{e_1}\bullet_\times\overline{e_2}\right)(t,s)=\left(\frac{\partial\psi}{\partial t}\bullet_\times\frac{\partial\psi}{\partial s}\right)(t,s)=\frac{u'(s)v'(t)}{s^2}.
$$

Finally,

$$\mathfrak{g}(t,s) = (\overline{e_2} \bullet_\times \overline{e_2})(t,s) = \left(\frac{\partial \psi}{\partial s} \bullet_\times \frac{\partial \psi}{\partial s}\right)(t,s) = \frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)} + \frac{1}{(\varphi\circ\psi)^2(t,s)} = \frac{1+(u'(s))^2}{s^2}.$$

$\square$

**Proposition 2.1.17.** *The following holds:*

$$H(t,s) = \frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{2s^2\omega^3(t,s)} - \frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\omega^3(t,s)},$$

*and*

$$K(t,s) = \frac{(sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2}{s^2\omega^4(t,s)}.$$

*and*

$$\begin{aligned}|A|^2(t,s) = &\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{s^2\omega^3(t,s)}\right.\\ &\left.-\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{s^2\omega^3(t,s)}\right)^2\\ &-\frac{2s^2(u'(s))^2\left((sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2\right)}{s^4\omega^6(t,s)}\\ &-\frac{2\left(s^2+(v'(t))^2\right)\left((sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2\right)}{s^4\omega^6(t,s)}.\end{aligned}$$

**Proof:**

$$\begin{aligned}H(t,s) &= \frac{1}{2}\mathrm{tr}(\mathcal{S})(t,s)\\ &= \frac{1}{2}(a_{tt}+a_{ss})\\ &= \frac{1}{2}\left(\frac{\mathcal{E}\mathfrak{g}+\mathcal{G}\mathfrak{e}-2\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\\ &= \frac{1}{2}\left(\frac{\left(\frac{sv''(t)-u'(s)(v'(t))^2}{s^2\omega(t,s)}\right)\left(\frac{1+(u'(s))^2}{s^2}\right)+\left(\frac{su''(s)-u'(s)-(u'(s))^3}{s^2\omega(t,s)}\right)\left(\frac{(v'(t))^2+s^2}{s^2}\right)}{\left(\frac{(v'(t))^2+s^2}{s^2}\right)\left(\frac{1+(u'(s))^2}{s^2}\right)-\frac{(u'(s)v'(t))^2}{s^4}}\right)\\ &\quad -\frac{1}{2}\frac{\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{s^4\omega(t,s)}}{\left(\frac{(v'(t))^2+s^2}{s^2}\right)\left(\frac{1+(u'(s))^2}{s^2}\right)-\frac{(u'(s)v'(t))^2}{s^4}}\\ &= \frac{1}{2}\left(\frac{\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)-2u'(s)(v'(t))^2(1+(u'(s))^2)}{s^4\omega(t,s)}}{\frac{1}{s^2}\omega^2(t,s)}\right)\\ &= \frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{2s^2\omega^3(t,s)}\\ &\quad -\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\omega^3(t,s)}.\end{aligned}$$

Also,

$$\begin{aligned}
K(t,s) &= \det(\mathcal{S}) \\
&= a_{tt}a_{ss} - a_{ts}a_{st} \\
&= \left(\frac{\mathcal{E}\mathfrak{g} - \mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{G}\mathfrak{e} - \mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s) - \left(\frac{\mathcal{F}\mathfrak{g} - \mathcal{G}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{F}\mathfrak{e} - \mathcal{E}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G}\mathfrak{g}\mathfrak{e} - \mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f} - \mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e} + (\mathcal{F}\mathfrak{f})^2 - \left(\mathcal{F}^2\mathfrak{g}\mathfrak{e} - \mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f} - \mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e} + \mathcal{E}\mathcal{F}\mathfrak{f}^2\right)}{(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2)^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G}(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2) - \mathcal{F}^2(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2)}{(\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2)^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G} - \mathcal{F}^2}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s) \\
&= \frac{\left(\frac{sv''(t) - u'(s)(v'(t))^2}{s^2\omega(t,s)}\right)\left(\frac{su''(s) - u'(s) - (u'(s))^3}{s^2\omega(t,s)}\right) - \frac{(v'(t))^2(1+(u'(s))^2)^2}{s^4\omega^2(t,s)}}{\frac{1}{s^2}\omega^2(t,s)} \\
&= \frac{(sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1+(u'(s))^2))^2}{s^2\omega^4(t,s)}.
\end{aligned}$$

Finally, being $\lambda_1(t,s)$ and $\lambda_2(t,s)$ the principal curvatures we have,

$$\begin{aligned}
|A|^2(t,s) &= \lambda_1^2(t,s) + \lambda_2^2(t,s) \\
&= 4\frac{(\lambda_1 + \lambda_2)^2(t,s)}{4} - 2\lambda_1(t,s)\lambda_2(t,s) \\
&= 4H^2(t,s) - 2K(t,s) \\
&= \Bigg(\frac{(sv''(t) - u'(s)(v'(t))^2)(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)((v'(t))^2 + s^2)}{s^2\omega^3(t,s)} \\
&\quad - \frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{s^2\omega^3(t,s)}\Bigg)^2 \\
&\quad - \frac{2s^2(u'(s))^2\left((sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1+(u'(s))^2))^2\right)}{s^4\omega^6(t,s)} \\
&\quad - \frac{2\left(s^2 + (v'(t))^2\right)\left((sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1+(u'(s))^2))^2\right)}{s^4\omega^6(t,s)}.
\end{aligned}$$

□

### 2.1.10 *Type II Translation Surfaces - Flat case*

By Proposition 2.1.17 we conclude that a **Type II Translation Surface** is **Flat** if it satisfies a differential equation described in the following proposition

**Proposition 2.1.18.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, is Flat if and only if,*

$$0 = (sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1+(u'(s))^2))^2.$$

**Proposition 2.1.19.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, Flat. Then $v(t) = \mu t + \tau$, with $I \subset \mathbb{R}$, or $v(t) = \mu \ln|t+\nu| + \tau$, with $I \subset \mathbb{R} - \{\nu\}$, for $\mu, \nu, \tau$ constants.*

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.1.18, the functions $u, v$ satisfy the equation

$$0 = (sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1+(u'(s))^2))^2,$$

or equivalently

$$0 = ((u'(s))^2 + su'(s)u''(s) + 1)(v'(t))^2 + s((u'(s))^3 + u'(s) - su''(s))v''(t). \tag{2.40}$$

**Claim:** If $(u'(s))^2 + su'(s)u''(s) + 1 \equiv 0$ in $J$ then $v(t) = \mu t + \tau$ in $I$. Indeed, if

$$(u'(s))^2 + su'(s)u''(s) + 1 = 0$$

then

$$(u'(s))^2 + \frac{s}{2}\frac{d}{ds}[(u'(s))^2] + 1 = 0,$$

and defining $w(s) = (u'(s))^2$ we obtain

$$w(s) + \frac{s}{2}w'(s) + 1 = 0$$

whose solutions are $w(s) = \frac{\alpha}{s^2} - 1$ and therefore $(u'(s))^2 = \frac{\alpha}{s^2} - 1$. From (2.40) we have

$$\begin{aligned} 0 &= ((u'(s))^2 + su'(s)u''(s) + 1)(v'(t))^2 + s((u'(s))^3 + u'(s) - su''(s))v''(t) \\ &= s((u'(s))^3 + u'(s) - su''(s))v''(t) \\ &= \frac{\alpha^2}{s^3}v''(t), \end{aligned}$$

and since $\alpha \neq 0$ (otherwise we would have $(u'(s))^2 = -1$, a contradiction) we obtain $v(t) = \mu t + \tau$ in $I$ and thus proof the claim.

If $(u'(s))^2 + su'(s)u''(s) + 1 \equiv 0$ in $J$ then, by the claim, $v(t) = \mu t + \tau$ in $I$. If there exists an open set $V \subset J$ such that $(u'(s))^2 + su'(s)u''(s) + 1 \neq 0$ then from (2.40) we have

$$0 = (v'(t))^2 + \frac{s(u'(s) - su''(s) + (u'(s))^3)}{(u'(s))^2 + su'(s)u''(s) + 1}v''(t). \tag{2.41}$$

Defining the family of irreducible polynomials

$$P_s(X,Y) = X + \frac{s(u'(s) - su''(s) + (u'(s))^3)}{2((u'(s))^2 + su'(s)u''(s) + 1)} Y,$$

we observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in V}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = X + \frac{s(u'(s) - su''(s) + (u'(s))^3)}{2((u'(s))^2 + su'(s)u''(s) + 1)} Y,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there exists a constant $\beta$ such that,

$$\beta = \frac{s(u'(s) - su''(s) + (u'(s))^3)}{2((u'(s))^2 + su'(s)u''(s) + 1)}$$

e então de $(2.41)$

$$0 = (v'(t))^2 + \frac{s(u'(s) - su''(s) + (u'(s))^3)}{2((u'(s))^2 + su'(s)u''(s) + 1)} v''(t) = (v'(t))^2 + \beta v''(t),$$

whose solutions are $v(t) = \beta \ln|t + \beta\eta| + \tau$ if $\beta \neq 0$, and $v(t) = \tau$ if $\beta = 0$. Therefore, there exist constants $\mu$, $\nu$ and $\tau$ such that $v(t) = \mu \ln|t+\nu| + \tau$ for all $t \in I - \{\nu\}$. On the other hand, if the curve $((v'(t))^2, v''(t))$ is constant there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.1.8.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t) + u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, Flat. The possibilities for $u$ and $v$ are:*

*i)* *$v$ is constant and $u$ is arbitrary.*

*ii)* *$v(t) = \mu t + \tau$, for constant $\tau$ and $\mu \neq 0$, and $I \subset \mathbb{R}$, and*

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{C^2(F+1)}{r^2} - 1}\, dr$$

*for constants $D$, $C > 0$, $F \geqslant 0$ and $J \subset \left(0, C\sqrt{F+1}\right]$*

*iii)* *$v(t) = \mu \ln|t+\nu| + \tau$, for $\mu \neq 0$, $\nu, \tau$ constants and $I \subset \mathbb{R} - \{\nu\}$, and the function $u$ satisfies the differential equation*

$$u''(s) = \frac{(1 + (u'(s))^2)(su'(s) - \mu)}{s(s + \mu u'(s))}.$$

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.1.18, the functions $u, v$ satisfy the equation

$$0 = (sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1 + (u'(s))^2))^2. \tag{2.42}$$

By Proposition 2.1.19 it holds that $v(t) = \mu t + \tau$, with $I \subset \mathbb{R}$, or $v(t) = \mu \ln |t + \nu| + \tau$, with $I \subset \mathbb{R} - \{\nu\}$, for $\mu, \nu, \tau$ constants. In both cases if $\mu = 0$ we have that $v$ is constant. Since $v$ is constant in $I$ we conclude from (2.42) that $u$ is arbitrary, that is, we obtain *i*). Suppose that $\mu \neq 0$. In the first case, i.e., $v(t) = \mu t + \tau$, we obtain from (2.42)

$$0 = su'(s)u''(s) + (u'(s))^2 + 1.$$

Defining $w(s) = (u'(s))^2$ we obtain the separable differential equation

$$w'(s) = -\frac{2(w(s)+1)}{s}.$$

Since $w(s) \geqslant 0$ then $w(s) + 1 > 0$ and therefore

$$\int_{s_0}^{s} \frac{w'(r)}{w(r)+1} dr = -2 \int_{s_0}^{s} \frac{1}{r} dr,$$

and by the Change of Variables Theorem,

$$\int_{w(s_0)}^{w(s)} \frac{dl}{l+1} = -2 \int_{s_0}^{s} \frac{1}{r} dr.$$

Thus,

$$\ln\left(\frac{w(s)+1}{w(s_0)+1}\right) = \ln\left(\frac{s_0^2}{s^2}\right)$$

i.e.,

$$\frac{w(s)+1}{w(s_0)+1} = \frac{s_0^2}{s^2}$$

that is, for constants $C > 0$ and $F \geqslant 0$

$$w(s) = \frac{C^2(F+1)}{s^2} - 1$$

in the interval $\left(0, C\sqrt{F+1}\right]$. By the definition of $w$

$$u'(s) = \pm\sqrt{\frac{C^2(F+1)}{s^2} - 1}$$

i.e.,

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{C^2(F+1)}{r^2} - 1} dr.$$

In the second case, i.e., $v(t) = \mu \ln |t + \nu| + \tau$, we obtain from (2.42)

$$0 = s(s + \mu u'(s))u''(s) + (1 + (u'(s))^2)(\mu - su'(s)). \tag{2.43}$$

**Claim:** $s(s+\mu u'(s)) \neq 0$ in $J$. Indeed, if there exists $s_1 \in J$ such that $s_1(s_1+\mu u'(s_1)) = 0$ we have (2.43)

$$
\begin{aligned}
0 =& s_1^2 + \mu s_1 u'(s_1) \\
0 =& \mu - s_1 u'(s_1).
\end{aligned}
$$

Isolating $s_1 u'(s_1)$ in the second equality and substituting it in the first we conclude $s_1^2 + \mu^2 = 0$, a contradiction and thus we proof the claim.

By the claim we conclude that (2.43)

$$
u''(s) = \frac{(1+(u'(s))^2)(su'(s)-\mu)}{s(s+\mu u'(s))}.
$$

□

### 2.1.11 *Type II Translation Surfaces - Constant Gauss Curvature case*

By Proposition 2.1.17 we conclude that a **Type II Translation Surface** has **Constant Gauss curvature** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.20.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, has constant Gaussian Curvature $c$ , if, and only if,*

$$
c = \frac{(sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1+(u'(s))^2))^2}{s^2\left(1+(u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^2}.
$$

**Proposition 2.1.21.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, with constant Gaussian Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, with $I \subset \mathbb{R}$, for $\mu, \tau$ constants.*

**Proof:** Suppose $\psi$ has constant Gaussian curvature $c \neq 0$. By Proposition 2.1.20, the functions $u, v$ satisfy the equation

$$
c = \frac{(sv''(t) - u'(s)(v'(t))^2)(su''(s) - u'(s) - (u'(s))^3) - (v'(t)(1+(u'(s))^2))^2}{s^2\left(1+(u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^2},
$$

or equivalently

$$
\begin{aligned}
0 = (v'(t))^4 &+ \frac{s^2(2c + (u'(s))^2 + 2c(u'(s))^2 + su'(s)u''(s) + 1)}{c}(v'(t))^2 \\
&+ \frac{s^3(u'(s) - su''(s) + (u'(s))^3)}{c}v''(t) + s^4(1+(u'(s))^2)^2.
\end{aligned}
$$

Define the family of irreducible polynomials

$$P_s(X,Y) = X^2 + A_s X + B_s Y + C_s,$$

where

$$\begin{aligned}
A_s =& \frac{s^2(2c + (u'(s))^2 + 2c(u'(s))^2 + su'(s)u''(s) + 1)}{c} \\
B_s =& \frac{s^3(u'(s) - su''(s) + (u'(s))^3)}{c} \\
C_s =& s^4(1 + (u'(s))^2)^2.
\end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = X^2 + A_s X + B_s Y + C_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Then, their coefficients are constant, that is, there are $K_1, K_2, K_3$ constants such that,

$$\begin{aligned}
A_s &= K_1 \\
B_s &= K_2 \\
C_s &= K_3,
\end{aligned}$$

i.e.,

$$\begin{aligned}
K_1 &= \frac{s^2(2c + (u'(s))^2 + 2c(u'(s))^2 + su'(s)u''(s) + 1)}{c} \\
K_2 &= \frac{s^3(u'(s) - su''(s) + (u'(s))^3)}{c} \\
K_3 &= s^4(1 + (u'(s))^2)^2,
\end{aligned}$$

then, setting $w(s) = u'(s)$, we obtain

$$\begin{aligned}
K_1 &= \frac{s^2(2c + w^2(s) + 2cw^2(s) + sw(s)w'(s) + 1)}{c} \\
K_2 &= \frac{s^3(w(s) - sw'(s) + w^3(s))}{c} \\
K_3 &= s^4(1 + w^2(s))^2,
\end{aligned}$$

thus

$$\begin{aligned}
K_1 &= \frac{s^2(2c + w^2(s) + 2cw^2(s) + sw(s)w'(s) + 1)}{c} \\
sw'(s) &= w(s) + w^3(s) - \frac{cK_2}{s^3} \\
0 &= w^4(s)s^4 + 2w^2(s)s^4 + s^4 - K_3,
\end{aligned}$$

and substituting the second equality into the first

$$\begin{aligned} 0 &= s^3w^4(s) + (2c+2)s^3w^2(s) + (2c+1)s^3 - cK_1s - cK_2w(s) \\ 0 &= w^4(s)s^4 + 2w^2(s)s^4 + s^4 - K_3, \end{aligned}$$

and multiplying the first by $s$ and subtracting the second from it

$$\begin{aligned} 0 &= 2cw^2(s)s^4 - cK_2w(s)s + 2cs^4 - cK_1s^2 + K_3 \\ 0 &= w^4(s)s^4 + 2w^2(s)s^4 + s^4 - K_3. \end{aligned} \tag{2.44}$$

Define $G : (0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G} : (0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$G(x,y) = 2cy^2x - cK_2y + 2cx^2 - cK_1x + K_3$$

and

$$\overline{G}(x,y) = y^4 + 2y^2x + x^2 - K_3.$$

Notice that from (2.44) $G(s^2, sw(s)) = 0 = \overline{G}(s^2, sw(s))$. Since the curve $(s^2, sw(s))$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A$.0.15. Therefore, the curve $((v'(t))^2, v''(t))$ is constant and then there exist $\mu,\eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$). We conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Proposition 2.1.22.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2\times\mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J\times I \subset (0,+\infty)\times\mathbb{R}$, with constant Gaussian Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, with $I \subset \mathbb{R}$, for $\mu \neq 0$ and $\tau$ constants. Moreover,*

*i) If $c \in \left(-\infty, -\frac{1}{4}\right) \bigcup (0,+\infty)$ then $u'(s) \neq 0$ in $J$.*

*ii) If $c \in \left[-\frac{1}{4}, 0\right)$ e $u'(s) = 0$ then*

$$s = |\mu|\sqrt{\frac{-(1+2c)\pm\sqrt{4c+1}}{2c}}.$$

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.1.21 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. Therefore, by Proposition 2.1.20, the functions $u, v$ satisfy the equation

$$0 = \mu s u'(s)u''(s) + (cs^2+\mu^2-\mu)(u'(s))^4 + (2cs^2+2(c+1)\mu^2-\mu)(u'(s))^2 + cs^2\left(1+\frac{\mu^2}{s^2}\right)^2 + \mu^2. \tag{2.45}$$

Observe that $\mu \neq 0$. Indeed, if $\mu = 0$ we obtain(2.45)

$$0 = cs^2(u'(s))^4 + 2cs^2(u'(s))^2 + cs^2$$

i.e., $0 = (u'(s))^4 + 2(u'(s))^2 + 1$, a contradiction. Furthermore, suppose that there exists $x \in J \subset (0,+\infty)$ such that $u'(x) = 0$. We have (2.45)

$$0 = c(x^2)^2 + (2c+1)\mu^2x^2 + c\mu^4 = \frac{c}{\mu^4}(\mu^2x^2)^2 + (2c+1)\mu^2x^2 + c\mu^4. \tag{2.46}$$

If $c \in (0,+\infty)$ the right-hand side of this equality is strictly positive, so it does not hold, that is, it has no roots. Also, the discriminant of the equation (2.46) is, considering $\mu^2x^2$ as the variable,

$$\Delta = 4c+1$$

therefore, if $c \in \left(-\infty, -\frac{1}{4}\right)$ we have no real roots. Finally, if $c \in \left[-\frac{1}{4}, 0\right)$ we have at most two distinct roots. Thus, we obtain de (2.46),

$$\mu^2x^2 = \frac{-(1+2c) \pm \sqrt{4c+1}}{\frac{2c}{\mu^4}}$$

that is,

$$x = |\mu|\sqrt{\frac{-(1+2c) \pm \sqrt{4c+1}}{2c}}.$$

□

**Theorem 2.1.9.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, and $J \times I \subset (0,+\infty) \times \mathbb{R}$ with constant Gaussian curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for constants $\mu \neq 0, \tau$, and $I \subset \mathbb{R}$. Moreover, $u$ satisfies the differential equation*

$$u''(s) = -\frac{(cs^2+\mu^2-\mu)(u'(s))^4 + (2cs^2+2(c+1)\mu^2-\mu)(u'(s))^2 + cs^2\left(1+\frac{\mu^2}{s^2}\right)^2 + \mu^2}{\mu s u'(s)} \tag{2.47}$$

*in*

$$J \subset (0,+\infty) - \left\{ |\mu|\sqrt{\frac{-(1+2c)+\sqrt{4c+1}}{2c}}, |\mu|\sqrt{\frac{-(1+2c)-\sqrt{4c+1}}{2c}} \right\},$$

*if $c \in \left[-\frac{1}{4}, 0\right)$, and in $J \subset (0,+\infty)$ otherwise.*

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.1.21 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. Therefore, by Proposition 2.1.20, the functions $u, v$ satisfy the equation

$$0 = \mu s u'(s)u''(s) + (cs^2+\mu^2-\mu)(u'(s))^4 + (2cs^2+2(c+1)\mu^2-\mu)(u'(s))^2 + cs^2\left(1+\frac{\mu^2}{s^2}\right)^2 + \mu^2,$$

and then, by Proposition 2.1.22,

$$u''(s) = -\frac{(cs^2+\mu^2-\mu)(u'(s))^4 + (2cs^2+2(c+1)\mu^2-\mu)(u'(s))^2 + cs^2\left(1+\frac{\mu^2}{s^2}\right)^2 + \mu^2}{\mu s u'(s)}$$

in

$$J \subset (0,+\infty) - \left\{ |\mu|\sqrt{\frac{-(1+2c) \pm \sqrt{4c+1}}{2c}} \right\},$$

if $c \in \left[-\frac{1}{4}, 0\right)$, and in $J \subset (0,+\infty)$ otherwise. □

**Corollary 2.1.3.** *Assume the conditions of the Theorem* 2.1.9. *The change of variable* $g(s) = (u'(s))^2$ *to the equation* (2.47) *reduces to the Ricatti differential equation*

$$g'(s) = -\frac{2}{\mu s}\left((cs^2+\mu^2-\mu)g^2(s) + (2cs^2+2(c+1)\mu^2-\mu)g(s) + cs^2\left(1+\frac{\mu^2}{s^2}\right)^2 + \mu^2\right). \quad (2.48)$$

**Proof:** Just observe that as

$$0 = \mu s u'(s)u''(s) + (cs^2+\mu^2-\mu)(u'(s))^4 + (2cs^2+2(c+1)\mu^2-\mu)(u'(s))^2 + cs^2\left(1+\frac{\mu^2}{s^2}\right)^2 + \mu^2,$$

then

$$0 = \frac{\mu s}{2}\frac{d}{ds}[(u'(s))^2] + (cs^2+\mu^2-\mu)(u'(s))^4 + (2cs^2+2(c+1)\mu^2-\mu)(u'(s))^2 + cs^2\left(1+\frac{\mu^2}{s^2}\right)^2 + \mu^2.$$

□

**Remark:** A way to investigate Ricatti's equation (2.48) is to reduce it to a separable differential equation. This is possible if a particular solution is known. This is essentially the approach contained in the Corollary 2.1.1.

### 2.1.12 *Type II Translation Surfaces - Minimal case*

**Lemma 2.1.9.** *Let* $\mu$, $C > 0$ *and* $F \in (0,1)$ *be constants. Consider the function* $g : J \subset (0,+\infty) \to \mathbb{R}$ *given by*

$$g(s) = \frac{F\left(\frac{s}{C}\right)^8\left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^3}{1-F\left(\frac{s}{C}\right)^8\left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^3}.$$

*Then* $g > 0$ *only in* $\left(0,\sqrt{\tilde{z}}\right) \cap J$, *where* $\tilde{z}$ *is the* ***only*** *positive root of the polynomial*

$$P(Z) = -F(\mu^2+C^2)^3Z^4 + C^8Z^3 + 3C^8\mu^2Z^2 + 3C^8\mu^4Z + C^8\mu^6.$$

**Proof:** To obtain $g > 0$, we necessarily need the denominator of $g$ to be strictly positive, that is,

$$0 < 1 - F\left(\frac{s}{C}\right)^8\left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^3 = \frac{-F(\mu^2+C^2)^3s^8 + C^8s^6 + 3C^8\mu^2s^4 + 3C^8\mu^4s^2 + C^8\mu^6}{C^8(\mu^2+s^2)^3}$$

i.e.,

$$0 < -F(\mu^2+C^2)^3s^8 + C^8s^6 + 3C^8\mu^2s^4 + 3C^8\mu^4s^2 + C^8\mu^6. \quad (2.49)$$

If $\mu = 0$ we obtain from (2.49) that $g > 0$ in the interval $\left(0,\sqrt{\frac{C^2}{F}}\right)\bigcap J$. If $\mu \neq 0$, define the polynomial

$$P(Z) = -F(\mu^2+C^2)^3Z^4 + C^8Z^3 + 3C^8\mu^2Z^2 + 3C^8\mu^4Z + C^8\mu^6$$

for $Z \geqslant 0$.Since $P(0) > 0$ and $P(+\infty) < 0$ there exists a unique $\tilde{z} \in (0,+\infty)$ such that $P(\tilde{z}) = 0$. Indeed, suppose there exists another $z \in (0,+\infty)$ that vanishes $P$. Then,

$$P(Z) = (Z-\tilde{z})(Z-z)(aZ^2+bZ+c)$$

thus

$$0 = -F(\mu^2 + C^2)^3 Z^4 + C^8 Z^3 + 3C^8\mu^2 Z^2 + 3C^8\mu^4 Z + C^8\mu^6 - (Z - \tilde{z})(Z - z)(aZ^2 + bZ + c)$$

that reorganizing,

$$\begin{aligned} 0 = &-(a + F(C^2 + \mu^2)^3)Z^4 + (C^8 - b + a(\tilde{z} + z))Z^3 + (3C^8\mu^2 - c + b(\tilde{z} + z) - a\tilde{z}z)Z^2 \\ &+ (3C^8\mu^4 + c(\tilde{z} + z) - b\tilde{z}z)Z + C^8\mu^6 - c\tilde{z}z \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} 0 &= a + F(C^2 + \mu^2)^3 \\ 0 &= C^8 - b + a(\tilde{z} + z) \\ 0 &= 3C^8\mu^2 - c + b(\tilde{z} + z) - a\tilde{z}z \\ 0 &= 3C^8\mu^4 + c(\tilde{z} + z) - b\tilde{z}z \\ 0 &= C^8\mu^6 - c\tilde{z}z. \end{aligned}$$

By the first and last equalities we obtain, respectively, $a = -F(C^2 + \mu^2)^3$ and $c = \frac{C^8\mu^6}{\tilde{z}z}$. Substituting in the others we obtain

$$\begin{aligned} 0 &= C^8 - b - F(C^2 + \mu^2)^3(\tilde{z} + z) \\ 0 &= 3C^8\mu^2 - \frac{C^8\mu^6}{\tilde{z}z} + b(\tilde{z} + z) + F(C^2 + \mu^2)^3\tilde{z}z \\ 0 &= 3C^8\mu^4 + \frac{C^8\mu^6}{\tilde{z}z}(\tilde{z} + z) - b\tilde{z}z. \end{aligned}$$

Isolating $b$ in the first equality and substituting it in the others we obtain

$$\begin{aligned} 0 &= \frac{\tilde{z}z(\mu^2 + C^2)^3(\tilde{z}^2 + z^2 + \tilde{z}z)F + C^8(-\tilde{z}z^2 - \tilde{z}^2 z + \mu^6 - 3\mu^2\tilde{z}z)}{\tilde{z}z} \\ 0 &= \frac{\tilde{z}^2 z^2(\mu^2 + C^2)^3(\tilde{z} + z)F + C^8(-\tilde{z}^2 z^2 + \mu^6\tilde{z} + \mu^6 z + 3\mu^4\tilde{z}z)}{\tilde{z}z}. \end{aligned}$$

Isolating $F$ in the second equality and substituting it into the first we obtain

$$0 = -C^8\frac{\mu^6\tilde{z}^3 + \mu^6 z^3 + \tilde{z}^3 z^3 + 3\mu^2\tilde{z}^2 z^3 + 3\mu^2\tilde{z}^3 z^2 + 3\mu^4\tilde{z}^2 z^2 + 3\mu^4\tilde{z}z^3 + 3\mu^4\tilde{z}^3 z + \mu^6\tilde{z}z^2 + \mu^6\tilde{z}^2 z}{\tilde{z}z(\tilde{z} + z)} < 0$$

a contradiction. Then $\tilde{z}$ is unique. Finally, since $P(0) > 0$, we conclude that $P(s^2) > 0$ in the interval $\left(0, \sqrt{\tilde{z}}\right) \cap J$, and consequently $g > 0$ in such interval. □

By Proposition 2.1.17 we conclude that a **Type II Translation Surface** is **Minimal** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.23.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t, s) = (v(t) + u(s), s, t)$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, is Minimal if and only if,*

$$\begin{aligned} 0 = &(sv''(t) - u'(s)(v'(t))^2)(1 + (u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)((v'(t))^2 + s^2) \\ &- 2u'(s)(v'(t))^2(1 + (u'(s))^2). \end{aligned}$$

**Proposition 2.1.24.** *Let $\psi:\Omega\subset\mathbb{R}^2\to H^2\times\mathbb{R}$ given by $\psi(t,s)=(v(t)+u(s),s,t)$, where $u\in C^2(J,\mathbb{R})$ non-constant, $v\in C^2(I,\mathbb{R})$ and $J\times I\subset(0,+\infty)\times\mathbb{R}$, Minimal. Then $v(t)=\mu t+\tau$, with $I\subset\mathbb{R}$, for $\mu,\tau$ constants.*

**Proof:** Suppose $\psi$ is Minimal. By Proposition 2.1.23, the functions $u,v$ satisfy the equation

$$\begin{aligned}0&=(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)\\&-2u'(s)(v'(t))^2(1+(u'(s))^2),\end{aligned}$$

or equivalently

$$0=v''(t)+\frac{-4u'(s)+su''(s)-4(u'(s))^3}{s(1+(u'(s))^2)}(v'(t))^2-\frac{s^2(u'(s)-su''(s)+(u'(s))^3)}{s(1+(u'(s))^2)}.$$

Define the family of irreducible polynomials

$$P_s(X,Y)=Y+A_sX+B_s,$$

where

$$\begin{aligned}A_s&=\frac{-4u'(s)+su''(s)-4(u'(s))^3}{s(1+(u'(s))^2)}\\B_s&=-\frac{s^2(u'(s)-su''(s)+(u'(s))^3)}{s(1+(u'(s))^2)},\end{aligned}$$

and observe that the curve $((v'(t))^2,v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda=1$ we conclude that

$$P_s(X,Y)=Y+A_sX+B_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1,K_2$ constants such that,

$$\begin{aligned}A_s&=K_1\\B_s&=K_2,\end{aligned}$$

that is,

$$\begin{aligned}K_1&=\frac{-4u'(s)+su''(s)-4(u'(s))^3}{s(1+(u'(s))^2)}\\K_2&=-\frac{s^2(u'(s)-su''(s)+(u'(s))^3)}{s(1+(u'(s))^2)},\end{aligned}$$

then, setting $w(s)=u'(s)$, we obtain

$$\begin{aligned}K_1&=\frac{-4w(s)+sw'(s)-4w^3(s)}{s(1+w^2(s))}\\K_2&=-\frac{s^2(w(s)-sw'(s)+w^3(s))}{s(1+w^2(s))},\end{aligned}\tag{2.50}$$

thus

$$s^2K_1 - K_2 = -3sw(s),$$

that is,

$$w(s) = -\frac{s^2K_1 - K_2}{3s}. \tag{2.51}$$

We have $w'(s) = -\frac{K_1 s^2 + K_1}{2s^2}$, which substituting into the first equality of (2.50) we obtain

$$0 = -2K_1^3 s^6 + 3K_1(4K_1K_2 + 3)s^4 - 9K_2(2K_1K_2 - 11)s^2 + 8K_2^2$$

and since $K_1, K_2$ are not mutually zero (otherwise, from (2.51), we would have $0 = w(s) = u'(s)$, i.e., $u$ constant) we conclude that $s^2$ is contained in the roots of the non-zero polynomial

$$P(Z) = -2K_1^3 ZZ^3 + 3K_1(4K_1K_2 + 3)Z^2 - 9K_2(2K_1K_2 - 11)Z + 8K_2^2,$$

a contradiction, since $s$ is not constant. Then, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.1.10.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t) + u(s), s, t)$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, Minimal. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover, $u(s) = D$ for constant $D$ and $J \subset (0, +\infty)$, or*

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{F\left(\frac{r}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+r^2}\right)^3}{1 - F\left(\frac{r}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+r^2}\right)^3}}\, dr,$$

*for $C > 0$, $D$ and $F \in (0,1)$ constants and $J \subset \left(0, \sqrt{\tilde{z}}\right)$, with $\tilde{z}$ being the **only** positive root of the polynomial*

$$P(Z) = -F(\mu^2 + C^2)^3 Z^4 + C^8 Z^3 + 3C^8 \mu^2 Z^2 + 3C^8 \mu^4 Z + C^8 \mu^6.$$

**Proof:** Suppose $\psi$ is Minimal. By Proposition 2.1.23, the functions $u, v$ satisfy the equation

$$\begin{aligned} 0 = {} & (sv''(t) - u'(s)(v'(t))^2)(1 + (u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)((v'(t))^2 + s^2) \\ & - 2u'(s)(v'(t))^2(1 + (u'(s))^2). \end{aligned} \tag{2.52}$$

If $u$ is constant in $J$ we have from (2.52) that $v''(t) = 0$ and therefore $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. If $u$ is non-constant in $J$ we have by Proposition 2.1.24 that $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Then from (2.52)

$$0 = s(s^2 + \mu^2)u''(s) - u'(s)((u'(s))^2 + 1)(4\mu^2 + s^2),$$

and defining $w(s) = u'(s)$,

$$w'(s) = \frac{4\mu^2 + s^2}{s(\mu^2 + s^2)} w(s)(w^2(s) + 1).$$

Observe that $w \equiv 0$ is a solution of this separable differential equation. By the Existence and Uniqueness Theorem any other solution does not reach zero. Since $u$ is non-constant we have $w \neq 0$ and thus we obtain

$$\frac{w'(s)}{w(s)(w^2(s)+1)} = \frac{4\mu^2+s^2}{s(\mu^2+s^2)},$$

then,

$$\int_{s_0}^{s} \frac{w'(r)}{w(r)(w^2(r)+1)}dr = \int_{s_0}^{s} \frac{4\mu^2+r^2}{r(\mu^2+r^2)}dr,$$

and by the Change of Variables Theorem

$$\int_{w(s_0)}^{w(s)} \frac{1}{l(l^2+1)}dl = \int_{s_0}^{s} \frac{4\mu^2+r^2}{r(\mu^2+r^2)}dr.$$

So,

$$\ln\left(\frac{w(s)}{\sqrt{w^2(s)+1}}\right) - \ln\left(\frac{w(s_0)}{\sqrt{w^2(s_0)+1}}\right) = \ln\left(\frac{s^4}{(\mu^2+s^2)^{\frac{3}{2}}}\right) - \ln\left(\frac{s_0^4}{(\mu^2+s_0^2)^{\frac{3}{2}}}\right),$$

that is,

$$\ln\left(\frac{w(s)}{\sqrt{w^2(s)+1}}\right) = \ln\left(\left(\frac{s}{s_0}\right)^4 \left(\frac{\mu^2+s_0^2}{\mu^2+s^2}\right)^{\frac{3}{2}} \frac{w(s_0)}{\sqrt{w^2(s_0)+1}}\right),$$

thus,

$$\frac{w(s)}{\sqrt{w^2(s)+1}} = \left(\frac{s}{s_0}\right)^4 \left(\frac{\mu^2+s_0^2}{\mu^2+s^2}\right)^{\frac{3}{2}} \frac{w(s_0)}{\sqrt{w^2(s_0)+1}},$$

then,

$$\frac{w^2(s)}{w^2(s)+1} = \left(\frac{s}{s_0}\right)^8 \left(\frac{\mu^2+s_0^2}{\mu^2+s^2}\right)^{3} \frac{w^2(s_0)}{w^2(s_0)+1},$$

consequently,

$$w^2(s) = \frac{\left(\frac{s}{s_0}\right)^8 \left(\frac{\mu^2+s_0^2}{\mu^2+s^2}\right)^{3} \frac{w^2(s_0)}{w^2(s_0)+1}}{1-\left(\frac{s}{s_0}\right)^8 \left(\frac{\mu^2+s_0^2}{\mu^2+s^2}\right)^{3} \frac{w^2(s_0)}{w^2(s_0)+1}}$$

i.e., there exist constants $C > 0$ and $F \in (0,1)$ such that

$$w^2(s) = \frac{F\left(\frac{s}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^{3}}{1-F\left(\frac{s}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^{3}},$$

that is,

$$(u'(s))^2 = \frac{F\left(\frac{s}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^{3}}{1-F\left(\frac{s}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^{3}}.$$

By Lemma 2.1.9 we have two distinct differential equations

$$u'(s) = \pm\sqrt{\frac{F\left(\frac{s}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^{3}}{1-F\left(\frac{s}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+s^2}\right)^{3}}},$$

in $J \subset \left(0, \sqrt{\tilde{z}}\right)$, being $\tilde{z}$ the **only** positive root of the polynomial

$$P(Z) = -F(\mu^2 + C^2)^3 Z^4 + C^8 Z^3 + 3C^8 \mu^2 Z^2 + 3C^8 \mu^4 Z + C^8 \mu^6.$$

Integrating these equations we conclude

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{F\left(\frac{r}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+r^2}\right)^3}{1 - F\left(\frac{r}{C}\right)^8 \left(\frac{\mu^2+C^2}{\mu^2+r^2}\right)^3}}\, dr.$$

### 2.1.13 *Type II Translation Surfaces - Constant Mean Curvature case*

By Proposition 2.1.17 we conclude that a **Type II Translation Surface** has **Constant Mean Curvature** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.25.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t) + u(s), s, t)$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, has constant Mean Curvature $c$ , if, and only if,*

$$c = \frac{(sv''(t) - u'(s)(v'(t))^2)(1 + (u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)((v'(t))^2 + s^2)}{2s^2 \left(1 + (u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}$$
$$- \frac{2u'(s)(v'(t))^2(1 + (u'(s))^2)}{2s^2 \left(1 + (u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}.$$

**Proposition 2.1.26.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t) + u(s), s, t)$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, with constant Mean Curvature $c$ neq$0$. Then $v(t) = \mu t + \tau$, with $I \subset \mathbb{R}$, for $\mu, \tau$ constants.*

**Proof:** Suppose $\psi$ has constant Mean Curvature $c \neq 0$. By Proposition 2.1.25, the functions $u, v$ satisfy the equation

$$c = \frac{(sv''(t) - u'(s)(v'(t))^2)(1 + (u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)((v'(t))^2 + s^2)}{2s^2 \left(1 + (u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}$$
$$- \frac{2u'(s)(v'(t))^2(1 + (u'(s))^2)}{2s^2 \left(1 + (u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}},$$

or equivalently

$$
\begin{aligned}
0 = {} & (v''(t))^2 + \frac{2(-4u'(s) + su''(s) - 4(u'(s))^3)}{s(1+(u'(s))^2)} v''(t)(v'(t))^2 + \frac{2s(-u'(s) + su''(s) - (u'(s))^3)}{1+(u'(s))^2} v''(t) \\
& - \frac{4c^2}{s^4(1+(u'(s))^2)^2}(v'(t))^6 \\
& + \frac{(u'(s) - su''(s) + (u'(s))^3 + 3u'(s)(1+(u'(s))^2))^2 - 12c^2(1+(u'(s))^2)}{s^2(1+(u'(s))^2)^2}(v'(t))^4 \\
& + \frac{2s^2((u'(s))^3 + u'(s) - su''(s))(u'(s) - su''(s) + (u'(s))^3 + 3u'(s)(1+(u'(s))^2))}{s^2(1+(u'(s))^2)^2}(v'(t))^2 \\
& - \frac{12c^2s^2(1+(u'(s))^2)^2)}{s^2(1+(u'(s))^2)^2}(v'(t))^2 + \frac{s^4((u'(s))^3 + u'(s) - su''(s))^2 - 4c^2s^4(1+(u'(s))^2)^3}{s^2(1+(u'(s))^2)}.
\end{aligned}
$$

Define the family of polynomials

$$P_s(X,Y) = Y^2 + A_sXY + B_sY + C_sX^3 + D_sX^2 + E_sX + F_s,$$

where

$$
\begin{aligned}
A_s = {} & \frac{2(-4u'(s) + su''(s) - 4(u'(s))^3)}{s(1+(u'(s))^2)} \\
B_s = {} & \frac{2s(-u'(s) + su''(s) - (u'(s))^3)}{1+(u'(s))^2} \\
C_s = {} & -\frac{4c^2}{s^4(1+(u'(s))^2)^2} \\
D_s = {} & \frac{(u'(s) - su''(s) + (u'(s))^3 + 3u'(s)(1+(u'(s))^2))^2 - 12c^2(1+(u'(s))^2)}{s^2(1+(u'(s))^2)^2} \\
E_s = {} & \frac{2s^2((u'(s))^3 + u'(s) - su''(s))(u'(s) - su''(s) + (u'(s))^3 + 3u'(s)(1+(u'(s))^2))}{s^2(1+(u'(s))^2)^2} \\
& - \frac{12c^2s^2(1+(u'(s))^2)^2)}{s^2(1+(u'(s))^2)^2} \\
F_s = {} & \frac{s^4((u'(s))^3 + u'(s) - su''(s))^2 - 4c^2s^4(1+(u'(s))^2)^3}{s^2(1+(u'(s))^2)},
\end{aligned}
$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible then there exist polynomials of degree 2 and degree 1 such that

$$P_s(X,Y) = (f + gX + hY)(i + jX + kY + lXY + mX^2 + nY^2)$$

that is,

$$0 = Y^2 + A_sXY + B_sY + C_sX^3 + D_sX^2 + E_sX + F_s - (f + gX + hY)(i + jX + kY + lXY + mX^2 + nY^2)$$

and reorganizing

$$\begin{aligned}0 =&(C_s - gm)X^3 - (gl + hm)X^2Y + (D_s - gj - fm)X^2 - (hl + gn)XY^2 + (A_s - fl - gk - hj)XY \\ &+ (E_s - gi - fj)X - hnY^2 + (1 - fn - hk)Y^2 + (B_s - hi - fk)Y + F_s - fi\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0 =&C_s - gm \\ 0 =&gl + hm \\ 0 =&D_s - gj - fm \\ 0 =&hl + gn\end{aligned} \qquad \begin{aligned}0 =&A_s - fl - gk - hj \\ 0 =&E_s - gi - fj \\ 0 =&hn\end{aligned} \qquad \begin{aligned}0 =&1 - fn - hk \\ 0 =&B_s - hi - fk \\ 0 =&F_s - fi.\end{aligned} \tag{2.53}$$

By the seventh equality $h = 0$ or $n = 0$. Suppose $h = 0$. We have,

$$\begin{aligned}0 =&C_s - gm \\ 0 =&gl \\ 0 =&D_s - gj - fm\end{aligned} \qquad \begin{aligned}0 =&gn \\ 0 =&A_s - fl - gk \\ 0 =&E_s - gi - fj\end{aligned} \qquad \begin{aligned}0 =&1 - fn \\ 0 =&B_s - fk \\ 0 =&F_s - fi.\end{aligned}$$

Since $g \neq 0$ (otherwise we would not have factorization) we obtain, by the fourth equality, $n = 0$ which substituting in the seventh equality we conclude $1 = 0$, a contradiction. Then, $h \neq 0$ and $n = 0$. From (2.53) we have,

$$\begin{aligned}0 =&C_s - gm \\ 0 =&gl + hm \\ 0 =&D_s - gj - fm\end{aligned} \qquad \begin{aligned}0 =&hl \\ 0 =&A_s - fl - gk - hj \\ 0 =&E_s - gi - fj\end{aligned} \qquad \begin{aligned}0 =&1 - hk \\ 0 =&B_s - hi - fk \\ 0 =&F_s - fi.\end{aligned}$$

By the fourth equality we obtain $l = 0$. Substituting into the second equality we conclude $m = 0$. Substituting into the first equality we conclude $C_s = 0$, a contradiction since $C_s < 0$ and thus we proof the claim.

By the claim the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = Y^2 + A_sXY + B_sY + C_sX^3 + D_sX^2 + E_sX + F_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Thus, their coefficients are constant, that is, there are $K_1, K_2, K_3$ constants such that,

$$\begin{aligned}A_s &= K_1 \\ B_s &= K_2 \\ C_s &= K_3,\end{aligned}$$

that is,

$$\begin{aligned} K_1 &= \frac{2(-4u'(s)+su''(s)-4(u'(s))^3)}{s(1+(u'(s))^2)} \\ K_2 &= \frac{2s(-u'(s)+su''(s)-(u'(s))^3)}{1+(u'(s))^2} \\ K_3 &= -\frac{4c^2}{s^4(1+(u'(s))^2)^2}, \end{aligned}$$

then, setting $w(s)=u'(s)$, we obtain

$$\begin{aligned} K_1 &= \frac{2(-4w(s)+sw'(s)-4w^3(s))}{s(1+w^2(s))} \\ K_2 &= \frac{2s(-w(s)+sw'(s)-w^3(s))}{1+w^2(s)} \\ K_3 &= -\frac{4c^2}{s^4(1+w^2(s))^2}, \end{aligned}$$

thus

$$\begin{aligned} sw'(s) &= \frac{(1+w^2(s))(8w(s)+sK_1)}{2} \\ K_2 &= \frac{2s(-w(s)+sw'(s)-w^3(s))}{1+w^2(s)} \\ K_3 &= -\frac{4c^2}{s^4(1+w^2(s))^2}, \end{aligned}$$

and substituting the first equality into the second we obtain

$$\begin{aligned} 0 = K_1^4K_3s^8 - 4K_1^2K_3(K_1K_2-18)s^6 + 6K_3(K_1^2K_2^2-24K_1K_2+216)s^4 - 4K_2^2K_3(K_1K_2-18)s^2 \\ + 5184c^2 + K_3K_2^4 \end{aligned}$$

and since $s$ is non-constant, we conclude that all the coefficients of the above polynomial are zero. We have,

$$\begin{aligned} 0 &= K_1^4K_3 \\ 0 &= 4K_1^2K_3(K_1K_2-18) \\ 0 &= 6K_3(K_1^2K_2^2-24K_1K_2+216) \end{aligned} \qquad \begin{aligned} 0 &= 4K_2^2K_3(K_1K_2-18) \\ 0 &= 5184c^2 + K_3K_2^4. \end{aligned}$$

Since $K_3 \neq 0$ we obtain, from the first equality, $K_1 = 0$ which substituting in the third equality we conclude $216 = 0$, a contradiction. Therefore, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.1.11.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, and $J \times I \subset (0,+\infty)\times\mathbb{R}$, with constant Mean Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, with $I \subset \mathbb{R}$, for constant $\mu, \tau$, and the function $u$ satisfies the differential equation*

$$u''(s) = \frac{u'(s)(1+(u'(s))^2)(4\mu^2+s^2) + 2cs^2\left(1+(u'(s))^2+\frac{\mu^2}{s^2}\right)^{\frac{3}{2}}}{s(s^2+\mu^2)}. \tag{2.54}$$

**Proof:** Suppose $\psi$ has constant mean curvature $c \neq 0$. By Proposition 2.1.25, the functions $u, v$ satisfy the equation

$$c = \frac{(sv''(t) - u'(s)(v'(t))^2)(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)((v'(t))^2 + s^2)}{2s^2\left(1+(u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}$$
$$- \frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\left(1+(u'(s))^2 + \left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}},$$

By Proposition 2.1.26 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. Thus, the equation reduces to

$$0 = -s(s^2+\mu^2)u''(s) + u'(s)(1+(u'(s))^2)(4\mu^2+s^2) + 2cs^2\left(1+(u'(s))^2 + \frac{\mu^2}{s^2}\right)^{\frac{3}{2}},$$

that is,

$$u''(s) = \frac{u'(s)(1+(u'(s))^2)(4\mu^2+s^2) + 2cs^2\left(1+(u'(s))^2+\frac{\mu^2}{s^2}\right)^{\frac{3}{2}}}{s(s^2+\mu^2)}.$$

$\square$

**Corollary 2.1.4.** *Assume the conditions of Theorem* 2.1.11. *By changing the variable* $g(s) = u'(s)$ *the equation* (2.54) *reduces to*

$$g'(s) = \frac{g(s)(1+g^2(s))(4\mu^2+s^2) + 2cs^2\left(1+g^2(s)+\frac{\mu^2}{s^2}\right)^{\frac{3}{2}}}{s(s^2+\mu^2)}.$$

*In particular, if* $\mu = 0$, *g satisfies the separable differential equation*

$$g'(s) = \frac{(g^2(s)+1)\left(g(s)+2c\sqrt{g^2(s)+1}\right)}{s}. \tag{2.55}$$

**Remark:** The equation (2.55) is the same as that obtained in the Corollary 2.1.2.

### 2.1.14 *Type II Translation Surfaces of Null Linear Weingarten Form*

By Proposition 2.1.17 we conclude that a **Type II Translation Surface** is of **Null Linear Weingarten Form** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.27.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, is of Null Linear Weingarten Form, if, and only if,*

$$\begin{aligned}0 = a &\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\right)\\ -a&\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\\ +b&\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2}{s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{2}}\right).\end{aligned}$$

**Proposition 2.1.28.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$ non-constant, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, of Null Linear Weingarten Form. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is of Null Linear Weingarten Form. Let $\tilde{J} \subset J$ be set open such that $u'(s) \neq 0$. By Proposition 2.1.27 the functions $u, v$ satisfy the equation,

$$\begin{aligned}0 = a &\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\right)\\ -a&\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\\ +b&\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2}{s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{2}}\right).\end{aligned}$$

or equivalently,

$$
\begin{aligned}
0 &= \frac{\left(-4u'(s)+su''(s)-4(u'(s))^3\right)^2}{s^2\left((u'(s))^2+1\right)^2}(v'(t))^6 + 2\left(\frac{-4u'(s)+su''(s)-4(u'(s))^3}{s\left((u'(s))^2+1\right)}\right)(v'(t))^4 v''(t) \\
&+ (v'(t))^2(v''(t))^2 + \frac{a^2\left((u'(s))^2+1\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)^2}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^4 \\
&+ \frac{2a^2\left((u'(s))^3+u'(s)-su''(s)\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^4 \\
&- \frac{4b^2\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)^2}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^4 \\
&+ \frac{8b^2 s\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2 v''(t) \\
&- \frac{2a^2 s\left((u'(s))^2+1\right)^2\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2 v''(t) \\
&- \frac{2a^2 s\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2 v''(t) \\
&+ s^2\left(-u'(s)+su''(s)-(u'(s))^3\right)\frac{-9u'(s)+3su''(s)-17(u'(s))^3-8(u'(s))^5+2s(u'(s))^2u''(s)}{\left((u'(s))^2+1\right)^2}(v'(t))^2 \\
&+ \frac{\left(a^2s^2\left((u'(s))^2+1\right)^3-4b^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2\right)}{a^2\left((u'(s))^2+1\right)^2}(v''(t))^2 \\
&+ 2s^3\left(-u'(s)+su''(s)-(u'(s))^3\right)v''(t) + s^4\frac{\left(-u'(s)+su''(s)-(u'(s))^3\right)^2}{(u'(s))^2+1}.
\end{aligned}
$$

Define the family of polynomials

$$P_s(X,Y) = A_sX^3 + B_sX^2Y + C_sX^2 + XY^2 + D_sXY + E_sX + F_sY^2 + G_sY + H_s$$

where,

$$
\begin{aligned}
A_s &= \frac{\left(-4u'(s)+su''(s)-4(u'(s))^3\right)^2}{s^2\left((u'(s))^2+1\right)^2} \\
B_s &= 2\left(\frac{-4u'(s)+su''(s)-4(u'(s))^3}{s\left((u'(s))^2+1\right)}\right) \\
C_s &= \frac{a^2\left((u'(s))^2+1\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)^2}{a^2\left((u'(s))^2+1\right)^2} \\
&+ \frac{2a^2\left((u'(s))^3+u'(s)-su''(s)\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2} \\
&- \frac{4b^2\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)^2}{a^2\left((u'(s))^2+1\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
D_s =& \frac{8b^2 s\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}\\
&-\frac{2a^2 s\left((u'(s))^2+1\right)^2\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}\\
&-\frac{2a^2 s\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}\\
E_s =& s^2\left(-u'(s)+su''(s)-(u'(s))^3\right)\frac{-9u'(s)+3su''(s)-17(u'(s))^3-8(u'(s))^5+2s(u'(s))^2u''(s)}{\left((u'(s))^2+1\right)^2}\\
F_s =& \frac{\left(a^2s^2\left((u'(s))^2+1\right)^3-4b^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2\right)}{\left(a^2\left((u'(s))^2+1\right)^2\right)}\\
G_s =& 2s^3\left(-u'(s)+su''(s)-(u'(s))^3\right)\\
H_s =& s^4\frac{\left(-u'(s)+su''(s)-(u'(s))^3\right)^2}{(u'(s))^2+1}
\end{aligned}
$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$..

**Claim 1:** $4C_s + B_s^2F_s - 2B_sD_s \neq 0$ for all $s \in \tilde{J} \subset J$. Indeed, if there exists $s_0 \in \tilde{J} \subset J$ such that

$$0 = 4C_{s_0} + B_{s_0}^2 F_{s_0} - 2B_{s_0}D_{s_0}$$

then, substituting the coefficients, we have

$$
\begin{aligned}
0 =& -16b^2\frac{(s_0^2(u''(s_0))^2+6(u'(s_0))^2+9(u'(s_0))^4+4(u'(s_0))^6}{a^2(1+(u'(s_0))^2)^4}\\
&-16b^2\frac{-4su'(s_0)u''(s_0)-4s_0(u'(s_0))^3u''(s_0)+1)^2}{a^2(1+(u'(s_0))^2)^4}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
0 =& s_0^2(u''(s_0))^2+6(u'(s_0))^2+9(u'(s_0))^4+4(u'(s_0))^6\\
&-4s_0u'(s_0)u''(s_0)-4s(u'(s_0))^3u''(s_0)+1
\end{aligned}
$$

which multiplying by $(u'(s_0))^2$

$$
\begin{aligned}
0 =& s_0^2(u'(s_0)u''(s_0))^2+6(u'(s_0))^4+9(u'(s_0))^6+4(u'(s_0))^8\\
&-4s_0(u'(s_0))^3u''(s_0)-4s_0(u'(s_0))^5u''(s_0)+(u'(s_0))^2.
\end{aligned}
$$

Defining $w(s) = (u'(s))^2$

$$0 = \frac{1}{4}(s_0w'(s_0))^2 - 2(w^2(s_0)+w(s_0))s_0w'(s_0) + 4w^4(s_0) + 9w^3(s_0) + 6w^2(s_0) + w(s_0)$$

therefore, by the discriminant formula,

$$0 \leqslant \Delta = -w(s_0)(w(s_0)+1)^2 < 0$$

a contradiction and thus we conclude the claim.

**Claim 2:** If $G_{s_0} + B_{s_0}F_{s_0}^2 - F_{s_0}D_{s_0} \neq 0$ or $H_{s_0} - A_{s_0}F_{s_0}^3 + C_{s_0}F_{s_0}^2 - E_{s_0}F_{s_0} \neq 0$ for some $s_0 \in \tilde{J} \subset J$, then the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$. Indeed, if $P_s$ is reducible for all $s \in \tilde{J} \subset J$ then there exist polynomials of degree 1 and degree 2 such that

$$\begin{aligned} 0 &= A_sX^3 + B_sX^2Y + C_sX^2 + XY^2 + D_sXY + E_sX + F_sY^2 + G_sY + H_s \\ &\quad - (f + gX + hY)(i + jX + kY + lXY + mX^2 + nY^2) \end{aligned}$$

that is,

$$\begin{aligned} 0 = {} & (A_s - gm)X^3 + (B_s - gl - hm)X^2Y + (C_s - gj - fm)X^2 + (1 - gn - hl)XY^2 \\ & + (D_s - fl - gk - hj)XY + (E_s - gi - fj)X - hnY^3 + (F_s - hk - fn)Y^2 + (G_s - hi - fk)Y + H_s - fi \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = B_s - gl - hm & 0 = D_s - fl - gk - hj & 0 = F_s - hk - fn \\ 0 = C_s - gj - fm & 0 = E_s - gi - fj & 0 = G_s - hi - fk \\ 0 = 1 - gn - hl & 0 = hn & 0 = H_s - fi. \end{array} \tag{2.56}$$

By the seventh equality $h$ or $n$ is zero. Suppose $n = 0$. We have,

$$\begin{array}{lll} 0 = A_s - gm & 0 = 1 - hl & 0 = F_s - hk \\ 0 = B_s - gl - hm & 0 = D_s - fl - gk - hj & 0 = G_s - hi - fk \\ 0 = C_s - gj - fm & 0 = E_s - gi - fj & 0 = H_s - fi, \end{array}$$

and by the fourth equality $l = \frac{1}{h}$ and by the seventh equality $k = \frac{F_s}{h}$, that is,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = B_s - \dfrac{g}{h} - hm & 0 = D_s - \dfrac{f}{h} - g\dfrac{F_s}{h} - hj & 0 = G_s - hi - f\dfrac{F_s}{h} \\ 0 = C_s - gj - fm & 0 = E_s - gi - fj & 0 = H_s - fi. \end{array}$$

By the second, fourth and sixth equality we obtain, respectively, $m = \frac{B_s}{h} - \frac{g}{h^2}$, $j = \frac{D_s}{h} - \frac{f}{h^2} - g\frac{F_s}{h^2}$ and $i = \frac{G_s}{h} - f\frac{F_s}{h^2}$. We have,

$$\begin{aligned} 0 &= A_s - g\left(\frac{B_s}{h} - \frac{g}{h^2}\right) \\ 0 &= B_s - \frac{g}{h} - h\left(\frac{B_s}{h} - \frac{g}{h^2}\right) \\ 0 &= C_s - g\left(\frac{D_s}{h} - \frac{f}{h^2} - g\frac{F_s}{h^2}\right) - f\left(\frac{B_s}{h} - \frac{g}{h^2}\right) \\ 0 &= D_s - \frac{f}{h} - g\frac{F_s}{h} - h\left(\frac{D_s}{h} - \frac{f}{h^2} - g\frac{F_s}{h^2}\right) \end{aligned}$$

$$\begin{aligned} 0 &= E_s - g\left(\frac{G_s}{h} - f\frac{F_s}{h^2}\right) - f\left(\frac{D_s}{h} - \frac{f}{h^2} - g\frac{F_s}{h^2}\right) \\ 0 &= G_s - h\left(\frac{G_s}{h} - f\frac{F_s}{h^2}\right) - f\frac{F_s}{h} \\ 0 &= H_s - f\left(\frac{G_s}{h} - f\frac{F_s}{h^2}\right). \end{aligned}$$

Since $B_s^2 = 4A_s$ we obtain, by the first equality, $\frac{g}{h} = \frac{B_s \pm \sqrt{B_s^2 - 4A_s}}{2} = \frac{B_s}{2}$. We have, substituting $g = \frac{hB_s}{2}$ in the third equality,

$$4C_s + B_s^2 F_s - 2B_s D_s = 0$$

for all $s \in \tilde{J} \subset J$, a contradiction with claim 1. Therefore, $h = 0$, and from (2.56),

$$\begin{array}{lll} 0 = A_s - gm & 0 = 1 - gn & 0 = F_s - fn \\ 0 = B_s - gl & 0 = D_s - fl - gk & 0 = G_s - fk \\ 0 = C_s - gj - fm & 0 = E_s - gi - fj & 0 = H_s - fi. \end{array}$$

By the first, second and fourth equalities we have $n = \frac{1}{g}$, $m = \frac{A_s}{g}$ and $l = \frac{B_s}{g}$. Therefore,

$$\begin{array}{lll} 0 = C_s - gj - f\dfrac{A_s}{g} & 0 = E_s - gi - fj & 0 = G_s - fk \\ 0 = D_s - f\dfrac{B_s}{g} - gk & 0 = F_s - \dfrac{f}{g} & 0 = H_s - fi. \end{array}$$

By the first, second and fourth equalities we obtain $j = \frac{C_s}{g} - f\frac{A_s}{g^2}$, $k = \frac{D_s}{g} - f\frac{B_s}{g^2}$ and $f = gF_s$. Substituting

$$\begin{aligned} 0 &= E_s - gi - F_s\left(C_s - F_s A_s\right) \\ 0 &= G_s - F_s\left(D_s - F_s B_s\right) \\ 0 &= H_s - giF_s \end{aligned}$$

then, by the first equality, $i = \frac{E_s}{g} - \frac{F_s}{g}\left(C_s - F_s A_s\right)$, that is,

$$\begin{aligned} 0 &= G_s - F_s\left(D_s - F_s B_s\right) \\ 0 &= H_s - F_s\left(E_s - F_s\left(C_s - F_s A_s\right)\right) \end{aligned}$$

for all $s \in \tilde{J} \subset J$, a contradiction with the hypothesis, and thus we conclude the claim.

**Claim 3:** If $G_s + B_s F_s^2 - F_s D_s = 0$ and $H_s - A_s F_s^3 + C_s F_s^2 - E_s F_s = 0$, for all $s \in \tilde{J} \subset J$, then the polynomials $P_s$ are written, uniquely, as

$$P_s(X,Y) = (F_s + X)(A_s X^2 + B_s XY + (C_s - A_s F_s)X + Y^2 + (D_s - B_s F_s)Y + E_s - F_s(C_s - A_s F_s))$$

in which both factors are irreducible. Indeed,

$$\begin{aligned} &P_s(X,Y) - (F_s + X)(A_s X^2 + B_s XY + (C_s - A_s F_s)X + Y^2 + (D_s - B_s F_s)Y + E_s - F_s(C_s - A_s F_s)) \\ &= (G_s - F_s(D_s - B_s F_s))Y + H_s - F_s(E_s - F_s(C_s - A_s F_s)) \\ &= 0. \end{aligned}$$

Finally, $X + F_s$ is irreducible and therefore it remains to show that

$$A_s X^2 + B_s XY + (C_s - A_s F_s)X + Y^2 + (D_s - B_s F_s)Y + E_s - F_s(C_s - A_s F_s)$$

is irreducible. Indeed, suppose that there are polynomials of degree 1 such that

$$A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + E_s - F_s(C_s - A_sF_s) = (f + gX + hY)(i + jX + kY)$$

that is,

$$\begin{aligned}0 &= (A_s - gj)X^2 + (B_s - gk - hj)XY + (C_s - gi - A_sF_s - fj)X + (1 - hk)Y^2 \\ &+ (D_s - hi - B_sF_s - fk)Y + E_s - F_s(C_s - A_sF_s - fi)\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0 &= A_s - gj \\ 0 &= B_s - gk - hj \\ 0 &= C_s - gi - A_sF_s - fj \\ 0 &= 1 - hk \\ 0 &= D_s - hi - B_sF_s - fk \\ 0 &= E_s - F_s(C_s - A_sF_s - fi).\end{aligned}$$

By the fourth equality we can assume, without loss of generality, $h = k = 1$. We have,

$$\begin{aligned}0 &= A_s - gj \\ 0 &= B_s - g - j \\ 0 &= C_s - gi - A_sF_s - fj \\ 0 &= D_s - i - B_sF_s - f \\ 0 &= E_s - F_s(C_s - A_sF_s - fi).\end{aligned}$$

By the second and fourth equalities we have $j = B_s - g$ and $i = D_s - B_sF_s - f$. Substituting we obtain

$$\begin{aligned}0 &= A_s - g(B_s - g) \\ 0 &= C_s - g(D_s - B_sF_s - f) - A_sF_s - f(B_s - g) \\ 0 &= E_s - F_s(C_s - A_sF_s - f(D_s - B_sF_s - f)).\end{aligned}$$

Since $B_s^2 = 4A_s$ we obtain, by the first equality, $g = \frac{B_s \pm \sqrt{B_s^2 - 4A_s}}{2} = \frac{B_s}{2}$. We have, substituting $g = \frac{B_s}{2}$ in the second equality,

$$4C_s + B_s^2F_s - 2B_sD_s = 0$$

for all $s \in \tilde{J} \subset J$, a contradiction with the claim 1 and thus we conclude the claim.

**Claim 4:** If $B_s$ and $G_s$ are constants in $\tilde{J} \subset J$ then $G_s$ is nonzero in $\tilde{J} \subset J$. Indeed, if they are constants then there exist constants $K_2$ and $K_7$ such that

$$\begin{aligned}K_2 &= 2\left(\frac{-4u'(s) + su''(s) - 4(u'(s))^3}{s((u'(s))^2 + 1)}\right) \\ K_7 &= 2s^3\left(-u'(s) + su''(s) - (u'(s))^3\right).\end{aligned}$$

Notice that $(K_2, K_7) \neq (0,0)$. Indeed, if $(K_2, K_7) = (0,0)$ then

$$\begin{aligned} 0 &= -4u'(s) + su''(s) - 4(u'(s))^3 \\ 0 &= -u'(s) + su''(s) - (u'(s))^3 \end{aligned}$$

for all $s \in \tilde{J} \subset J$. Therefore, subtracting the second equality from the first we obtain $0 = u'(s)(1 + (u'(s))^2)$ $s \in \tilde{J} \subset J$, and then $u$ is constant in $\tilde{J} \subset J$, a contradiction. If $G_s$ is zero in $\tilde{J} \subset J$ then $K_7 = 0$ and so

$$\begin{aligned} K_2 =& 2\left(\frac{-4u'(s) + su''(s) - 4(u'(s))^3}{s\,((u'(s))^2+1)}\right) \\ 0 =& -u'(s) + su''(s) - (u'(s))^3. \end{aligned}$$

isolating $u''(s)$ in the second equality and substituting it in the first we obtain $u'(s) = -\frac{sK_2}{6}$ and therefore

$$0 = -u'(s) + su''(s) - (u'(s))^3 = -\left(-\frac{sK_2}{6}\right) + s\left(-\frac{K_2}{6}\right) - \left(-\frac{sK_2}{6}\right)^3 = \frac{s^3K_2^3}{216} \neq 0$$

a contradiction and thus we conclude the claim.

**Claim 5:** There are no constants $K_2, K_7, K_8$ such that

$$\begin{aligned} B_s &= K_2 \\ G_s &= K_7 \\ H_s &= K_8 \end{aligned}$$

for all $s \in \tilde{J} \subset J$. Indeed, if there exist then

$$\begin{aligned} K_2 &= 2\left(\frac{-4u'(s) + su''(s) - 4(u'(s))^3}{s\,((u'(s))^2+1)}\right) \\ K_7 &= 2s^3\left(-u'(s) + su''(s) - (u'(s))^3\right) \\ K_8 &= s^4\frac{\left(-u'(s) + su''(s) - (u'(s))^3\right)^2}{(u'(s))^2+1}. \end{aligned}$$

Isolating $u''(s)$ in the first equality and substituting it in the others we obtain

$$\begin{aligned} 0 &= 6(u'(s))^3s^3 + K_2(u'(s))^2s^4 + 6u'(s)s^3 + K_2s^4 - K_7 \\ 0 &= 9(u'(s))^6s^4 + 3K_2(u'(s))^5s^5 + \frac{K_2^2}{4}(u'(s))^4s^6 + 18(u'(s))^4s^4 + 6K_2(u'(s))^3s^5 + \frac{K_2^2}{2}(u'(s))^2s^6 \\ &\quad + 9(u'(s))^2s^4 - K_8(u'(s))^2 + 3K_2u'(s)s^5 + \frac{K_2^2}{4}s^6 - K_8 \end{aligned}$$

and multiplying the second equality by $s^2$

$$\begin{aligned} 0 &= 6(u'(s))^3s^3 + K_2(u'(s))^2s^4 + 6u'(s)s^3 + K_2s^4 - K_7 \\ 0 &= 9(u'(s))^6s^6 + 3K_2(u'(s))^5s^7 + \frac{K_2^2}{4}(u'(s))^4s^8 + 18(u'(s))^4s^6 + 6K_2(u'(s))^3s^7 + \frac{K_2^2}{2}(u'(s))^2s^8 \\ &\quad + 9(u'(s))^2s^6 - K_8(u'(s))^2s^2 + 3K_2u'(s)s^7 + \frac{K_2^2}{4}s^8 - K_8s^2. \end{aligned} \tag{2.57}$$

Define $G : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{G} : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$G(x,y) = 6y^3 + K_2 y^2 x + 6yx + K_2 x^2 - K_7$$

and

$$\begin{aligned}\overline{G}(x,y) = {} & 9y^6 + 3K_2 y^5 x + \frac{K_2^2}{4} y^4 x^2 + 18 y^4 x + 6K_2 y^3 x^2 + \frac{K_2^2}{2} y^2 x^3 + 9y^2 x^2 \\ & - K_8 y^2 + 3K_2 y x^3 + \frac{K_2^2}{4} x^4 - K_8 x.\end{aligned}$$

Note that, from (2.57), $G(s^2,(su'(s))^2) = 0 = \overline{G}(s^2,(su'(s))^2)$. By claim 4 we have $K_7 \neq 0$ (and consequently $K_8 \neq 0$). Since the curve $(s^2,(su'(s))^2)$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A$.0.12 and thus we conclude the claim.

**Claim 6:** There are no constants $M_2, M_3, M_4$ such that

$$\begin{aligned} B_s &= M_2 \\ G_s &= M_3 F_s \\ H_s &= M_4 F_s \end{aligned}$$

for all $s \in \tilde{J} \subset J$. Indeed, if there exist then

$$\begin{aligned} u''(s) &= \frac{s(1+(u'(s))^2)M_2 + 8(u'(s)+(u'(s))^3)}{2s} \\ 2s^3\left(-u'(s)+su''(s)-(u'(s))^3\right) &= M_3 \frac{\left(a^2s^2\left((u'(s))^2+1\right)^3 - 4b^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2\right)}{\left(a^2\left((u'(s))^2+1\right)^2\right)} \\ s^4\frac{\left(-u'(s)+su''(s)-(u'(s))^3\right)^2}{(u'(s))^2+1} &= M_4 \frac{\left(a^2s^2\left((u'(s))^2+1\right)^3 - 4b^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2\right)}{\left(a^2\left((u'(s))^2+1\right)^2\right)} \end{aligned}$$

and replacing the first equality in the others

$$\begin{aligned} 0 = {} & 6a^2(u'(s))^3 s + a^2 M_2 (u'(s))^2 s^2 + M_3(36b^2 - a^2)(u'(s))^2 + (6a^2 + 12M_2M_3b^2)u'(s)s \\ & + (a^2M_2 + M_3M_2^2b^2)s^2 - a^2M_3 \\ 0 = {} & 36a^2(u'(s))^4 s^2 + 12a^2M_2(u'(s))^3 s^3 + a^2M_2^2(u'(s))^2 s^4 + 36a^2(u'(s))^2 s^2 \\ & + M_4(144b^2 - 4a^2)(u'(s))^2 + 12a^2M_2u'(s)s^3 + 48b^2M_2M_4u'(s)s + a^2M_2^2s^4 + 4b^2M_2^2M_4s^2 - 4a^2M_4 \end{aligned}$$

and multiplying both equalities by $s^2$

$$\begin{aligned} 0 = {} & 6a^2(u'(s))^3 s^3 + a^2 M_2 (u'(s))^2 s^4 + M_3(36b^2 - a^2)(u'(s))^2 s^2 + (6a^2 + 12M_2M_3b^2)u'(s)s^3 \\ & + (a^2M_2 + M_3M_2^2b^2)s^4 - a^2M_3 s^2 \\ 0 = {} & 36a^2(u'(s))^4 s^4 + 12a^2M_2(u'(s))^3 s^5 + a^2M_2^2(u'(s))^2 s^6 + 36a^2(u'(s))^2 s^4 \\ & + M_4(144b^2 - 4a^2)(u'(s))^2 s^2 + 12a^2M_2u'(s)s^5 + 48b^2M_2M_4u'(s)s^3 + a^2M_2^2s^6 \\ & + 4b^2M_2^2M_4s^4 - 4a^2M_4s^2. \end{aligned} \tag{2.58}$$

Define $H : (0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{H} : (0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned} H(x,y) &= 6a^2y^3 + a^2M_2y^2x + M_3(36b^2-a^2)y^2 + (6a^2+12M_2M_3b^2)yx \\ &\quad + (a^2M_2 + M_3M_2^2b^2)x^2 - a^2M_3x \end{aligned}$$

and

$$\begin{aligned} \overline{H}(x,y) &= 36a^2y^4 + 12a^2M_2y^3x + a^2M_2^2y^2x^2 + 36a^2y^2x + M_4(144b^2-4a^2)y^2 \\ &\quad + 12a^2M_2yx^2 + 48b^2M_2M_4yx + a^2M_2^2x^3 + 4b^2M_2^2M_4x^2 - 4a^2M_4x. \end{aligned}$$

Observe that, from (2.58), $H(s^2,(su'(s))^2) = 0 = \overline{H}(s^2,(su'(s))^2)$. By claim 4 we have $M_3 \neq 0$. Since the curve $(s^2,(su'(s))^2)$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $H$ and $\overline{H}$ must have at least one factor in common, a contradiction with Lemma $A.0.13$ and thus we conclude the claim.

- If $G_{s_0} + B_{s_0}F_{s_0}^2 - F_{s_0}D_{s_0} \neq 0$ or $H_{s_0} - A_{s_0}F_{s_0}^3 + C_{s_0}F_{s_0}^2 - E_{s_0}F_{s_0} \neq 0$ for some $s_0 \in \tilde{J} \subset J$, by claim 2, the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$, that is, for $s \in V \subset \tilde{J} \subset J$. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = A_sX^3 + B_sX^2Y + C_sX^2 + XY^2 + D_sXY + E_sX + F_sY^2 + G_sY + H_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, a contradiction with claim 1. Thus, the curve $((v'(t))^2, v''(t))$ is constant and thus there exist $\mu,\eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$.

- If $G_s + B_sF_s^2 - F_sD_s = 0$ and $H_s - A_sF_s^3 + C_sF_s^2 - E_sF_s = 0$, for all $s \in \tilde{J} \subset J$, we obtain, by the claim 3,

$$P_s(X,Y) = (F_s + X)(A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + E_s - F_s(C_s - A_sF_s)),$$

for all $s \in \tilde{J} \subset J$. Suppose that $F_s$ and the curve $((v'(t))^2, v''(t))$ are non-constant. Then, there exist $s_0, s_1 \in \tilde{J} \subset J$ such that $F_{s_0} \neq F_{s_1}$ and therefore, by continuity, there exists a neighborhood $V$ of $s_1 \in \tilde{J} \subset J$ such that $F_s \neq F_{s_0}$ for all $s \in V$. By the theory of Plane Algebraic Curves, $P_s$, for $s \in V$, must have a factor in common with $P_{s_0}$ and since such factor cannot be $X + F_s$ it is that of degree 2. Since the polynomial of degree 2 has a monic coefficient we obtain that

$$A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + E_s - F_s(C_s - A_sF_s)$$

is constant in the variable $s$, that is, the polynomials are all equal in $V$, and then, their coefficients are constant in $V$. Since, by hypothesis, $G_s + B_sF_s^2 - F_sD_s = 0$, that is, $G_s = (D_s - B_sF_s)F_s$, and

$H_s - A_s F_s^3 + C_s F_s^2 - E_s F_s = 0$, that is, $H_s = (E_s - F_s(C_s - A_s F_s))F_s$ we obtain

$$B_s = M_2$$
$$G_s = M_3 F_s$$
$$H_s = M_4 F_s,$$

a contradiction with claim 6. Thus, either $F_s$ is constant in $\tilde{J} \subset J$ or the curve $((v'(t))^2, v''(t))$ is constant. If $((v'(t))^2, v''(t))$ is constant then there exist $\mu, \eta$ such that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. If $F_s$ is constant in $\tilde{J} \subset J$ we obtain that if $(v'(t))^2 + F_s = 0$ for all $t \in I$ then $v'(t)$ is constant, that is, $v(t)$ is a straight line. Suppose there exists $t_0 \in J$ such that $(v'(t_0))^2 + F_s \neq 0$. Then there exists maximal interval $(\alpha, \beta) \subset I$ containing $t_0$ such that $(v'(t))^2 + F_s \neq 0$ for all $t \in (\alpha, \beta)$. Since the curve $((v'(t))^2, v''(t))$ vanishes the family of polynomials $P_s$ we conclude that its restriction to the interval $(\alpha, \beta)$ vanishes

$$A_s X^2 + B_s XY + (C_s - A_s F_s)X + Y^2 + (D_s - B_s F_s)Y + E_s - F_s(C_s - A_s F_s)$$

which, by claim 3, are irreducible and have a monic coefficient. Since, by hypothesis, $G_s + B_s F_s^2 - F_s D_s = 0$, that is, $G_s = (D_s - B_s F_s)F_s$, and $H_s - A_s F_s^3 + C_s F_s^2 - E_s F_s = 0$, that is, $H_s = (E_s - F_s(C_s - A_s F_s))F_s$ we obtain

$$B_s = M_2$$
$$G_s = M_3 F_s$$
$$H_s = M_4 F_s,$$

a contradiction with claim 6 and then we conclude by the theory of Plane Algebraic Curves that the curve $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta)$. Therefore, $(v'(t))^2$ is constant and different from $-F_s$ by the definition of $(\alpha, \beta)$. Finally, note that $\alpha \notin I$ because otherwise we would have $(v'(\alpha))^2 = -F_s$, a contradiction with the continuity of $v'(t)$. In the same way, $\beta \notin I$, and being $(\alpha, \beta)$ maximal we conclude that $(\alpha, \beta) = I$, that is, $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta) = I$ and therefore $v(t)$ is a line in $I$. $\square$

**Theorem 2.1.12.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t) + u(s), s, t)$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, of Null Linear Weingarten Form. Then:*

*i) $v$ is constant and $u$ is constant.*

*ii) $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Moreover, the function $u$ satisfies the differential equation*

$$u''(s) = -\frac{2b\mu^2(1 + (u'(s))^2) - au'(s)(1 + (u'(s))^2)(4\mu^2 + s^2)\sqrt{\frac{\mu^2}{s^2} + (u'(s))^2 + 1}}{as(s^2 + \mu^2)\sqrt{\frac{\mu^2}{s^2} + (u'(s))^2 + 1} - 2bs\mu^2 u'(s)}.$$

**Proof:** Suppose that $\psi$ is of Null Linear Weingarten Form. By Proposition 2.1.27 the functions $u$ and $v$ satisfy (assuming without loss of generality $b = 1$)

$$\begin{aligned}0 = a&\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\right)\\ -a&\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\\ +&\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2}{s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^2}\right).\end{aligned} \tag{2.59}$$

**-** $i)$ If $u$ is constant in $J$ we have (2.59)

$$0 = \frac{asv''(t)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+1\right)^{\frac{3}{2}}} - \frac{(v'(t))^2}{s^2\left(\left(\frac{v'(t)}{s}\right)^2+1\right)^2}, \tag{2.60}$$

that is,

$$asv''(t)\sqrt{\left(\frac{v'(t)}{s}\right)^2+1} = 2(v'(t))^2\left(\left(\frac{v'(t)}{s}\right)^2+1\right)$$

and squaring it

$$0 = ((v'(t))^2+s^2)\left(\frac{a^2s^4(v''(t))^2-4(v'(t))^6-4s^2(v'(t))^4}{s^4}\right)$$

i.e.,

$$0 = a^2s^4(v''(t))^2-4(v'(t))^6-4s^2(v'(t))^4$$

and thus the curve $((v'(t))^2,(v''(t))^2)$ is contained in the intersection of the roots of the family of irreducible polynomials

$$P_s(X,Y) = a^2s^4Y-4X^3-4s^2X^2.$$

By the theory of Plane Algebraic Curves such a curve is constant or the polynomials $P_s$ are multiples by constants. Since the second possibility does not occur we obtain that $((v'(t))^2,(v''(t))^2)$ is constant and therefore there exist $\mu,\eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. Substituting in (2.60) we obtain

$$0 = \frac{asv''(t)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+1\right)^{\frac{3}{2}}} - \frac{(v'(t))^2}{s^2\left(\left(\frac{v'(t)}{s}\right)^2+1\right)^2} = \frac{\mu^2s^2}{(\mu^2+s^2)^2}$$

therefore $\mu = 0$, that is, $v$ is constant in $I$.

- $ii)$ If $u$ is non-constant in $J$ we have by Proposition 2.1.28 that $v(t) = \mu t + \tau$, for constant $\mu, \tau$, and $I \subset \mathbb{R}$. Then (2.59)

$$\begin{aligned} 0 = &\left( as(s^2+\mu^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1} - 2s\mu^2 u'(s) \right) u''(s) \\ &+ 2\mu^2(1+(u'(s))^2) - au'(s)(1+(u'(s))^2)(4\mu^2+s^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}. \end{aligned} \tag{2.61}$$

We have two situations. If $\mu = 0$ we have from (2.61) that

$$0 = \left( as^3\sqrt{(u'(s))^2+1} \right) u''(s) - as^2u'(s)(1+(u'(s))^2)\sqrt{(u'(s))^2+1},$$

that is,

$$u''(s) = \frac{u'(s)(1+(u'(s))^2)}{s}.$$

If $\mu \neq 0$ we have from (2.61) that $as(s^2+\mu^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1} - 2s\mu^2u'(s) \neq 0$ in $J$. Indeed, if there exists $s_0 \in J$ such that $as_0(s_0^2+\mu^2)\sqrt{\frac{\mu^2}{s_0^2}+(u'(s_0))^2+1} - 2s_0\mu^2u'(s_0) = 0$ we obtain, applying $s_0$ in (2.61),

$$\begin{aligned} 0 =& as_0(s_0^2+\mu^2)\sqrt{\frac{\mu^2}{s_0^2}+(u'(s_0))^2+1} - 2s_0\mu^2u'(s_0) \\ 0 =& 2\mu^2(1+(u'(s_0))^2) - au'(s_0)(1+(u'(s_0))^2)(4\mu^2+s_0^2)\sqrt{\frac{\mu^2}{s_0^2}+(u'(s_0))^2+1}. \end{aligned}$$

Isolating the square root in the first equality and substituting it in the second we obtain

$$0 = -2\mu^2(1+(u'(s_0))^2)\left( \frac{s_0^2(u'(s_0))^2+4\mu^2(u'(s_0))^2+\mu^2+s_0^2}{\mu^2+s_0^2} \right) \neq 0$$

a contradiction. Then, from (2.61), we conclude

$$u''(s) = -\frac{2\mu^2(1+(u'(s))^2) - au'(s)(1+(u'(s))^2)(4\mu^2+s^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}}{as(s^2+\mu^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1} - 2s\mu^2u'(s)}$$

in $J$ and thus we conclude the proof. $\square$

### 2.1.15 *Type II Translation Surfaces of Linear Weingarten Form*

By Proposition 2.1.17 we conclude that a **Type II Translation Surface** is of **Linear Weingarten Form** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.29.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, is of Linear Weingarten Form, if and only if*

$$c = a\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\right)$$
$$-a\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}$$
$$+b\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2}{s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^2}\right).$$

**Proposition 2.1.30.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, of Linear Weingarten Form with constant $c \neq 0$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is of Linear Weingarten Form. By Proposition 2.1.29 the functions $u, v$ satisfy the equation,

$$c = a\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}\right)$$
$$-a\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{2s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}}$$
$$+b\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2}{s^2\left(\left(\frac{v'(t)}{s}\right)^2+(u'(s))^2+1\right)^2}\right)$$

or equivalently

$$
\begin{aligned}
0 = &-\frac{4c^2}{a^2 s^4\left((u'(s))^2+1\right)^2}(v'(t))^8+(v'(t))^2(v''(t))^2\\
&-\frac{4b^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2-a^2s^2\left((u'(s))^2+1\right)^3}{a^2\left((u'(s))^2+1\right)^2}(v''(t))^2\\
&-\frac{4c\left(4c\left((u'(s))^2+1\right)-2b\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)\right)}{a^2s^2\left((u'(s))^2+1\right)^2}(v'(t))^6\\
&+\frac{a^2\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)^2}{a^2s^2\left((u'(s))^2+1\right)^2}(v'(t))^6\\
&-\frac{8bc\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2s\left((u'(s))^2+1\right)^2}(v'(t))^4v''(t)\\
&-\frac{2a^2\left((u'(s))^2+1\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2s\left((u'(s))^2+1\right)^2}(v'(t))^4v''(t)\\
&-\frac{\left(4c\left((u'(s))^2+1\right)-2b\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)\right)^2}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^4\\
&+\frac{a^2\left((u'(s))^2+1\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)^2}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^4\\
&+\frac{2a^2\left((u'(s))^3+u'(s)-su''(s)\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^4\\
&-\frac{8c^2\left((u'(s))^2+1\right)^2}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^4+2s^3(4bc+a^2)\frac{u'(s)+su''(s)-(u'(s))^3}{a^2}v''(t)\\
&-\frac{2a^2s\left((u'(s))^2+1\right)^2\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2v''(t)\\
&-\frac{2a^2s\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2v''(t)\\
&-\frac{16cbs\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2v''(t)\\
&+\frac{8b^2s\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2v''(t)\\
&-\frac{4cs^2\left((u'(s))^2+1\right)^2\left(4c\left((u'(s))^2+1\right)-2b\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right)-\left((u'(s))^2+1\right)^2\right)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2\\
&+\frac{a^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2\\
&+\frac{2a^2s^2\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}(v'(t))^2\\
&-\frac{4c^2s^4\left((u'(s))^2+1\right)^4-a^2s^4\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)^2}{a^2\left((u'(s))^2+1\right)^2}.
\end{aligned}
$$

Define the family of polynomials

$$P_s(X,Y) = A_sX^4 + B_sX^3 + C_sX^2Y + D_sX^2 + XY^2 + E_sXY + F_sX + G_sY^2 + H_sY + I_s$$

where

$$A_s = -\frac{4c^2}{a^2s^4\left((u'(s))^2+1\right)^2}$$

$$B_s = -\frac{4c\left(4c\left((u'(s))^2+1\right) - 2b\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right) - \left((u'(s))^2+1\right)^2\right)\right)}{a^2s^2\left((u'(s))^2+1\right)^2}$$
$$+\frac{a^2\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)^2}{a^2s^2\left((u'(s))^2+1\right)^2}$$

$$C_s = -\frac{8bc\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2s\left((u'(s))^2+1\right)^2}$$
$$-\frac{2a^2\left((u'(s))^2+1\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2s\left((u'(s))^2+1\right)^2}$$

$$D_s = -\frac{\left(4c\left((u'(s))^2+1\right) - 2b\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right) - \left((u'(s))^2+1\right)^2\right)\right)^2}{a^2\left((u'(s))^2+1\right)^2}$$
$$+\frac{a^2\left((u'(s))^2+1\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)^2}{a^2\left((u'(s))^2+1\right)^2}$$
$$+\frac{2a^2\left((u'(s))^3+u'(s)-su''(s)\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}$$
$$-\frac{8c^2\left((u'(s))^2+1\right)^2}{a^2\left((u'(s))^2+1\right)^2}$$

$$E_s = -\frac{2a^2s\left((u'(s))^2+1\right)^2\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}$$
$$-\frac{2a^2s\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}$$
$$-\frac{16cbs\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}$$
$$+\frac{8b^2s\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right) - \left((u'(s))^2+1\right)^2\right)\left((u'(s))^3+u'(s)-su''(s)\right)}{a^2\left((u'(s))^2+1\right)^2}$$

$$F_s = -\frac{4cs^2\left((u'(s))^2+1\right)^2\left(4c\left((u'(s))^2+1\right) - 2b\left(u'(s)\left((u'(s))^3+u'(s)-su''(s)\right) - \left((u'(s))^2+1\right)^2\right)\right)}{a^2\left((u'(s))^2+1\right)^2}$$
$$+\frac{a^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2}{a^2\left((u'(s))^2+1\right)^2}$$
$$+\frac{2a^2s^2\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)\left(u'(s)-su''(s)+(u'(s))^3+3u'(s)\left((u'(s))^2+1\right)\right)}{a^2\left((u'(s))^2+1\right)^2}$$

$$G_s = -\frac{\left(4b^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2 - a^2s^2\left((u'(s))^2+1\right)^3\right)}{a^2\left((u'(s))^2+1\right)^2}$$
$$H_s = 2s^3(4bc+a^2)\frac{u'(s)+su''(s)-(u'(s))^3}{a^2}$$
$$I_s = -\frac{\left(4c^2s^4\left((u'(s))^2+1\right)^4 - a^2s^4\left((u'(s))^2+1\right)\left((u'(s))^3+u'(s)-su''(s)\right)^2\right)}{a^2\left((u'(s))^2+1\right)^2}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim 1:** Let $V \subset J$ be open set. Then $A_s, H_s$ are not mutually constant in $V$. Indeed, if there exist constants $K_1, K_7$ such that

$$A_s = K_1$$
$$H_s = K_7,$$

that is,

$$\begin{aligned} K_1 &= -\frac{4c^2}{a^2s^4\left((u'(s))^2+1\right)^2} \\ K_7 &= 2s^3(4bc+a^2)\frac{u'(s)+su''(s)-(u'(s))^3}{a^2}. \end{aligned} \tag{2.62}$$

By the first equality we obtain

$$0 = a^2s^4K_1(u'(s))^4 + 2a^2K_1(u'(s))^2s^4 + a^2K_1s^4 + 4c^2 \tag{2.63}$$

that is, $(u'(s))^2$ is a root of the nonzero polynomial

$$Q(Z) = a^2s^4K_1Z^2 + 2a^2K_1s^4Z + a^2K_1s^4 + 4c^2$$

and therefore $(u'(s))^2 = \frac{-a^2s^2K_1 \pm \sqrt{-4a^2c^2K_1}}{a^2s^2K_1}$. Notice that differentiating this equality in both sides we obtain

$$u'(s)u''(s) = -\frac{1}{s}\left(1+(u'(s))^2\right). \tag{2.64}$$

By the second equality of (2.62)

$$K_7u'(s) = 2s^3(4bc+a^2)\frac{(u'(s))^2+su'(s)u''(s)-(u'(s))^4}{a^2}$$

which substituting the equality obtained in (2.64)

$$0 = (2a^2+8bc)(u'(s))^4s^3 + (4a^2+16bc)(u'(s))^2s^3 + a^2K_7u'(s) + (2a^2+8bc)s^3$$

and multiplying by $s$ we obtain

$$0 = (2a^2+8bc)(u'(s))^4s^4 + (4a^2+16bc)(u'(s))^2s^4 + a^2K_7su'(s) + (2a^2+8bc)s^4. \tag{2.65}$$

Multiplying the equality obtained $(2.63)$ by $2a^2+8bc$ and the equality obtained in $(2.65)$ by $a^2K_1$ we have

$$0 = a^2K_1(2a^2+8bc)(u'(s))^4s^4 + 2a^2K_1(2a^2+8bc)(u'(s))^2s^4 + a^2K_1(2a^2+8bc)s^4 + 4c^2(2a^2+8bc)$$
$$0 = a^2K_1(2a^2+8bc)(u'(s))^4s^4 + a^2K_1(4a^2+16bc)(u'(s))^2s^4 + a^4K_1K_7su'(s) + a^2K_1(2a^2+8bc)s^4$$

and subtracting the second equality from the first

$$0 = (8K_1a^4+32bcK_1a^2)(u'(s))^2s^4 + a^4K_1K_7su'(s) + (4K_1a^4+16bcK_1a^2)s^4 + 8a^2c^2 + 32bc^3. \quad (2.66)$$

Define $G:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$G(x,y) = a^2K_1y^4 + 2a^2K_1y^2x + a^2K_1x^2 + 4c^2$$

and

$$\overline{G}(x,y) = (8K_1a^4+32bcK_1a^2)y^2x + a^4K_1K_7y + (4K_1a^4+16bcK_1a^2)x^2 + 8a^2c^2 + 32bc^3.$$

Observe that, from $(2.63)$ and $(2.66)$, $G(s^2,su'(s)) = 0 = \overline{G}(s^2,su'(s))$. Since the curve $(s^2,su'(s))$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A.0.16$ and thus we conclude the claim.

**Claim 2:** If $H_{s_0} - G_{s_0}(E_{s_0} - C_{s_0}G_{s_0}) \neq 0$ or $I_{s_0} - G_{s_0}(F_{s_0} - G_{s_0}(D_{s_0} - G_{s_0}(B_{s_0} - A_{s_0}G_{s_0}))) \neq 0$, for some $s_0 \in J$, then the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$. Indeed, observe that

$$P_s(X,Y) = (G_s+X)Y^2 + (C_sX^2+E_sX+H_s)Y + A_sX^4 + B_sX^3 + D_sX^2 + F_sX + I_s$$

then if given $s\in V$ the polynomial $P_s$ is reducible we have two possible situations. In the first case

$$P_s(X,Y) = (R(X)Y+S(X))\,(T(X)Y+U(X))$$

so $R(X)T(X) = X+G_s$ and then we can assume without loss of generality $T(X)=1$ and $R(X) = X+G_s$, that is,

$$P_s(X,Y) = ((G_s+X)Y+S(X))\,(Y+U(X))$$

thus $U(X)(G_s+X)+S(X) = C_sX^2+E_sX+H_s$ and $S(X)U(X) = A_sX^4+B_sX^3+D_sX^2+F_sX+I_s$. Since $A_s\neq 0$ we have that

$$\deg\,(S(X)U(X)) = \deg\left(A_sX^4+B_sX^3+D_sX^2+F_sX+I_s\right) = 4$$

and

$$\deg\,(U(X)(G_s+X)+S(X)) = \deg\left(C_sX^2+E_sX+H_s\right) \leqslant 2$$

and then we have the possibilities

$$\begin{aligned}
&\deg(S(X)) = 0 \text{ and } \deg(U(X)) = 4 \text{ then } \deg\left(U(X)(G_s+X)+S(X)\right) = 5 > 2\\
&\deg(S(X)) = 1 \text{ and } \deg(U(X)) = 3 \text{ then } \deg\left(U(X)(G_s+X)+S(X)\right) = 4 > 2\\
&\deg(S(X)) = 2 \text{ and } \deg(U(X)) = 2 \text{ then } \deg\left(U(X)(G_s+X)+S(X)\right) = 3 > 2\\
&\deg(S(X)) = 3 \text{ and } \deg(U(X)) = 1 \text{ then } \deg\left(U(X)(G_s+X)+S(X)\right) = 3 > 2\\
&\deg(S(X)) = 4 \text{ and } \deg(U(X)) = 1 \text{ then } \deg\left(U(X)(G_s+X)+S(X)\right) = 4 > 2,
\end{aligned}$$

that is, the first case does not occur. In the second case

$$P_s(X,Y) = U(X)\left(R(X)Y^2 + S(X)Y + T(X)\right)$$

where $U(X)$ is non-constant and $T(X)$ is non-zero. Therefore, $R(X)U(X) = X + G_s$ and then we can assume, without loss of generality, $R(X) = 1$ and $U(X) = X + G_s$, that is,

$$\begin{aligned}
(X+G_s)S(X) &= C_sX^2 + E_sX + H_s\\
(X+G_s)T(X) &= A_sX^4 + B_sX^3 + D_sX^2 + F_sX + I_s
\end{aligned}$$

and consequently $\deg(S(X)) \leqslant 1$ and $\deg(T(X)) = 3$ (since $A_s \neq 0$). Therefore,

$$\begin{aligned}
S(X) &= fX + g\\
T(X) &= kX^3 + lX^2 + mX + n
\end{aligned}$$

with $k \neq 0$, then

$$P_s(X,Y) = (X+G_s)\left(Y^2 + (fX+g)Y + kX^3 + lX^2 + mX + n\right)$$

that is,

$$0 = P_s(X,Y) - (X+G_s)\left(Y^2 + (fX+g)Y + kX^3 + lX^2 + mX + n\right)$$

that reorganizing

$$\begin{aligned}
0 = {} & (A_s - k)X^4 + (B_s - l - G_sk)X^3 + (C_s - f)X^2Y + (D_s - m - G_sl)X^2\\
& + (E_s - g - G_sf)XY + (F_s - n - G_sm)X + (H_s - G_sg)Y + I_s - G_sn
\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll}
0 = A_s - k & 0 = D_s - m - G_sl & \\
0 = B_s - l - G_sk & 0 = E_s - g - G_sf & 0 = H_s - G_sg\\
0 = C_s - f & 0 = F_s - n - G_sm & 0 = I_s - G_sn.
\end{array}$$

We have,

$$\begin{array}{lll}
k = A_s & 0 = D_s - m - G_sl & \\
0 = B_s - l - G_sk & 0 = E_s - g - G_sf & 0 = H_s - G_sg\\
f = C_s & 0 = F_s - n - G_sm & 0 = I_s - G_sn,
\end{array}$$

that is,

$$0 = B_s - l - G_s A_s \qquad 0 = E_s - g - G_s C_s \qquad 0 = H_s - G_s g$$
$$0 = D_s - m - G_s l \qquad 0 = F_s - n - G_s m \qquad 0 = I_s - G_s n.$$

We have,

$$l = B_s - G_s A_s \qquad g = E_s - G_s C_s \qquad 0 = H_s - G_s g$$
$$0 = D_s - m - G_s l \qquad n = F_s - G_s m \qquad 0 = I_s - G_s n,$$

That is,

$$0 = D_s - m - G_s(B_s - G_s A_s)$$
$$0 = H_s - G_s(E_s - G_s C_s)$$
$$0 = I_s - G_s(F_s - G_s m).$$

Thus,

$$m = D_s - G_s(B_s - G_s A_s)$$
$$0 = H_s - G_s(E_s - G_s C_s)$$
$$0 = I_s - G_s(F_s - G_s m),$$

obtaining

$$0 = H_s - G_s(E_s - G_s C_s)$$
$$0 = I_s - G_s(F_s - G_s(D_s - G_s(B_s - G_s A_s)))$$

for all $s \in V$, a contradiction with the hypothesis, and thus we proof the claim.

**Claim 3:** If $H_s - G_s(E_s - C_s G_s) = 0$ and $I_s - G_s(F_s - G_s(D_s - G_s(B_s - A_s G_s))) = 0$, for all $s \in J$, then the polynomials $P_s$ are uniquely written as

$$\begin{aligned} P_s(X,Y) = {} & (X + G_s)\left(A_s X^3 + (B_s - A_s G_s)X^2 + C_s XY + (D_s - G_s(B_s - A_s G_s))X + Y^2\right) \\ & + (X + G_s)\left((E_s - C_s G_s)Y + F_s - G_s(D_s - G_s(B_s - A_s G_s))\right) \end{aligned}$$

in which both factors are irreducible. Indeed, defining the right-side as

$$\begin{aligned} \Phi_s(X,Y) = {} & (X + G_s)\left(A_s X^3 + (B_s - A_s G_s)X^2 + C_s XY + (D_s - G_s(B_s - A_s G_s))X + Y^2\right) \\ & + (X + G_s)\left((E_s - C_s G_s)Y + F_s - G_s(D_s - G_s(B_s - A_s G_s))\right) \end{aligned}$$

que conclude

$$\begin{aligned} P_s(X,Y) - \Phi_s(X,Y) &= (H_s - G_s(E_s - C_s G_s))Y + I_s - G_s(F_s - G_s(Z_s - G_s(B_s - A_s G_s))) \\ &= 0. \end{aligned}$$

Finally, $X + G_s$ is irreducible and therefore it remains to show that

$$\begin{aligned} & A_s X^3 + (B_s - A_s G_s)X^2 + C_s XY + (D_s - G_s(B_s - A_s G_s))X + Y^2 \\ & + (E_s - C_s G_s)Y + F_s - G_s(D_s - G_s(B_s - A_s G_s)) \end{aligned}$$

is irreducible. Indeed, suppose that there exist polynomials of degree 2 and degree 1 such that

$$\begin{aligned} (f + gX + hY)(i + jX + kY + lXY + mX^2 + nY^2) = {} & A_s X^3 + (B_s - A_s G_s)X^2 + C_s XY \\ & + (D_s - G_s(B_s - A_s G_s))X + Y^2 \\ & + (E_s - C_s G_s)Y + F_s - G_s(D_s - G_s(B_s - A_s G_s)) \end{aligned}$$

that is,

$$\begin{aligned}0 &= (A_s - gm)X^3 - (gl + hm)X^2Y + (B_s - A_sG_s - gj - fm)X^2 - (hl + gn)XY^2\\&+ (C_s - fl - gk - hj)XY + (D_s - G_s(B_s - A_sG_s) - gi - fj)X - hnY^3 + (1 - fn - hk)Y^2\\&+ (E_s - hi - C_sG_s - fk)Y + F_s - fi - G_s(B_s - A_sG_s)\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll}0 = A_s - gm & & \\ 0 = gl + hm & 0 = C_s - fl - gk - hj & 0 = 1 - fn - hk\\ 0 = B_s - A_sG_s - gj - fm & 0 = D_s - G_s(B_s - A_sG_s) - gi - fj & 0 = E_s - hi - C_sG_s - fk\\ 0 = hl + gn & 0 = hn & 0 = F_s - fi - G_s(B_s - A_sG_s).\end{array} \tag{2.67}$$

By the seventh equality we obtain $h = 0$ or $n = 0$. Let us suppose $h = 0$. We have

$$\begin{array}{lll}0 = A_s - gm & 0 = gn & 0 = 1 - fn\\ 0 = gl & 0 = C_s - fl - gk & 0 = E_s - C_sG_s - fk\\ 0 = B_s - A_sG_s - gj - fm & 0 = D_s - G_s(B_s - A_sG_s) - gi - fj & 0 = F_s - fi - G_s(B_s - A_sG_s).\end{array}$$

Since $g \neq 0$ (otherwise we would not have factorization) we obtain by the fourth equality that $n = 0$ which substituting in the seventh equality we obtain $1 = 0$, that is, a contradiction. Thus $n = 0$ and therefore from (2.67) we have

$$\begin{array}{lll}0 = A_s - gm & 0 = hl & 0 = 1 - hk\\ 0 = gl + hm & 0 = C_s - fl - gk - hj & 0 = E_s - hi - C_sG_s - fk\\ 0 = B_s - A_sG_s - gj - fm & 0 = D_s - G_s(B_s - A_sG_s) - gi - fj & 0 = F_s - fi - G_s(B_s - A_sG_s).\end{array}$$

By the seventh equality we have $h = \frac{1}{k}$ which substituting into the fourth equality we obtain $l = 0$ and then substituting this into the second equality we obtain $m = 0$, that is, we have no factorization and thus we proof the claim.

**Claim 4:** Let $V \subset J$ be open set. Then there are no constants $M_1, M_9$ such that

$$A_s = M_1$$
$$I_s = M_9 G_s,$$

in $V$. Indeed, if there are such constants $M_1, M_9$ we have

$$\begin{aligned}M_1 &= -\frac{4c^2}{a^2s^4\left((u'(s))^2+1\right)^2}\\ M_9\frac{4b^2s^2\left((u'(s))^3+u'(s)-su''(s)\right)^2 - a^2s^2\left((u'(s))^2+1\right)^3}{a^2\left((u'(s))^2+1\right)^2} &= \frac{4c^2s^4\left((u'(s))^2+1\right)^4}{a^2\left((u'(s))^2+1\right)^2}\\ &- \frac{a^2s^4(u'(s))^2\left((u'(s))^3+u'(s)-su''(s)\right)^2}{a^2\left((u'(s))^2+1\right)^2}\\ &- \frac{a^2s^4\left((u'(s))^3+u'(s)-su''(s)\right)^2}{a^2\left((u'(s))^2+1\right)^2}.\end{aligned} \tag{2.68}$$

By the first equality we obtain

$$0 = a^2s^4M_1(u'(s))^4 + 2a^2M_1(u'(s))^2s^4 + a^2M_1s^4 + 4c^2 \tag{2.69}$$

that is, $(u'(s))^2$ is a root of the nonzero polynomial

$$Q(Z) = a^2s^4M_1Z^2 + 2a^2M_1s^4Z + a^2M_1s^4 + 4c^2$$

and thus $(u'(s))^2 = \frac{-a^2s^2M_1 \pm \sqrt{-4a^2c^2M_1}}{a^2s^2M_1}$. Observe that differentiating this equality in both sides we obtain

$$2u'(s)u''(s) = -\frac{2}{s}\left(1 + (u'(s))^2\right). \tag{2.70}$$

By the second equality of (2.68) we obtain

$$\begin{aligned}
0 = &-((1+(u'(s))^2)a^2s^6 + 4M_9b^2s^4)(u''(s))^2\\
&+ (2a^2s^5u'(s)(1+(u'(s))^2)^2 + 8M_9b^2s^3u'(s)(1+(u'(s))^2))u''(s)\\
&+ M_9(a^2s^2(1+(u'(s))^2)^3 - 4b^2s^2(u'(s))^2(1+(u'(s))^2)^2)\\
&+ 4c^2s^4(1+(u'(s))^2)^4 - a^2s^4(u'(s))^2(1+(u'(s))^2)^3
\end{aligned}$$

which multiplying by $4(u'(s))^2$

$$\begin{aligned}
0 = &-4((1+(u'(s))^2)a^2s^6 + 4M_9b^2s^4)(u'(s)u''(s))^2\\
&+ 4(2a^2s^5u'(s)(1+(u'(s))^2)^2 + 8M_9b^2s^3u'(s)(1+(u'(s))^2))(u'(s))^2u''(s)\\
&+ 4M_9(u'(s))^2(a^2s^2(1+(u'(s))^2)^3 - 4b^2s^2(u'(s))^2(1+(u'(s))^2)^2)\\
&+ 4(u'(s))^2\left(4c^2s^4(1+(u'(s))^2)^4 - a^2s^4(u'(s))^2(1+(u'(s))^2)^3\right).
\end{aligned}$$

and then substituting the equality (2.70)

$$\begin{aligned}
0 = &\,(16c^2-4a^2)(u'(s))^{10}s^4 + (64c^2-20a^2)(u'(s))^8s^4 + M_9(4a^2-16b^2)(u'(s))^8s^2\\
&+ (96c^2-40a^2)(u'(s))^6s^4 + M_9(12a^2-64b^2)(u'(s))^6s^2 + (64c^2-40a^2)(u'(s))^4s^4\\
&+ M_9(12a^2-96b^2)(u'(s))^4s^2 + (16c^2-20a^2)(u'(s))^2s^4\\
&+ M_9(4a^2-64b^2)(u'(s))^2s^2 - 4a^2s^4 - 16M_9b^2s^2
\end{aligned}$$

and finally multiplying by $s^6$

$$\begin{aligned}
0 = &\,(16c^2-4a^2)(u'(s))^{10}s^{10} + (64c^2-20a^2)(u'(s))^8s^{10} + M_9(4a^2-16b^2)(u'(s))^8s^8\\
&+ (96c^2-40a^2)(u'(s))^6s^{10} + M_9(12a^2-64b^2)(u'(s))^6s^8 + (64c^2-40a^2)(u'(s))^4s^{10}\\
&+ M_9(12a^2-96b^2)(u'(s))^4s^8 + (16c^2-20a^2)(u'(s))^2s^{10}\\
&+ M_9(4a^2-64b^2)(u'(s))^2s^8 - 4a^2s^{10} - 16M_9b^2s^8.
\end{aligned} \tag{2.71}$$

On the other hand, observe that $M_9 \neq 0$, otherwise by the second equality of (2.68),

$$\begin{aligned}
0 = &-a^2s^6(1+(u'(s))^2)(u''(s))^2 + 2a^2s^5u'(s)(1+(u'(s))^2)^2u''(s)\\
&+ 4c^2s^4(1+(u'(s))^2)^4 - a^2s^4(u'(s))^2(1+(u'(s))^2)^3
\end{aligned}$$

which multiplying by $(u'(s))^2$

$$\begin{aligned}0=&-a^2s^6(1+(u'(s))^2)(u'(s)u''(s))^2+2a^2s^5(1+(u'(s))^2)^2(u'(s))^3u''(s)\\&+(u'(s))^2(4c^2s^4(1+(u'(s))^2)^4-a^2s^4(u'(s))^2(1+(u'(s))^2)^3)\end{aligned}$$

and substituting (2.70) we obtain

$$0=-s^4(1+(u'(s))^2)^4(a^2+(a^2-4c^2)(u'(s))^2)$$

i.e.,

$$0=a^2+(a^2-4c^2)(u'(s))^2$$

that is, $u'(s)$ is constant, a contradiction with the first equality of (2.68) (otherwise we would have $s$ constant). Then, $M_9\neq 0$. Define $H:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{H}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$H(x,y)=a^2M_1y^2+2a^2M_1xy+a^2M_1x^2+4c^2$$

and

$$\begin{aligned}\overline{H}(x,y)=&(16c^2-4a^2)y^5+(64c^2-20a^2)xy^4+M_9(4a^2-16b^2)y^4+(96c^2-40a^2)x^2y^3\\&+M_9(12a^2-64b^2)xy^3+(64c^2-40a^2)x^3y^2+M_9(12a^2-96b^2)x^2y^2+(16c^2-20a^2)x^4y\\&+M_9(4a^2-64b^2)x^3y-4a^2x^5-16M_9b^2x^4.\end{aligned}$$

Notice that, from (2.69) and (2.71), $H(s^2,(su'(s))^2)=0=\overline{H}(s^2,(su'(s))^2)$. Since the curve $(s^2,(su'(s))^2)$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $H$ and $\overline{H}$ must have at least one factor in common, a contradiction with Lemma $A$.0.17 and thus we conclude the claim.

- If $H_{s_0}-G_{s_0}(E_{s_0}-C_{s_0}G_{s_0})\neq 0$ or $I_{s_0}-G_{s_0}(F_{s_0}-G_{s_0}(D_{s_0}-G_{s_0}(B_{s_0}-A_{s_0}G_{s_0})))\neq 0$, for some $s_0\in J$, by claim 2, the polynomials $\{P_s\}_{s\in V}$ are irreducible in a neighborhood $V$ of $s_0$, that is, for $s\in V\subset J$. As observed earlier, the curve $((v'(t))^2,v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda=1$ we conclude that

$$P_s(X,Y)=A_sX^4+B_sX^3+C_sX^2Y+D_sX^2+XY^2+E_sXY+F_sX+G_sY^2+H_sY+I_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Thus, their coefficients are constant, a contradiction with claim 1. Then, the curve $((v'(t))^2,v''(t))$ is constant and therefore there exist $\mu,\eta$ such that $v'(t)=\mu$ and $v''(t)=\eta$ (in this case necessarily $\eta=0$) and we conclude that $v(t)=\mu t+\tau$ in $I\subset\mathbb{R}$.

- If $H_s - G_s(E_s - C_sG_s) = 0$ and $I_s - G_s(F_s - G_s(Z_s - G_s(B_s - A_sG_s))) = 0$, for all $s \in J$, we obtain, by claim 3, that

$$
\begin{aligned}
P_s(X,Y) = {} & (X+G_s)\left(A_sX^3 + (B_s - A_sG_s)X^2 + C_sXY + (D_s - G_s(B_s - A_sG_s))X + Y^2\right) \\
& + (X+G_s)\left((E_s - C_sG_s)Y + F_s - G_s(D_s - G_s(B_s - A_sG_s))\right),
\end{aligned}
$$

for all $s \in J$. Suppose that $G_s$ and the curve $((v'(t))^2, v''(t))$ are non-constant. Then, there exist $s_0, s_1 \in J$ such that $G_{s_0} \neq G_{s_1}$ and therefore, by continuity, there exists a neighborhood $V$ of $s_1 \in J$ such that $G_s \neq G_{s_0}$ for all $s \in V$. By the theory of Plane Algebraic Curves, $P_s$, for $s \in V$, must have a factor in common with $P_{s_0}$ and since such a factor cannot be $X + G_s$ it is that of degree 3. Since the polynomial of degree 3 has a monic coefficient we obtain that

$$A_sX^3 + (B_s - A_sG_s)X^2 + C_sXY + (D_s - G_s(B_s - A_sG_s))X + Y^2 + (E_s - C_sG_s)Y + F_s - G_s(D_s - G_s(B_s - A_sG_s))$$

is constant in the variable $s$, that is, the polynomials are all equal in $V$, and therefore, their coefficients are constant in $V$. Since, by hypothesis, $I_s - G_s(F_s - G_s(D_s - G_s(B_s - A_sG_s))) = 0$, that is, $I_s = G_s(F_s - G_s(D_s - G_s(B_s - A_sG_s)))$, we obtain

$$
\begin{aligned}
A_s &= M_1 \\
I_s &= M_9G_s,
\end{aligned}
$$

a contradiction with claim 4. Therefore, either $G_s$ is constant in $J$ or the curve $((v'(t))^2, v''(t))$ is constant. If $((v'(t))^2, v''(t))$ is constant then there exist $\mu, \eta$ such that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. If $G_s$ is constant in $J$ we obtain that if $(v'(t))^2 + G_s = 0$ for all $t \in I$ then $v'(t)$ is constant, that is, $v(t)$ is a straight line. Suppose there exists $t_0 \in J$ such that $(v'(t_0))^2 + G_s \neq 0$. Then, there exists maximal interval $(\alpha, \beta) \subset I$ containing $t_0$ such that $(v'(t))^2 + G_s \neq 0$ for all $t \in (\alpha, \beta)$. Since the curve $((v'(t))^2, v''(t))$ vanishes the family of polynomials $P_s$ we conclude that, its restriction to the interval $(\alpha, \beta)$ vanishes

$$A_sX^3 + (B_s - A_sG_s)X^2 + C_sXY + (D_s - G_s(B_s - A_sG_s))X + Y^2 + (E_s - C_sG_s)Y + F_s - G_s(D_s - G_s(B_s - A_sG_s))$$

which, by claim 3, are irreducible and have a monic coefficient. Since, by hypothesis, $I_s - G_s(F_s - G_s(D_s - G_s(B_s - A_sG_s))) = 0$, that is, $I_s = G_s(F_s - G_s(D_s - G_s(B_s - A_sG_s)))$, we obtain

$$
\begin{aligned}
A_s &= M_1 \\
I_s &= M_9G_s,
\end{aligned}
$$

a contradiction with claim 4 and then we conclude by the theory of Plane Algebraic Curves that the curve $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta)$. Therefore, $(v'(t))^2$ is constant and different from $-G_s$ by the definition of $(\alpha, \beta)$. Finally, note that $\alpha \notin I$ because otherwise we would have $(v'(\alpha))^2 = -G_s$, a contradiction with the continuity of $v'(t)$. In the same way, $\beta \notin I$, and being $(\alpha, \beta)$ maximal we conclude that $(\alpha, \beta) = I$, that is, $((v'(t))^2, v''(t))$ is constant in $(\alpha, \beta) = I$ and therefore $v(t)$ is a line in $I$. $\square$

**Theorem 2.1.13.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s), s, t)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, of Linear Weingarten Form with constant $c \neq 0$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover:*

*i) If $\mu = 0$ then*

$$u''(s) = \frac{((u'(s))^2+1)(au'(s)+2c\sqrt{(u'(s))^2+1})}{as}.$$

*ii) If $\mu \neq 0$ and the constants $a,b,c$ verifies*

$$\left(\frac{a}{b},\frac{c}{b}\right) \in \left(-\infty,-\sqrt[4]{2}\right] \bigcup \left[\sqrt[4]{2},+\infty\right) \times \left[-\left(\frac{3a^4}{4b^4}+\frac{5a^2}{4b^2}+\frac{1}{2}\right),+\infty\right),$$

*or*

$$\left(\frac{a}{b},\frac{c}{b}\right) \in \left[-\sqrt[4]{2},\sqrt[4]{2}\right] \times \left[-\left(\frac{a^4}{b^4}+\frac{5a^2}{4b^2}\right),+\infty\right)$$

*then*

$$\begin{aligned} u''(s) = {} & \frac{2b\mu^2(1+(u'(s))^2)}{as(s^2+\mu^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}-2b\mu^2 su'(s)} \\ & + \frac{au'(s)(1+(u'(s))^2)(4\mu^2+s^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}+2cs^2\left(\frac{\mu^2}{s^2}+(u'(s))^2+1\right)^2}{as(s^2+\mu^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}-2b\mu^2 su'(s)}. \end{aligned}$$

**Proof:** Suppose that $\psi$ is of Linear Weingarten Form. By Proposition 2.1.30 we have $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Therefore, by Proposition 2.1.29 (assuming without loss of generality $b = 1$)

$$\begin{aligned} c = {} & \frac{a(-u'(s)\mu^2(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)(\mu^2+s^2)-2u'(s)\mu^2(1+(u'(s))^2))}{2s^2\left(\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1\right)^{\frac{3}{2}}} \\ & + \frac{-u'(s)\mu^2(su''(s)-u'(s)-(u'(s))^3)-(\mu(1+(u'(s))^2))^2}{s^2\left(\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1\right)^2}, \end{aligned}$$

that is,

$$\begin{aligned} c = {} & \frac{a\sqrt{\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1}\left(-u'(s)\mu^2(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)(\mu^2+s^2)\right)}{2s^2\left(\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1\right)^2} \\ & + \frac{a\sqrt{\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1}\left(-2u'(s)\mu^2(1+(u'(s))^2)\right)}{2s^2\left(\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1\right)^2} \\ & + \frac{2(-u'(s)\mu^2(su''(s)-u'(s)-(u'(s))^3)-(\mu(1+(u'(s))^2))^2)}{2s^2\left(\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1\right)^2}, \end{aligned}$$

i.e.,

$$
\begin{aligned}
0 =& a\sqrt{\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1}\left(-u'(s)\mu^2(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)(\mu^2+s^2)\right)\\
&+a\sqrt{\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1}\left(-2u'(s)\mu^2(1+(u'(s))^2)\right)\\
&+2(-u'(s)\mu^2(su''(s)-u'(s)-(u'(s))^3)-(\mu(1+(u'(s))^2))^2)-2cs^2\left(\left(\frac{\mu}{s}\right)^2+(u'(s))^2+1\right)^2
\end{aligned}
$$

thus,

$$
\begin{aligned}
0 =& \left(as(s^2+\mu^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}-2\mu^2su'(s)\right)u''(s)\\
&-\left(2\mu^2(1+(u'(s))^2)+au'(s)(1+(u'(s))^2)(4\mu^2+s^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}\right)\\
&-2cs^2\left(\frac{\mu^2}{s^2}+(u'(s))^2+1\right)^2.
\end{aligned}
\tag{2.72}
$$

**-** *i*) Let $\mu=0$. From (2.72)

$$
0=\left(as^3\sqrt{(u'(s))^2+1}\right)u''(s)-\left(as^2u'(s)(1+(u'(s))^2)\sqrt{(u'(s))^2+1}+2cs^2\left((u'(s))^2+1\right)^2\right),
$$

that is,

$$
u''(s)=\frac{as^2u'(s)(1+(u'(s))^2)\sqrt{(u'(s))^2+1}+2cs^2\left((u'(s))^2+1\right)^2}{as^3\sqrt{(u'(s))^2+1}},
$$

i.e.,

$$
u''(s)=\frac{((u'(s))^2+1)(au'(s)+2c\sqrt{(u'(s))^2+1})}{as}.
$$

**-** *ii*) Let $\mu\neq 0$. We have from (2.72) that $as(s^2+\mu^2)\sqrt{\frac{\mu^2}{s^2}+(u'(s))^2+1}-2\mu^2su'(s)\neq 0$ in the interval $J$. Indeed, if there exists $s_0$ in $J$ such that $as_0(s_0^2+\mu^2)\sqrt{\frac{\mu^2}{s_0^2}+(u'(s_0))^2+1}-2\mu^2s_0u'(s_0)=0$ we obtain, applying $s_0$ in (2.72),

$$
\begin{aligned}
&0=as_0(s_0^2+\mu^2)\sqrt{\frac{\mu^2}{s_0^2}+(u'(s_0))^2+1}-2\mu^2s_0u'(s_0)\\
&0=2\mu^2(1+(u'(s_0))^2)+au'(s_0)(1+(u'(s_0))^2)(4\mu^2+s_0^2)\sqrt{\frac{\mu^2}{s_0^2}+(u'(s_0))^2+1}\\
&\quad+2cs_0^2\left(\frac{\mu^2}{s_0^2}+(u'(s_0))^2+1\right)^2.
\end{aligned}
$$

Isolating the square root in the first equality and substituting it in the second we obtain

$$\sqrt{\frac{\mu^2}{s_0^2} + (u'(s_0))^2 + 1} = \frac{2\mu^2 u'(s_0)}{a(\mu^2 + s_0^2)}$$
$$0 = \frac{4\mu^4 + \mu^2 s_0^2 + cs_0^4 + c\mu^2 s_0^2}{\mu^2 + s_0^2}(u'(s_0))^4 + \frac{5\mu^4 + 2\mu^2 s_0^2 + 2c\mu^4 + 2cs_0^4 + 4c\mu^2 s_0^2}{\mu^2 + s_0^2}(u'(s_0))^2 + \frac{\mu^2 s_0^2 + c\mu^4 + cs_0^4 + 2c\mu^2 s_0^2}{s_0}$$

that by squaring the first equality and arranging it again

$$0 = \left(1 - \frac{4\mu^4}{a^2(\mu^2 + s_0^2)^2}\right)(u'(s_0))^2 + 1 + \frac{\mu^2}{s_0^2}$$
$$0 = \frac{4\mu^4 + \mu^2 s_0^2 + cs_0^4 + c\mu^2 s_0^2}{\mu^2 + s_0^2}(u'(s_0))^4 + \frac{5\mu^4 + 2\mu^2 s_0^2 + 2c\mu^4 + 2cs_0^4 + 4c\mu^2 s_0^2}{\mu^2 + s_0^2}(u'(s_0))^2 + \frac{\mu^2 s_0^2 + c\mu^4 + cs_0^4 + 2c\mu^2 s_0^2}{s_0}.$$

Notice, in the first equality, that $1 - \frac{4\mu^4}{a^2(\mu^2+s_0^2)^2} \neq 0$ (otherwise $1 + \frac{\mu^2}{s_0^2} = 0$, a contradiction) and so we can isolate $(u'(s_0))^2$ in the first equality and substitute it in the second concluding that

$$0 = 4\mu^6 \frac{a^4 s_0^8 + \mu^2(4c + 5a^2 + 4a^4)s_0^6 + 2\mu^4(4c + 5a^2 + 3a^4 + 2)s_0^4 + \mu^6(4c + 5a^2 + 4a^4)s_0^2 + a^4\mu^8}{s_0^4(-2\mu^2 + a\mu^2 + as_0^2)^2(2\mu^2 + a\mu^2 + as_0^2)^2},$$

isto é,

$$0 = a^4 s_0^8 + \mu^2(4c + 5a^2 + 4a^4)s_0^6 + 2\mu^4(4c + 5a^2 + 3a^4 + 2)s_0^4 + \mu^6(4c + 5a^2 + 4a^4)s_0^2 + a^4\mu^8.$$

Define $M(c,a) = 4c + 5a^2$. We have

$$0 = a^4 s_0^8 + \mu^2(M(c,a) + 4a^4)s_0^6 + 2\mu^4(M(c,a) + 3a^4 + 2)s_0^4 + \mu^6(M(c,a) + 4a^4)s_0^2 + a^4\mu^8. \quad (2.73)$$

If $(a,c) \in \left(-\infty, -\sqrt[4]{2}\right] \bigcup \left[\sqrt[4]{2}, +\infty\right) \times \left[-\frac{3a^4+5a^2+2}{4}, +\infty\right)$ observe that $M(c,a) \geqslant -(3a^4+2) \geqslant -4a^4$ and then from (2.73)

$$0 = a^4 s_0^8 + \mu^2(M(c,a) + 4a^4)s_0^6 + 2\mu^4(M(c,a) + 3a^4 + 2)s_0^4 + \mu^6(M(c,a) + 4a^4)s_0^2 + a^4\mu^8 > 0,$$

a contradiction. If $(a,c) \in \left[-\sqrt[4]{2}, \sqrt[4]{2}\right] \times \left[-\frac{4a^4+5a^2}{4}, +\infty\right)$ observe that $M(c,a) \geqslant -4a^4 \geqslant -(3a^4+2)$ and then from (2.73)

$$0 = a^4 s_0^8 + \mu^2(M(c,a) + 4a^4)s_0^6 + 2\mu^4(M(c,a) + 3a^4 + 2)s_0^4 + \mu^6(M(c,a) + 4a^4)s_0^2 + a^4\mu^8 > 0,$$

a contradiction. Thus, from (2.72), we conclude

$$u''(s) = \frac{2\mu^2(1 + (u'(s))^2) + au'(s)(1 + (u'(s))^2)(4\mu^2 + s^2)\sqrt{\frac{\mu^2}{s^2} + (u'(s))^2 + 1} + 2cs^2\left(\frac{\mu^2}{s^2} + (u'(s))^2 + 1\right)^2}{as(s^2 + \mu^2)\sqrt{\frac{\mu^2}{s^2} + (u'(s))^2 + 1} - 2\mu^2 su'(s)}$$

in $J$. $\square$

In the previous theorem (Theorem 2.1.13), we made a rather unusual choice for the constants $a,b,c$, which determine the Weingarten relation. Unlike the result obtained in Theorem 2.1.6 (a similar case for Type I Translation), in which we obtained a differential equation (or explicit solution) for $u$, in this one, the choice of constants is merely technical. It is easy to observe in (2.72) that, since the goal is to isolate $u''$, such a choice guarantees us that.

### 2.1.16 *Type II Translation Surfaces with Prescribed Length of Second Fundamental Form*

By Proposition 2.1.17 we conclude that a **Type II Translation Surface** has **Prescribed Length of Second Fundamental Form** if it satisfies a differential equation described in the following proposition.

**Proposition 2.1.31.** *Let $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, $c \in C(J,(0,+\infty))$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s),s,t)$, has Prescribed Length of Second Fundamental Form $c(s)$, if, and only if,*

$$
\begin{aligned}
c(s) = &\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{s^2\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}\right.\\
&\left.-\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{s^2\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}\right)^2\\
&-\frac{2s^2(u'(s))^2\left((sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2\right)}{s^4\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^3}\\
&-\frac{2\left(s^2+(v'(t))^2\right)\left((sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2\right)}{s^4\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^3}.
\end{aligned}
\tag{2.74}
$$

**Proposition 2.1.32.** *Let $u \in C^2(J,\mathbb{R})$ be non-constant, $v \in C^2(I,\mathbb{R})$, $c \in C(J,(0,+\infty))$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t)+u(s),s,t)$ with Prescribed Length of Second Fundamental Form $c(s)$. Then $v(t) = \mu t + \tau$, for constant $\mu,\tau$, and $I \subset \mathbb{R}$.*

**Proof:** Suppose that $\psi$ has Prescribed Length of Second Fundamental Form $c(s) > 0$. By Proposition

2.1.31, the functions $u, v$ satisfy the equation (2.74) or equivalently

$$\begin{aligned}
0 = &(v'(t))^6 - s^4 \frac{\left((u'(s))^2+1\right)^2}{c(s)} (v''(t))^2 - 2s^3 u'(s) \frac{-7(u'(s))^2 - 4(u'(s))^4}{c(s)} (v'(t))^2 v''(t) \\
&- 2s^3 u'(s) \frac{su'(s)u''(s) - 3}{c(s)} (v'(t))^2 v''(t) + s^2 \frac{3c(s) - s^2(u''(s))^2 - 18(u'(s))^2 - 32(u'(s))^4}{c(s)} (v'(t))^4 \\
&+ s^2 \frac{-16(u'(s))^6 + 3c(s)(u'(s))^2 + 6su'(s)u''(s) + 8s(u'(s))^3 u''(s) - 2}{c(s)} (v'(t))^4 \\
&+ s^4 \frac{3c(s) - 2s^2(u''(s))^2 - 12(u'(s))^2 - 18(u'(s))^4 - 8(u'(s))^6 + 6c(s)(u'(s))^2}{c(s)} (v'(t))^2 \\
&+ s^4 \frac{3c(s)(u'(s))^4 + 8su'(s)u''(s) - 2}{c(s)} (v'(t))^2 \\
&+ \frac{s^2 \left(c(s)s^4 \left((u'(s))^2+1\right)^3 - s^4 \left((u'(s))^3 + u'(s) - su''(s)\right)^2\right)}{c(s)}
\end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = X^3 + B_s Y^2 + C_s XY + D_s X^2 + E_s X + F_s,$$

where

$$\begin{aligned}
B_s =& - s^4 \frac{\left((u'(s))^2+1\right)^2}{c(s)} \\
C_s =& - 2s^3 u'(s) \frac{-7(u'(s))^2 - 4(u'(s))^4 + su'(s)u''(s) - 3}{c(s)} \\
D_s =& s^2 \frac{3c(s) - s^2(u''(s))^2 - 18(u'(s))^2 - 32(u'(s))^4}{c(s)} \\
&+ s^2 \frac{-16(u'(s))^6 + 3c(s)(u'(s))^2 + 6su'(s)u''(s) + 8s(u'(s))^3 u''(s) - 2}{c(s)} \\
E_s =& s^4 \frac{3c(s) - 2s^2(u''(s))^2 - 12(u'(s))^2 - 18(u'(s))^4 - 8(u'(s))^6 + 6c(s)(u'(s))^2}{c(s)} \\
&+ s^4 \frac{3c(s)(u'(s))^4 + 8su'(s)u''(s) - 2}{c(s)} \\
F_s =& s^2 \frac{\left(c(s)s^4 \left((u'(s))^2+1\right)^3 - s^4 \left((u'(s))^3 + u'(s) - su''(s)\right)^2\right)}{c(s)}
\end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s \in J}$.

**Claim 1:** Let $V \subset J$ be an open set such that $u'(s) \neq 0$ in $V$ (such open set exists since $u$ is non-constant in $J$). If there exist $K_1, K_2, K_3, K_4$ constants such that in $V \subset J$

$$\begin{aligned}
B_s &= K_1 \\
C_s &= K_2 \\
D_s &= K_3 \\
E_s &= K_4,
\end{aligned}$$

then, $K_2 \neq 0$. Indeed,

$$\begin{aligned}
K_1 =& -s^4\frac{\left((u'(s))^2+1\right)^2}{c(s)}\\
0 =& -2s^3u'(s)\frac{-7(u'(s))^2-4(u'(s))^4+su'(s)u''(s)-3}{c(s)}\\
K_3 =& s^2\frac{3c(s)-s^2(u''(s))^2-18(u'(s))^2-32(u'(s))^4}{c(s)}\\
&+s^2\frac{-16(u'(s))^6+3c(s)(u'(s))^2+6su'(s)u''(s)+8s(u'(s))^3u''(s)-2}{c(s)}\\
K_4 =& s^4\frac{3c(s)-2s^2(u''(s))^2-12(u'(s))^2-18(u'(s))^4-8(u'(s))^6+6c(s)(u'(s))^2}{c(s)}\\
&+s^4\frac{3c(s)(u'(s))^4+8su'(s)u''(s)-2}{c(s)}.
\end{aligned} \tag{2.75}$$

By the first equality we obtain $c(s)=-s^4\frac{(1+(u'(s))^2)^2}{K_1}$ which substituting in the second

$$(u'(s))^2u''(s)=\frac{2u'(s)(4(u'(s))^4+7(u'(s))^2+3)}{2s}.$$

Multiplying the last two equalities of (2.76) by $(u'(s))^4$ and substituting the two relations obtained we will have

$$\begin{aligned}
0 =& 4(u'(s))^2K_1(3s^4(u'(s))^4+3s^4(u'(s))^2-K_3s^2(u'(s))^2+8K_1(u'(s))^2+9K_1)\\
0 =& 2(u'(s))^2K_1(3s^4(u'(s))^6+6s^4(u'(s))^4+8K_1(u'(s))^4+3s^4(u'(s))^2+(26K_1-K_4)(u'(s))^2+18K_1),
\end{aligned}$$

that is,

$$\begin{aligned}
0 =& 3s^4(u'(s))^4+3s^4(u'(s))^2-K_3s^2(u'(s))^2+8K_1(u'(s))^2+9K_1\\
0 =& 3s^4(u'(s))^6+6s^4(u'(s))^4+8K_1(u'(s))^4+3s^4(u'(s))^2+(26K_1-K_4)(u'(s))^2+18K_1
\end{aligned}$$

then subtracting twice the first equality from the second equality

$$\begin{aligned}
0 =& 3s^4(u'(s))^4+3s^4(u'(s))^2-K_3s^2(u'(s))^2+8K_1(u'(s))^2+9K_1\\
0 =& 3s^4(u'(s))^6+6s^4(u'(s))^4+8K_1(u'(s))^4+3s^4(u'(s))^2+(26K_1-K_4)(u'(s))^2+18K_1\\
0 =& 3s^4(u'(s))^4+8K_1(u'(s))^2-3s^4+2K_3s^2+10K_1-K_4.
\end{aligned}$$

Subtracting the third equality from the first equality we obtain $(u'(s))^2=\frac{3s^4-2K_3s^2-K_1+K_4}{3s^4-s^2K_3}$ which substituting into the first equality we obtain

$$\begin{aligned}
0 =& \frac{54s^{10}-72K_3s^8+(27K_3^2+126K_1+27K_4)s^6-K_3(2K_3^2+108K_1+18K_4)s^4}{s^2(3s^2-K_3)^2}\\
&+\frac{(24K_1K_3^2-21K_1^2+18K_1K_4+K_3^2K_4+3K_4^2)s^2+8K_1^2K_3-8K_1K_3K_4}{s^2(3s^2-K_3)^2}
\end{aligned}$$

that is,

$$\begin{aligned}
0 =& 54s^{10}-72K_3s^8+(27K_3^2+126K_1+27K_4)s^6-K_3(2K_3^2+108K_1+18K_4)s^4\\
&+(24K_1K_3^2-21K_1^2+18K_1K_4+K_3^2K_4+3K_4^2)s^2+8K_1^2K_3-8K_1K_3K_4
\end{aligned}$$

i.e., $s^2$ is contained in the roots of the nonzero polynomial

$$\begin{aligned} Q(Z) =& 54Z^5 - 72K_3Z^4 + (27K_3^2 + 126K_1 + 27K_4)Z^3 - K_3(2K_3^2 + 108K_1 + 18K_4)Z^2 \\ &+ (24K_1K_3^2 - 21K_1^2 + 18K_1K_4 + K_3^2K_4 + 3K_4^2)Z + 8K_1^2K_3 - 8K_1K_3K_4 \end{aligned}$$

therefore $s$ is constant, a contradiction, and thus we proof the claim.

**Claim 2:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible then there exist polynomials of degree 2 and degree 1 such that

$$P_s(X,Y) = (X^2 + fXY + gY^2 + hX + iY + j)(X + kY + l)$$

that is,

$$0 = X^3 + B_sY^2 + C_sXY + D_sX^2 + E_sX + F_s - (X^2 + fXY + gY^2 + hX + iY + j)(X + kY + l)$$

and reorganizing

$$\begin{aligned} 0 =& (-f-k)X^2Y + (D_s - l - h)X^2 + (-g - fk)XY^2 + (C_s - i - fl - hk)XY + (E_s - j - hl)X \\ &+ (-gk)Y^3 + (B_s - ki - gl)Y^2 + (-li - jk)Y + F_s - jl \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{rrr} -f-k=0 & C_s - i - fl - hk = 0 & B_s - ki - gl = 0 \\ D_s - l - h = 0 & E_s - j - hl = 0 & -li - jk = 0 \\ -g - fk = 0 & -gk = 0 & F_s - jl = 0. \end{array}$$

By the first, third and sixth equalities we obtain that $f = g = k = 0$. Consequently by the seventh equality, $B_s = 0$, a contradiction since $B_s < 0$ and thus we proof the claim.

By claim 2 the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = X^3 + B_sY^2 + C_sXY + D_sX^2 + E_sX + F_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Then, their coefficients are constant, that is, there are $K_1, K_2, K_3, K_4$ constants such that,

$$\begin{aligned} B_s &= K_1 \\ C_s &= K_2 \\ D_s &= K_3 \\ E_s &= K_4, \end{aligned}$$

that is,

$$
\begin{aligned}
K_1 =& -s^4\frac{\left((u'(s))^2+1\right)^2}{c(s)}\\
K_2 =& -2s^3u'(s)\frac{-7(u'(s))^2-4(u'(s))^4+su'(s)u''(s)-3}{c(s)}\\
K_3 =& s^2\frac{3c(s)-s^2(u''(s))^2-18(u'(s))^2-32(u'(s))^4}{c(s)}\\
&+s^2\frac{-16(u'(s))^6+3c(s)(u'(s))^2+6su'(s)u''(s)+8s(u'(s))^3u''(s)-2}{c(s)}\\
K_4 =& s^4\frac{3c(s)-2s^2(u''(s))^2-12(u'(s))^2-18(u'(s))^4-8(u'(s))^6+6c(s)(u'(s))^2}{c(s)}\\
&+s^4\frac{3c(s)(u'(s))^4+8su'(s)u''(s)-2}{c(s)}.
\end{aligned}
\tag{2.76}
$$

By the first equality we obtain $c(s)=-s^4\frac{(1+(u'(s))^2)^2}{K_1}$ which substituting in the second

$$
(u'(s))^2u''(s)=\frac{2u'(s)K_1(4(u'(s))^4+7(u'(s))^2+3)+sK_2(1+(u'(s))^2)^2}{2sK_1}.
$$

Multiplying the last two equalities of (2.76) by $(u'(s))^4$ and substituting the two relations obtained we will have

$$
\begin{aligned}
0 =& 12K_1s^4(u'(s))^6+12K_1s^4(u'(s))^4+(K_2^2-4K_1K_3)s^2(u'(s))^4+32K_1^2(u'(s))^4+16K_1K_2s(u'(s))^3\\
&+2K_2^2s^2(u'(s))^2+36K_1^2(u'(s))^2+12K_1K_2su'(s)+K_2^2s^2\\
0 =& 6K_1s^4(u'(s))^8+12K_1s^4(u'(s))^6+16K_1^2(u'(s))^6+8K_1K_2s(u'(s))^5+6K_1s^4(u'(s))^4+K_2^2s^2(u'(s))^4\\
&+(52K_1^2-2K_1K_4)(u'(s))^4+20K_1K_2s(u'(s))^3+2K_2^2s^2(u'(s))^2+36K_1^2(u'(s))^2\\
&+12K_1K_2u'(s)s+K_2^2s^2,
\end{aligned}
$$

and multiplying the first by $s^4$ and the second by $s^6$

$$
\begin{aligned}
0 =& 12K_1s^8(u'(s))^6+12K_1s^8(u'(s))^4+(K_2^2-4K_1K_3)s^6(u'(s))^4+32K_1^2s^4(u'(s))^4+16K_1K_2s^5(u'(s))^3\\
&+2K_2^2s^6(u'(s))^2+36K_1^2s^4(u'(s))^2+12K_1K_2s^5u'(s)+K_2^2s^6\\
0 =& 6K_1s^{10}(u'(s))^8+12K_1s^{10}(u'(s))^6+16K_1^2s^6(u'(s))^6+8K_1K_2s^7(u'(s))^5+6K_1s^{10}(u'(s))^4\\
&+K_2^2s^8(u'(s))^4+(52K_1^2-2K_1K_4)s^6(u'(s))^4+20K_1K_2s^7(u'(s))^3+2K_2^2s^8(u'(s))^2+36K_1^2s^6(u'(s))^2\\
&+12K_1K_2u'(s)s^7+K_2^2s^8.
\end{aligned}
\tag{2.77}
$$

Define $G:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$
\begin{aligned}
G(x,y) =& 12K_1xy^6+12K_1x^2y^4+(K_2^2-4K_1K_3)xy^4+32K_1^2y^4+16K_1K_2xy^3+2K_2^2x^2y^2+36K_1^2xy^2\\
&+12K_1K_2x^2y+K_2^2x^3
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{G}(x,y) =& 6K_1xy^8+12K_1x^2y^6+16K_1^2y^6+8K_1K_2xy^5+6K_1x^3y^4+K_2^2x^2y^4+(52K_1^2-2K_1K_4)xy^4\\
&+20K_1K_2x^2y^3+2K_2^2x^3y^2+36K_1^2x^2y^2+12K_1K_2x^3y+K_2^2x^4.
\end{aligned}
$$

Notice that from (2.77) $G(s^2, su'(s)) = 0 = \overline{G}(s^2, su'(s))$. Since the curve $(s^2, su'(s))$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common and since, by claim 1, $K_2 \neq 0$ we obtain a contradiction with Lemma $A$.0.14. Therefore, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. $\square$

**Theorem 2.1.14.** *Let $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$, $c \in C(J, (0, +\infty))$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$. Let $\psi : \Omega \subset \mathbb{R}^2 \to H^2 \times \mathbb{R}$ given by $\psi(t,s) = (v(t) + u(s), s, t)$ with Prescribed Length of the Second Fundamental $c(s)$. Then:*

*i) $v(t) = \mu t + \tau$, in $I \subset \mathbb{R}$, for $\mu, \tau$ constants (with $\mu \neq 0$), $u$ is constant and furthermore,*

$$c(s) = \frac{2s^2\mu^2}{(\mu^2 + s^2)^2}.$$

*ii) $v(t) = \mu t + \tau$, with $I \subset \mathbb{R}$, for constants $\mu, \tau$. Furthermore, if $\mu = 0$ then $u$ satisfies the differential equation*

$$u''(s) = \frac{u'(s)(1 + (u'(s))^2) + \sqrt{c(s)(1 + (u'(s))^2)^3}}{s},$$

*or*

$$u''(s) = \frac{u'(s)(1 + (u'(s))^2) - \sqrt{c(s)(1 + (u'(s))^2)^3}}{s}.$$

*If $\mu \neq 0$ and*

$$c(s) \geqslant \frac{\mu^2(2s^4 + 9\mu^2 s^2 + 8\mu^4)}{s^2(\mu^2 + s^2)^2},$$

*then $u$ satisfies the differential equation*

$$u''(s) = \frac{su'(s)((s^4 + 4s^2\mu^2 + 4\mu^4)(u'(s))^2 + (s^4 + 4s^2\mu^2 + 3\mu^4)) + \sqrt{\tilde{\Delta}(\mu^2, s^2, c(s), (u'(s))^2)}}{s^2(s^2 + \mu^2)^2},$$

*or*

$$u''(s) = \frac{su'(s)((s^4 + 4s^2\mu^2 + 4\mu^4)(u'(s))^2 + (s^4 + 4s^2\mu^2 + 3\mu^4)) - \sqrt{\tilde{\Delta}(\mu^2, s^2, c(s), (u'(s))^2)}}{s^2(s^2 + \mu^2)^2},$$

*where*

$$\begin{aligned}\tilde{\Delta}(x,y,z,w) =& (zy^5 + 2(z-1)xy^4 + (z-9)x^2y^3 - 8x^3y^2)w^3 + y(x+y)(3zy^3 + 6(z-1)xy^2 \\ &+ (3z-18)x^2y - 8x^3)w^2 + 3y(x+y)^2(zy^2 + 2(z-1)xy + (z-3)x^2)w \\ &+ (x+y)^2(zy^2 + 2(z-1)xy + zx^2).\end{aligned}$$

**Proof:** Suppose that $\psi$ has Prescribed Length of Second Fundamental Form $c(s)$. By Proposition 2.1.31, the functions $u, v$ satisfy the equation

$$c(s)=\left(\frac{(sv''(t)-u'(s)(v'(t))^2)(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)((v'(t))^2+s^2)}{s^2\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}\right.$$
$$\left.-\frac{2u'(s)(v'(t))^2(1+(u'(s))^2)}{s^2\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^{\frac{3}{2}}}\right)^2$$
$$-\frac{2s^2(u'(s))^2\left((sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2\right)}{s^4\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^3}$$
$$-\frac{2\left(s^2+(v'(t))^2\right)\left((sv''(t)-u'(s)(v'(t))^2)(su''(s)-u'(s)-(u'(s))^3)-(v'(t)(1+(u'(s))^2))^2\right)}{s^4\left(1+(u'(s))^2+\left(\frac{v'(t)}{s}\right)^2\right)^3}, \tag{2.78}$$

**-** $i)$ If $u$ is constant in $J$ we have from (2.78)

$$c(s)=\frac{(sv''(t))^2+2s^2(v'(t))^2\left(1+\left(\frac{v'(t)}{s}\right)^2\right)}{s^4\left(1+\left(\frac{v'(t)}{s}\right)^2\right)^3}$$

that is,

$$0=-\frac{c(s)}{s^2}(v'(t))^6+(2-3c(s))(v'(t))^4+s^2(2-3c(s))(v'(t))^2+s^2(v''(t))^2-s^4c(s)$$

and multiplying by $\frac{s^2}{c(s)}$

$$0=-(v'(t))^6+\frac{s^2(2-3c(s))}{c(s)}(v'(t))^4+\frac{s^4(2-3c(s))}{c(s)}(v'(t))^2+\frac{s^4}{c(s)}(v''(t))^2-s^6. \tag{2.79}$$

Define the family of polynomials

$$P_s(X,Y)=-X^3+\frac{s^2(2-3c(s))}{c(s)}X^2+\frac{s^4(2-3c(s))}{c(s)}X+\frac{s^4}{c(s)}Y-s^6$$

and observe that the curve $((v'(t))^2,(v''(t))^2)$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible then there exist polynomials of degree 2 and degree 1 such that

$$P_s(X,Y)=(f+gX+hY)(i+jX+kY+lXY+mX^2+nY^2)$$

that is,

$$0 = -X^3 + \frac{s^2(2-3c(s))}{c(s)}X^2 + \frac{s^4(2-3c(s))}{c(s)}X + \frac{s^4}{c(s)}Y - s^6$$
$$-(f+gX+hY)(i+jX+kY+lXY+mX^2+nY^2)$$

that reorganizing,

$$\begin{aligned}0 &= -(gm+1)X^3 - (gl+hm)X^2Y - \left(gj+fm+\frac{s^2(3c(s)-2)}{c(s)}\right)X^2 - (hl+gn)XY^2\\ &- (fl+gk+hj)XY - \left(gi+fj+\frac{s^4(3c(s)-2)}{c(s)}\right)X - hnY^3 - (hk+fn)Y^2\\ &+ \left(\frac{s^4}{c(s)} - hi - fk\right)Y - (s^6+fi),\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}&0 = gm+1 && 0 = fl+gk+hj && 0 = hk+fn\\ &0 = gl+hm && 0 = gi+fj+\frac{s^4(3c(s)-2)}{c(s)} && 0 = \frac{s^4}{c(s)} - hi - fk\\ &0 = gj+fm+\frac{s^2(3c(s)-2)}{c(s)} && 0 = hn && 0 = s^6+fi.\\ &0 = hl+gn\end{aligned} \tag{2.80}$$

By the seventh equality $h=0$ or $n=0$. Suppose $h=0$. We have,

$$\begin{aligned}&0 = gm+1 && 0 = gn && 0 = fn\\ &0 = gl && 0 = fl+gk && 0 = \frac{s^4}{c(s)} - fk\\ &0 = gj+fm+\frac{s^2(3c(s)-2)}{c(s)} && 0 = gi+fj+\frac{s^4(3c(s)-2)}{c(s)} && 0 = s^6+fi.\end{aligned}$$

By the first equality $g \neq 0$ then, by the second and fourth equalities, $l=n=0$. Substituting in the fifth equality we obtain $k=0$, a contradiction since we will not have monomials in $Y$. Then $h \neq 0$ and $n=0$. From (2.80) we have,

$$\begin{aligned}&0 = gm+1 && 0 = hl && 0 = hk\\ &0 = gl+hm && 0 = fl+gk+hj && 0 = \frac{s^4}{c(s)} - hi - fk\\ &0 = gj+fm+\frac{s^2(3c(s)-2)}{c(s)} && 0 = gi+fj+\frac{s^4(3c(s)-2)}{c(s)} && 0 = s^6+fi.\end{aligned}$$

By the fourth equality we obtain $l=0$. Substituting in the second equality we obtain $m=0$, a contradiction (since by the first equality we would have $0=1$) and thus we conclude the claim.

By the claim the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously the curve $((v'(t))^2,(v''(t))^2)$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Since the coefficients of $P_s$ are not constant we conclude that the curve $((v'(t))^2,(v''(t))^2)$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. Then, from (2.79)

$$0 = (\mu^2 + s^2)\frac{-(\mu^2 + s^2)^2 c(s) + 2s^2\mu^2}{s^2}$$

i.e.,

$$0 = \frac{-(\mu^2 + s^2)^2 c(s) + 2s^2\mu^2}{s^2}$$

thus

$$c(s) = \frac{2s^2\mu^2}{(\mu^2 + s^2)^2},$$

with $\mu \neq 0$ (otherwise $c \equiv 0$).

**-** *ii*) If $u$ is non-constant in $J$ we have, by Proposition 2.1.32, that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. Then, from (2.78)

$$\begin{aligned} c(s) = {} & \frac{((-u'(s)\mu^2)(1 + (u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(\mu^2 + s^2) - 2u'(s)\mu^2(1 + (u'(s))^2))^2}{s^4\left(1 + (u'(s))^2 + \left(\frac{\mu}{s}\right)^2\right)^3} \\ & - \frac{2\left(s^2 + s^2(u'(s))^2 + \mu^2\right)\left((-u'(s)\mu^2)(su''(s) - u'(s) - (u'(s))^3) - (\mu(1 + (u'(s))^2))^2\right)}{s^4\left(1 + (u'(s))^2 + \left(\frac{\mu}{s}\right)^2\right)^3}, \end{aligned}$$

that is,

$$\begin{aligned} 0 = {} & s^2(s^2 + \mu^2)^2(u''(s))^2 - 2su'(s)((s^4 + 4s^2\mu^2 + 4\mu^4)(u'(s))^2 + (s^4 + 4s^2\mu^2 + 3\mu^4))u''(s) \\ & + ((s^2 + 4\mu^2)^2 - c(s)s^4)(u'(s))^6 + ((2 - 3c(s))s^4 + (18 - 3c(s))\mu^2 s^2 + 32\mu^4)(u'(s))^4 \\ & + ((1 - 3c(s))s^4 + (12 - 6c(s))\mu^2 s^2 + (18 - 3c(s))\mu^4)(u'(s))^2 \\ & - (\mu^2 + s^2)\frac{c(s)s^4 + 2(c(s) - 1)\mu^2 s^2 + \mu^4 c(s)}{s^2}. \end{aligned} \tag{2.81}$$

If $\mu \neq 0$ and $c(s) \geqslant \frac{\mu^2(2s^4 + 9\mu^2 s^2 + 8\mu^4)}{s^2(\mu^2 + s^2)^2}$ define

$$P_s(Z) = A_s Z^2 + B_s Z + C_s,$$

where

$$\begin{aligned} A_s &= s^2(s^2 + \mu^2)^2 \\ B_s &= -2su'(s)((s^4 + 4s^2\mu^2 + 4\mu^4)(u'(s))^2 + (s^4 + 4s^2\mu^2 + 3\mu^4)) \\ C_s &= ((s^2 + 4\mu^2)^2 - c(s)s^4)(u'(s))^6 + ((2 - 3c(s))s^4 + (18 - 3c(s))\mu^2 s^2 + 32\mu^4)(u'(s))^4 \\ & \quad + ((1 - 3c(s))s^4 + (12 - 6c(s))\mu^2 s^2 + (18 - 3c(s))\mu^4)(u'(s))^2 \\ & \quad - (\mu^2 + s^2)\frac{c(s)s^4 + 2(c(s) - 1)\mu^2 s^2 + \mu^4 c(s)}{s^2}. \end{aligned}$$

Notice that the discriminant $\Delta(s) > 0$. Indeed,

$$\begin{aligned}
\Delta(s) &= B_s^2 - 4A_sC_s \\
&= 4(s^2(u'(s))^2+\mu^2+s^2)((\mu^2+s^2)^2(s^2(u'(s))^2+\mu^2+s^2)^2c(s)) \\
&\quad -4(s^2(u'(s))^2+\mu^2+s^2)(2s^6\mu^2+9s^4\mu^4+8s^2\mu^6)(u'(s))^4 \\
&\quad -4(s^2(u'(s))^2+\mu^2+s^2)((4s^6\mu^2+13s^4\mu^4+9s^2\mu^6)(u'(s))^2+2s^2\mu^2(\mu^2+s^2)^2) \\
&\geqslant 4(s^2(u'(s))^2+\mu^2+s^2)((\mu^2+s^2)^2(s^2(u'(s))^2+\mu^2+s^2)^2\frac{\mu^2(2s^4+9\mu^2s^2+8\mu^4)}{s^2(\mu^2+s^2)^2} \\
&\quad -4(s^2(u'(s))^2+\mu^2+s^2)(2s^6\mu^2+9s^4\mu^4+8s^2\mu^6)(u'(s))^4 \\
&\quad -4(s^2(u'(s))^2+\mu^2+s^2)((4s^6\mu^2+13s^4\mu^4+9s^2\mu^6)(u'(s))^2+2s^2\mu^2(\mu^2+s^2)^2) \\
&= 4\mu^4(\mu^2+s^2)(s^2(u'(s))^2+\mu^2+s^2)\frac{s^2(16\mu^2+9s^2)(u'(s))^2+(8\mu^2+9s^2(\mu^2+s^2))}{s^2} > 0.
\end{aligned}$$

Therefore, using the formula for the roots of a quadratic equation we obtain, from (2.81), the differential equations

$$u''(s) = \frac{su'(s)((s^4+4s^2\mu^2+4\mu^4)(u'(s))^2+(s^4+4s^2\mu^2+3\mu^4))+\sqrt{\tilde{\Delta}(\mu^2,s^2,c(s),(u'(s))^2)}}{s^2(s^2+\mu^2)^2},$$

or

$$u''(s) = \frac{su'(s)((s^4+4s^2\mu^2+4\mu^4)(u'(s))^2+(s^4+4s^2\mu^2+3\mu^4))-\sqrt{\tilde{\Delta}(\mu^2,s^2,c(s),(u'(s))^2)}}{s^2(s^2+\mu^2)^2},$$

where

$$\begin{aligned}
\tilde{\Delta}(x,y,z,w) =& (zy^5+2(z-1)xy^4+(z-9)x^2y^3-8x^3y^2)w^3+y(x+y)(3zy^3+6(z-1)xy^2 \\
&+(3z-18)x^2y-8x^3)w^2+3y(x+y)^2(zy^2+2(z-1)xy+(z-3)x^2)w \\
&+(x+y)^2(zy^2+2(z-1)xy+zx^2).
\end{aligned}$$

If $\mu = 0$ we have, from (2.81),

$$0 = s^6(u''(s))^2 - 2s^5u'(s)(1+(u'(s))^2)u''(s) - s^4(1+(u'(s))^2)^2((c(s)-1)(u'(s))^2+c(s))$$

which dividing by $s^4$

$$0 = s^2(u''(s))^2 - 2su'(s)(1+(u'(s))^2)u''(s) - (1+(u'(s))^2)^2((c(s)-1)(u'(s))^2+c(s))$$

and, as before, the discriminant of this equation is $\Delta(s) = 4s^2c(s)(1+(u'(s))^2)^3 > 0$. Thus, as before,

$$u''(s) = \frac{u'(s)(1+(u'(s))^2)+\sqrt{c(s)(1+(u'(s))^2)^3}}{s},$$

or

$$u''(s) = \frac{u'(s)(1+(u'(s))^2)-\sqrt{c(s)(1+(u'(s))^2)^3}}{s}.$$

□

## 2.2 Translation Surfaces in $H^3$ (Half-Space Model)

As discussed at the beginning of the section **Translation Surfaces in $H^2 \times \mathbb{R}$ (Half-Plane Model)**, we will restrict ourselves to curves that are contained in the orthogonal planes of the space and that are determined by real functions. In $H^3$ this reduces us to two types, the other possibilities being obtained by symmetries of the curves that determine them. More precisely:

$$\psi_I(t,s) := (t,0,v(t)) + (0,u(s),s) = (t,v(t)+u(s),s)$$
$$\psi_{II}(t,s) := (t,v(t),0) + (0,s,u(s)) = (t,s,v(t)+u(s)).$$

Here we will focus on the first case, **Type I**. First, we will define the space. Then, we will find the curvatures of this family of surfaces.

**Definition 2.2.1** ($H^3$)**.** The hyperbolic space is

$$H^3 = \{(x,y,z) \in \mathbb{R}^3 | z > 0\},$$

with the Riemannian metric,

$$(u \bullet_\times v)_{(x,y,z)} = \frac{\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle}{z^2},$$

where $(x,y,z) \in H^3$, $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3)$ since $u,v \in T_pH^3 \simeq \mathbb{R}^3$.

Consider $\{e_1,e_2,e_3\}$ a canonical basis of $\mathbb{R}^3$. Then,

$$E_1(x,y,z) = \varphi(x,y,z)e_1$$
$$E_2(x,y,z) = \varphi(x,y,z)e_2$$
$$E_3(x,y,z) = \varphi(x,y,z)e_3$$

generates a global orthonormal frame over $H^3$, where $\varphi : H^3 \to \mathbb{R}$ is given by $\varphi(x,y,z) = z$. Then, we will obtain the Connection, in this frame, at $H^3$. This is done by obtaining the Christoffel Symbols.

**Lemma 2.2.1.** $[E_i,E_j] \equiv 0$ *with the exception of:*

$$[E_1,E_3] \equiv -E_1 \qquad [E_2,E_3] \equiv -E_2.$$

**Proof:** The cases $[E_1,E_1] \equiv [E_2,E_2] \equiv [E_3,E_3] \equiv 0$are evident. Let $f : H^3 \to \mathbb{R}$ be a smooth function. We have,

$$\begin{aligned}
([E_1,E_3]f)(x,y,z) &= E_1(E_3f)(x,y,z) - E_3(E_1f)(x,y,z) \\
&= \left(E_1\left(\varphi\frac{\partial f}{\partial z}\right)\right)(x,y,z) - \left(E_3\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z) \\
&= \left(\varphi^2\frac{\partial^2 f}{\partial x\partial z}\right)(x,y,z) - \left(\varphi^2\frac{\partial^2 f}{\partial z\partial x} + \varphi\frac{\partial f}{\partial x}\right)(x,y,z) \\
&= -\left(\varphi\frac{\partial f}{\partial x}\right)(x,y,z) \\
&= -E_1(x,y,z)f,
\end{aligned}$$

then $[E_1,E_3] \equiv -E_1$ and similarly $[E_2,E_3] \equiv -E_2$. Finally,

$$\begin{aligned}([E_1,E_2]f)(x,y,z) &= E_1(E_2 f)(x,y,z) - E_2(E_1 f)(x,y,z)\\ &= \left(E_1\left(\varphi\frac{\partial f}{\partial y}\right)\right)(x,y,z) - E_2\left(\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z)\\ &= \left(\varphi\frac{\partial}{\partial x}\left(\varphi\frac{\partial f}{\partial y}\right)\right)(x,y,z) - \left(\varphi\frac{\partial}{\partial y}\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z)\\ &= \varphi(x,y,z)\left(\frac{\partial\varphi}{\partial x}\frac{\partial f}{\partial y} + \varphi\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial f}{\partial x} - \varphi\frac{\partial^2 f}{\partial y\partial x}\right)(x,y,z)\\ &= 0,\end{aligned}$$

as we wanted. □

From the previous lemma, we can find the Christoffel expressions:

**Lemma 2.2.2** (Christoffel). $\Gamma^k_{ji} \equiv 0$ *with the exception of:*

$$\Gamma^3_{11} \equiv \Gamma^3_{22} \equiv 1, \qquad \Gamma^1_{13} \equiv \Gamma^2_{23} \equiv -1.$$

**Proof:** By Lemma 2.2.1 and by the expression (Lemma 1.1.1)

$$\Gamma^k_{ji} = -\frac{1}{2}\left([E_i,E_k]\bullet_\times E_j + [E_j,E_k]\bullet_\times E_i + [E_i,E_j]\bullet_\times E_k\right),$$

the null cases are evident. We will calculate $\Gamma^3_{11}$ and the cases $\Gamma^3_{22}$,$\Gamma^1_{13}$, $\Gamma^2_{23}$ will be analogous. We have

$$\begin{aligned}\Gamma^3_{11}(x,y,z) &= -\frac{1}{2}\left([E_1,E_3]\bullet_\times E_1 + [E_1,E_3]\bullet_\times E_1 + [E_1,E_1]\bullet_\times E_3\right)(x,y,z)\\ &= -([E_1,E_3]\bullet_\times E_1)(x,y,z)\\ &= (E_1\bullet_\times E_1)(x,y,z)\\ &= 1.\end{aligned}$$

□

Since $\nabla^\times_{E_j}E_i = \sum_k \Gamma^k_{ji}E_k$ we obtain:

**Lemma 2.2.3** (Connection). *The following holds:*

$$\begin{array}{lll}\nabla^\times_{E_1}E_1 = E_3 & \nabla^\times_{E_2}E_1 = 0 & \nabla^\times_{E_3}E_1 = 0\\ \nabla^\times_{E_1}E_2 = 0 & \nabla^\times_{E_2}E_2 = E_3 & \nabla^\times_{E_3}E_2 = 0\\ \nabla^\times_{E_1}E_3 = -E_1 & \nabla^\times_{E_2}E_3 = -E_2 & \nabla^\times_{E_3}E_3 = 0.\end{array}$$

## 2.2.1 Curvatures of Type I Translation Surfaces

This section will be devoted to determining the curvatures of a Type I Translation Surface in $H^3$. Define the **Type I Translation Surface** $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ by $\psi(t,s) = (t, v(t)+u(s), s)$, $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. We have,

$$\begin{aligned}\frac{\partial\psi}{\partial t}(t,s) &= \left(1, v'(t), 0\right)\\ &= \frac{1}{\varphi\circ\psi(t,s)}E_1\circ\psi(t,s) + \frac{v'(t)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\psi}{\partial s}(t,s) &= \big(0,u'(s),1\big)\\ &= \frac{u'(s)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)+\frac{1}{\varphi\circ\psi(t,s)}E_3\circ\psi(t,s).\end{aligned}$$

The normal $N$ at $\psi(t,s)$ will be given by

$$N(t,s) = -\frac{v'(t)}{\omega(t,s)}E_1\circ\psi(t,s)+\frac{1}{\omega(t,s)}E_2\circ\psi(t,s)-\frac{u'(s)}{\omega(t,s)}E_3\circ\psi(t,s),$$

where $\omega(t,s)=\sqrt{(v'(t))^2+(u'(s))^2+1}$.

By Lemma 1.1.2 we conclude that to know the Weingarten map $\mathcal{S}$ we do not need to differentiate the normal field, we just need to know $\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}$, $\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}$ and $\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}$. We have,

**Lemma 2.2.4.** *The following holds:*

$$\begin{aligned}
\textit{\textbf{i)}}\ \left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s) =&\frac{v''(t)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)+\frac{1+(v'(t))^2}{(\varphi\circ\psi)^2(t,s)}E_3\circ\psi(t,s).\\
\textit{\textbf{ii)}}\ \left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s) =&-\frac{1}{(\varphi\circ\psi)^2(t,s)}E_1\circ\psi(t,s)-\frac{v'(t)}{(\varphi\circ\psi)^2(t,s)}E_2\circ\psi(t,s)\\
&+\frac{v'(t)u'(s)}{(\varphi\circ\psi)^2(t,s)}E_3\circ\psi(t,s).\\
\textit{\textbf{iii)}}\ \left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s) =&\left(\frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)-\frac{u'(s)}{(\varphi\circ\psi)^2(t,s)}\right)E_2\circ\psi(t,s)\\
&+\left(\frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)}+\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\right)E_3\circ\psi(t,s).
\end{aligned}$$

**Proof:**

$$\begin{aligned}
\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s) =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_1\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\
&+\frac{v'(t)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_2\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{v'(t)}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\
=&\frac{1}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1+\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\
&+\frac{v'(t)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2+\frac{v''(t)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)
\end{aligned}$$

but,

$$\begin{aligned}
\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1 =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_1\right)\circ\psi(t,s)+\frac{v'(t)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_1\right)\circ\psi(t,s)\\
=&\frac{1}{\varphi\circ\psi(t,s)}E_3\circ\psi(t,s)
\end{aligned}$$

and

$$\begin{aligned}
\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2 =&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_2\right)\circ\psi(t,s)+\frac{v'(t)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_2\right)\circ\psi(t,s)\\
=&\frac{v'(t)}{\varphi\circ\psi(t,s)}E_3\circ\psi(t,s)
\end{aligned}$$

thus,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=\frac{v''(t)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)+\frac{1+(v'(t))^2}{(\varphi\circ\psi)^2(t,s)}E_3\circ\psi(t,s).$$

Also,

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=&\frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_2\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\&+\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_1}}E_3\circ\psi\right)(t,s)+\overline{e_1}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_3\circ\psi(t,s)\\=&\frac{u'(s)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2+\frac{\partial}{\partial t}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\&+\frac{1}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3+\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3=&\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_3\right)\circ\psi(t,s)+\frac{v'(t)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_3\right)\circ\psi(t,s)\\=&-\frac{1}{\varphi\circ\psi(t,s)}E_1\circ\psi(t,s)-\frac{v'(t)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)\end{aligned}$$

then,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=-\frac{1}{(\varphi\circ\psi)^2(t,s)}E_1\circ\psi(t,s)-\frac{v'(t)}{(\varphi\circ\psi)^2(t,s)}E_2\circ\psi(t,s)+\frac{v'(t)u'(s)}{(\varphi\circ\psi)^2(t,s)}E_3\circ\psi(t,s).$$

Finally,

$$\begin{aligned}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s)=&\frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_2}}E_2\circ\psi\right)(t,s)+\overline{e_2}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\&+\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_2}}E_3\circ\psi\right)(t,s)+\overline{e_2}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_3\circ\psi(t,s)\\=&\frac{u'(s)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2+\frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\&+\frac{1}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3+\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2=\frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_2\right)\circ\psi(t,s)+\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_3}E_2\right)\circ\psi(t,s)=\frac{u'(s)}{\varphi\circ\psi(t,s)}E_3\circ\psi(t,s)$$

and

$$\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3=\frac{u'(s)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_3\right)\circ\psi(t,s)+\frac{1}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_3}E_3\right)\circ\psi(t,s)=-\frac{u'(s)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)$$

therefore,

$$\begin{aligned}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s)=&\left(\frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)-\frac{u'(s)}{(\varphi\circ\psi)^2(t,s)}\right)E_2\circ\psi(t,s)\\&+\left(\frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)}+\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\right)E_3\circ\psi(t,s).\end{aligned}$$

□

With this we have,

**Lemma 2.2.5.** *The following holds:*

$$\textit{\textbf{i)}}\mathcal{E}(t,s)=\frac{sv''(t)-u'(s)(1+(v'(t))^2)}{s^2\omega(t,s)}.$$
$$\textit{\textbf{ii)}}\mathcal{F}(t,s)=\frac{v'(t)(u'(s))^2}{s^2\omega(t,s)}.$$
$$\textit{\textbf{iii)}}\mathcal{G}(t,s)=\frac{su''(s)-u'(s)-(u'(s))^3}{s^2\omega(t,s)}.$$
$$\textit{\textbf{iv)}}\mathfrak{e}(t,s)=\frac{1+(v'(t))^2}{s^2}.$$
$$\textit{\textbf{v)}}\mathfrak{f}(t,s)=\frac{u'(s)v'(t)}{s^2}.$$
$$\textit{\textbf{vi)}}\mathfrak{g}(t,s)=\frac{1+(u'(s))^2}{s^2}.$$

**Proof:**

$$\begin{aligned}\mathcal{E}(t,s)=\left(N\bullet_\times\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=&\frac{v''(t)}{\omega(t,s)\varphi\circ\psi(t,s)}-\frac{u'(s)(1+(v'(t))^2)}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}\\=&\frac{sv''(t)-u'(s)(1+(v'(t))^2)}{s^2\omega(t,s)}.\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{F}(t,s)=\left(N\bullet_\times\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=&\frac{v'(t)}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}-\frac{v'(t)}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}-\frac{v'(t)(u'(s))^2}{\omega(t,s)(\varphi\circ\psi)^2(t,s)}\\=&\frac{v'(t)(u'(s))^2}{s^2\omega(t,s)}.\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{G}(t,s)=&\left(N\bullet_\times\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s)\\=&\frac{1}{\omega(t,s)}\left(\frac{\partial}{\partial s}\left(\frac{u'(s)}{\varphi\circ\psi(t,s)}\right)-\frac{u'(s)}{(\varphi\circ\psi)^2(t,s)}\right)-\frac{u'(s)}{\omega(t,s)}\left(\frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)}+\frac{\partial}{\partial s}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\right)\\=&\frac{su''(s)-u'(s)-(u'(s))^3}{s^2\omega(t,s)}.\end{aligned}$$

The metric coefficients are,

$$\begin{aligned}\mathfrak{e}(t,s)=\left(\overline{e_1}\bullet_\times\overline{e_1}\right)(t,s)=\left(\frac{\partial\psi}{\partial t}\bullet_\times\frac{\partial\psi}{\partial t}\right)(t,s)=&\frac{1}{(\varphi\circ\psi)^2(t,s)}+\frac{(v'(t))^2}{(\varphi\circ\psi)^2(t,s)}\\=&\frac{1+(v'(t))^2}{s^2}.\end{aligned}$$

Also,

$$\mathfrak{f}(t,s)=\left(\overline{e_1}\bullet_\times\overline{e_2}\right)(t,s)=\left(\frac{\partial\psi}{\partial t}\bullet_\times\frac{\partial\psi}{\partial s}\right)(t,s)=\frac{v'(t)u'(s)}{s^2}.$$

Finally,

$$\mathfrak{g}(t,s)=\left(\overline{e_2}\bullet_\times\overline{e_2}\right)(t,s)=\left(\frac{\partial\psi}{\partial s}\bullet_\times\frac{\partial\psi}{\partial s}\right)(t,s)=\frac{1}{(\varphi\circ\psi)^2(t,s)}+\frac{(u'(s))^2}{(\varphi\circ\psi)^2(t,s)}=\frac{1+(u'(s))^2}{s^2}.$$

$\square$

**Proposition 2.2.1.** *The following holds:*

$$H(t,s) = \frac{(sv''(t) - u'(s)(1+(v'(t))^2))(1+(u'(s))^2)}{2\omega^3(t,s)} + \frac{(su''(s) - u'(s) - (u'(s))^3)(1+(v'(t))^2) - 2(v'(t))^2(u'(s))^3}{2\omega^3(t,s)},$$

*and*

$$K(t,s) = \frac{(sv''(t) - u'(s)(1+(v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4}{\omega^4(t,s)},$$

*and*

$$\begin{aligned} |A|^2(t,s) = &\left(\frac{(sv''(t) - u'(s)(1+(v'(t))^2))(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(1+(v'(t))^2)}{\omega^3(t,s)}\right. \\ &\left. - \frac{2(v'(t))^2(u'(s))^3}{\omega^3(t,s)}\right)^2 \\ &- \frac{2(v'(t))^2((sv''(t) - u'(s)(1+(v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4)}{\omega^6(t,s)} \\ &- \frac{2(1+(u'(s))^2)((sv''(t) - u'(s)(1+(v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4)}{\omega^6(t,s)}. \end{aligned}$$

**Proof:**

$$\begin{aligned} H(t,s) &= \frac{1}{2}\mathrm{tr}(\mathcal{S})(t,s) \\ &= \frac{1}{2}(a_{tt} + a_{ss}) \\ &= \frac{1}{2}\left(\frac{\mathcal{E}\mathfrak{g} + \mathcal{G}\mathfrak{e} - 2\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s) \\ &= \frac{1}{2}\left(\frac{\left(\frac{sv''(t) - u'(s)(1+(v'(t))^2)}{s^2\omega(t,s)}\right)\left(\frac{1+(u'(s))^2}{s^2}\right) + \left(\frac{su''(s) - u'(s) - (u'(s))^3}{s^2\omega(t,s)}\right)\left(\frac{1+(v'(t))^2}{s^2}\right) - \frac{2(v'(t))^2(u'(s))^3}{s^4\omega(t,s)}}{\left(\frac{1+(v'(t))^2}{s^2}\right)\left(\frac{1+(u'(s))^2}{s^2}\right) - \frac{(u'(s)v'(t))^2}{s^4}}\right) \\ &= \frac{1}{2}\left(\frac{\frac{(sv''(t) - u'(s)(1+(v'(t))^2))(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(1+(v'(t))^2)) - 2(v'(t))^2(u'(s))^3}{s^4\omega(t,s)}}{\frac{\omega^2(t,s)}{s^4}}\right) \\ &= \frac{(sv''(t) - u'(s)(1+(v'(t))^2))(1+(u'(s))^2)}{2\omega^3(t,s)} \\ &\quad + \frac{(su''(s) - u'(s) - (u'(s))^3)(1+(v'(t))^2) - 2(v'(t))^2(u'(s))^3}{2\omega^3(t,s)}. \end{aligned}$$

Also,

$$
\begin{aligned}
K(t,s) &= \det(\mathcal{S}) \\
&= a_{tt}a_{ss} - a_{ts}a_{st} \\
&= \left(\frac{\mathcal{E}\mathfrak{g}-\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{G}\mathfrak{e}-\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s) - \left(\frac{\mathcal{F}\mathfrak{g}-\mathcal{G}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{F}\mathfrak{e}-\mathcal{E}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G}\mathfrak{g}\mathfrak{e}-\mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f}-\mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e}+(\mathcal{F}\mathfrak{f})^2-\left(\mathcal{F}^2\mathfrak{g}\mathfrak{e}-\mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f}-\mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e}+\mathcal{E}\mathcal{F}\mathfrak{f}^2\right)}{(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G}(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)-\mathcal{F}^2(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)}{(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)^2}\right)(t,s) \\
&= \left(\frac{\mathcal{E}\mathcal{G}-\mathcal{F}^2}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s) \\
&= \frac{\left(\frac{sv''(t)-u'(s)(1+(v'(t))^2)}{s^2\omega(t,s)}\right)\left(\frac{su''(s)-u'(s)-(u'(s))^3}{s^2\omega(t,s)}\right)-\frac{(v'(t))^2(u'(s))^4}{s^4\omega^2(t,s)}}{\frac{1}{s^4}\omega^2(t,s)} \\
&= \frac{(sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4}{\omega^4(t,s)}.
\end{aligned}
$$

Finally, being $\lambda_1(t,s)$ and $\lambda_2(t,s)$ the principal curvatures we have,

$$
\begin{aligned}
|A|^2(t,s) &= \lambda_1^2(t,s)+\lambda_2^2(t,s) \\
&= 4\frac{(\lambda_1+\lambda_2)^2(t,s)}{4} - 2\lambda_1(t,s)\lambda_2(t,s) \\
&= 4H^2(t,s) - 2K(t,s) \\
&= \Bigg(\frac{(sv''(t)-u'(s)(1+(v'(t))^2))(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)(1+(v'(t))^2)}{\omega^3(t,s)} \\
&\quad - \frac{2(v'(t))^2(u'(s))^3}{\omega^3(t,s)}\Bigg)^2 \\
&\quad - \frac{2(v'(t))^2((sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4)}{\omega^6(t,s)} \\
&\quad - \frac{2(1+(u'(s))^2)((sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4)}{\omega^6(t,s)}.
\end{aligned}
$$

□

### 2.2.2 *Type I Translation Surfaces - Flat case*

By Proposition 2.2.1 we conclude that a **Type I Translation Surface** is **Flat** if it satisfies a differential equation described in the following proposition.

**Proposition 2.2.2.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, is Flat if, and only if,*

$$0 = (sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4.$$

**Proposition 2.2.3.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t) + u(s), s)$, where $u \in C^2(J,\mathbb{R})$ non-constant, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, Flat. Then $v(t) = \mu t^2 + \eta t + \tau$, for $\mu, \eta, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.2.2, the functions $u, v$ satisfy the equation

$$0 = (sv''(t) - u'(s)(1 + (v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4.$$

or equivalently

$$\begin{aligned}0 =& (u'(s)((u'(s))^3 + u'(s) - su''(s)) - (u'(s))^4)(v'(t))^2 - s((u'(s))^3 + u'(s) - su''(s))v''(t) \\ &+ u'(s)((u'(s))^3 + u'(s) - su''(s)).\end{aligned}$$

Define the family of irreducible polynomials

$$P_s(X,Y) = A_sX + B_sY + C_s,$$

where

$$\begin{aligned}A_s &= u'(s)((u'(s))^3 + u'(s) - su''(s)) - (u'(s))^4 \\ B_s &= -s((u'(s))^3 + u'(s) - su''(s)) \\ C_s &= u'(s)((u'(s))^3 + u'(s) - su''(s)),\end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** There exists an set open $V \subset J$ such that $B_s \neq 0$ in $V$. Indeed, if $B_s \equiv 0$ in $J$ then $C_s \equiv 0$ and $A_s = -(u'(s))^4$ in $J$ and therefore

$$0 = P_s(v'(t), v''(t)) = -(u'(s))^4,$$

a contradiction, since $u$ is non-constant, and thus we proof the claim.

By the claim there exists an open set $V \subset J$ such that $B_s \neq 0$ in $V$. Define the family of irreducible polynomials

$$\overline{P}_s(X,Y) = \frac{A_s}{B_s}X + Y + \frac{C_s}{B_s},$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in V}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$\overline{P}_s(X,Y) = \frac{A_s}{B_s}X + Y + \frac{C_s}{B_s}, \tag{2.82}$$

is constant in the variable $s$, that is, the polynomials $\overline{P}_s$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2$ constants such that

$$K_1 = \frac{A_s}{B_s} = -\frac{u'(s)(u'(s)-su''(s))}{s((u'(s))^3+u'(s)-su''(s))}$$
$$K_2 = \frac{C_s}{B_s} = -\frac{u'(s)}{s},$$

and setting $w(s) = u'(s)$,

$$K_1 = -\frac{w(s)(w(s)-sw'(s))}{s((w)^3(s)+w(s)-sw'(s))}$$
$$K_2 = -\frac{w(s)}{s}.$$

By the second equality we obtain $w(s) = -K_2 s$, with $K_2 \neq 0$ (otherwise $B_s \equiv 0$), which substituting into the first equality

$$K_1 = -\frac{-K_2 s(-K_2 s + sK_2)}{s(-(K_2 s)^3 - K_2 s + sK_2)} = \frac{0}{s^4 K_2^3} = 0$$

and then from (2.82) we conclude

$$0 = \overline{P}_s((v'(t))^2, v''(t)) = K_1(v'(t))^2 + v''(t) + K_2 = v''(t) + K_2$$

that is, $v(t) = \mu t^2 + \eta t + \tau$, for constants $\mu, \eta, \tau$ with $\mu \neq 0$ in $I \subset \mathbb{R}$. On the other hand, if the curve $((v'(t))^2, v''(t))$ is constant there exist $\eta, \gamma$ such that $v'(t) = \eta$ and $v''(t) = \gamma$ (in this case necessarily $\gamma = 0$) and we conclude that $v(t) = \eta t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.2.1.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t) + u(s), s)$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$, Flat. Then:*

*i) $u$ is constant and $v$ is arbitrary.*

*ii) $u(s) = \mu s^2 + \kappa$ and $v(t) = \mu t^2 + \eta t + \tau$, where $\mu \neq 0$, $\eta$, $\tau$ and $\kappa$ are constants and $J \subset (0, +\infty)$ and $I \subset \mathbb{R}$.*

*iii) $u(s) = D \pm \displaystyle\int_{s_0}^{s} \sqrt{\frac{F(\eta^2+1)r^2}{C^2\left(\left(1-\frac{r^2}{C^2}\right)F+\eta^2+1\right)}}\,dr$ and $v(t) = \eta t + \tau$, being $\eta$, $\tau$, $C > 0$, $F > 0$ and $D$ constants and $J \subset \left(0, \sqrt{\frac{C^2(F+\eta^2+1)}{F}}\right)$ and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.2.2, the functions $u, v$ satisfy the equation

$$0 = (sv''(t) - u'(s)(1+(v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4. \tag{2.83}$$

**-** *i)***:** If $u$ is constant it is easy to verify from (2.83) that $v$ is arbitrary.

**-** *ii*) **e** *iii*)**:** If $u$ is non-constant we obtain from Proposition 2.2.3 that $v(t) = \mu t^2 + \eta t + \tau$, for $\mu, \eta, \tau$ constants, and $I \subset \mathbb{R}$. Then, from (2.83), we obtain

$$\begin{aligned} 0 &= -4\mu^2 u'(s)(su''(s) - u'(s))t^2 - 4\mu\eta(su''(s) - u'(s))t \\ &\quad + (u'(s))^4 - ((\eta^2+1)su'(s) - 2s^2\mu)u''(s) + (\eta^2+1)(u'(s))^2 - 2\mu su'(s) - 2\mu s(u'(s))^3, \end{aligned}$$

thus, $t$ vanishes the family of polynomials

$$Q_s(Z) = A_s Z^2 + B_s Z + C_s,$$

where

$$\begin{aligned} A_s &= -4\mu^2 u'(s)(su''(s) - u'(s)) \\ B_s &= -4\mu\eta(su''(s) - u'(s)) \\ C_s &= (u'(s))^4 - ((\eta^2+1)su'(s) - 2s^2\mu)u''(s) + (\eta^2+1)(u'(s))^2 - 2\mu su'(s) - 2\mu s(u'(s))^3. \end{aligned}$$

Since $t$ is not constant and for each fixed $s$ we have $Q_s(t) = 0$ we conclude that the coefficients of $Q_s$ are all zero. Then,

$$\begin{aligned} 0 &= -4\mu^2 u'(s)(su''(s) - u'(s)) \\ 0 &= -4\mu\eta(su''(s) - u'(s)) \\ 0 &= (u'(s))^4 - ((\eta^2+1)su'(s) - 2s^2\mu)u''(s) + (\eta^2+1)(u'(s))^2 - 2\mu su'(s) - 2\mu s(u'(s))^3. \end{aligned}$$

and $u$ being non-constant (in an open set $V \subset J$)

$$\begin{aligned} 0 &= -4\mu^2(su''(s) - u'(s)) \\ 0 &= -4\mu\eta(su''(s) - u'(s)) \\ 0 &= (u'(s))^4 - ((\eta^2+1)su'(s) - 2s^2\mu)u''(s) + (\eta^2+1)(u'(s))^2 - 2\mu su'(s) - 2\mu s(u'(s))^3. \end{aligned} \tag{2.84}$$

If $\mu \neq 0$ we obtain from (2.84)

$$\begin{aligned} u''(s) &= \frac{u'(s)}{s} \\ 0 &= (u'(s))^4 - ((\eta^2+1)su'(s) - 2s^2\mu)u''(s) + (\eta^2+1)(u'(s))^2 - 2\mu su'(s) - 2\mu s(u'(s))^3 \end{aligned}$$

and substituting the first equality into the second we obtain

$$0 = (u'(s))^3(u'(s) - 2\mu s)$$

that is,

$$u'(s) = 2\mu s$$

therefore $u(s) = \mu s^2 + \kappa$. If $\mu = 0$ we obtain from (2.84)

$$0 = -s(\eta^2+1)u'(s)u''(s) + (u'(s))^4 + (\eta^2+1)(u'(s))^2$$

then
$$0=-\frac{s(\eta^2+1)}{2}\frac{d}{ds}[(u'(s))^2]+(u'(s))^4+(\eta^2+1)(u'(s))^2.$$
Finally, defining $w(s)=(u'(s))^2$ we obtain the separable differential equation
$$w'(s)=\frac{2(w^2(s)+(\eta^2+1)w(s))}{s(\eta^2+1)}. \tag{2.85}$$
Observe that $w\equiv 0$ is a solution of (2.85). By the Existence and Uniqueness Theorem any other solution does not reach zero. Since $u$ is non-constant we have $w\neq 0$. Thus, from (2.85), we obtain
$$\int_{s_0}^{s}\frac{w'(r)}{2(w^2(r)+(\eta^2+1)w(r))}dr=\int_{s_0}^{s}\frac{1}{r(\eta^2+1)}dr,$$
and by the Change of Variables Theorem
$$\frac{1}{2}\int_{w(s_0)}^{w(s)}\frac{dl}{l^2+(\eta^2+1)l}=\frac{1}{\eta^2+1}\int_{s_0}^{s}\frac{1}{r}dr.$$
Consequently,
$$\frac{1}{\eta^2+1}\left(\ln\left(\frac{w(s)}{w(s)+\eta^2+1}\right)-\ln\left(\frac{w(s_0)}{w(s_0)+\eta^2+1}\right)\right)=\frac{2\left(\ln(s)-\ln(s_0)\right)}{\eta^2+1},$$
that is,
$$\ln\left(\frac{w(s)(w(s_0)+\eta^2+1)}{w(s_0)(w(s)+\eta^2+1)}\right)=2\ln\left(\frac{s}{s_0}\right),$$
and applying the exponential
$$\frac{w(s)(w(s_0)+\eta^2+1)}{w(s_0)(w(s)+\eta^2+1)}=\left(\frac{s}{s_0}\right)^2,$$
thus,
$$\left(\left(1-\frac{s^2}{s_0^2}\right)w(s_0)+\eta^2+1\right)w(s)=\frac{w(s_0)(\eta^2+1)s^2}{s_0^2}$$
and for all $s\neq\sqrt{\frac{s_0^2(w(s_0)+\eta^2+1)}{w(s_0)}}$
$$w(s)=\frac{w(s_0)(\eta^2+1)s^2}{s_0^2\left(\left(1-\frac{s^2}{s_0^2}\right)w(s_0)+\eta^2+1\right)}$$
and since we want $w>0$, there are constants $C>0$ and $F>0$ such that
$$w(s)=\frac{F(\eta^2+1)s^2}{C^2\left(\left(1-\frac{s^2}{C^2}\right)F+\eta^2+1\right)}$$
in the interval $\left(0,\sqrt{\frac{C^2(F+\eta^2+1)}{F}}\right)$. Thus, by the definition of $w$, we obtain
$$u'(s)=\pm\sqrt{\frac{F(\eta^2+1)s^2}{C^2\left(\left(1-\frac{s^2}{C^2}\right)F+\eta^2+1\right)}}$$

in $J \subset \left(0, \sqrt{\frac{C^2(F+\eta^2+1)}{F}}\right)$. Concluding,

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{F(\eta^2+1)r^2}{C^2\left(\left(1-\frac{r^2}{C^2}\right)F+\eta^2+1\right)}}dr.$$

$\square$

## 2.2.3 *Type I Translation Surfaces - Constant Gauss Curvature case*

By Proposition 2.2.1 we conclude that a **Type I Translation Surface** has **Constant Gauss curvature** if it satisfies a differential equation described in the following proposition.

**Proposition 2.2.4.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, has constant Gaussian Curvature c, if, and only if,*

$$c = \frac{(sv''(t) - u'(s)(1+(v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4}{(1+(v'(t))^2+(u'(s))^2)^2}$$

**Proposition 2.2.5.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$ is non-constant, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, with constant Gaussian Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.2.4, the functions $u, v$ satisfy the equation

$$c = \frac{(sv''(t) - u'(s)(1+(v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4}{(1+(v'(t))^2+(u'(s))^2)^2},$$

or equivalently

$$\begin{aligned} 0 =& (v'(t))^4 - \frac{(u'(s)((u'(s))^3 + u'(s) - su''(s)) - (u'(s))^4 - 2c((u'(s))^2+1))}{c}(v'(t))^2 \\ &+ \frac{s((u'(s))^3 + u'(s) - su''(s))}{c}v''(t) - \frac{(u'(s)((u'(s))^3 + u'(s) - su''(s)) - c((u'(s))^2+1)^2)}{c}. \end{aligned} \tag{2.86}$$

Define the family of irreducible polynomials

$$P_s(X,Y) = X^2 + A_sX + B_sY + C_s,$$

where,

$$\begin{aligned} A_s &= -\frac{(u'(s)((u'(s))^3 + u'(s) - su''(s)) - (u'(s))^4 - 2c((u'(s))^2+1))}{c} \\ B_s &= \frac{s((u'(s))^3 + u'(s) - su''(s))}{c} \\ C_s &= -\frac{(u'(s)((u'(s))^3 + u'(s) - su''(s)) - c((u'(s))^2+1)^2)}{c}, \end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = X^2 + A_sX + B_sY + C_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Then, their coefficients are constant, that is, there are $K_1, K_2, K_3$ constants such that,

$$\begin{aligned} A_s &= K_1 \\ B_s &= K_2 \\ C_s &= K_3, \end{aligned}$$

that is,

$$\begin{aligned} K_1 &= -\frac{(u'(s)((u'(s))^3 + u'(s) - su''(s)) - (u'(s))^4 - 2c((u'(s))^2 + 1))}{c} \\ K_2 &= \frac{s((u'(s))^3 + u'(s) - su''(s))}{c} \\ K_3 &= -\frac{(u'(s)((u'(s))^3 + u'(s) - su''(s)) - c((u'(s))^2 + 1)^2)}{c}, \end{aligned}$$

then, setting $w(s) = u'(s)$, we obtain

$$\begin{aligned} K_1 &= -\frac{(w(s)(w^3(s) + w(s) - sw'(s)) - w^4(s) - 2c(w^2(s) + 1))}{c} \\ K_2 &= \frac{s(w^3(s) + w(s) - sw'(s))}{c} \\ K_3 &= -\frac{(w(s)(w^3(s) + w(s) - sw'(s)) - c(w^2(s) + 1)^2)}{c}, \end{aligned}$$

thus

$$\begin{aligned} K_1 &= -\frac{(w(s)(w^3(s) + w(s) - sw'(s)) - w^4(s) - 2c(w^2(s) + 1))}{c} \\ w'(s) &= \frac{s(w(s) + w^3(s)) - cK_2}{s^2} \\ K_3 &= -\frac{(w(s)(w^3(s) + w(s) - sw'(s)) - c(w^2(s) + 1)^2)}{c}, \end{aligned} \tag{2.87}$$

and replacing the second equality in the others

$$\begin{aligned} 0 &= sw^4(s) + 2csw^2(s) + (2c - cK_1)s - cK_2w(s) \\ 0 &= sw^4(s) + 2sw^2(s) + (1 - K_3)s - K_2w(s). \end{aligned} \tag{2.88}$$

**Claim:** $K_2 \neq 0$. Indeed, we can see by the first equality of $(2.88)$ that if $K_2 = 0$ we would have

$$0 = w^2(s) + 2cw^2(s) + 2c - cK_1,$$

that is, $w^2$ would vanish the polynomial

$$Q(Z) = Z^2 + 2cZ + 2c - cK_1.$$

Since $w$ is non-constant, the polynomial $Q$ should be zero, that is, all its coefficients should be zero, a situation that does not happen, a contradiction. Therefore $K_2 \neq 0$ and thus we conclude the claim.

Differentiating the second equality of (2.88) in $s$ we obtain

$$0 = (4sw^3(s) + 4sw(s) - K_2)w'(s) + w^4(s) + 2w^2(s) + 1 - K_3. \tag{2.89}$$

Substituting the second equality of (2.87) in (2.89) and multiplying it by $s^6$, we will have, combining with the second equality of (2.88) multiplied by $s^3$,

$$\begin{aligned} 0 =& 4s^6w^6(s) + 9s^6w^4(s) - K_2(1+4c)s^5w^3(s) + 6s^6w^2(s) - K_2(1+4c)s^5w(s) + (1-K_3)s^6 + cK_2^2s^4 \\ 0 =& s^4w^4(s) + 2s^4w^2(s) + (1-K_3)s^4 - K_2s^3w(s). \end{aligned} \tag{2.90}$$

Define $G : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$, $\overline{G} : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$ by

$$G(x,y) = 4y^6 + 9xy^4 - K_2(1+4c)xy^3 + 6x^2y^2 - K_2(1+4c)x^2y + (1-K_3)x^3 + cK_2^2x^2$$

and

$$\overline{G}(x,y) = y^4 + 2xy^2 - K_2xy + (1-K_3)x^2.$$

Notice that, from (2.90), $G(s^2, sw(s)) = 0 = \overline{G}(s^2, sw(s))$. Since the curve $(s^2, sw(s))$ is non-constant we conclude from the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A.0.1$. Therefore, the curve $((v'(t))^2, v''(t))$ is constant and then there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.2.2.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t) + u(s), s)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, and $J \times I \subset (0,+\infty)\times\mathbb{R}$, with constant Gaussian curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for constant $\mu, \tau$, and $I \subset \mathbb{R}$. Moreover, the possibilities for the function $u$ are:*

*i)*

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{C^2(1+\mu^2+F)}{r^2} - (1+\mu^2)}\,dr,$$

*for constants $D$, $C > 0$, $F > 0$ e $J \subset \left(0, C\sqrt{\frac{1+\mu^2+F}{1+\mu^2}}\right)$.*

*ii)*

$$u(s) = D \pm \sqrt{\frac{c(1+\mu^2)}{1-c}}\,s,$$

*for constant $D$, $c \in (0,1)$ e $J \subset (0,+\infty)$.*

*iii*)

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{(1+\mu^2)(M_{F,\mu,c}r^2 - C^2(M_{F,\mu,c}+F))}{\frac{(c-1)C^2(M_{F,\mu,c}+F)}{c} - M_{F,\mu,c}r^2}}dr,$$

*for constants D, $C > 0$, $F > 0$, and $c \neq 1$, where*

$$M_{F,\mu,c} = ((1+\mu^2+F)c - F).$$

*Furthermore, the interval is given by*

$$\operatorname{sgn}(M_{F,\mu,c})s^2 \in \left(\frac{c-1}{c}\frac{C^2(M_{F,\mu,c}+F)}{\operatorname{sgn}(M_{F,\mu,c})M_{F,\mu,c}}, \frac{C^2(M_{F,\mu,c}+F)}{\operatorname{sgn}(M_{F,\mu,c})M_{F,\mu,c}}\right).$$

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.2.4, the functions $u, v$ satisfy the equation

$$c = \frac{(sv''(t) - u'(s)(1+(v'(t))^2))(su''(s) - u'(s) - (u'(s))^3) - (v'(t))^2(u'(s))^4}{(1+(v'(t))^2+(u'(s))^2)^2}. \tag{2.91}$$

It is easy to check that $u$ is non-constant and $u' \neq 0$. Indeed, if $u$ is constant or $u'(s) = 0$ for some $s$, we obtain from (2.91) that $c = 0$, a contradiction. By Proposition 2.2.5 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. Therefore, from (2.91), we have

$$0 = s(1+\mu^2)u'(s)u''(s) + (1+\mu^2+(u'(s))^2)((1+\mu^2+(u'(s))^2)c - (u'(s))^2),$$

that is,

$$0 = \frac{s(1+\mu^2)}{2}\frac{d}{ds}\left[(u'(s))^2\right] + (1+\mu^2+(u'(s))^2)((1+\mu^2+(u'(s))^2)c - (u'(s))^2).$$

Define $w(s) = (u'(s))^2$. We have,

$$w'(s) = -\frac{2(1+\mu^2+w(s))((1+\mu^2+w(s))c - w(s))}{s(1+\mu^2)}. \tag{2.92}$$

$-i)$ : If $c = 1$ the equation in (2.92) reduces to

$$w'(s) = -\frac{2(1+\mu^2+w(s))}{s}$$

thus

$$\int_{s_0}^{s} \frac{w'(r)}{1+\mu^2+w(r)}dr = -\int_{s_0}^{s} \frac{2}{r}dr$$

that by the Change of Variables Theorem

$$\int_{w(s_0)}^{w(s)} \frac{1}{1+\mu^2+l}dl = \int_{s_0}^{s} \frac{2}{r}dr.$$

Consequently,

$$\ln\left(\frac{1+\mu^2+w(s)}{1+\mu^2+w(s_0)}\right) = \ln\left(\frac{s_0^2}{s^2}\right)$$

i.e.,

$$\frac{1+\mu^2+w(s)}{1+\mu^2+w(s_0)} = \frac{s_0^2}{s^2}$$

that is, for constants $C > 0$ and $F > 0$,

$$w(s) = \frac{C^2(1+\mu^2+F)}{s^2} - (1+\mu^2)$$

in $\left(0, C\sqrt{\frac{1+\mu^2+F}{1+\mu^2}}\right)$. By the definition of $w$, and since $u' \neq 0$,

$$u'(s) = \pm\sqrt{\frac{C^2(1+\mu^2+F)}{s^2} - (1+\mu^2)}$$

i.e.,

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{C^2(1+\mu^2+F)}{r^2} - (1+\mu^2)}dr.$$

$-ii)$ e $iii)$ : If $c \neq 1$ one can see that $w \equiv \frac{c(1+\mu^2)}{1-c}$ is a solution of (2.92) and since we want $w(s) = (u'(s))^2 \geqslant 0$ we will have, if $c \in (0,1)$,

$$u(s) = D \pm \sqrt{\frac{c(1+\mu^2)}{1-c}}s.$$

By the Existence and Uniqueness Theorem any other solution does not reach $\frac{c(1+\mu^2)}{1-c}$ and therefore $((1+\mu^2+w(s))c - w(s)) \neq 0$. Thus, from (2.92),

$$\int_{s_0}^{s} \frac{w'(r)}{2(1+\mu^2+w(r))((1+\mu^2+w(r))c-w(r))}dr = -\frac{2}{1+\mu^2}\int_{s_0}^{s}\frac{1}{r}dr,$$

and by the Change of Variables Theorem,

$$\int_{w(s_0)}^{w(s)} \frac{dl}{2(1+\mu^2+l)((1+\mu^2+l)c-l)} = -\frac{2}{1+\mu^2}\int_{s_0}^{s}\frac{1}{r}dr.$$

Consequently,

$$\frac{1}{1+\mu^2}\left(\ln\left(\frac{1+\mu^2+w(s)}{(1+\mu^2+w(s))c-w(s)}\right) - \ln\left(\frac{1+\mu^2+w(s_0)}{(1+\mu^2+w(s_0))c-w(s_0)}\right)\right) = \frac{1}{1+\mu^2}\ln\left(\frac{s_0^2}{s^2}\right)$$

i.e.,

$$\frac{(1+\mu^2+w(s))((1+\mu^2+w(s_0))c-w(s_0))}{(1+\mu^2+w(s_0))((1+\mu^2+w(s))c-w(s))} = \frac{s_0^2}{s^2}$$

that is, for constants $C > 0$ and $F > 0$,

$$\begin{aligned} 0 =& ((1+\mu^2+F)c-F)s^2 - (c-1)C^2(1+\mu^2+F))w(s) + (1+\mu^2)(((1+\mu^2+F)c-F)s^2 \\ & - cC^2(1+\mu^2+F). \end{aligned} \tag{2.93}$$

Note that the coefficient accompanying $w(s)$ in (2.93) is non-zero. Indeed, if there exists $s_1$ that vanishes it out we would have from (2.93)

$$\begin{aligned}0 =&((1+\mu^2+F)c-F)s_1^2-(c-1)C^2(1+\mu^2+F)\\0 =&((1+\mu^2+F)c-F)s_1^2-cC^2(1+\mu^2+F)\end{aligned}$$

and then substituting the second equality into the first we conclude $0 = C^2(1+\mu^2+F) \neq 0$, a contradiction. Thus, from (2.93)

$$w(s) = -(1+\mu^2)\frac{((1+\mu^2+F)c-F)s^2-cC^2(1+\mu^2+F)}{((1+\mu^2+F)c-F)s^2-(c-1)C^2(1+\mu^2+F)}.$$

By the definition of $w$, and since $u' \neq 0$,

$$u'(s) = \pm\sqrt{-(1+\mu^2)\frac{((1+\mu^2+F)c-F)s^2-cC^2(1+\mu^2+F)}{((1+\mu^2+F)c-F)s^2-(c-1)C^2(1+\mu^2+F)}},$$

that is, for constants $D$, $C > 0$ and $F > 0$,

$$u(s) = D \pm \int_{s_0}^{s}\sqrt{-(1+\mu^2)\frac{((1+\mu^2+F)c-F)r^2-cC^2(1+\mu^2+F)}{((1+\mu^2+F)c-F)r^2-(c-1)C^2(1+\mu^2+F)}}dr.$$

The domain of $u$ is determined by analyzing in which interval

$$\frac{((1+\mu^2+F)c-F)s^2-cC^2(1+\mu^2+F)}{((1+\mu^2+F)c-F)s^2-(c-1)C^2(1+\mu^2+F)} < 0. \tag{2.94}$$

If the numerator in (2.94) is positive, then the denominator is

$$\begin{aligned}((1+\mu^2+F)c-F)s^2-(c-1)C^2(1+\mu^2+F) &= ((1+\mu^2+F)c-F)s^2-cC^2(1+\mu^2+F)\\&\quad+C^2(1+\mu^2+F)\\&> 0\end{aligned}$$

and then the fraction in (2.94) is positive, a situation we do not want. Thus, the numerator in (2.94) is necessarily negative, that is,

$$((1+\mu^2+F)c-F)s^2-cC^2(1+\mu^2+F) < 0.$$

If the denominator were negative then the fraction in (2.94) would be positive, which we do not want. Therefore, the denominator is bounded by

$$0 < ((1+\mu^2+F)c-F)s^2-(c-1)C^2(1+\mu^2+F)$$

Bringing together the two inequalities we conclude

$$(c-1)C^2(1+\mu^2+F) < ((1+\mu^2+F)c-F)s^2 < cC^2(1+\mu^2+F).$$

Define $M_{F,\mu,c} = ((1+\mu^2+F)c - F)$. Then,

$$\frac{(c-1)C^2(M_{F,\mu,c}+F)}{c} < M_{F,\mu,c}s^2 < C^2(M_{F,\mu,c}+F),$$

equivalently,

$$\frac{(c-1)C^2(M_{F,\mu,c}+F)}{c} < \left(\operatorname{sgn}(M_{F,\mu,c})\right)^2 M_{F,\mu,c}s^2 < C^2(M_{F,\mu,c}+F),$$

thus

$$\frac{c-1}{c}\frac{C^2(M_{F,\mu,c}+F)}{\operatorname{sgn}(M_{F,\mu,c})M_{F,\mu,c}} < \operatorname{sgn}(M_{F,\mu,c})s^2 < \frac{C^2(M_{F,\mu,c}+F)}{\operatorname{sgn}(M_{F,\mu,c})M_{F,\mu,c}}.$$

□

### 2.2.4 *Type I Translation Surfaces - Minimal case*

By Proposition 2.2.1 we conclude that a **Type I Translation Surface** is **Minimal** if it satisfies a differential equation described in the following proposition.

**Proposition 2.2.6.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, is Minimal if, and only if,*

$$\begin{aligned}0 &= (sv''(t) - u'(s)(1+(v'(t))^2))(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(1+(v'(t))^2)\\ &\quad - 2(v'(t))^2(u'(s))^3.\end{aligned}$$

**Proposition 2.2.7.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$ non-constant, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, Minimal. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is Minimal. By Proposition 2.2.6, the functions $u, v$ satisfy the equation

$$\begin{aligned}0 &= (sv''(t) - u'(s)(1+(v'(t))^2))(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(1+(v'(t))^2)\\ &\quad - 2(v'(t))^2(u'(s))^3,\end{aligned}$$

or equivalently

$$0 = \frac{-2u'(s) + su''(s) - 4(u'(s))^3}{s(1+(u'(s))^2)}(v'(t))^2 + v''(t) + \frac{-2u'(s) + su''(s) - 2(u'(s))^3}{s(1+(u'(s))^2)}.$$

Define the family of irreducible polynomials

$$P_s(X,Y) = A_sX + Y + B_s,$$

where,

$$\begin{aligned}A_s &= \frac{-2u'(s) + su''(s) - 4(u'(s))^3}{s(1+(u'(s))^2)}\\ B_s &= \frac{-2u'(s) + su''(s) - 2(u'(s))^3}{s(1+(u'(s))^2)},\end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = A_sX + Y + B_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Thus, their coefficients are constant, that is, there are $K_1, K_2$ constants such that,

$$\begin{aligned} K_1 &= \frac{-2u'(s) + su''(s) - 4(u'(s))^3}{s(1+(u'(s))^2)} \\ K_2 &= \frac{-2u'(s) + su''(s) - 2(u'(s))^3}{s(1+(u'(s))^2)}, \end{aligned}$$

Defining $w(s) = u'(s)$ we obtain

$$\begin{aligned} K_1 &= \frac{-2w(s) + sw'(s) - 4w^3(s)}{s(1+w^2(s))} \\ K_2 &= \frac{-2w(s) + sw'(s) - 2w^3(s)}{s(1+w^2(s))}. \end{aligned} \tag{2.95}$$

then

$$K_2 - K_1 = \frac{2w^3(s)}{s(1+w^2(s))},$$

consequently

$$0 = 2w^3(s) - (K_2 - K_1)sw^2 - s(K_2 - K_1). \tag{2.96}$$

Define $F : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,y) = 2y^3 - (K_2 - K_1)xy^2 - x(K_2 - K_1).$$

Notice that $F(s, w(s)) = 0$. We have,

$$\begin{aligned} \frac{\partial F}{\partial x}(x,y) &= -(K_2 - K_1)(1+y^2) \\ \frac{\partial F}{\partial y}(x,y) &= -2y((K_2 - K_1)x - 3y). \end{aligned}$$

**Claim 1:** $K_2 \neq K_1$. Indeed, if $K_2 = K_1$ then from (2.96) we obtain that $u$ is constant, a contradiction.

**Claim 2:** There exists $s_0 \in I$ such that $\frac{\partial F}{\partial y}(s_0, w(s_0)) \neq 0$. Indeed, if $\frac{\partial F}{\partial y}(s, w(s)) = 0$ for all $s \in I$ we have

$$0 = \frac{\partial F}{\partial y}(s, w(s)) = -2w(s)((K_2 - K_1)s - 3w(s)),$$

and since $w$ is non-zero (otherwise $u$ would be constant) we conclude that for some neighborhood $V \subset I$

$$w(s) = \frac{(K_2 - K_1)s}{3}$$

which substituting in (2.96)

$$0 = \frac{(K_2 - K_1)^3}{27} s^3 + (K_2 - K_1)s$$

and therefore $s$ is contained in the set of roots of the nonzero polynomial

$$Q(Z) = \frac{(K_2 - K_1)^3}{27} Z^3 + (K_2 - K_1)Z$$

i.e., $s$ is constant, a contradiction and thus we proof the claim.

By claim 2 we can apply the Implicit Function Theorem obtaining

$$w'(s) = \frac{\frac{\partial F}{\partial x}(s,w(s))}{\frac{\partial F}{\partial y}(s,w(s))} = \frac{(K_2 - K_1)(1 + w^2(s))}{2w(s)((K_2 - K_1)s - 3w(s))}$$

which substituting in the first equality of (2.95)

$$\begin{aligned} 0 &= (8K_2 - 14K_1)sw^4(s) - 24w^5(s) - 2K_1(K_1 - K_2)s^2w^3(s) - 12w^3(s) \\ &\quad + (3K_2 - 9K_1)sw^2(s) - 2K_1(K_1 - K_2)s^2w(s) + (K_1 - K_2)s. \end{aligned} \tag{2.97}$$

Define $G : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} G(x,y) &= (8K_2 - 14K_1)xy^4 - 24y^5 - 2K_1(K_1 - K_2)x^2y^3 - 12y^3 \\ &\quad + (3K_2 - 9K_1)xy^2 - 2K_1(K_1 - K_2)x^2y + (K_1 - K_2)x. \end{aligned}$$

Observe that, from (2.96) and (2.97), $F(s,w(s)) = 0 = G(s,w(s))$. Since the curve $(s,w(s))$ is non-constant we conclude by the theory of Plane Algebraic Curves that the polynomials $F$ and $G$ must have at least one factor in common, a contradiction with Lemma $A.0.7$. Then, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.2.3.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t) + u(s), s)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty) \times \mathbb{R}$, Minimal. Then:*

*i) $u$ is constant and $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$.*

*ii) $u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{F(\mu^2+1)r^4}{C^4\left((2\mu^2+1)\left(1-\frac{r^4}{C^4}\right)F + \mu^2 + 1\right)}}\,dr$ e $v(t) = \mu t + \tau$, where $\mu$, $\tau$, $C > 0$, $F > 0$ and $D$ constants and the intervals $J \subset \left(0, \sqrt[4]{\frac{C^4(F(2\mu^2+1)+\mu^2+1)}{F(2\mu^2+1)}}\right)$ and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is Minimal. By Proposition 2.2.6, the functions $u, v$ satisfy the equation

$$\begin{aligned} 0 &= (sv''(t) - u'(s)(1 + (v'(t))^2))(1 + (u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(1 + (v'(t))^2)) \\ &\quad - 2(v'(t))^2(u'(s))^3. \end{aligned} \tag{2.98}$$

**-** $i$)**:** If $u$ is constant it is easy to see from (2.98) that $v''(t) = 0$ in $I$ and therefore $v(t) = \mu t + \tau$.

**-** $ii$)**:** If $u$ is non-constant we obtain by Proposition 2.2.7 that $v(t) = \mu t + \tau$ for constant $\mu, \tau$, and $I \subset \mathbb{R}$. Therefore, from (2.98), we obtain

$$0 = s(\mu^2+1)u''(s) - 2u'(s)((2\mu^2+1)(u'(s))^2 + \mu^2 + 1)$$

and multiplying by $2u'(s)$

$$0 = s(\mu^2+1)2u'(s)u''(s) - 4(u'(s))^2((2\mu^2+1)(u'(s))^2 + \mu^2 + 1)$$

that is,

$$0 = s(\mu^2+1)\frac{d}{ds}[(u'(s))^2] - 4(u'(s))^2((2\mu^2+1)(u'(s))^2 + \mu^2 + 1).$$

Finally, defining $w(s) = (u'(s))^2$, we obtain the separable differential equation

$$w'(s) = \frac{4w(s)((2\mu^2+1)w(s) + \mu^2 + 1)}{s(\mu^2+1)}. \tag{2.99}$$

Notice that $w \equiv 0$ is a solution of (2.99). By the Existence and Uniqueness Theorem any other solution does not reach zero. Since $u$ is non-constant we conclude that $u' \not\equiv 0$ and therefore $w(s) > 0$ in $J$. Thus, from (2.99), we obtain

$$\int_{s_0}^{s} \frac{w'(r)}{w(r)((2\mu^2+1)w(r) + \mu^2 + 1)}dr = \int_{s_0}^{s} \frac{4}{r(\mu^2+1)}dr,$$

and by the Change of Variables Theorem

$$\int_{w(s_0)}^{w(s)} \frac{dl}{l((2\mu^2+1)l + \mu^2 + 1)} = \frac{4}{\mu^2+1}\int_{s_0}^{s} \frac{1}{r}dr.$$

Thus,

$$\frac{1}{\mu^2+1}\left(\ln\left(\frac{w(s)}{(2\mu^2+1)w(s) + \mu^2 + 1}\right) - \ln\left(\frac{w(s_0)}{(2\mu^2+1)w(s_0) + \mu^2 + 1}\right)\right) = \frac{4(\ln(s) - \ln(s_0))}{\mu^2+1},$$

that is,

$$\ln\left(\frac{w(s)((2\mu^2+1)w(s_0) + \mu^2 + 1)}{w(s_0)((2\mu^2+1)w(s) + \mu^2 + 1)}\right) = \ln\left(\frac{s}{s_0}\right)^4,$$

and applying the exponential

$$\frac{w(s)((2\mu^2+1)w(s_0) + \mu^2 + 1)}{w(s_0)((2\mu^2+1)w(s) + \mu^2 + 1)} = \left(\frac{s}{s_0}\right)^4,$$

then,

$$\left((2\mu^2+1)\left(1 - \frac{s^4}{s_0^4}\right)w(s_0) + \mu^2 + 1\right)w(s) = \frac{w(s_0)(\mu^2+1)s^4}{s_0^4}$$

and for all $s \neq \sqrt[4]{\frac{s_0^4(w(s_0)(2\mu^2+1)+\mu^2+1)}{w(s_0)(2\mu^2+1)}}$

$$w(s) = \frac{w(s_0)(\mu^2+1)s^4}{s_0^4\left((2\mu^2+1)\left(1-\frac{s^4}{s_0^4}\right)w(s_0)+\mu^2+1\right)}$$

and since we want $w > 0$, there are constants $C > 0$ and $F > 0$ such that

$$w(s) = \frac{F(\mu^2+1)s^4}{C^4\left((2\mu^2+1)\left(1-\frac{s^4}{C^4}\right)F+\mu^2+1\right)}$$

in the interval $\left(0, \sqrt[4]{\frac{C^4(F(2\mu^2+1)+\mu^2+1)}{F(2\mu^2+1)}}\right)$. Thus, by the definition of $w$, we obtain

$$u'(s) = \pm\sqrt{\frac{F(\mu^2+1)s^4}{C^4\left((2\mu^2+1)\left(1-\frac{s^4}{C^4}\right)F+\mu^2+1\right)}}$$

in $J \subset \left(0, \sqrt[4]{\frac{C^4(F(2\mu^2+1)+\mu^2+1)}{F(2\mu^2+1)}}\right)$. Consequently,

$$u(s) = D \pm \int_{s_0}^{s}\sqrt{\frac{F(\mu^2+1)r^4}{C^4\left((2\mu^2+1)\left(1-\frac{r^4}{C^4}\right)F+\mu^2+1\right)}}dr.$$

□

### 2.2.5 *Type I Translation Surfaces - Constant Mean Curvature case*

By Proposition 2.2.1 we conclude that a **Type I Translation Surface** has **Constant Mean curvature** if it satisfies a differential equation described in the following proposition.

**Proposition 2.2.8.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, and $J \times I \subset (0,+\infty)\times\mathbb{R}$, has constant Mean Curvature c, if and only if,*

$$\begin{aligned} c =& \frac{(sv''(t)-u'(s)(1+(v'(t))^2))(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)(1+(v'(t))^2)}{2\left(1+(v'(t))^2+(u'(s))^2\right)^{\frac{3}{2}}} \\ &-\frac{2(v'(t))^2(u'(s))^3}{2\left(1+(v'(t))^2+(u'(s))^2\right)^{\frac{3}{2}}}. \end{aligned}$$

**Proposition 2.2.9.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$ is non-constant, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$, with constant Mean Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for constant $\mu, \tau$, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ has constant Mean Curvature $c \neq 0$. Squaring the both sides of the equation on Proposition 2.2.8 and rewriting we conclude

$$\begin{aligned}
0 = & -\frac{4c^2}{s^2((u'(s))^2+1)^2}(v'(t))^6 \\
& + \frac{(u'(s) - su''(s) + 3(u'(s))^3 + u'(s)(1+(u'(s))^2))^2 - 12c^2(1+(u'(s))^2)}{s^2((u'(s))^2+1)^2}(v'(t))^4 \\
& + \frac{2(-2u'(s) + su''(s) - 4(u'(s))^3)}{s((u'(s))^2+1)}(v'(t))^2 v''(t) + \frac{2(-2u'(s) + su''(s) - 2(u'(s))^3)}{s((u'(s))^2+1)}v''(t) \\
& + \frac{2(2u'(s) - su''(s) + 2(u'(s))^3)(2u'(s) - su''(s) + 4(u'(s))^3) - 12c^2(1+(u'(s))^2)^2}{s^2((u'(s))^2+1)^2}(v'(t))^2 \\
& + (v''(t))^2 + \frac{(u'(s) - su''(s) + (u'(s))^3 + u'(s)(1+(u'(s))^2))^2 - 4c^2(1+(u'(s))^2)^3}{s^2((u'(s))^2+1)^2}.
\end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = A_s X^3 + B_s X^2 + C_s XY + D_s X + Y^2 + E_s Y + F_s,$$

where,

$$\begin{aligned}
A_s &= -\frac{4c^2}{s^2((u'(s))^2+1)^2} \\
B_s &= \frac{(u'(s) - su''(s) + 3(u'(s))^3 + u'(s)(1+(u'(s))^2))^2 - 12c^2(1+(u'(s))^2)}{s^2((u'(s))^2+1)^2} \\
C_s &= \frac{2(-2u'(s) + su''(s) - 4(u'(s))^3)}{s((u'(s))^2+1)} \\
D_s &= \frac{2(2u'(s) - su''(s) + 2(u'(s))^3)(2u'(s) - su''(s) + 4(u'(s))^3) - 12c^2(1+(u'(s))^2)^2}{s^2((u'(s))^2+1)^2} \\
E_s &= \frac{2(-2u'(s) + su''(s) - 2(u'(s))^3)}{s((u'(s))^2+1)} \\
F_s &= \frac{(u'(s) - su''(s) + (u'(s))^3 + u'(s)(1+(u'(s))^2))^2 - 4c^2(1+(u'(s))^2)^3}{s^2((u'(s))^2+1)^2},
\end{aligned}$$

and notice that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible then there exist polynomials of degree 2 and degree 1 such that

$$P_s(X,Y) = (f + gX + hY)(i + jX + kY + lXY + mX^2 + nY^2),$$

that is,

$$0 = A_s X^3 + B_s X^2 + C_s XY + D_s X + Y^2 + E_s Y + F_s - (f + gX + hY)(i + jX + kY + lXY + mX^2 + nY^2),$$

and reorganizing

$$\begin{aligned}
0 = & (A_s - gm)X^3 - (gl + hm)X^2Y + (B_s - gj - fm)X^2 - (hl + gn)XY^2 \\
& + (C_s - fl - gk - hj)XY + (D_s - ig - fj)X - hnY^3 + (1 - fn - hk)Y^2 \\
& + (E_s - ih - fk)Y + F_s - if
\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = gl + hm & 0 = C_s - fl - gk - hj & 0 = 1 - fn - hk \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = E_s - ih - fk \\ 0 = hl + gn & 0 = hn & 0 = F_s - if. \end{array} \tag{2.100}$$

By the seventh equality we conclude that $h = 0$ or $n = 0$. Suppose that $h = 0$. We have

$$\begin{array}{lll} 0 = A_s - gm & 0 = gn & 0 = 1 - fn \\ 0 = gl & 0 = C_s - fl - gk & 0 = E_s - fk \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

and since $g \neq 0$ (otherwise we would not have factorization) we obtain from the second and fourth equalities that $l = n = 0$. We have,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = E_s - fk \\ 0 = C_s - gk & 0 = 1 & 0 = F_s - if, \end{array}$$

and therefore by the fifth equality we obtain a contradiction. Therefore, we conclude $n = 0$. From (2.100) we have,

$$\begin{array}{lll} 0 = A_s - gm & 0 = hl & 0 = 1 - hk \\ 0 = gl + hm & 0 = C_s - fl - gk - hj & 0 = E_s - ih - fk \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

and, by the seventh equality, we can assume without loss of generality $h = k = 1$. We have,

$$\begin{array}{lll} 0 = A_s - gm & 0 = l & \\ 0 = gl + m & 0 = C_s - fl - g - j & 0 = E_s - i - f \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

and by the fourth equality $l = 0$. We obtain,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = m & 0 = C_s - g - j & 0 = E_s - i - f \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

and substituting the second equality into the first equality we conclude $A_s \equiv 0$, a contradiction, and thus proof the claim.

By the claim the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = A_sX^3 + B_sX^2 + C_sXY + D_sX + Y^2 + E_sY + F_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_3, K_5$ constants such that,

$$\begin{aligned} A_s &= K_1 \\ B_s &= K_3 \\ E_s &= K_5, \end{aligned}$$

that is,

$$\begin{aligned} K_1 &= -\frac{4c^2}{s^2((u'(s))^2+1)^2} \\ K_3 &= \frac{2(-2u'(s)+su''(s)-4(u'(s))^3)}{s((u'(s))^2+1)} \\ K_5 &= \frac{2(-2u'(s)+su''(s)-2(u'(s))^3)}{s((u'(s))^2+1)}, \end{aligned}$$

then, setting $w(s) = u'(s)$, we obtain

$$\begin{aligned} K_1 &= -\frac{4c^2}{s^2(w^2(s)+1)^2} \\ K_3 &= \frac{2(-2w(s)+sw'(s)-4w^3(s))}{s(w^2(s)+1)} \\ K_5 &= \frac{2(-2w(s)+sw'(s)-2w^3(s))}{s(w^2(s)+1)}, \end{aligned}$$

then

$$\begin{aligned} 0 &= K_1w^4(s)s^2 + 2K_1w^2(s)s^2 + K_1s^2 + 4c^2 \\ sw'(s) &= \frac{4w(s)+8w^3(s)+sK_3(w^2(s)+1)}{2} \\ K_5 &= \frac{2(-2w(s)+sw'(s)-2w^3(s))}{s(w^2(s)+1)}, \end{aligned}$$

and substituting the second equality into the third

$$\begin{aligned} 0 &= K_1w^4(s)s^2 + 2K_1w^2(s)s^2 + K_1s^2 + 4c^2 \\ 0 &= (K_5-K_3)w^2(s)s - 4w^3(s) + (K_5-K_3)s. \end{aligned} \tag{2.101}$$

Define $G : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$, $\overline{G} : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$ by

$$G(x,y) = K_1x^2y^4 + 2K_1x^2y^2 + K_1x^2 + 4c^2$$

and

$$\overline{G}(x,y) = (K_5-K_3)xy^2 - 4y^3 + (K_5-K_3)x.$$

Notice that, from (2.101), we have $K_5 \neq K_3$ (otherwise $u$ would be constant). Also, from (2.101), $G(s,w(s)) = 0 = \overline{G}(s,w(s))$. Since the curve $(s,w(s))$ is non-constant we conclude from the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A$.0.8. Therefore, the curve $((v'(t))^2, v''(t))$ is constant and then there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. □

**Theorem 2.2.4.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, and $J \times I \subset (0,+\infty) \times \mathbb{R}$, with constant Mean Curvature $c \neq 0$. Then $v(t) = \mu t + \tau$, for constant $\mu, \tau$, and $I \subset \mathbb{R}$. Moreover, the function $u$ satisfies the differential equation*

$$u''(s) = \frac{c(1+\mu^2+(u'(s))^2)^{\frac{3}{2}} + u'(s)(1+\mu^2+2\mu^2(u'(s))^2+(u'(s))^2)}{s(1+\mu^2)}.$$

**Proof:** Suppose $\psi$ has constant Mean Curvature $c \neq 0$. By Proposition 2.2.8 the functions $u, v$ satisfy the equation

$$\begin{aligned} c =& \frac{(sv''(t) - u'(s)(1+(v'(t))^2))(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(1+(v'(t))^2)}{2\,(1+(v'(t))^2+(u'(s))^2)^{\frac{3}{2}}} \\ &- \frac{2(v'(t))^2(u'(s))^3}{2\,(1+(v'(t))^2+(u'(s))^2)^{\frac{3}{2}}}. \end{aligned} \tag{2.102}$$

Notice that $u$ is non-constant in $J$. Indeed, if $u$ is constant in $J$ the equation in (2.102) reduces to

$$\frac{2c}{s} = \frac{v''(t)}{(1+(v'(t))^2)^{\frac{3}{2}}}.$$

and since the variables are independent there exists $C \neq 0$ such that

$$\begin{aligned} C =& \frac{2c}{s} \\ C =& \frac{v''(t)}{(1+(v'(t))^2)^{\frac{3}{2}}} \end{aligned}$$

a contradiction, since $s$ is not constant. Since $u$ is non-constant we obtain by Proposition 2.2.9 that $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Therefore, from (2.102),

$$u''(s) = \frac{c(1+\mu^2+(u'(s))^2)^{\frac{3}{2}} + u'(s)(1+\mu^2+2\mu^2(u'(s))^2+(u'(s))^2)}{s(1+\mu^2)}.$$

□

### 2.2.6 *Type I Translation Surfaces with Prescribed Length of Second Fundamental Form.*

By Proposition 2.2.1 we conclude that a **Type I Translation Surface** has **Prescribed Length of Second Fundamental Form** if it satisfies a differential equation described in the following proposition.

**Proposition 2.2.10.** *Let $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, $c \in C(J,(0,+\infty))$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. The surface $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$, has Prescribed Length of Second Fundamental Form $c(s)$, if and only if,*

$$
\begin{aligned}
c(s) = &\left(\frac{(sv''(t)-u'(s)(1+(v'(t))^2))(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)(1+(v'(t))^2)}{\omega^3(t,s)}\right.\\
&\left.-\frac{2(v'(t))^2(u'(s))^3}{\omega^3(t,s)}\right)^2\\
&-\frac{2(v'(t))^2((sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4)}{\omega^6(t,s)}\\
&-\frac{2(1+(u'(s))^2)((sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4)}{\omega^6(t,s)}.
\end{aligned}
$$

**Proposition 2.2.11.** *Let $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, $c \in C(J,(0,+\infty))$ and $J \times I \subset (0,+\infty)\times\mathbb{R}$. Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t)+u(s), s)$ with Prescribed Length of Second Fundamental Form $c(s)$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ has Prescribed Length of Second Fundamental Form $c(s) > 0$. By Proposition 2.2.10, the functions $u, v$ satisfy the equation

$$
\begin{aligned}
c(s) = &\left(\frac{(sv''(t)-u'(s)(1+(v'(t))^2))(1+(u'(s))^2)+(su''(s)-u'(s)-(u'(s))^3)(1+(v'(t))^2)}{\omega^3(t,s)}\right.\\
&\left.-\frac{2(v'(t))^2(u'(s))^3}{\omega^3(t,s)}\right)^2\\
&-\frac{2(v'(t))^2((sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4)}{\omega^6(t,s)}\\
&-\frac{2(1+(u'(s))^2)((sv''(t)-u'(s)(1+(v'(t))^2))(su''(s)-u'(s)-(u'(s))^3)-(v'(t))^2(u'(s))^4)}{\omega^6(t,s)}.
\end{aligned}
\tag{2.103}
$$

Notice that $u$ is non-constant. Indeed, if $u$ is constant we obtain from (2.103)

$$(sv''(t))^2 = c(s)(1+(v'(t))^2)^3 \tag{2.104}$$

thus for $s_0 \in J$ we have

$$
\begin{aligned}
(sv''(t))^2 &= c(s)(1+(v'(t))^2)^3\\
(s_0v''(t))^2 &= c(s)(1+(v'(t))^2)^3
\end{aligned}
$$

thus, subtracting the second from the first, we obtain $(s_0^2 - s^2)v''(t) = 0$, in other words, $v''(t) = 0$ in $I$ and therefore from (2.104) we conclude $c(s) = 0$ in $J$, a contradiction. Then, $u$ is non-constant. From

(2.103) we obtain equivalently

$$
\begin{aligned}
0 &= -\frac{c(s)}{s^2(1+(u'(s))^2)^2}(v'(t))^6 + \frac{2u'(s)(-5(u'(s))^2-4(u'(s))^4+su'(s)u''(s)-1)}{s(1+(u'(s))^2)^2}(v'(t))^2v''(t) \\
&+ (v''(t))^2 - \frac{2u'(s)}{s}v''(t) + \frac{-3c(s)+s^2(u''(s))^2+2(u'(s))^2+16(u'(s))^4+16(u'(s))^6}{s^2(1+(u'(s))^2)^2}(v'(t))^4 \\
&+ \frac{-3c(s)(u'(s))^2-2su'(s)u''(s)-8s(u'(s))^3u''(s)}{s^2(1+(u'(s))^2)^2}(v'(t))^4 + \frac{-3c(s)+2s^2(u''(s))^2+4(u'(s))^2}{s^2(1+(u'(s))^2)^2}(v'(t))^2 \\
&+ \frac{20(u'(s))^4+16(u'(s))^6-6c(s)(u'(s))^2-3c(s)(u'(s))^4-4su'(s)u''(s)-10s(u'(s))^3u''(s)}{s^2(1+(u'(s))^2)^2}(v'(t))^2 \\
&+ \frac{-c(s)+s^2(u''(s))^2+2(u'(s))^2+4(u'(s))^4+2(u'(s))^6-3c(s)(u'(s))^2-3c(s)(u'(s))^4}{s^2(1+(u'(s))^2)^2} \\
&+ \frac{-c(s)(u'(s))^6-2su'(s)u''(s)-2s(u'(s))^3u''(s)}{s^2(1+(u'(s))^2)^2}.
\end{aligned}
\tag{2.105}
$$

Define the family of polynomials

$$P_s(X,Y) = A_sX^3 + B_sX^2 + C_sXY + D_sX + Y^2 + E_sY + F_s,$$

where,

$$
\begin{aligned}
A_s &= -\frac{c(s)}{s^2(1+(u'(s))^2)^2} \\
B_s &= \frac{-3c(s)+s^2(u''(s))^2+2(u'(s))^2+16(u'(s))^4+16(u'(s))^6}{s^2(1+(u'(s))^2)^2} \\
&+ \frac{-3c(s)(u'(s))^2-2su'(s)u''(s)-8s(u'(s))^3u''(s)}{s^2(1+(u'(s))^2)^2} \\
C_s &= \frac{2u'(s)(-5(u'(s))^2-4(u'(s))^4+su'(s)u''(s)-1)}{s(1+(u'(s))^2)^2} \\
D_s &= \frac{-3c(s)+2s^2(u''(s))^2+4(u'(s))^2}{s^2(1+(u'(s))^2)^2} \\
&+ \frac{20(u'(s))^4+16(u'(s))^6-6c(s)(u'(s))^2-3c(s)(u'(s))^4-4su'(s)u''(s)-10s(u'(s))^3u''(s)}{s^2(1+(u'(s))^2)^2} \\
E_s &= -\frac{2u'(s)}{s} \\
F_s &= \frac{-c(s)+s^2(u''(s))^2+2(u'(s))^2+4(u'(s))^4+2(u'(s))^6-3c(s)(u'(s))^2-3c(s)(u'(s))^4}{s^2(1+(u'(s))^2)^2} \\
&+ \frac{-c(s)(u'(s))^6-2su'(s)u''(s)-2s(u'(s))^3u''(s)}{s^2(1+(u'(s))^2)^2},
\end{aligned}
$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible then there exist polynomials of degree 2 and degree 1 such that

$$P_s(X,Y) = (f+gX+hY)(i+jX+kY+lXY+mX^2+nY^2),$$

that is,

$$0 = A_sX^3 + B_sX^2 + C_sXY + D_sX + Y^2 + E_sY + F_s - (f + gX + hY)(i + jX + kY + lXY + mX^2 + nY^2),$$

and reorganizing

$$\begin{aligned}0 = (A_s - gm)X^3 - (gl + hm)X^2Y + (B_s - gj - fm)X^2 - (hl + gn)XY^2 + (C_s - fl - gk - hj)XY \\ + (D_s - ig - fj)X - hnY^3 + (1 - fn - hk)Y^2 + (E_s - ih - fk)Y + F_s - if\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = gl + hm & 0 = C_s - fl - gk - hj & 0 = 1 - fn - hk \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = E_s - ih - fk \\ 0 = hl + gn & 0 = hn & 0 = F_s - if. \end{array} \tag{2.106}$$

By the seventh equality we conclude that $h = 0$ or $n = 0$. Suppose $h = 0$. We have,

$$\begin{array}{lll} 0 = A_s - gm & 0 = gn & 0 = 1 - fn \\ 0 = gl & 0 = C_s - fl - gk & 0 = E_s - fk \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

and since $g \neq 0$ (otherwise we would not have factorization) we obtain from the second and fourth equalities that $l = n = 0$. We have,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = E_s - fk \\ 0 = C_s - gk & 0 = 1 & 0 = F_s - if, \end{array}$$

then by the fifth equality we obtain a contradiction. Therefore, we conclude $n = 0$. From (2.106) we have,

$$\begin{array}{lll} 0 = A_s - gm & 0 = hl & 0 = 1 - hk \\ 0 = gl + hm & 0 = C_s - fl - gk - hj & 0 = E_s - ih - fk \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

and, by the seventh equality, we can assume without loss of generality $h = k = 1$. We have,

$$\begin{array}{lll} 0 = A_s - gm & 0 = l & \\ 0 = gl + m & 0 = C_s - fl - g - j & 0 = E_s - i - f \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

and by the fourth equality $l = 0$. We obtain,

$$\begin{array}{lll} 0 = A_s - gm & & \\ 0 = m & 0 = C_s - g - j & 0 = E_s - i - f \\ 0 = B_s - gj - fm & 0 = D_s - ig - fj & 0 = F_s - if, \end{array}$$

that substituting the second equality into the first equality we conclude $A_s \equiv 0$, a contradiction, and thus proof the claim.

By the claim the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Suppose that $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = A_sX^3 + B_sX^2 + C_sXY + D_sX + Y^2 + E_sY + F_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there are $K_3, K_5$ constants such that,

$$C_s = K_3$$
$$E_s = K_5,$$

that is,

$$K_3 = \frac{2u'(s)(-5(u'(s))^2 - 4(u'(s))^4 + su'(s)u''(s) - 1)}{s(1+(u'(s))^2)^2}$$
$$K_5 = -\frac{2u'(s)}{s},$$

then, setting $w(s) = u'(s)$, we obtain

$$K_3 = \frac{2w(s)(-5w^2(s) - 4w^4(s) + sw(s)w'(s) - 1)}{s(1+w^2(s))^2}$$
$$K_5 = -\frac{2w(s)}{s},$$

thus

$$K_3 = \frac{2w(s)(-5w^2(s) - 4w^4(s) + sw(s)w'(s) - 1)}{s(1+w^2(s))^2}$$
$$w(s) = -\frac{sK_5}{2},$$

and substituting the second equality into the first

$$0 = \frac{K_5^4(K_3 - 4K_5)}{16}s^5 + \frac{K_5^2(K_3 - 2K_5)}{2}s^3 + (K_3 - K_5)s. \tag{2.107}$$

If $K_3 = K_5$ we obtain from (2.107) that $s$ is contained in the set of roots of the non-zero polynomial (because if $K_5 = 0$ then $u$ is constant, a contradiction)

$$Q(Z) = \frac{3K_5^5}{16}Z^5 + \frac{K_5^3}{2}Z^3,$$

i.e., $s$ is constant, a contradiction. If $K_3 \neq K_5$ we obtain from (2.107) that $s$ is contained in the set of roots of the nonzero polynomial

$$\overline{Q}(Z) = \frac{K_5^4(K_3 - 4K_5)}{16}Z^5 + \frac{K_5^2(K_3 - 2K_5)}{2}Z^3 + (K_3 - K_5)Z,$$

i.e., $s$ is constant, a contradiction. Then there are no such constants $K_3, K_5$ and thus, the curve $((v'(t))^2, v''(t))$ is constant and therefore there exist $\mu, \eta$ such that $v'(t) = \mu$ and $v''(t) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$. $\square$

**Theorem 2.2.5.** *Let $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$, $c \in C(J, (0, +\infty))$ and $J \times I \subset (0, +\infty) \times \mathbb{R}$. Let $\psi : \Omega \subset \mathbb{R}^2 \to H^3$ given by $\psi(t,s) = (t, v(t) + u(s), s)$ with Prescribed Length of Second Fundamental Form $c(s)$. Then $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover, if $c(s) \geqslant 1 + 8\mu^2$ we have*

$$u''(s) = \frac{u'(s)((u'(s))^2 + \mu^2 + 4\mu^2(u'(s))^2 + 1) + \sqrt{\tilde{\Delta}(\mu^2, c(s), (u'(s))^2)}}{s(1+\mu^2)}$$

*or*

$$u''(s) = \frac{u'(s)((u'(s))^2 + \mu^2 + 4\mu^2(u'(s))^2 + 1) - \sqrt{\tilde{\Delta}(\mu^2, c(s), (u'(s))^2)}}{s(1+\mu^2)}$$

*where*

$$\tilde{\Delta}(x,y,z) = (1+x+z)((1+x+z)y - ((1+8x)z^2 + (1+x)z)).$$

**Proof:** Suppose $\psi$ has Prescribed Length of Second Fundamental Form $c(s)$. By Proposition 2.2.11 we have $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Then by Proposition 2.2.10,

$$\begin{aligned} c(s) = {} & \frac{((-u'(s)(1+\mu^2))(1+(u'(s))^2) + (su''(s) - u'(s) - (u'(s))^3)(1+\mu^2)) - 2\mu^2(u'(s))^3)^2}{(1+(u'(s))^2+\mu^2)^3} \\ & - \frac{2(1+(u'(s))^2+\mu^2)((-u'(s)(1+\mu^2))(su''(s) - u'(s) - (u'(s))^3) - \mu^2(u'(s))^4)}{(1+(u'(s))^2+\mu^2)^3}. \end{aligned} \tag{2.108}$$

i.e.,

$$\begin{aligned} 0 = {} & s^2(1+\mu^2)^2(u''(s))^2 - 2s(1+\mu^2)((u'(s))^2 + \mu^2 + 4\mu^2(u'(s))^2 + 1)u''(s) \\ & + ((4\mu^2+2)^2 - c(s) - 2)(u'(s))^6 + (1+\mu^2)(16\mu^2 + 4 - 3c(s))(u'(s))^4 \\ & - (1+\mu^2)^2(3c(s) - 2)(u'(s))^2 - c(s)(1+\mu^2)^3. \end{aligned} \tag{2.109}$$

Define

$$P_s(Z) = A_s Z^2 + B_s Z + C_s$$

where

$$\begin{aligned} A_s &= s^2(1+\mu^2)^2 \\ B_s &= -2s(1+\mu^2)((u'(s))^2 + \mu^2 + 4\mu^2(u'(s))^2 + 1) \\ C_s &= ((4\mu^2+2)^2 - c(s) - 2)(u'(s))^6 + (1+\mu^2)(16\mu^2 + 4 - 3c(s))(u'(s))^4 \\ &\quad - (1+\mu^2)^2(3c(s) - 2)(u'(s))^2 - c(s)(1+\mu^2)^3. \end{aligned}$$

Observe that $P_s(u''(s)) = 0$ and furthermore, notice that the discriminant $\Delta(s) > 0$. Indeed, since

$c(s) \geqslant 1+8\mu^2$, we have

$$\begin{aligned}\Delta(s) = B_s^2 - 4A_sC_s &= 4s^2(1+\mu^2)^2(1+\mu^2+(u'(s))^2)((1+\mu^2+(u'(s))^2)c(s)\\ &\quad -((1+8\mu^2)(u'(s))^4+(1+\mu^2)(u'(s))^2))\\ &\geqslant 4s^2(1+\mu^2)^2(1+\mu^2+(u'(s))^2)((1+\mu^2+(u'(s))^2)(1+8\mu^2)\\ &\quad -((1+8\mu^2)(u'(s))^4+(1+\mu^2)(u'(s))^2))\\ &= 4s^2(1+\mu^2)^3((u'(s))^2+9\mu^2+8\mu^4+16\mu^2(u'(s))^2+1) > 0.\end{aligned}$$

Therefore, using the formula for the roots of the quadratic equation we obtain, from (2.109), the differential equations

$$u''(s) = \frac{u'(s)((u'(s))^2+\mu^2+4\mu^2(u'(s))^2+1)+\sqrt{\tilde{\Delta}(\mu^2,c(s),(u'(s))^2)}}{s(1+\mu^2)}$$

or

$$u''(s) = \frac{u'(s)((u'(s))^2+\mu^2+4\mu^2(u'(s))^2+1)-\sqrt{\tilde{\Delta}(\mu^2,c(s),(u'(s))^2)}}{s(1+\mu^2)}$$

where

$$\tilde{\Delta}(x,y,z) = (1+x+z)((1+x+z)y-((1+8x)z^2+(1+x)z)).$$

$\square$

## 2.3 Helical Surfaces in $S^2 \times \mathbb{R}$ (Stereographic Model)

Typically, a Helical Surface is thought of as the surface determined by moving a curve in a fixed direction at constant speed and, furthermore, rotating it as the motion occurs. In Euclidean space, considering a curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ given by $\alpha(t) = (\alpha_1(t), 0, \alpha_3(t))$, we could define a Helical Surface as $\psi(t,s) = (\cos(s)\alpha_1(t), \mathrm{sen}(s)\alpha_1(t), \alpha_3(t) + ks)$, where $k$ is constant (speed of motion). In $S^2 \times \mathbb{R}$, given the way we will define this space, we will consider a slightly different Helical Surface. Our curve $\alpha$ will be a graph of a function and the speed will not necessarily be constant. More precisely:

$$\psi(t,s) := (t\cos s, t\,\mathrm{sen}\, s, v(t) + u(s)).$$

First, we will define the space. Then, we will find the curvatures of this family of surfaces.

**Definition 2.3.1** ($S^2 \times \mathbb{R}$)**.** The Spherical space is

$$S^2 = \{(x,y) \in \mathbb{R}^2\},$$

with the Riemannian metric,

$$(u \bullet v)_{(x,y)} = \frac{4\langle (u_1,u_2),(v_1,v_2)\rangle}{(1+x^2+y^2)^2},$$

where $(x,y) \in S^2$, $u = (u_1,u_2)$ and $v = (v_1,v_2)$ since $u, v \in T_pS^2 \simeq \mathbb{R}^2$.

We consider in $S^2 \times \mathbb{R}$ the product metric, that is,

$$(u \bullet_\times v)_{(x,y,z)} = \frac{4\langle (u_1,u_2),(v_1,v_2)\rangle}{(1+x^2+y^2)^2} + u_3v_3,$$

where $(x,y,z) \in S^2 \times \mathbb{R}$, $u = (u_1,u_2,u_3)$ e $v = (v_1,v_2,v_3)$ since $T_p\left(S^2 \times \mathbb{R}\right) \simeq \mathbb{R}^3$.

Consider $\{e_1,e_2,e_3\}$ a canonical basis of $\mathbb{R}^3$. Then,

$$E_1(x,y,z) = \varphi(x,y,z)e_1$$
$$E_2(x,y,z) = \varphi(x,y,z)e_2$$
$$E_3(x,y,z) = e_3$$

generates a global orthonormal frame over $S^2 \times \mathbb{R}$, where $\varphi : S^2 \times \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x,y,z) = \frac{1+x^2+y^2}{2}$. Then, we will obtain the Connection, in this frame, at $S^2 \times \mathbb{R}$. This is done by obtaining the Christoffel Symbols.

**Lemma 2.3.1.** $[E_i,E_j] \equiv 0$ *with the exception of:*

$$[E_1,E_2](x,y,z) = xE_2(x,y,z) - yE_1(x,y,z),$$

*for all* $(x,y,z) \in S^2 \times \mathbb{R}$.

**Proof:** The cases $[E_1,E_1]\equiv[E_2,E_2]\equiv[E_3,E_3]\equiv 0$ are evidents. Let $f:S^2\times\mathbb{R}\to\mathbb{R}$ be a smooth function. We have,

$$\begin{aligned}([E_1,E_3]f)(x,y,z) &= E_1(E_3f)(x,y,z)-E_3(E_1f)(x,y,z)\\ &= \left(E_1\frac{\partial f}{\partial z}\right)(x,y,z)-\left(E_3\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z)\\ &= \left(\varphi\frac{\partial^2 f}{\partial x\partial z}\right)(x,y,z)-\left(\varphi\frac{\partial^2 f}{\partial z\partial x}\right)(x,y,z)=0,\end{aligned}$$

then $[E_1,E_3]\equiv 0$ and similarly $[E_2,E_3]\equiv 0$. Finally,

$$\begin{aligned}([E_1,E_2]f)(x,y,z) &= E_1(E_2f)(x,y,z)-E_2(E_1f)(x,y,z)\\ &= \left(E_1\left(\varphi\frac{\partial f}{\partial y}\right)\right)(x,y,z)-E_2\left(\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z)\\ &= \left(\varphi\frac{\partial}{\partial x}\left(\varphi\frac{\partial f}{\partial y}\right)\right)(x,y,z)-\left(\varphi\frac{\partial}{\partial y}\left(\varphi\frac{\partial f}{\partial x}\right)\right)(x,y,z)\\ &= \varphi(x,y,z)\left(\frac{\partial\varphi}{\partial x}\frac{\partial f}{\partial y}+\varphi\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial f}{\partial x}-\varphi\frac{\partial^2 f}{\partial y\partial x}\right)(x,y,z)\\ &= \varphi(x,y,z)\left(x\frac{\partial f}{\partial y}(x,y,z)-y\frac{\partial f}{\partial x}(x,y,z)\right)\\ &= (xE_2(x,y,z)-yE_1(x,y,z))\,f,\end{aligned}$$

as we wanted. □

From the previous lemma, we can find the Christoffel expressions:

**Lemma 2.3.2** (Christoffel)**.** $\Gamma_{ji}^k\equiv 0$ *with the exception of:*

$$\Gamma_{12}^1(x,y,z)=-y,\qquad \Gamma_{11}^2(x,y,z)=y,\qquad \Gamma_{22}^1(x,y,z)=x,\qquad \Gamma_{21}^2(x,y,z)=-x.$$

**Proof:** By Lemma 2.3.1 and by the expression (Lemma 1.1.1)

$$\Gamma_{ji}^k=-\frac{1}{2}\left([E_i,E_k]\bullet_\times E_j+[E_j,E_k]\bullet_\times E_i+[E_i,E_j]\bullet_\times E_k\right),$$

the null cases are evident. We will calculate $\Gamma_{12}^1$ and the cases $\Gamma_{11}^2$,$\Gamma_{22}^1$, $\Gamma_{21}^2$ are similar. We have

$$\begin{aligned}\Gamma_{12}^1(x,y,z) &= -\frac{1}{2}\left([E_2,E_1]\bullet_\times E_1+[E_1,E_1]\bullet_\times E_2+[E_2,E_1]\bullet_\times E_1\right)(x,y,z)\\ &= -([E_2,E_1]\bullet_\times E_1)(x,y,z)\\ &= ([E_1,E_2]\bullet_\times E_1)(x,y,z)\\ &= (xE_2(x,y,z)-yE_1(x,y,z))\bullet_\times E_1(x,y,z)\\ &= -y.\end{aligned}$$

□

Since $\nabla^\times_{E_j}E_i=\sum_k\Gamma_{ji}^kE_k$ we obtain:

**Lemma 2.3.3** (Connection)**.** *The following holds:*

$$\begin{array}{lll}\nabla^\times_{E_1}E_1(x,y,z)=yE_2(x,y,z) & \nabla^\times_{E_2}E_1(x,y,z)=-xE_2(x,y,z) & \nabla^\times_{E_3}E_1(x,y,z)=0\\ \nabla^\times_{E_1}E_2(x,y,z)=-yE_1(x,y,z) & \nabla^\times_{E_2}E_2(x,y,z)=xE_1(x,y,z) & \nabla^\times_{E_3}E_2(x,y,z)=0\\ \nabla^\times_{E_1}E_3(x,y,z)=0 & \nabla^\times_{E_2}E_3(x,y,z)=0 & \nabla^\times_{E_3}E_3(x,y,z)=0.\end{array}$$

### 2.3.1 *Curvatures of Helical Surfaces*

This section will be dedicated to determining the curvatures of the Helical Surfaces in $S^2 \times \mathbb{R}$. Define the **Helical Surface** $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ by $\psi(t,s) = (t\cos(s), t\mathrm{sen}(s), v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$. We have,

$$\begin{aligned}\frac{\partial \psi}{\partial t}(t,s) &= \big(\cos(s), \mathrm{sen}(s), v'(t)\big)\\ &= \frac{\cos(s)}{\varphi \circ \psi(t,s)} E_1 \circ \psi(t,s) + \frac{\mathrm{sen}(s)}{\varphi \circ \psi(t,s)} E_2 \circ \psi(t,s) + v'(t) E_3 \circ \psi(t,s),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \psi}{\partial s}(t,s) &= \big(-t\,\mathrm{sen}(s), t\cos(s), u'(s)\big)\\ &= -\frac{t\,\mathrm{sen}(s)}{\varphi \circ \psi(t,s)} E_1 \circ \psi(t,s) + \frac{t\cos(s)}{\varphi \circ \psi(t,s)} E_2 \circ \psi(t,s) + u'(s) E_3 \circ \psi(t,s).\end{aligned}$$

The normal $N$ at $\psi(t,s)$ will be given by

$$\begin{aligned}N(t,s) = &\frac{tv'(t)\cos(s) - u'(s)\,\mathrm{sen}(s)}{\omega(t,s)} E_1 \circ \psi(t,s) + \frac{tv'(t)\,\mathrm{sen}(s) + u'(s)\cos(s)}{\omega(t,s)} E_2 \circ \psi(t,s)\\ &- \frac{t}{\omega(t,s)\varphi \circ \psi(t,s)} E_3 \circ \psi(t,s),\end{aligned}$$

where $\omega(t,s) = \sqrt{(tv'(t))^2 + (u'(s))^2 + \frac{t^2}{(\varphi\circ\psi)^2(t,s)}}$.

By Lemma 1.1.2 we conclude that to know the Weingarten map $\mathcal{S}$ we do not need to differentiate the normal field, we just need to know $\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}$, $\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}$ e $\nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}$. We have,

**Lemma 2.3.4.** *The following holds:*

***i)*** $$\begin{aligned}\left(\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s) =& \cos(s) \frac{\partial}{\partial t}\left(\frac{1}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) + \mathrm{sen}(s) \frac{\partial}{\partial t}\left(\frac{1}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s)\\ &+ v''(t) E_3 \circ \psi(t,s).\end{aligned}$$

***ii)*** $$\left(\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s) = -\mathrm{sen}(s) \frac{\partial}{\partial t}\left(\frac{t}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) + \cos(s) \frac{\partial}{\partial t}\left(\frac{t}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s).$$

***iii)*** $$\begin{aligned}\left(\nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}\right)(t,s) =& t\cos(s)\left(\frac{t^2 - \varphi \circ \psi(t,s)}{(\varphi \circ \psi)^2(t,s)}\right) E_1 \circ \psi(t,s) + t\,\mathrm{sen}(s)\left(\frac{t^2 - \varphi \circ \psi(t,s)}{(\varphi \circ \psi)^2(t,s)}\right) E_2 \circ \psi(t,s)\\ &+ u''(s) E_3 \circ \psi(t,s).\end{aligned}$$

**Proof:**

$$\begin{aligned}
\left(\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s) =& \frac{\cos(s)}{\varphi \circ \psi(t,s)} \left(\nabla_{\overline{e_1}} E_1 \circ \psi\right)(t,s) + \overline{e_1}\left(\frac{\cos(s)}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) \\
&+ \frac{\operatorname{sen}(s)}{\varphi \circ \psi(t,s)} \left(\nabla_{\overline{e_1}} E_2 \circ \psi\right)(t,s) + \overline{e_1}\left(\frac{\operatorname{sen}(s)}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s) \\
&+ v'(t) \left(\nabla_{\overline{e_1}} E_3 \circ \psi\right)(t,s) + \overline{e_1}\left(v'(t)\right) E_3 \circ \psi(t,s) \\
=& \frac{\cos(s)}{\varphi \circ \psi(t,s)} \nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_1 + \cos(s) \frac{\partial}{\partial t}\left(\frac{1}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) \\
&+ \frac{\operatorname{sen}(s)}{\varphi \circ \psi(t,s)} \nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_2 + \operatorname{sen}(s) \frac{\partial}{\partial t}\left(\frac{1}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s) \\
&+ v'(t) \nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_3 + v''(t) E_3 \circ \psi(t,s)
\end{aligned}$$

but,

$$\begin{aligned}
\nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_1 =& \frac{\cos(s)}{\varphi \circ \psi(t,s)} \left(\nabla^{\times}_{E_1} E_1\right) \circ \psi(t,s) + \frac{\operatorname{sen}(s)}{\varphi \circ \psi(t,s)} \left(\nabla^{\times}_{E_2} E_1\right) \circ \psi(t,s) + v'(t) \left(\nabla^{\times}_{E_3} E_1\right) \circ \psi(t,s) \\
=& \frac{t \operatorname{sen}(s) \cos(s)}{\varphi \circ \psi(t,s)} E_2 \circ \psi(t,s) - \frac{t \cos(s) \operatorname{sen}(s)}{\varphi \circ \psi(t,s)} E_2 \circ \psi(t,s) \\
=& 0,
\end{aligned}$$

and similarly $\nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_2 = 0 = \nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_3$, then,

$$\begin{aligned}
\left(\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s) =& \cos(s) \frac{\partial}{\partial t}\left(\frac{1}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) + \operatorname{sen}(s) \frac{\partial}{\partial t}\left(\frac{1}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s) \\
&+ v''(t) E_3 \circ \psi(t,s).
\end{aligned}$$

Also,

$$\begin{aligned}
\left(\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s) =& -\frac{t \operatorname{sen}(s)}{\varphi \circ \psi(t,s)} \left(\nabla_{\overline{e_1}} E_1 \circ \psi\right)(t,s) + \overline{e_1}\left(-\frac{t \operatorname{sen}(s)}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) \\
&+ \frac{t \cos(s)}{\varphi \circ \psi(t,s)} \left(\nabla_{\overline{e_1}} E_2 \circ \psi\right)(t,s) + \overline{e_1}\left(\frac{t \cos(s)}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s) \\
&+ u'(s) \left(\nabla_{\overline{e_1}} E_3 \circ \psi\right)(t,s) + \overline{e_1}\left(u'(s)\right) E_3 \circ \psi(t,s) \\
=& -\frac{t \operatorname{sen}(s)}{\varphi \circ \psi(t,s)} \nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_1 - \operatorname{sen}(s) \frac{\partial}{\partial t}\left(\frac{t}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) \\
&+ \frac{t \cos(s)}{\varphi \circ \psi(t,s)} \nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_2 + \cos(s) \frac{\partial}{\partial t}\left(\frac{t}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s) \\
&+ u'(s) \nabla^{\times}_{\frac{\partial \psi}{\partial t}(t,s)} E_3 + \frac{\partial}{\partial t}\left(u'(s)\right) E_3 \circ \psi(t,s)
\end{aligned}$$

thus,

$$\left(\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s) = -\operatorname{sen}(s) \frac{\partial}{\partial t}\left(\frac{t}{\varphi \circ \psi(t,s)}\right) E_1 \circ \psi(t,s) + \cos(s) \frac{\partial}{\partial t}\left(\frac{t}{\varphi \circ \psi(t,s)}\right) E_2 \circ \psi(t,s).$$

Finally,

$$\begin{aligned}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s) =&-\frac{t\,\mathrm{sen}(s)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_2}}E_1\circ\psi\right)(t,s)+\overline{e_2}\left(-\frac{t\,\mathrm{sen}(s)}{\varphi\circ\psi(t,s)}\right)E_1\circ\psi(t,s)\\&+\frac{t\cos(s)}{\varphi\circ\psi(t,s)}\left(\nabla_{\overline{e_2}}E_2\circ\psi\right)(t,s)+\overline{e_2}\left(\frac{t\cos(s)}{\varphi\circ\psi(t,s)}\right)E_2\circ\psi(t,s)\\&+u'(s)\left(\nabla_{\overline{e_2}}E_3\circ\psi\right)(t,s)+\overline{e_2}\left(u'(s)\right)E_3\circ\psi(t,s)\\=&-\frac{t\,\mathrm{sen}(s)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_1-\frac{t}{\varphi\circ\psi(t,s)}\frac{\partial}{\partial s}\left(\mathrm{sen}(s)\right)E_1\circ\psi(t,s)\\&+\frac{t\cos(s)}{\varphi\circ\psi(t,s)}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2+\frac{t}{\varphi\circ\psi(t,s)}\frac{\partial}{\partial s}\left(\cos(s)\right)E_2\circ\psi(t,s)\\&+u'(s)\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3+u''(s)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\begin{aligned}\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_1 &=-\frac{t\,\mathrm{sen}(s)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_1}E_1\right)\circ\psi(t,s)+\frac{t\cos(s)}{\varphi\circ\psi(t,s)}\left(\nabla^{\times}_{E_2}E_1\right)\circ\psi(t,s)+u'(s)\left(\nabla^{\times}_{E_3}E_1\right)\circ\psi(t,s)\\&=-\frac{t^2\,\mathrm{sen}^2(s)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)-\frac{t^2\cos^2(s)}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)\\&=-\frac{t^2}{\varphi\circ\psi(t,s)}E_2\circ\psi(t,s)\end{aligned}$$

and similarly $\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2=\frac{t^2}{\varphi\circ\psi(t,s)}E_1\circ\psi(t,s)$ and $\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3=0$, thus,

$$\begin{aligned}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s) =&t\cos(s)\left(\frac{t^2-\varphi\circ\psi(t,s)}{(\varphi\circ\psi)^2(t,s)}\right)E_1\circ\psi(t,s)+t\,\mathrm{sen}(s)\left(\frac{t^2-\varphi\circ\psi(t,s)}{(\varphi\circ\psi)^2(t,s)}\right)E_2\circ\psi(t,s)\\&+u''(s)E_3\circ\psi(t,s).\end{aligned}$$

□

With this we have,

**Lemma 2.3.5.** *The following holds:*

$$\begin{aligned}&\textbf{\textit{i)}}\mathcal{E}(t,s)=-\frac{4t^2v'(t)+2t(1+t^2)v''(t)}{\omega(t,s)(1+t^2)^2}.\\&\textbf{\textit{ii)}}\mathcal{F}(t,s)=-\frac{4tu'(s)}{\omega(t,s)(1+t^2)^2}.\\&\textbf{\textit{iii)}}\mathcal{G}(t,s)=\frac{2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)}{\omega(t,s)(1+t^2)^2}.\\&\textbf{\textit{iv)}}\mathfrak{e}(t,s)=\frac{4+(v'(t))^2(1+t^2)^2}{(1+t^2)^2}.\\&\textbf{\textit{v)}}\mathfrak{f}(t,s)=u'(s)v'(t).\\&\textbf{\textit{vi)}}\mathfrak{g}(t,s)=\frac{4t^2+(u'(s))^2(1+t^2)^2}{(1+t^2)^2}.\end{aligned}$$

**Proof:**

$$\begin{aligned}
\mathcal{E}(t,s) = \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s) &= \left(\frac{tv'(t)\cos(s) - u'(s)\,\mathrm{sen}(s)}{\omega(t,s)}\right)\cos(s)\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\\
&\quad + \left(\frac{tv'(t)\,\mathrm{sen}(s) + u'(s)\cos(s)}{\omega(t,s)}\right)\mathrm{sen}(s)\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\\
&\quad - \frac{tv''(t)}{\omega(t,s)\varphi\circ\psi(t,s)}\\
&= \frac{tv'(t)}{\omega(t,s)}\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right) - \frac{tv''(t)}{\omega(t,s)\varphi\circ\psi(t,s)}\\
&= -\frac{4t^2v'(t)}{\omega(t,s)(1+t^2)^2} - \frac{2tv''(t)}{\omega(t,s)(1+t^2)}\\
&= -\frac{4t^2v'(t) + 2t(1+t^2)v''(t)}{\omega(t,s)(1+t^2)^2}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{F}(t,s) = \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s) &= \left(\frac{tv'(t)\cos(s) - u'(s)\,\mathrm{sen}(s)}{\omega(t,s)}\right)\left(-\,\mathrm{sen}(s)\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\right)\\
&\quad + \left(\frac{tv'(t)\,\mathrm{sen}(s) + u'(s)\cos(s)}{\omega(t,s)}\right)\cos(s)\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\\
&= \frac{u'(s)}{\omega(t,s)}\frac{\partial}{\partial t}\left(\frac{1}{\varphi\circ\psi(t,s)}\right)\\
&= -\frac{4tu'(s)}{\omega(t,s)(1+t^2)^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{G}(t,s) = \left(N \bullet_\times \nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}\right)(t,s) &= \left(\frac{tv'(t)\cos(s) - u'(s)\,\mathrm{sen}(s)}{\omega(t,s)}\right)t\cos(s)\left(\frac{t^2 - \varphi\circ\psi(t,s)}{(\varphi\circ\psi)^2(t,s)}\right)\\
&\quad + \left(\frac{tv'(t)\,\mathrm{sen}(s) + u'(s)\cos(s)}{\omega(t,s)}\right)t\,\mathrm{sen}(s)\left(\frac{t^2 - \varphi\circ\psi(t,s)}{(\varphi\circ\psi)^2(t,s)}\right)\\
&\quad - \frac{tu''(s)}{\omega(t,s)\varphi\circ\psi(t,s)}\\
&= \frac{t^2v'(t)}{\omega(t,s)}\left(\frac{t^2 - \varphi\circ\psi(t,s)}{(\varphi\circ\psi)^2}\right) - \frac{tu''(s)}{\omega(t,s)\varphi\circ\psi(t,s)}\\
&= \frac{2t^2(t^2-1)v'(t) - t(1+t^2)u''(s)}{\omega(t,s)(1+t^2)^2}.
\end{aligned}$$

The metric coefficients are,

$$\begin{aligned}
\mathfrak{e}(t,s) = (\overline{e_1} \bullet_\times \overline{e_1})(t,s) = \left(\frac{\partial \psi}{\partial t} \bullet_\times \frac{\partial \psi}{\partial t}\right)(t,s) &= \frac{\cos^2(s)}{(\varphi\circ\psi)^2(t,s)} + \frac{\mathrm{sen}^2(s)}{(\varphi\circ\psi)^2(t,s)} + (v'(t))^2\\
&= \frac{1}{(\varphi\circ\psi)^2(t,s)} + (v'(t))^2\\
&= \frac{4 + (v'(t))^2(1+t^2)^2}{(1+t^2)^2}.
\end{aligned}$$

Also,

$$\begin{aligned}\mathfrak{f}(t,s) &= (\overline{e_1} \bullet_\times \overline{e_2})(t,s)\\ &= \left(\frac{\partial\psi}{\partial t} \bullet_\times \frac{\partial\psi}{\partial s}\right)(t,s)\\ &= v'(t)u'(s).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathfrak{g}(t,s) &= (\overline{e_2} \bullet_\times \overline{e_2})(t,s)\\ &= \left(\frac{\partial\psi}{\partial s} \bullet_\times \frac{\partial\psi}{\partial s}\right)(t,s)\\ &= \frac{t^2\operatorname{sen}^2(s)}{(\varphi\circ\psi)^2(t,s)} + \frac{t^2\cos^2(s)}{(\varphi\circ\psi)^2(t,s)} + (v'(t))^2\\ &= \frac{t^2}{(\varphi\circ\psi)^2(t,s)} + (u'(s))^2\\ &= \frac{4t^2 + (u'(s))^2(1+t^2)^2}{(1+t^2)^2}.\end{aligned}$$

□

**Proposition 2.3.1.** *The following holds:*

$$\begin{aligned}H(t,s) = {} & \frac{2t^2(t^2-1)v'(t)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\ & - \frac{t(1+t^2)u''(s)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\ & + \frac{8tv'(t)(u'(s))^2(1+t^2)^2}{8\omega^3(t,s)(1+t^2)^2},\end{aligned}$$

*and*

$$K(t,s) = -\frac{\left(4t^2v'(t)+2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)\right)+16t^2(u'(s))^2}{4\omega^4(t,s)(1+t^2)^2},$$

*and*

$$\begin{aligned}|A|^2(t,s) = 4\Bigg( & \frac{2t^2(t^2-1)v'(t)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\ & - \frac{t(1+t^2)u''(s)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\ & + \frac{8tv'(t)(u'(s))^2(1+t^2)^2}{8\omega^3(t,s)(1+t^2)^2}\Bigg)^2\\ & + \frac{\left(4t^2v'(t)+2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)\right)+16t^2(u'(s))^2}{2\omega^4(t,s)(1+t^2)^2}.\end{aligned}$$

**Proof:**

$$
\begin{aligned}
H(t,s) &= \frac{1}{2}\mathrm{tr}(\mathcal{S})(t,s)\\
&= \frac{1}{2}(a_{tt}+a_{ss})\\
&= \frac{1}{2}\left(\frac{\mathcal{E}\mathfrak{g}+\mathcal{G}\mathfrak{e}-2\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\\
&= \frac{1}{2}\left(\frac{-\left(\frac{4t^2v'(t)+2t(1+t^2)v''(t)}{\omega(t,s)(1+t^2)^2}\right)\left(\frac{4t^2+(u'(s))^2(1+t^2)^2}{(1+t^2)^2}\right)+\left(\frac{2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)}{\omega(t,s)(1+t^2)^2}\right)\left(\frac{4+(v'(t))^2(1+t^2)^2}{(1+t^2)^2}\right)}{\left(\frac{4+(v'(t))^2(1+t^2)^2}{(1+t^2)^2}\right)\left(\frac{4t^2+(u'(s))^2(1+t^2)^2}{(1+t^2)^2}\right)-(u'(s)v'(t))^2}\right)\\
&+\frac{1}{2}\left(\frac{2\left(\frac{4tu'(s)}{\omega(t,s)(1+t^2)^2}\right)(u'(s)v'(t))}{\left(\frac{4+(v'(t))^2(1+t^2)^2}{(1+t^2)^2}\right)\left(\frac{4t^2+(u'(s))^2(1+t^2)^2}{(1+t^2)^2}\right)-(u'(s)v'(t))^2}\right)\\
&= \frac{2t^2(t^2-1)v'(t)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\
&-\frac{t(1+t^2)u''(s)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\
&+\frac{8tv'(t)(u'(s))^2(1+t^2)^2}{8\omega^3(t,s)(1+t^2)^2}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
K(t,s) &= \det(\mathcal{S})\\
&= a_{tt}a_{ss}-a_{ts}a_{st}\\
&= \left(\frac{\mathcal{E}\mathfrak{g}-\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{G}\mathfrak{e}-\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)-\left(\frac{\mathcal{F}\mathfrak{g}-\mathcal{G}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{F}\mathfrak{e}-\mathcal{E}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\\
&= \left(\frac{\mathcal{E}\mathcal{G}\mathfrak{g}\mathfrak{e}-\mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f}-\mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e}+(\mathcal{F}\mathfrak{f})^2-\left(\mathcal{F}^2\mathfrak{g}\mathfrak{e}-\mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f}-\mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e}+\mathcal{E}\mathcal{F}\mathfrak{f}^2\right)}{(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)^2}\right)(t,s)\\
&= \left(\frac{\mathcal{E}\mathcal{G}(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)-\mathcal{F}^2(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)}{(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)^2}\right)(t,s)\\
&= \left(\frac{\mathcal{E}\mathcal{G}-\mathcal{F}^2}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\\
&= \frac{\left(-\frac{4t^2v'(t)+2t(1+t^2)v''(t)}{\omega(t,s)(1+t^2)^2}\right)\left(\frac{2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)}{\omega(t,s)(1+t^2)^2}\right)-\left(-\frac{4tu'(s)}{\omega(t,s)(1+t^2)^2}\right)^2}{\frac{4\omega^2(t,s)}{(1+t^2)^2}}\\
&= -\frac{\left(4t^2v'(t)+2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)\right)+16t^2(u'(s))^2}{4\omega^4(t,s)(1+t^2)^2}.
\end{aligned}
$$

Finally, being $\lambda_1(t,s)$ and $\lambda_2(t,s)$ the principal curvatures we have,

$$\begin{aligned}
|A|^2(t,s) &= \lambda_1^2(t,s)+\lambda_2^2(t,s)\\
&= 4\frac{(\lambda_1+\lambda_2)^2(t,s)}{4}-2\lambda_1(t,s)\lambda_2(t,s)\\
&= 4H^2(t,s)-2K(t,s)\\
&= 4\Bigg(\frac{2t^2(t^2-1)v'(t)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\
&-\frac{t(1+t^2)u''(s)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))}{8\omega^3(t,s)(1+t^2)^2}\\
&+\frac{8tv'(t)(u'(s))^2(1+t^2)^2}{8\omega^3(t,s)(1+t^2)^2}\Bigg)^2\\
&+\frac{\left(4t^2v'(t)+2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)\right)+16t^2(u'(s))^2}{2\omega^4(t,s)(1+t^2)^2}.
\end{aligned}$$

□

### 2.3.2 *Flat Helical Surfaces.*

**Lemma 2.3.6.** *Let $\mu\neq 0$, $C\in\mathbb{R}-\{-1,0,1\}$ and $F\geqslant 0$ be constants. Consider the function $w: I\subset\mathbb{R}-\{-1,0,1\}\to\mathbb{R}$ given by*

$$w(t)=\frac{1}{(t^2+1)^2}\left(8\mu^2\ln\left|\frac{(1-C^2)t}{(1-t^2)C}\right|+F\right)$$

*Then, $w(t)\geqslant 0$ only in $((-\infty,-r_2]\bigcup[r_2,+\infty))\bigcap I$ where*

$$r_2=\sqrt{\left(\left|\frac{1-C^2}{2C}\right|e^{\frac{F}{8\mu^2}}\right)^2+1}-\left|\frac{1-C^2}{2C}\right|e^{\frac{F}{8\mu^2}},$$

*and $r_2<1$.*

**Proof:** $w(t)\geqslant 0$ if and only if,

$$8\mu^2\ln\left|\frac{(1-C^2)t}{(1-t^2)C}\right|+F\geqslant 0$$

i.e.,

$$\left|\frac{(1-C^2)t}{(1-t^2)C}\right|\geqslant e^{-\frac{F}{8\mu^2}}$$

that is,

$$\left|\frac{1-t^2}{t}\right|\leqslant\left|\frac{1-C^2}{C}\right|e^{\frac{F}{8\mu^2}}$$

thus,

$$|1-t^2|\leqslant\left|\frac{1-C^2}{C}\right|e^{\frac{F}{8\mu^2}}|t|$$

and since $1-|t|^2 \leqslant |1-t^2|$ we have

$$0 \leqslant |t|^2 + \left|\frac{1-C^2}{C}\right| e^{\frac{F}{8\mu^2}} |t| - 1.$$

Define the function $q(x) = x^2 + \left|\frac{1-C^2}{C}\right| e^{\frac{F}{8\mu^2}} x - 1$. Notice that $\Delta > 0$ and that the concavity is upwards. Then, we always have two distinct roots. We know that $q$ is negative in the interval $(r_1, r_2)$ with $r_1$ and $r_2$ being the roots of $q$. Since we are only interested in $x > 0$ we obtain that $q(x) \geqslant 0$ only in $[r_2, +\infty)$, with $r_2$ being the positive root of $q$, in other words,

$$0 \leqslant q(|t|) = |t|^2 + \left|\frac{1-C^2}{C}\right| e^{\frac{F}{8\mu^2}} |t| - 1,$$

if and only if $t \in ((-\infty, -r_2] \bigcup [r_2, +\infty)) \bigcap I$ where

$$r_2 = \sqrt{\left(\left|\frac{1-C^2}{2C}\right| e^{\frac{F}{8\mu^2}}\right)^2 + 1} - \left|\frac{1-C^2}{2C}\right| e^{\frac{F}{8\mu^2}}.$$

Finally, notice that $r_2 < 1$. Actually, it suffices to see that $\sqrt{m^2+1} - m < 1$ for $m > 0$. □

By Proposition 2.3.1 we conclude that a **Helical Surface** is **Flat** if it satisfies a differential equation described in the following proposition.

**Proposition 2.3.2.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\mathrm{sen} s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, is Flat, if and only if*

$$0 = \left(4t^2 v'(t) + 2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t) - t(1+t^2)u''(s)\right) + 16t^2(u'(s))^2.$$

**Proposition 2.3.3.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\mathrm{sen} s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, Flat. Then $u(s) = \mu s + \tau$, for $\mu, \tau$ constants, and $J \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.3.2, the functions $u, v$ satisfy the equation

$$0 = 16t^2(u'(s))^2 - t(1+t^2)(4t^2v'(t) + 2tv''(t)(1+t^2))u''(s) + 2t^2v'(t)(t^2-1)(4t^2v'(t) + 2tv''(t)(1+t^2)), \tag{2.110}$$

that is,

$$0 = (u'(s))^2 - \frac{1}{8}(1+t^2)(v''(t) + 2tv'(t) + t^2v''(t))u''(s) + \frac{1}{4}tv'(t)(t^2-1)(v''(t) + 2tv'(t) + t^2v''(t)).$$

Define the family of irreducible polynomials

$$P_t(X,Y) = X + A_tY + B_t,$$

where,

$$\begin{aligned}A_t =& -\frac{1}{8}(1+t^2)(v''(t)+2tv'(t)+t^2v''(t))\\ B_t =& \frac{1}{4}tv'(t)(t^2-1)(v''(t)+2tv'(t)+t^2v''(t)),\end{aligned}$$

and notice that the curve $((u'(s))^2, u''(s))$ is contained in the intersection of the roots of $\{P_t\}_{t\in I}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_t$ are multiples by constants. Suppose that $\{P_t\}_{t\in I}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_t(X,Y) = X + A_tY + B_t,$$

is constant in the variable $t$, that is, the polynomials $P_t$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2$ constants such that,

$$\begin{aligned}K_1 =& A_t\\ K_2 =& B_t,\end{aligned}$$

that is,

$$\begin{aligned}K_1 =& -\frac{1}{8}(1+t^2)(v''(t)+2tv'(t)+t^2v''(t))\\ K_2 =& \frac{1}{4}tv'(t)(t^2-1)(v''(t)+2tv'(t)+t^2v''(t)).\end{aligned} \tag{2.111}$$

By the first equality we obtain $v''(t) = -\frac{2(4K_1+tv'(t)+t^3v'(t))}{(1+t^2)^2}$ which substituting in the second

$$K_2 = -2t(t^2-1)v'(t)\frac{K_1}{1+t^2}. \tag{2.112}$$

We have two situations. If $1 \in I$ or $-1 \in I$ then $K_2 = 0$ and therefore $K_1v'(t) = 0$. In this case, $K_1 = 0$ or $v' = 0$ (which from (2.111) we get $K_1 = 0$). We have,

$$0 = P_t((u'(s))^2, u''(s)) = (u'(s))^2 + K_1u''(s) + K_2 = (u'(s))^2,$$

i.e., $u$ is a straight line in $J$. In the second situation, $\pm 1 \notin I$, we obtain from (2.112)

$$v'(t)K_1 = -\frac{K_2(1+t^2)}{2t(t^2-1)}.$$

If $K_1 = 0$ then $K_2 = 0$ and as before $u$ is a straight line. If $K_1 \neq 0$ we obtain

$$v'(t) = -\frac{K_2(1+t^2)}{2t(t^2-1)K_1} \Rightarrow v''(t) = -\frac{K_2}{2K_1}\left(\frac{2t^2(t^2-1)-(3t^2-1)(t^2+1)}{t^2(t^2-1)^2}\right)$$

which substituting in (2.110) we have

$$\begin{aligned}0 =& \frac{(K_2^2+K_1K_2u''(s))t^8+(16(u'(s))^2K_1^2-4K_1K_2u''(s)-4K_2^2)t^6}{K_1^2(t^2-1)^2}\\ &+\frac{-(32(u'(s))^2K_1^2+10K_1K_2u''(s)+10K_2^2)t^4+(16(u'(s))^2K_1^2-4K_1K_2u''(s)-4K_2^2)t^2}{K_1^2(t^2-1)^2}\\ &+\frac{K_2^2+K_1K_2u''(s)}{K_1^2(t^2-1)^2}\end{aligned}$$

i.e.,

$$\begin{aligned}0 &= (K_2^2 + K_1K_2u''(s))t^8 + (16(u'(s))^2K_1^2 - 4K_1K_2u''(s) - 4K_2^2)t^6\\ &- (32(u'(s))^2K_1^2 + 10K_1K_2u''(s) + 10K_2^2)t^4 + (16(u'(s))^2K_1^2 - 4K_1K_2u''(s) - 4K_2^2)t^2\\ &+ K_2^2 + K_1K_2u''(s)\end{aligned}$$

and since $t$ is not constant we conclude that this polynomial has all zero coefficients, that is,

$$\begin{aligned}0 &= K_2^2 + K_1K_2u''(s)\\ 0 &= 16(u'(s))^2K_1^2 - 4K_1K_2u''(s) - 4K_2^2\\ 0 &= 32(u'(s))^2K_1^2 + 10K_1K_2u''(s) + 10K_2^2\\ 0 &= 16(u'(s))^2K_1^2 - 4K_1K_2u''(s) - 4K_2^2\\ 0 &= K_2^2 + K_1K_2u''(s).\end{aligned}$$

Isolating $u''$ in the first equality and substituting it in the others we obtain $(u'(s))^2K_1^2 = 0$ therefore, $u$ is a straight line.
Finally, if the curve $((u'(s))^2, u''(s))$ is constant there exist $\mu, \eta$ such that $u'(s) = \mu$ and $u''(s) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$. □

**Theorem 2.3.1.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\,\mathrm{sen}s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$, $v \in C^2(I, \mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, Flat. Then:*

*i) $u(s) = \tau$, for constant $\tau$ and $J \subset \mathbb{R}$, and*

$$v(t) = D \pm \sqrt{F}\arctan(t)$$

*in $I \subset \mathbb{R} - \{0\}$ and constants $D$, $F \geqslant 0$.*

*ii) $u(s) = \mu s + \tau$, for $\mu \neq 0$, $\tau$ constants and $J \subset \mathbb{R}$, and*

$$v(t) = D \pm \int_{t_0}^{t} \frac{\sqrt{8\mu^2 \ln\left|\frac{(1-C^2)r}{(1-r^2)C}\right| + F}}{r^2+1}dr,$$

*for constant $C \notin \{-1,0,1\}$, $D$ and $F \geqslant 0$, and $I \subset ((-\infty, -r_2]\bigcup[r_2, +\infty))\bigcap(\mathbb{R} - \{-1,0,1\})$ where*

$$r_2 = \sqrt{\left(\left|\frac{1-C^2}{2C}\right|e^{\frac{F}{8\mu^2}}\right)^2 + 1} - \left|\frac{1-C^2}{2C}\right|e^{\frac{F}{8\mu^2}},$$

*and $r_2 < 1$.*

**Proof:** Suppose $\psi$ is Flat. By Proposition 2.3.2, the functions $u, v$ satisfy the equation

$$0 = 16t^2(u'(s))^2 - t(1+t^2)(4t^2v'(t) + 2tv''(t)(1+t^2))u''(s) + 2t^2v'(t)(t^2-1)(4t^2v'(t) + 2tv''(t)(1+t^2)).$$

By the Proposition 2.3.3 $u(s) = \mu s + \tau$, for $\mu, \tau$ constants, and $J \subset \mathbb{R}$ then

$$0 = 2t(t^2-1)(t^2+1)v'(t)v''(t) + 8\mu^2 + 4t^2(v'(t))^2(t^2-1),$$

that is,

$$0 = t(t^2-1)(t^2+1)\frac{d}{dt}[(v'(t))^2] + 8\mu^2 + 4t^2(v'(t))^2(t^2-1),$$

and defining $w(t) = (v'(t))^2$

$$0 = t(t^2-1)(t^2+1)w'(t) + 4t^2(t^2-1)w(t) + 8\mu^2. \tag{2.113}$$

We have two possible situations. If $1 \in I$ or $-1 \in I$ we obtain, applying to (2.113), that $\mu = 0$ and therefore

$$0 = (t^2+1)w'(t) + 4tw(t). \tag{2.114}$$

Observe that $w \equiv 0$ is a solution of this separable differential equation, and consequently $v$ is constant. By the Existence and Uniqueness Theorem any other solution does not reach zero. Let $w \neq 0$ be. We obtain from (2.114)

$$\frac{w'(t)}{w(t)} = -\frac{4t}{t^2+1}$$

then

$$\int_{t_0}^{t} \frac{w'(r)}{w(r)} dr = -4\int_{t_0}^{t} \frac{r}{r^2+1} dr$$

by the Change of Variables Theorem

$$\int_{w(t_0)}^{w(t)} \frac{1}{l} dl = -4\int_{t_0}^{t} \frac{r}{r^2+1} dr$$

that is,

$$\ln\left(\frac{w(t)}{w(t_0)}\right) = \ln\left(\left(\frac{t_0^2+1}{t^2+1}\right)^2\right)$$

and applying the exponential,

$$w(t) = \left(\frac{t_0^2+1}{t^2+1}\right)^2 w(t_0),$$

i.e., there is constant $F > 0$ such that

$$w(t) = \frac{F}{(t^2+1)^2},$$

consequently, by the definition of $w$ we have two distinct differential equations

$$v'(t) = \pm\sqrt{\frac{F}{(t^2+1)^2}} = \pm\frac{\sqrt{F}}{t^2+1}$$

thus,

$$v(t) = D \pm \sqrt{F}\arctan(t)$$

in $I \subset \mathbb{R} - \{0\}$ and thus we conclude the first situation. In the second situation, if $\pm 1 \notin I$ we obtain from (2.113)

$$0 = w'(t) + \frac{4t}{t^2+1}w(t) + \frac{8\mu^2}{t(t^2-1)(t^2+1)}.$$

and $\mu = 0$ we obtain the previous situation. Let us then consider $\mu \neq 0$. Multiplying by the integration factor $(t^2+1)^2$ we have

$$0 = \frac{d}{dt}[(t^2+1)^2 w(t)] + \frac{8\mu^2(t^2+1)}{t(t^2-1)},$$

which integrating

$$(t^2+1)^2 w(t) - (t_0^2+1)^2 w(t_0) = -8\mu^2 \int_{t_0}^{t} \frac{r^2+1}{r(r^2-1)} dr$$

i.e.,

$$(t^2+1)^2 w(t) - (t_0^2+1)^2 w(t_0) = -8\mu^2 \left( \ln\left|\frac{1-t^2}{t}\right| - \ln\left|\frac{1-t_0^2}{t_0}\right| \right)$$

then,

$$(t^2+1)^2 w(t) = 8\mu^2 \ln\left|\frac{(1-t_0^2)t}{(1-t^2)t_0}\right| + (t_0^2+1)^2 w(t_0)$$

thus,

$$w(t) = \frac{1}{(t^2+1)^2}\left(8\mu^2 \ln\left|\frac{(1-t_0^2)t}{(1-t^2)t_0}\right| + (t_0^2+1)^2 w(t_0)\right)$$

that is, there exist constants $C \notin \{-1,0,1\}$ and $F \geqslant 0$ such that

$$w(t) = \frac{1}{(t^2+1)^2}\left(8\mu^2 \ln\left|\frac{(1-C^2)t}{(1-t^2)C}\right| + F\right).$$

By the definition of $w$ and by Lemma 2.3.6 we have two distinct differential equations

$$v'(t) = \pm\frac{\sqrt{8\mu^2 \ln\left|\frac{(1-C^2)t}{(1-t^2)C}\right| + F}}{t^2+1}$$

in $((-\infty,-r_2]\bigcup[r_2,+\infty))\bigcap I$ where

$$r_2 = \sqrt{\left(\left|\frac{1-C^2}{2C}\right| e^{\frac{F}{8\mu^2}}\right)^2 + 1} - \left|\frac{1-C^2}{2C}\right| e^{\frac{F}{8\mu^2}},$$

and $r_2 < 1$. Finally,

$$v(t) = D \pm \int_{t_0}^{t} \frac{\sqrt{8\mu^2 \ln\left|\frac{(1-C^2)r}{(1-r^2)C}\right| + F}}{r^2+1} dr.$$

□

### 2.3.3 Helical Surfaces of Constant Gauss Curvature.

By Proposition 2.3.1 we conclude that a **Helical Surface** has **Constant Gauss Curvature** if it satisfies a differential equation described in the following proposition.

**Proposition 2.3.4.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\text{sen} s, v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, has constant Gaussian Curvature, if and only if*

$$c = -\frac{\left(4t^2v'(t)+2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)\right)+16t^2(u'(s))^2}{4\left((tv'(t))^2+(u'(s))^2+\frac{4t^2}{(1+t^2)^2}\right)^2(1+t^2)^2}$$

**Proposition 2.3.5.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\text{sen} s, v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, with constant Gaussian Curvature $c \neq 0$. Then $u(s) = \mu s + \tau$, for $\mu, \tau$ constants, and $J \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.3.4, the functions $u, v$ satisfy the equation

$$c = -\frac{\left(4t^2v'(t)+2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t)-t(1+t^2)u''(s)\right)+16t^2(u'(s))^2}{4\left((tv'(t))^2+(u'(s))^2+\frac{4t^2}{(1+t^2)^2}\right)^2(1+t^2)^2},$$

or equivalently

$$\begin{aligned} 0 =& (u'(s))^4 + \frac{2t^2(4c+c(v'(t))^2+2ct^2(v'(t))^2+ct^4(v'(t))^2+2)}{c(t^2+1)^2}(u'(s))^2 \\ &- \frac{t^2(v''(t)+2tv'(t)+t^2v''(t))}{2c(t^2+1)}u''(s) \\ &+ \frac{4c(t^2+1)^2\left(t^2(v'(t))^2+\frac{4t^2}{(t^2+1)^2}\right)^2+2t^2v'(t)(t^2-1)(4t^2v'(t)+2tv''(t)(t^2+1))}{4c(t^2+1)^2}. \end{aligned} \tag{2.115}$$

Define the family of polynomials

$$P_t(X,Y) = X^2 + A_tX + B_tY + C_t,$$

where,

$$\begin{aligned} A_t &= \frac{2t^2(4c+c(v'(t))^2+2ct^2(v'(t))^2+ct^4(v'(t))^2+2)}{c(t^2+1)^2} \\ B_t &= -\frac{t^2(v''(t)+2tv'(t)+t^2v''(t))}{2c(t^2+1)} \\ C_t &= \frac{4c(t^2+1)^2\left(t^2(v'(t))^2+\frac{4t^2}{(t^2+1)^2}\right)^2+2t^2v'(t)(t^2-1)(4t^2v'(t)+2tv''(t)(t^2+1))}{4c(t^2+1)^2}, \end{aligned}$$

and notice that the curve $((u'(s))^2, u''(s))$ is contained in the intersection of the roots of $\{P_t\}_{t\in I}$. If $B_t \equiv 0$ in $I$ we have

$$0 = P_t((u'(s))^2, u''(s)) = (u'(s))^4 + (u'(s))^2A_t + C_t,$$

that is, for any fixed $t$ $u'(s)$ is constant because it is zero of a polynomial in one variable. Therefore $u$ is a straight line. If there exists an open $V \subset I$ such that $B_t \neq 0$ we conclude that the family of polynomials $\{P_t\}_{t\in V}$ is irreducible. By the theory of Plane Algebraic Curves we conclude that the curve $((u'(s))^2, u''(s))$ is constant or the polynomials $P_t$ are multiples by constants. Suppose that $\{P_t\}_{t\in V}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_t(X,Y) = X^2 + A_tX + B_tY + C_t,$$

is constant in the variable $t$, that is, the polynomials $P_t$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2, K_3$ constants such that,

$$\begin{aligned} A_t &= K_1 \\ B_t &= K_2 \\ C_t &= K_3, \end{aligned}$$

that is,

$$\begin{aligned} K_1 &= \frac{2t^2(4c + c(v'(t))^2 + 2ct^2(v'(t))^2 + ct^4(v'(t))^2 + 2)}{c(t^2+1)^2} \\ K_2 &= -\frac{t^2(v''(t) + 2tv'(t) + t^2v''(t))}{2c(t^2+1)} \\ K_3 &= \frac{4c(t^2+1)^2\left(t^2(v'(t))^2 + \frac{4t^2}{(t^2+1)^2}\right)^2 + 2t^2v'(t)(t^2-1)(4t^2v'(t) + 2tv''(t)(t^2+1))}{4c(t^2+1)^2}, \end{aligned}$$

then, defining $w(t) = v'(t)$, we obtain

$$\begin{aligned} K_1 &= \frac{2t^2(4c + cw^2(t) + 2ct^2w^2(t) + ct^4w^2(t) + 2)}{c(t^2+1)^2} \\ K_2 &= -\frac{t^2(w'(t) + 2tw(t) + t^2w'(t))}{2c(t^2+1)} \\ K_3 &= \frac{4c(t^2+1)^2\left(t^2w^2(t) + \frac{4t^2}{(t^2+1)^2}\right)^2 + 2t^2w(t)(t^2-1)(4t^2w(t) + 2tw'(t)(t^2+1))}{4c(t^2+1)^2}, \end{aligned}$$

thus

$$\begin{aligned} 0 &= 2cw^2(t)t^6 + 4cw^2(t)t^4 + 2cw^2(t)t^2 - cK_1t^4 + (8c - 2cK_1 + 4)t^2 - cK_1 \\ w'(s) &= -\frac{2(t^3w(t) + cK_2(t^2+1))}{t^2(t^2+1)} \\ K_3 &= \frac{4c(t^2+1)^2\left(t^2w^2(t) + \frac{4t^2}{(t^2+1)^2}\right)^2 + 2t^2w(t)(t^2-1)(4t^2w(t) + 2tw'(t)(t^2+1))}{4c(t^2+1)^2}, \end{aligned}$$

and substituting the second equality into the third

$$
\begin{aligned}
0 &= 2cw^2(t)t^6 + 4cw^2(t)t^4 + 2cw^2(t)t^2 - cK_1t^4 + (8c - 2cK_1 + 4)t^2 - cK_1\\
0 &= ct^{12}w^4(t) + 4ct^{10}w^4(t) - 2cK_2t^9w(t) + 6ct^8w^4(t) + 8ct^8w^2(t) - cK_3t^8 - 4cK_2t^7w(t) + 4ct^6w^4(t)\\
&\quad + 16t^6w^2(t) - 4cK_3t^6 + ct^4w^4(t) + 8ct^4w^2(t) + (16c - 6cK_3)t^4 + 4cK_2t^3w(t) - 4cK_3t^2\\
&\quad + 2cK_2tw(t) - cK_3.
\end{aligned} \tag{2.116}
$$

**Claim:** $c \neq -\frac{1}{2}$ and $c \neq \frac{1}{K_1-2}$. Indeed, if $c = -\frac{1}{2}$ we obtain from (2.116)

$$
\begin{aligned}
0 &= \frac{(t^2+1)^2(2t^2w^2(t) - K_1)}{2}\\
0 &= -\frac{t^4(t^2+1)^4}{2}w^4(t) - 4t^4(t^2+1)^2w^2(t) + 4K_2t(t^2-1)(t^2+1)^3w(t) + \frac{K_3}{2}t^8 + 2K_3t^6\\
&\quad + (3K_3 - 8)t^4 + 2K_3t^2 + \frac{K_3}{2}.
\end{aligned}
$$

By the first equality if $K_1 < 0$ we have a contradiction. If $K_1 \geqslant 0$ we have $w(t) = \pm\frac{1}{t}\sqrt{\frac{K_1}{2}}$. Taking the positive solution and substituting it into the second equality we have

$$
\begin{aligned}
0 &= \left(\frac{K_3}{2} - \frac{K_1^2}{8} + K_2\sqrt{\frac{K_1}{2}}\right)t^8 + \left(2K_3 - 2K_1 - \frac{K_1^2}{2} + 2K_2\sqrt{\frac{K_1}{2}}\right)t^6 + \left(3K_3 - 4K_1 - \frac{3K_1^2}{4} - 8\right)t^4\\
&\quad + \left(2K_3 - 2K_1 - \frac{K_1^2}{2} + 2K_2\sqrt{\frac{K_1}{2}}\right)t^2 + \frac{K_3}{2} - \frac{K_1^2}{8} - K_2\sqrt{\frac{K_1}{2}}
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{aligned}
0 &= \frac{K_3}{2} - \frac{K_1^2}{8} + K_2\sqrt{\frac{K_1}{2}}\\
0 &= 2K_3 - 2K_1 - \frac{K_1^2}{2} + 2K_2\sqrt{\frac{K_1}{2}}\\
0 &= 3K_3 - 4K_1 - \frac{3K_1^2}{4} - 8\\
0 &= \frac{K_3}{2} - \frac{K_1^2}{8} - K_2\sqrt{\frac{K_1}{2}}.
\end{aligned}
$$

Isolating $K_3$ in the third equality and then substituting it in the first and last equalities added together we conclude $K_1 + 2 = 0$, a contradiction. The case in which the solution $w$ is negative is analogous and therefore $c \neq -\frac{1}{2}$. If $c = \frac{1}{K_1-2}$ we proceed analogously and conclude $K_2 = 0$, a contradiction, and therefore $c \neq \frac{1}{K_1-2}$. Thus, we have proof the claim.

Define $G : \mathbb{R}^2 \to \mathbb{R}$, $\overline{G} : \mathbb{R}^2 \to \mathbb{R}$ by

$$G(x,y) = 2cy^2x^2 + 4cy^2x + 2cy^2 - cK_1x^2 + (8c - 2cK_1 + 4)x - cK_1$$

and

$$
\begin{aligned}
\overline{G}(x,y) &= cx^4y^4 + 4cx^3y^4 - 2cK_2x^4y + 6cx^2y^4 + 8cx^3y^2 - cK_3x^4 - 4cK_2x^3y + 4cxy^4 + 16cx^2y^2\\
&\quad - 4cK_3x^3 + cy^4 + 8cxy^2 + (16c - 6cK_3)x^2 + 4cK_2xy - 4cK_3x + 2cK_2y - cK_3.
\end{aligned}
$$

Observe that from (2.116) $G(t^2, tw(t)) = 0 = \overline{G}(t^2, tw(t))$. Since the curve $(t^2, tw(t))$ is non-constant we conclude from the theory of Plane Algebraic Curves that the polynomials $G$ and $\overline{G}$ must have at least one factor in common, a contradiction with Lemma $A.0.6$ (using the claim). Therefore, the curve $((u'(s))^2, u''(s))$ is constant and then there exist $\mu, \eta$ such that $u'(s) = \mu$ and $u''(s) = \eta$ (in this case necessarily $\eta = 0$) and we conclude that $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$. $\square$

**Theorem 2.3.2.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\,\mathrm{sen}\, s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$, and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, with constant Gaussian Curvature $c \neq 0$. Then $u(s) = \mu s + \tau$, for constant $\mu, \tau$, and $J \subset \mathbb{R}$. Moreover, the function $v$ satisfies the differential equation*

$$
\begin{aligned}
(1-t^2)v'(t)v''(t) = ct(1+t^2)(v'(t))^4 + 2\left(\frac{4ct + t(t^2-1)}{1+t^2} + \frac{c\mu^2(1+t^2)}{t}\right)(v'(t))^2 \\
+ \frac{4\mu^2}{t(1+t^2)} + \frac{c(\mu^2(1+t^2) + \frac{4t^2}{1+t^2})^2}{t^3(1+t^2)}.
\end{aligned}
\tag{2.117}
$$

**Proof:** Suppose $\psi$ has constant Gaussian Curvature $c \neq 0$. By Proposition 2.3.4, the functions $u, v$ satisfy the equation

$$
c = -\frac{\left(4t^2v'(t) + 2t(1+t^2)v''(t)\right)\left(2t^2(t^2-1)v'(t) - t(1+t^2)u''(s)\right) + 16t^2(u'(s))^2}{4\left((tv'(t))^2 + (u'(s))^2 + \frac{4t^2}{(1+t^2)^2}\right)^2(1+t^2)^2},
\tag{2.118}
$$

By Proposition 2.3.5 we have that $u(s) = \mu s + \tau$, for constant $\mu, \tau$, and $J \subset \mathbb{R}$. Then, from (2.118),

$$
\begin{aligned}
(1-t^2)v'(t)v''(t) = ct(1+t^2)(v'(t))^4 + 2\left(\frac{4ct + t(t^2-1)}{1+t^2} + \frac{c\mu^2(1+t^2)}{t}\right)(v'(t))^2 \\
+ \frac{4\mu^2}{t(1+t^2)} + \frac{c(\mu^2(1+t^2) + \frac{4t^2}{1+t^2})^2}{t^3(1+t^2)}.
\end{aligned}
$$

$\square$

**Corollary 2.3.1.** *Assume the conditions of Theorem 2.3.2. Then:*

*i) If $\pm 1 \notin I$ the change of variable $w(t) = (v'(t))^2$ reduces the equation (2.117) to the* ***Ricatti*** *differential equation*

$$
\begin{aligned}
w'(t) = \frac{2}{1-t^2}\left(ct(1+t^2)w^2(t) + 2\left(\frac{4ct + t(t^2-1)}{1+t^2} + \frac{c\mu^2(1+t^2)}{t}\right)w(t)\right) \\
+ \frac{2}{1-t^2}\left(\frac{4\mu^2}{t(1+t^2)} + \frac{c(\mu^2(1+t^2) + \frac{4t^2}{1+t^2})^2}{t^3(1+t^2)}\right)
\end{aligned}
$$

*ii) If $1 \in I$ or $-1 \in I$ then $\mu \neq 0$ e $c < 0$.*

**Proof:** To obtain item *i)* just observe that $w'(t) = 2v'(t)v''(t)$. To conclude item *ii)* notice that if 1 or $-1$ belongs to $I$ then we can apply such points to (2.118) obtaining

$$
0 = c(v_1')^4 + 2c(1+\mu^2)(v_1')^2 + \mu^2 + c(1+\mu^2)^2.
$$

If $\mu = 0$ we would have from this equality that $0 = (v_1')^2 + 1$, a contradiction. If $c > 0$ we would have that such equality is positive, a contradiction. $\square$

### 2.3.4 *Minimal Helical Surfaces*

By Proposition 2.3.1 we conclude that a **Helical Surface** is **Minimal** if it satisfies a differential equation described in the following proposition.

**Proposition 2.3.6.** *The surface $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\,\mathrm{sen} s, v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, is Minimal, if and only if*

$$\begin{aligned}0 &= 2t^2(t^2-1)v'(t)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))\\ &\quad -t(1+t^2)u''(s)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))\\ &\quad +8tv'(t)(u'(s))^2(1+t^2)^2.\end{aligned}$$

**Proposition 2.3.7.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\,\mathrm{sen} s, v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, Minimal. Then $u(s) = \mu s^2 + \eta s + \tau$, for $\mu, \eta, \tau$ constants, and $J \subset \mathbb{R}$.*

**Proof:** Suppose $\psi$ is Minimal. By Proposition 2.3.6, the functions $u, v$ satisfy the equation

$$\begin{aligned}0 &= 2t^2(t^2-1)v'(t)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))\\ &\quad -t(1+t^2)u''(s)(4+(v'(t))^2(1+t^2)^2-(4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))\\ &\quad +8tv'(t)(u'(s))^2(1+t^2)^2,\end{aligned} \tag{2.119}$$

that is,

$$\begin{aligned}0 &= t(t^2+1)^3(4t^2v'(t)+2tv''(t)(t^2+1))(u'(s))^2u''(s)\\ &\quad +(8tv'(t)(t^2+1)^2-2t^2v'(t)(t^2-1)(t^2+1)^2(4t^2v'(t)+2tv''(t)(t^2+1)))(u'(s))^2\\ &\quad -t(t^2+1)((v'(t))^2(t^2+1)^2-4t^2(4t^2v'(t)+2tv''(t)(t^2+1))+4)u''(s)\\ &\quad +2t^2v'(t)(t^2-1)((v'(t))^2(t^2+1)^2-4t^2(4t^2v'(t)+2tv''(t)(t^2+1))+4).\end{aligned}$$

Define the family of polynomials

$$P_t(X,Y) = A_tXY + B_tX + C_tY + D_t, \tag{2.120}$$

where

$$\begin{aligned}A_t &= t(t^2+1)^3(4t^2v'(t)+2tv''(t)(t^2+1))\\ B_t &= (8tv'(t)(t^2+1)^2-2t^2v'(t)(t^2-1)(t^2+1)^2(4t^2v'(t)+2tv''(t)(t^2+1)))\\ C_t &= -t(t^2+1)((v'(t))^2(t^2+1)^2-4t^2(4t^2v'(t)+2tv''(t)(t^2+1))+4)\\ D_t &= 2t^2v'(t)(t^2-1)((v'(t))^2(t^2+1)^2-4t^2(4t^2v'(t)+2tv''(t)(t^2+1))+4),\end{aligned}$$

and notice that the curve $((u'(s))^2, u''(s))$ is contained in the intersection of the roots of $\{P_t\}_{t\in I}$. We have two cases. Suppose $A_t \equiv 0$. Then, by the definition of $A_t$, we obtain

$$0 = 4t^2v'(t) + 2tv''(t)(t^2+1)$$

that is,

$$v''(t) = -\frac{2tv'(t)}{t^2+1}$$

whose solution is $v'(t) = \frac{a}{t^2+1}$. Substituting these two equalities into (2.119) we have

$$\begin{aligned} 0 = &\frac{t}{t^2+1}((8a(u'(s))^2 - a^2u''(s) - 4u''(s))t^4 + 2a(a^2+4)t^3 + 2(8a(u'(s))^2 - a^2u''(s) - 4u''(s))t^2) \\ &+ \frac{t}{t^2+1}(-a(a^2+4)t + 8a(u'(s))^2 - a^2u''(s) - 4u''(s)) \end{aligned}$$

that is,

$$\begin{aligned} 0 = &(8a(u'(s))^2 - a^2u''(s) - 4u''(s))t^4 + 2a(a^2+4)t^3 + 2(8a(u'(s))^2 - a^2u''(s) - 4u''(s))t^2 \\ &- a(a^2+4)t + 8a(u'(s))^2 - a^2u''(s) - 4u''(s). \end{aligned}$$

Since $t$ is not constant we conclude that the polynomial above is zero, that is, its coefficients are all zero, then

$$\begin{aligned} 0 &= 8a(u'(s))^2 - a^2u''(s) - 4u''(s) \\ 0 &= a(a^2+4), \end{aligned}$$

and by the second equality $a = 0$. Substituting in the first one we conclude that $u$ is a straight line and so the result holds. Let us now consider the second case. Suppose there is an open set $V \subset I$ such that $A_t \neq 0$ in $I$. Define the family of polynomials

$$\overline{P}_t(X,Y) = XY + \frac{B_t}{A_t}X + \frac{C_t}{A_t}Y + \frac{D_t}{A_t},$$

and observe that the curve $((u'(s))^2, u''(s))$ is contained in the intersection of the roots of $\{\overline{P}_t\}_{t\in V}$. Suppose that $A_tD_t - B_tC_t \equiv 0$ in $V$. We have, substituting the coefficients,

$$0 = -\frac{16t^3(t^2+1)v'(t)v''(t) - 2v'(t)((t^4+2t^2+1)(v'(t))^2 - 16t^4t^4v'(t) + 4)}{t^2(t^2+1)^3(v''(t) + 2tv'(t) + t^2v''(t))^2},$$

and since $v'$ is nonzero in some open set $\hat{V} \subset V$ (otherwise $A_t \equiv 0$ in $V$) we obtain

$$0 = 8t^3(t^2+1)v''(t) - ((t^4+2t^2+1)(v'(t))^2 - 16t^4t^4v'(t) + 4),$$

that is,

$$v''(t) = \frac{(t^4+2t^2+1)(v'(t))^2 - 16t^4t^4v'(t) + 4}{8t^3(t^2+1)},$$

which substituting in (2.119)

$$\begin{aligned} 0 = &\frac{(t^2+1)^3(t^4+2t^2+1)(v'(t))^2 + 4}{4t}(u'(s))^2u''(s) \\ &- \frac{v'(t)(t^2+1)^2((t^6+t^4-t^2-1)(v'(t))^2 + 4(t^2-4t-1))}{2}(u'(s))^2. \end{aligned}$$

If $u'$ is zero in $J$ then $u$ is constant. If there exists an open set $U \subset J$ such that $u' \neq 0$ in $U$ then

$$u''(s) = \frac{2tv'(t)(t^2+1)^2((t^6+t^4-t^2-1)(v'(t))^2 + 4(t^2-4t-1))}{(t^2+1)^3(t^4+2t^2+1)(v'(t))^2 + 4}$$

and since the variables on the right and left sides of the equality are independent, we conclude that $u''$ is constant, that is, $u$ is a parabola. In the case where there exists $t_0$ such that $A_{t_0}D_{t_0} - B_{t_0}C_{t_0} \neq 0$ we have:

**Claim 1:** and there exists an open set $\tilde{V} \subset V$ such that $A_tD_t - B_tC_t \neq 0$ then $\overline{P}_t$ are irreducible in $\tilde{V}$. Indeed, suppose that $\overline{P}_t$ is reducible. Then there exist polynomials of degree 1 such that

$$\overline{P}_t(X,Y) = (a + bX + cY)(d + fX + gY)$$

that is,

$$0 = XY + \frac{B_t}{A_t}X + \frac{C_t}{A_t}Y + \frac{D_t}{A_t} - (a + bX + cY)(d + fX + gY)$$

that reorganizing,

$$0 = -bfX^2 + (1 - cf - bg)XY + \left(\frac{B_t}{A_t} - bd - af\right)X - cgY^2 + \left(\frac{C_t}{A_t} - cd - ag\right)Y + \frac{D_t}{A_t} - ad,$$

that occurs if, and only if,

$$\begin{aligned}
0 &= bf \\
0 &= 1 - cf - bg \\
0 &= \frac{B_t}{A_t} - bd - af \\
0 &= cg \\
0 &= \frac{C_t}{A_t} - cd - ag \\
0 &= \frac{D_t}{A_t} - ad.
\end{aligned}$$

By the first equality we can assume, without loss of generality, $b = 0$. Then, since $c \neq 0$ (otherwise we would not have factorization), we obtain by the fourth equality $g = 0$. We have,

$$\begin{aligned}
0 &= 1 - cf \\
0 &= \frac{B_t}{A_t} - af \\
0 &= \frac{C_t}{A_t} - cd \\
0 &= \frac{D_t}{A_t} - ad.
\end{aligned}$$

By the first and second equalities we obtain $c = \frac{1}{f}$ and $a = \frac{B_t}{fA_t}$. We have,

$$\begin{aligned}
0 &= \frac{C_t}{A_t} - \frac{d}{f} \\
0 &= \frac{D_t}{A_t} - \frac{dB_t}{fA_t},
\end{aligned}$$

and isolating $\frac{d}{f}$ in the first equality and substituting it in the second we conclude that $A_tD_t - B_tC_t = 0$ in $\tilde{V}$, a contradiction, and thus we conclude the claim.

By Claim 1, the family of polynomials $\{\overline{P}_t\}_{t\in\tilde{V}}$ is irreducible in a neighborhood $\tilde{V}$ of $t_0$. As observed previously, the curve $((u'(s))^2, u''(s))$ is contained in the intersection of the roots of $\{\overline{P}_t\}_{t\in\tilde{V}}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $\overline{P}_t$ are multiples by constants. If the curve is constant we obtain that $u'$ is constant and consequently $u$ is a straight line. If $\{\overline{P}_t\}_{t\in\tilde{V}}$ are all multiples we conclude that the only possible multiple is $\lambda = 1$ and then

$$\overline{P}_t(X,Y) = XY + \frac{B_t}{A_t}X + \frac{C_t}{A_t}Y + \frac{D_t}{A_t},$$

is constant in the variable $t$, that is, the polynomials $\overline{P}_t$ are all equal. Therefore, there exist constants $K_2, K_3, K_4$ such that

$$\overline{P}_t(X,Y) = XY + K_2X + K_3Y + K_4, \tag{2.121}$$

where

$$\begin{aligned}
K_2 &= \frac{B_t}{A_t} = -\frac{2v'(t)(-t^2v''(t) - 2t^3v'(t) + 2t^5v'(t) + t^6v''(t) - 2)}{t(t^2+1)(v''(t) + 2tv'(t) + t^2v''(t))} \\
K_3 &= \frac{C_t}{A_t} = \frac{-2t^2(v'(t))^2 - t^4(v'(t))^2 - (v'(t))^2 + 8t^3v''(t) + 16t^4v'(t) + 8t^5v''(t) - 4}{2t(t^2+1)^2(v''(t) + 2tv'(t) + t^2v''(t))} \\
K_4 &= \frac{D_t}{A_t} = \frac{v'(t)(t^2-1)(2t^2(v'(t))^2 + t^4(v'(t))^2 + (v'(t))^2 - 8t^3v''(t) - 16t^4v'(t) - 8t^5v''(t) + 4)}{t(t^2+1)^3(v''(t) + 2tv'(t) + t^2v''(t))}.
\end{aligned}$$

Rewriting the three equalities we obtain

$$\begin{aligned}
0 =& t(t^2+1)(K_2 - 2tv'(t) + 2t^3v'(t) + t^2K_2)v''(t) + 2K_2t^2(t^2+1)v'(t) - 2v'(t)(2t^3v'(t) - 2t^5v'(t) + 2) \\
0 =& 2t(t^2+1)(K_3 + 2t^2K_3 + t^4K_3 - 4t^2)v''(t) + 2t^2(v'(t))^2 + t^4(v'(t))^2 + (v'(t))^2 \\
& -16t^4v'(t) + 4K_3t^2(t^2+1)^2 + 4 \\
0 =& (t^2+1)(K_4 - 8t^3v'(t) + 8t^5v'(t) + 3t^2K_4 + 3t^4K_4 + t^6K_4)v''(t) + 2K_4t(t^2+1)^3v'(t) \\
& -v'(t)(t^2-1)((t^4 + 2t^4 + 1)(v'(t))^2 - 16t^4v'(t) + 4).
\end{aligned} \tag{2.122}$$

Notice in the second equality of (2.122) that $K_3 + 2t^2K_3 + t^4K_3 - 4t^2 \neq 0$ in $\overline{V}$. Indeed, if there exists $t_1 \in \overline{V}$ such that $K_3 + 2t_1^2K_3 + t_1^4K_3 - 4t_1^2 = 0$ we obtain by the second equality of (2.122) that

$$\begin{aligned}
0 &= K_3 + 2t_1^2K_3 + t_1^4K_3 - 4t_1^2 \\
0 &= 2t_1^2(v'(t_1))^2 + t_1^4(v'(t_1))^2 + (v'(t_1))^2 - 16t_1^4v'(t_1) + 4K_3t_1^2(t_1^2+1)^2 + 4,
\end{aligned}$$

and then, by the first equality $K_3 = \frac{4t_1^2}{t_1^4 + 2t_1^2 + 1}$ which substituting into the second we obtain

$$0 = (t_1^4 + 2t_1^2 + 1)(v'(t_1))^2 + 4 \neq 0$$

a contradiction. Thus, from the second equality of (2.122) we conclude that

$$v''(t) = \frac{2t^2(v'(t))^2 + t^4(v'(t))^2 + (v'(t))^2 - 16t^4v'(t) + 4K_3t^2(t^2+1)^2 + 4}{2t(t^2+1)(K_3 + 2t^2K_3 + t^4K_3 - 4t^2)}.$$

Substituting in the other two equalities of (2.122) we will have

$$\begin{aligned}0 =& -\frac{2t(t^2-1)(t^2+1)^2(v'(t))^3+K_2(t^2+1)^3(v'(t))^2}{2(K_3+2t^2K_3+t^4K_3-4t^2)}\\ &-\frac{(8K_3t^4+8t^3+16(K_3-2)t^2-8t+8K_3)v'(t)+4K_2(t^2+1)}{2(K_3+2t^2K_3+t^4K_3-4t^2)}\\ 0 =& -\frac{(t^2+1)^2(2t^2(v'(t))^2+t^4(v'(t))^2+(v'(t))^2+4)(2K_3t(t^2-1)v'(t)+K_4(t^2+1))}{2t(K_3+2t^2K_3+t^4K_3-4t^2)}\end{aligned}$$

that is,

$$\begin{aligned}0 =&2t(t^2-1)(t^2+1)^2(v'(t))^3+K_2(t^2+1)^3(v'(t))^2\\ &+(8K_3t^4+8t^3+16(K_3-2)t^2-8t+8K_3)v'(t)+4K_2(t^2+1)\\ 0 =&2K_3t(t^2-1)v'(t)+K_4(t^2+1).\end{aligned} \tag{2.123}$$

**Claim 2:** $K_4=0$. Indeed, observe that if $-1\in\tilde{V}$ or $1\in\tilde{V}$ we conclude that $K_4=0$ by the second equality of (2.123). If $\pm 1\notin\tilde{V}$ we obtain from (2.123)

$$\begin{aligned}0 =&2t(t^2-1)(t^2+1)^2(v'(t))^3+K_2(t^2+1)^3(v'(t))^2\\ &+(8K_3t^4+8t^3+16(K_3-2)t^2-8t+8K_3)v'(t)+4K_2(t^2+1)\\ K_3v'(t) =& -\frac{K_4(t^2+1)}{2t(t^2-1)},\end{aligned}$$

that is, if $K_3=0$ we obtain $K_4=0$, and if $K_3\neq 0$ we can isolate $v'(t)$ in the second equality and substitute it in the first one obtaining

$$\begin{aligned}0 =&K_2(K_2K_3-K_4)t^8-16K_3^3K_4t^7+4(4K_3^2+K_4^2)(K_2K_3-K_4)t^6-16K_3^2K_4(K_3-4)t^5\\ &-2(16K_3^2-3K_4^2)(K_2K_3-K_4)t^4+16K_3^2K_4(K_3-4)t^3+4(4K_3^2+K_4^2)(K_2K_3-K_4)t^2\\ &+16K_3^3K_4t+K_4^2(K_2K_3-K_4).\end{aligned}$$

Since $t$ is not constant we conclude that this polynomial is zero, that is, all its coefficients are zero. From the second coefficient we conclude that $K_4=0$ and thus we conclude the claim.

**Claim 3:** If $K_3\neq 0$ then $K_2=0$. Indeed, by Claim 2 and (2.123)

$$\begin{aligned}0 =&2t(t^2-1)(t^2+1)^2(v'(t))^3+K_2(t^2+1)^3(v'(t))^2\\ &+(8K_3t^4+8t^3+16(K_3-2)t^2-8t+8K_3)v'(t)+4K_2(t^2+1)\\ 0 =&2K_3t(t^2-1)v'(t),\end{aligned}$$

and since $K_3\neq 0$ we obtain from the second equality that $v'(t)=0$ which substituting into the first we have $4K_2(t^2+1)=0$, that is, $K_2=0$ and thus we conclude the claim.

By Claim 2 and (2.121)

$$0=\overline{P}_t((u'(s))^2,u''(s))=(u'(s))^2u''(s)+K_2(u'(s))^2+K_3u''(s). \tag{2.124}$$

We have two situations for this differential equation. If $K_3 \neq 0$ by Claim 3 we obtain $K_2 = 0$ and thus from (2.124)

$$0 = ((u'(s))^2 + K_3)u''(s),$$

which multiplying by $2u'$,

$$0 = ((u'(s))^2 + K_3)\frac{d}{dt}[(u'(s))^2],$$

and defining $w(s) = (u'(s))^2$

$$0 = (w(s) + K_3)w'(s),$$

Suppose $w$ is non-constant. Then there exists an open set $U \subset J$ such that $w'(s) \neq 0$ and hence $w(s) = -K_3$ in $U \subset J$, a contradiction. Therefore, $w(s)$ is constant in $J$ and hence $u$ is a straight line. If $K_3 = 0$ we obtain from (2.124)

$$0 = (u'(s))^2(u''(s) + K_2)$$

and defining $W(s) = u'(s)$

$$0 = W^2(s)(W'(s) + K_2).$$

Notice that $W \equiv 0$ is a solution and therefore $u$ is constant. If there exists an open set such that $W \neq 0$ then $W'(s) = -K_2$ and therefore there exist constants $\mu, \eta$ and $\tau$ such that $w(s) = 2\mu s + \eta$ and consequently $u(s) = \mu s^2 + \eta s + \tau$ whose maximal interval is $J \subset \mathbb{R}$. □

**Theorem 2.3.3.** *Let $\psi : \Omega \subset \mathbb{R}^2 \to S^2 \times \mathbb{R}$ given by $\psi(t,s) = (t\cos s, t\,\mathrm{sen}s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$, $v \in C^2(I,\mathbb{R})$ and $J \times I \subset \mathbb{R} \times \mathbb{R}^*$, Minimal. Then $u(s) = \mu s^2 + \eta s + \tau$, for $\mu, \eta, \tau$ constants, and $J \subset \mathbb{R}$. Moreover, if $\mu = 0$ then $v$ satisfies the differential equation*

$$\begin{aligned}(1-t^2)v'(t)v''(t) =& v'(t)\frac{(t^3-t^5-t^7+t)(v'(t))^2 + 4t^3(t^2-1)(2\eta^2t^2+\eta^2t^4+\eta^2+4t^2)v'(t)}{2t^2(t^2+1)(2\eta^2t^2+\eta^2t^4+\eta^2+4t^2)}\\ &- 2v'(t)\frac{2\eta^2t^2+\eta^2t^4+\eta^2+t^3-t}{t^2(t^2+1)(2\eta^2t^2+\eta^2t^4+\eta^2+4t^2)}.\end{aligned}$$

**Proof:** Suppose that $\psi$ is Minimal. By Proposition 2.3.6, the functions $u, v$ satisfy the equation

$$\begin{aligned}0 =& 2t^2(t^2-1)v'(t)(4+(v'(t))^2(1+t^2)^2 - (4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))\\ &- t(1+t^2)u''(s)(4+(v'(t))^2(1+t^2)^2 - (4t^2v'(t)+2t(1+t^2)v''(t))(4t^2+(u'(s))^2(1+t^2)^2))\\ &+ 8tv'(t)(u'(s))^2(1+t^2)^2.\end{aligned} \tag{2.125}$$

By the Proposition 2.3.7 $u(s) = \mu s^2 + \eta s + \tau$, for $\mu, \eta, \tau$ constants, and $J \subset \mathbb{R}$. If $\mu = 0$ we have, from (2.125),

$$\begin{aligned}(1-t^2)v'(t)v''(t) =& v'(t)\frac{(t^3-t^5-t^7+t)(v'(t))^2 + 4t^3(t^2-1)(2\eta^2t^2+\eta^2t^4+\eta^2+4t^2)v'(t)}{2t^2(t^2+1)(2\eta^2t^2+\eta^2t^4+\eta^2+4t^2)}\\ &- 2v'(t)\frac{2\eta^2t^2+\eta^2t^4+\eta^2+t^3-t}{t^2(t^2+1)(2\eta^2t^2+\eta^2t^4+\eta^2+4t^2)}.\end{aligned}$$

□

# CHAPTER 3

# Solitons and the Weingarten Flow

In the last decades, in the field of Differential Geometry, a subject that has received a lot of attention is that of geometric flows. Among these flows, we can point one that has an extensive literature, the Mean Curvature Flow. One way to define it, in $\mathbb{R}^3$, is: Let $M$ be a manifold of dimension 2 and $\varphi_0 : M \to \mathbb{R}^3$ be a surface. The Mean Curvature Flow is a smooth family of surfaces $\varphi_r : M \to \mathbb{R}^3$, for $r \in [0,R)$, such that, setting $\varphi(p,r) := \varphi_r(p)$, the map $\varphi : M \times [0,R) \to \mathbb{R}^3$ is a solution of the system of partial differential equations:

$$\begin{aligned} \left\langle \frac{\partial \varphi}{\partial r}, N \right\rangle (p,r) &= H(p,r) \\ \varphi(p,0) &= \varphi_0(p), \end{aligned} \tag{3.1}$$

where $N(p,r)$ and $H(p,r)$ are, respectively, the unit normal vector and the mean curvature of the surface $\varphi_r$ at the point $p \in M$. In other words, this flow evolves a surface, moving each of its points in the normal direction and with a speed equal to its mean curvature at that point.

An initial surface $\varphi_0$ that has been attracting the interest of researchers are the Translating Solitons or, simply, Solitons ([5]). The Translating Solitons of this flow are surfaces $\varphi_0 : M \to \mathbb{R}^3$ such that there exists a non-zero vector $\alpha \in \mathbb{R}^3$ such that the mean curvature verifies the equation

$$H(p) = \langle \alpha, N(p) \rangle \tag{3.2}$$

for all $p \in M$. One can see that (check the end of the next section) defining $\varphi(p,r) = \varphi_0(p) + \alpha r$, such a family is a Mean Curvature Flow. This flow is simply the translating of the surface $\varphi_0$, in the space $\mathbb{R}^3$, with constant speed $\alpha$.

In the previous chapter, we were mainly focused with studying Translation Surfaces that verify polynomial relations. Given a function $\Psi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ we tried to obtain properties of families of surfaces that verify $\Psi(H,K) \equiv 0$. If we consider $\Psi(X,Y) = X$ we would have in (3.2)

$$\Psi(H,K)(p) = \langle \alpha, N(p) \rangle. \tag{3.3}$$

We will see that this is not a mere coincidence. We can modify the flow (3.1) by introducing the Weingarten Flow, in a such way that (3.3) continues to have the property of being a soliton, for arbitrary $\Psi$. In order to achieve this, in $(3.1)$, we replace $H$ with $\Psi(H,K)$ and, then, we will have, for each $\Psi$, a different flow. This flow has attracted the attention of some researchers ([3], [4]). In the next section, we will introduce the Weingarten Flow in $\mathbb{R}^3$. Then, following the structure of the previous chapter, we will analyze cases of Translation Surfaces that are Solitons. More precisely, we will consider the equations

$$\begin{aligned} K &= \langle \alpha, N \rangle \\ H &= \langle \alpha, N \rangle \\ aH + bK &= \langle \alpha, N \rangle \\ |A|^2 \coloneqq 4H^2 - 2K &= \langle \alpha, N \rangle . \end{aligned}$$

# 3.1 Weingarten Flow

**Definition 3.1.1.** Let be $M$ a manifold of dimension 2, $\varphi_0 : M \to \mathbb{R}^3$ a surface and $\Psi \in C^\infty(\mathbb{R}^2, \mathbb{R})$. The Weingarten flow of $(\Psi, \varphi_0)$ is a smooth family of surfaces $\varphi_r : M \to \mathbb{R}^3$, for $r \in [0, R)$, such that defining $\varphi(p,r) := \varphi_r(p)$ the map $\varphi : M \times [0,R) \to \mathbb{R}^3$ is a solution of the system of partial differential equations:

$$\begin{aligned} \left\langle \frac{\partial \varphi}{\partial r}, N \right\rangle (p,r) &= \Psi(H,K)(p,r) \\ \varphi(p,0) &= \varphi_0(p), \end{aligned} \tag{3.4}$$

where $N(p,r), H(p,r)$ and $K(p,r)$ are respectively the normal vector, the mean curvature and the Gaussian curvature of the surface $\varphi_r$ at $p \in M$.

**Example 3.1.1.** *Let $M$ be a manifold of dimension 2, $\varphi_0 : M \to \mathbb{R}^3$ a surface, and $\overline{\Psi} \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $\overline{\Psi}(H_{\varphi_0}, K_{\varphi_0}) \equiv 0$, that is, $\varphi_0$ is a Weingarten surface. Then, by definition, the flow $\varphi(p,r) = \varphi_0(p)$ satisfies the system (3.4).*

**Example 3.1.2** (Self-Shrinker)**.** *Let be $M$ a 2-dimensional manifold, $\overline{\Psi} \in C^\infty(\mathbb{R}^2, \mathbb{R})$ given by $\overline{\Psi}(X,Y) = aX + bY^{\frac{1}{2}}$ and $\varphi_0 : M \to \mathbb{R}^3$ surface such that,*

$$\overline{\Psi}(H,K)(p) = -\langle \varphi_0(p), N(p) \rangle,$$

*that is,*

$$aH(p) + bK^{\frac{1}{2}}(p) = -\langle \varphi_0(p), N(p) \rangle, \tag{3.5}$$

*for all $p \in M$. Such surfaces are called Self-Shrinker. Defining $\varphi(p,r) = \sqrt{1-2r}\,\varphi_0(p)$ for $r < \frac{1}{2}$ and $\varphi(p,0) = \varphi_0(p)$ we conclude that $\varphi$ is a Weingarten flow of $(\overline{\Psi}, \varphi_0)$. Indeed,*

$$\frac{\partial \varphi}{\partial r}(p,r) = -\frac{1}{\sqrt{1-2r}}\varphi_0(p).$$

*Since $\varphi$ is homothety, it follows that*

$$N(p,r) = N(p,0), \qquad H(p,r) = \frac{H(p,0)}{\sqrt{1-2r}}, \qquad K(p,r) = \frac{K(p,0)}{1-2r},$$

*for all $r \leqslant \frac{1}{2}$. Hence,*

$$\begin{aligned} \left\langle \frac{\partial \varphi}{\partial r}, N \right\rangle (p,r) &= -\frac{1}{\sqrt{1-2r}} \langle \varphi_0, N(p,0) \rangle \\ &= \frac{aH(p,0) + bK^{\frac{1}{2}}(p,0)}{\sqrt{1-2r}} \\ &= aH(p,r) + bK^{\frac{1}{2}}(p,r) \\ &= \overline{\Psi}(H,K)(p,r). \end{aligned}$$

*Finally, we observe that taking $a = 1$ and $b = 0$ in $\overline{\Psi}$ we obtain in (3.5) the usual definition of Self-Shrinker.*

**Example 3.1.3.** *Let be $M$ a 2-dimensional manifold, $\overline{\Psi} \in C^\infty(\mathbb{R}^2,\mathbb{R})$ given by $\overline{\Psi}(X,Y) = aX^2 + bY$ and $\varphi_0 : M \to \mathbb{R}^3$ surface such that,*

$$\overline{\Psi}(H,K)(p) = -\langle \varphi_0(p), N(p)\rangle,$$

*that is,*

$$aH^2(p) + bK(p) = -\langle \varphi_0(p), N(p)\rangle,$$

*for all $p \in M$. Defining $\varphi(p,r) = \sqrt{1-3r}\varphi_0(p)$ for $r < \frac{1}{2}$ and $\varphi(p,0) = \varphi_0(p)$ we conclude that $\varphi$ is a Weingarten flow of $(\overline{\Psi}, \varphi_0)$. Indeed,*

$$\frac{\partial \varphi}{\partial r}(p,r) = -\frac{1}{(1-3r)^{\frac{2}{3}}}\varphi_0(p).$$

*Since $\varphi$ is homothety, it follows that*

$$N(p,r) = N(p,0), \qquad H(p,r) = \frac{H(p,0)}{(1-3r)^{\frac{1}{3}}}, \qquad K(p,r) = \frac{K(p,0)}{(1-3r)^{\frac{2}{3}}},$$

*for all $r \leqslant \frac{1}{3}$. Hence,*

$$\begin{aligned}\left\langle \frac{\partial \varphi}{\partial r}, N\right\rangle(p,r) &= -\frac{1}{(1-3r)^{\frac{2}{3}}}\langle \varphi_0, N(p,0)\rangle \\ &= \frac{aH^2(p,0) + bK(p,0)}{(1-3r)^{\frac{2}{3}}} \\ &= aH^2(p,r) + bK(p,r) \\ &= \overline{\Psi}(H,K)(p,r).\end{aligned}$$

*Finally, we observe that taking $a = 4$ and $b = -2$ in $\overline{\Psi}$ we obtain that $\overline{\Psi}(H,K) \equiv |A|^2$.*

**Example 3.1.4** (Translating Soliton)**.** *Let $M$ be a 2-dimensional manifold, $\Psi \in C^\infty(\mathbb{R}^2,\mathbb{R})$, $\alpha \in \mathbb{R}^3$ nonzero and $\varphi_0 : M \to \mathbb{R}^3$ a surface such that,*

$$\Psi(H,K)(p) = \langle \alpha, N(p)\rangle,$$

*for all $p \in M$. Such surfaces we call Translating Soliton. Defining $\varphi(p,r) = \varphi_0(p) + r\alpha$ we conclude that $\varphi$ is a Weingarten flow of $(\Psi, \varphi_0)$. Indeed, since $N(p,r) = N(p,0)$, $H(p,r) = H(p,0)$ and $K(p,r) = K(p,0)$ we obtain,*

$$\begin{aligned}\left\langle \frac{\partial \varphi}{\partial r}, N\right\rangle(p,r) &= \langle \alpha, N(p,r)\rangle \\ &= \langle \alpha, N(p,0)\rangle \\ &= \Psi(H(p,0), K(p,0)) \\ &= \Psi(H(p,r), K(p,r)) \\ &= \Psi(H,K)(p,r).\end{aligned}$$

### 3.1.1 *Translation Surfaces in* $E^3$

As done in the sections dedicated to the spaces $H^2 \times \mathbb{R}$, $H^3$ and $S^2 \times \mathbb{R}$ we will repeat the same approach here. Since the Euclidean space is simpler than the previous ones and obtaining the Connection is trivial, we will only state the lemmas. To obtain such results the reader can follow in an analogous way to the other spaces contained in the previous chapter. Consider $\{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$. Then,

$$E_1(x,y,z) = e_1$$
$$E_2(x,y,z) = e_2$$
$$E_3(x,y,z) = e_3$$

generates a global orthonormal frame over $\mathbb{R}^3$.

**Lemma 3.1.1.** $[E_i, E_j] \equiv 0$ *for all* $i, j = 1,2,3$.

**Lemma 3.1.2.** $\Gamma_{ji}^k \equiv 0$ *for all* $i, j, k = 1,2,3$.

Since $\nabla^{\times}_{E_j} E_i = \sum_k \Gamma_{ji}^k E_k$ we obtain:

**Lemma 3.1.3** (Connection)**.** $\nabla^{\times}_{E_j} E_i \equiv 0$ *for all* $i, j = 1,2,3$.

### 3.1.2 *Curvatures of Translation Surfaces*

This section will be devoted to determining the curvatures of a Translation Surface in $\mathbb{R}^3$. Define the **Translation Surface** $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$ and $v \in C^2(I, \mathbb{R})$. We have,

$$\frac{\partial \psi}{\partial t}(t,s) = \big(1, 0, v'(t)\big) = E_1 \circ \psi(t,s) + v'(t) E_3 \circ \psi(t,s),$$

and

$$\frac{\partial \psi}{\partial s}(t,s) = \big(0, 1, u'(s)\big) = E_2 \circ \psi(t,s) + u'(s) E_3 \circ \psi(t,s).$$

The normal $N$ at $\psi(t,s)$ will be given by

$$N(t,s) = -\frac{v'(t)}{\omega(t,s)} E_1 \circ \psi(t,s) - \frac{u'(s)}{\omega(t,s)} E_2 \circ \psi(t,s) + \frac{1}{\omega(t,s)} E_3 \circ \psi(t,s), \tag{3.6}$$

where $\omega(t,s) = \sqrt{(v'(t))^2 + (u'(s))^2 + 1}$. By Lemma 1.1.2 we conclude that to know the Weingarten map $\mathcal{S}$ we do not need to differentiate the normal field, we just need to know $\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}$, $\nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}$ and $\nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}$. We have,

**Lemma 3.1.4.** *The following holds:*

***i)*** $\left(\nabla_{\overline{e_1}} \dfrac{\partial \psi}{\partial t}\right)(t,s) = v''(t) E_3 \circ \psi(t,s).$ ***ii)*** $\left(\nabla_{\overline{e_1}} \dfrac{\partial \psi}{\partial s}\right)(t,s) = 0.$

***iii)*** $\left(\nabla_{\overline{e_2}} \dfrac{\partial \psi}{\partial s}\right)(t,s) = u''(s) E_3 \circ \psi(t,s).$

**Proof:**

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s) &= \left(\nabla_{\overline{e_1}}E_1\circ\psi\right)(t,s)+v'(t)\left(\nabla_{\overline{e_1}}E_3\circ\psi\right)(t,s)+\overline{e_1}\left(v'(t)\right)E_3\circ\psi(t,s)\\ &=\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1+v'(t)\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3+v''(t)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_1=\left(\nabla^{\times}_{E_1}E_1\right)\circ\psi(t,s)+v'(t)\left(\nabla^{\times}_{E_3}E_1\right)\circ\psi(t,s)=0$$

and, similarly, $\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3=0$, thus,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial t}\right)(t,s)=v''(t)E_3\circ\psi(t,s).$$

Also,

$$\begin{aligned}\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s) &= \left(\nabla_{\overline{e_1}}E_2\circ\psi\right)(t,s)+u'(s)\left(\nabla_{\overline{e_1}}E_3\circ\psi\right)(t,s)+\overline{e_1}\left(u'(s)\right)E_3\circ\psi(t,s)\\ &=\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2+u'(s)\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_3+\frac{\partial}{\partial t}\left(u'(s)\right)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\nabla^{\times}_{\frac{\partial\psi}{\partial t}(t,s)}E_2=\left(\nabla^{\times}_{E_1}E_2\right)\circ\psi(t,s)+u'(s)\left(\nabla^{\times}_{E_3}E_2\right)\circ\psi(t,s)=0$$

then,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=0.$$

Finally,

$$\begin{aligned}\left(\nabla_{\overline{e_2}}\frac{\partial\psi}{\partial s}\right)(t,s) &= \left(\nabla_{\overline{e_2}}E_2\circ\psi\right)(t,s)+u'(s)\left(\nabla_{\overline{e_2}}E_3\circ\psi\right)(t,s)+\overline{e_2}\left(u'(s)\right)E_3\circ\psi(t,s)\\ &=\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2+u'(s)\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3+u''(s)E_3\circ\psi(t,s)\end{aligned}$$

but,

$$\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_2=\left(\nabla^{\times}_{E_2}E_2\right)\circ\psi(t,s)+u'(s)\left(\nabla^{\times}_{E_3}E_2\right)\circ\psi(t,s)=0$$

and, similarly, $\nabla^{\times}_{\frac{\partial\psi}{\partial s}(t,s)}E_3=0$, thus,

$$\left(\nabla_{\overline{e_1}}\frac{\partial\psi}{\partial s}\right)(t,s)=u''(s)E_3\circ\psi(t,s).$$

□

With this we have

**Lemma 3.1.5.** *The following holds:*

| | | |
|---|---|---|
| ***i)*** $\mathcal{E}(t,s)=\dfrac{v''(t)}{\omega(t,s)}.$ | ***iii)*** $\mathcal{G}(t,s)=\dfrac{u''(s)}{\omega(t,s)}.$ | ***v)*** $\mathfrak{f}(t,s)=u'(s)v'(t).$ |
| ***ii)*** $\mathcal{F}(t,s)=0.$ | ***iv)*** $\mathfrak{e}(t,s)=1+(v'(t))^2.$ | ***vi)*** $\mathfrak{g}(t,s)=1+(u'(s))^2.$ |

**Proof:**

$$\mathcal{E}(t,s) = \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial t}\right)(t,s) = \frac{v''(t)}{\omega(t,s)}.$$

Also,

$$\mathcal{F}(t,s) = \left(N \bullet_\times \nabla_{\overline{e_1}} \frac{\partial \psi}{\partial s}\right)(t,s) = 0.$$

Finally,

$$\mathcal{G}(t,s) = \left(N \bullet_\times \nabla_{\overline{e_2}} \frac{\partial \psi}{\partial s}\right)(t,s) = \frac{u''(s)}{\omega(t,s)}.$$

The metric coefficients are,

$$\mathfrak{e}(t,s) = \left(\overline{e_1} \bullet_\times \overline{e_1}\right)(t,s) = \left(\frac{\partial \psi}{\partial t} \bullet_\times \frac{\partial \psi}{\partial t}\right)(t,s) = 1 + (v'(t))^2.$$

Also,

$$\mathfrak{f}(t,s) = \left(\overline{e_1} \bullet_\times \overline{e_2}\right)(t,s) = \left(\frac{\partial \psi}{\partial t} \bullet_\times \frac{\partial \psi}{\partial s}\right)(t,s) = v'(t)u'(s).$$

Finally,

$$\mathfrak{g}(t,s) = \left(\overline{e_2} \bullet_\times \overline{e_2}\right)(t,s) = \left(\frac{\partial \psi}{\partial s} \bullet_\times \frac{\partial \psi}{\partial s}\right)(t,s) = 1 + (u'(s))^2.$$

$\square$

**Proposition 3.1.1.** *The following holds:*

$$H(t,s) = \frac{v''(t)\left(1+(u'(s))^2\right) + u''(s)\left(1+(v'(t))^2\right)}{2\omega^3(t,s)},$$

*and*

$$K(t,s) = \frac{u''(s)v''(t)}{\omega^4(t,s)}.$$

**Proof:**

$$\begin{aligned}
H(t,s) &= \frac{1}{2}\mathrm{tr}(\mathcal{S})(t,s)\\
&= \frac{1}{2}(a_{tt} + a_{ss})\\
&= \frac{1}{2}\left(\frac{\mathcal{E}\mathfrak{g} + \mathcal{G}\mathfrak{e} - 2\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g} - \mathfrak{f}^2}\right)(t,s)\\
&= \frac{1}{2}\left(\frac{\left(\frac{v''(t)}{\omega(t,s)}\right)\left(1+(u'(s))^2\right) + \left(\frac{u''(s)}{\omega(t,s)}\right)\left(1+(v'(t))^2\right)}{\left(1+(v'(t))^2\right)\left(1+(u'(s))^2\right) - (u'(s)v'(t))^2}\right)\\
&= \frac{1}{2}\left(\frac{\frac{v''(t)\left(1+(u'(s))^2\right)}{\omega(t,s)} + \frac{u''(s)\left(1+(v'(t))^2\right)}{\omega(t,s)}}{\omega^2(t,s)}\right)\\
&= \frac{1}{2}\left(\frac{v''(t)\left(1+(u'(s))^2\right) + u''(s)\left(1+(v'(t))^2\right)}{\omega^3(t,s)}\right)\\
&= \frac{v''(t)\left(1+(u'(s))^2\right) + u''(s)\left(1+(v'(t))^2\right)}{2\omega^3(t,s)}.
\end{aligned}$$

Also,

$$\begin{aligned}
K(t,s) &= \det(\mathcal{S})\\
&= a_{tt}a_{ss} - a_{ts}a_{st}\\
&= \left(\frac{\mathcal{E}\mathfrak{g}-\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{G}\mathfrak{e}-\mathcal{F}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s) - \left(\frac{\mathcal{F}\mathfrak{g}-\mathcal{G}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\left(\frac{\mathcal{F}\mathfrak{e}-\mathcal{E}\mathfrak{f}}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\\
&= \left(\frac{\mathcal{E}\mathcal{G}\mathfrak{g}\mathfrak{e} - \mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f} - \mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e} + (\mathcal{F}\mathfrak{f})^2 - \left(\mathcal{F}^2\mathfrak{g}\mathfrak{e} - \mathcal{E}\mathcal{F}\mathfrak{g}\mathfrak{f} - \mathcal{F}\mathcal{G}\mathfrak{f}\mathfrak{e} + \mathcal{E}\mathcal{F}\mathfrak{f}^2\right)}{(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)^2}\right)(t,s)\\
&= \left(\frac{\mathcal{E}\mathcal{G}(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2) - \mathcal{F}^2(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)}{(\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2)^2}\right)(t,s)\\
&= \left(\frac{\mathcal{E}\mathcal{G}-\mathcal{F}^2}{\mathfrak{e}\mathfrak{g}-\mathfrak{f}^2}\right)(t,s)\\
&= \frac{\left(\frac{v''(t)}{\omega(t,s)}\right)\left(\frac{u''(s)}{\omega(t,s)}\right)}{\omega^2(t,s)}\\
&= \frac{u''(s)v''(t)}{\omega^4(t,s)}.
\end{aligned}$$

□

### 3.1.3 *Translating Solitons with* $\Psi(X,Y) = Y$

**Proposition 3.1.2.** *Let be $\Psi(X,Y) = Y$ and $\alpha \in \mathbb{R}^3$ non-zero. Then $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, is a Translating Soliton if and only if*

$$v''(t)u''(s) = ((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3).$$

**Proof:** By the definition of Translating Soliton, that is,

$$\Psi(H,K)(t,s) = \langle \alpha, N(t,s)\rangle,$$

i.e.,

$$K(t,s) = \langle \alpha, N(t,s)\rangle,$$

and by Proposition 3.1.1 and (3.6) we obtain

$$v''(t)u''(s) = ((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3).$$

□

**Proposition 3.1.3.** *Let be $\Psi(X,Y) = Y$, $\alpha \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for $\mu, \tau$ constants.*

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y) = Y$ and $\alpha \in \mathbb{R}^3$ non-zero. By Proposition 3.1.2 the functions $u, v$ satisfy the equation

$$v''(t)u''(s) = ((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3). \tag{3.7}$$

Suppose that $u, v$ are not straight lines, that is, there exist open set $U \subset J$ and $V \subset I$ such that $u''(s) \neq 0$ in $U$ and $v''(t) \neq 0$ in $V$. Choose $s_0 \in J$ such that if $\alpha_2 \neq 0$ then $u'(s_0) \neq \frac{\alpha_3}{\alpha_2}$ and if $\alpha_2 = 0$ choose any $s_0$ (note that such a choice is possible since $u'$ is non-constant). We obtain from (3.7)

$$v''(t)u''(s_0) = ((v'(t))^2 + (u'(s_0))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s_0) + \alpha_3), \tag{3.8}$$

hence from (3.7) e (3.8)

$$\begin{aligned} v''(t)u''(s)u''(s_0) =& u''(s_0)((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3) \\ v''(t)u''(s)u''(s_0) =& u''(s)((v'(t))^2 + (u'(s_0))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s_0) + \alpha_3), \end{aligned}$$

thus

$$\begin{aligned} u''(s)((v'(t))^2 + (u'(s_0))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s_0) + \alpha_3) = & -u''(s_0)((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}\alpha_1 v'(t) \\ & - u''(s_0)((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}\alpha_2 u'(s) \\ & + u''(s_0)((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}\alpha_3 \end{aligned}$$

which squaring and rewriting

$$\begin{aligned} 0 =& \alpha_1^2((u''(s_0))^2 - (u''(s))^2)(v'(t))^8 \\ & + 2\alpha_1((u''(s))^2(\alpha_3 - u'(s_0)\alpha_2) - (u''(s_0))^2(\alpha_3 - u'(s)\alpha_2))(v'(t))^7 \\ & + ((u''(s_0))^2(3\alpha_1^2((u'(s))^2 + 1) + (\alpha_3 - \alpha_2 u'(s))^2))(v'(t))^6 \\ & - (u''(s))^2(3\alpha_1^2((u'(s_0))^2 + 1) + (\alpha_3 - \alpha_2 u'(s_0))^2)(v'(t))^6 \\ & + 6\alpha_1((u''(s))^2((u'(s_0))^2 + 1)(\alpha_3 - \alpha_2 u'(s_0)) - (u''(s_0))^2((u'(s))^2 + 1)(\alpha_3 - \alpha_2 u'(s)))(v'(t))^5 \\ & + 3(u''(s_0))^2(((u'(s))^2 + 1)(\alpha_3 - \alpha_2 u'(s))^2 + \alpha_1^2(u'(s)^2 + 1)^2)(v'(t))^4 \\ & - 3(u''(s))^2(((u'(s_0))^2 + 1)(\alpha_3 - \alpha_2 u'(s_0))^2 + \alpha_1^2(u'(s_0)^2 + 1)^2)(v'(t))^4 \\ & + 6\alpha_1((u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))(1 + (u'(s_0))^2)^2 - (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))(1 + (u'(s))^2)^2)(v'(t))^3 \\ & + (u''(s_0))^2(3(\alpha_3 - \alpha_2 u'(s))^2((u'(s))^2 + 1)^2 + \alpha_1^2((u'(s))^2 + 1)^3)(v'(t))^2 \\ & - (u''(s))^2(3(\alpha_3 - \alpha_2 u'(s_0))^2((u'(s_0))^2 + 1)^2 + \alpha_1^2((u'(s_0))^2 + 1)^3)(v'(t))^2 \\ & + 2\alpha_1((u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))((u'(s_0))^2 + 1)^3 - (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))((u'(s))^2 + 1)^3)v'(t) \\ & + (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))^2((u'(s))^2 + 1)^3 - (u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))^2((u'(s_0))^2 + 1)^3, \end{aligned}$$

that is, $v'(t)$ vanishes the polynomials

$$P_s(Z) = A_s Z^8 + B_s Z^7 + C_s Z^6 + D_s Z^5 + E_s Z^4 + F_s Z^3 + G_s Z^2 + H_s Z + I_s,$$

where

$$
\begin{aligned}
A_s &= \alpha_1^2((u''(s_0))^2 - (u''(s))^2) \\
B_s &= 2\alpha_1((u''(s))^2(\alpha_3 - u'(s_0)\alpha_2) - (u''(s_0))^2(\alpha_3 - u'(s)\alpha_2)) \\
C_s &= (u''(s_0))^2(3\alpha_1^2((u'(s))^2+1) + (\alpha_3 - \alpha_2 u'(s))^2) \\
&\quad - (u''(s))^2(3\alpha_1^2((u'(s_0))^2+1) + (\alpha_3 - \alpha_2 u'(s_0))^2) \\
D_s &= 6\alpha_1((u''(s))^2((u'(s_0))^2+1)(\alpha_3 - \alpha_2 u'(s_0)) - (u''(s_0))^2((u'(s))^2+1)(\alpha_3 - \alpha_2 u'(s))) \\
E_s &= 3(u''(s_0))^2(((u'(s))^2+1)(\alpha_3 - \alpha_2 u'(s))^2 + \alpha_1^2(u'(s)^2+1)^2) \\
&\quad - 3(u''(s))^2(((u'(s_0))^2+1)(\alpha_3 - \alpha_2 u'(s_0))^2 + \alpha_1^2(u'(s_0)^2+1)^2) \\
F_s &= 6\alpha_1((u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))(1+(u'(s_0))^2)^2 - (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))(1+(u'(s))^2)^2) \\
G_s &= (u''(s_0))^2(3(\alpha_3 - \alpha_2 u'(s))^2((u'(s))^2+1)^2 + \alpha_1^2((u'(s))^2+1)^3) \\
&\quad - (u''(s))^2(3(\alpha_3 - \alpha_2 u'(s_0))^2((u'(s_0))^2+1)^2 + \alpha_1^2((u'(s_0))^2+1)^3) \\
H_s &= 2\alpha_1((u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))((u'(s_0))^2+1)^3 - (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))((u'(s))^2+1)^3) \\
I_s &= (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))^2((u'(s))^2+1)^3 - (u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))^2((u'(s_0))^2+1)^3.
\end{aligned}
$$

Since $v'(t)$ is non-constant (otherwise $v'' \equiv 0$) and for each fixed $s$ we have $P_s(v'(t)) = 0$ we conclude that the coefficients of $P_s$ are all zero. We obtain from the coefficients $A_s, B_s, C_s, I_s$

$$
\begin{aligned}
0 &= \alpha_1^2((u''(s_0))^2 - (u''(s))^2) \\
0 &= 2\alpha_1((u''(s))^2(\alpha_3 - u'(s_0)\alpha_2) - (u''(s_0))^2(\alpha_3 - u'(s)\alpha_2)) \\
0 &= (u''(s_0))^2(3\alpha_1^2((u'(s))^2+1) + (\alpha_3 - \alpha_2 u'(s))^2) - (u''(s))^2(3\alpha_1^2((u'(s_0))^2+1) + (\alpha_3 - \alpha_2 u'(s_0))^2) \\
0 &= (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))^2((u'(s))^2+1)^3 - (u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))^2((u'(s_0))^2+1)^3.
\end{aligned}
\tag{3.9}
$$

If $\alpha_1 = 0$ we obtain from (3.9)

$$
\begin{aligned}
0 &= (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))^2 - (u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))^2 \\
0 &= (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))^2((u'(s))^2+1)^3 - (u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))^2((u'(s_0))^2+1)^3,
\end{aligned}
$$

that is,

$$
\begin{aligned}
(u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))^2 &= (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))^2 \\
0 &= (u''(s_0))^2(\alpha_3 - \alpha_2 u'(s))^2((u'(s))^2+1)^3 - (u''(s))^2(\alpha_3 - \alpha_2 u'(s_0))^2((u'(s_0))^2+1)^3,
\end{aligned}
$$

and substituting the first equality into the second equality we obtain

$$
\begin{aligned}
0 &= (u''(s))^2(u'(s))^2(3(u'(s))^2 + (u'(s))^4 + 3(u'(s_0))^2 + (u'(s_0))^4 + (u'(s))^2(u'(s_0))^2 + 3)(\alpha_2 u'(s_0) - \alpha_3)^2 \\
&\quad -(u''(s))^2(u'(s_0))^2(3(u'(s))^2 + (u'(s))^4 + 3(u'(s_0))^2 + (u'(s_0))^4 + (u'(s))^2(u'(s_0))^2 + 3)(\alpha_2 u'(s_0) - \alpha_3)^2
\end{aligned}
$$

and since $u''(s) \neq 0$ in $U \subset J$ we obtain $(u''(s))^2((u'(s))^2 - (u'(s_0))^2) \neq 0$ in $U \subset J$. Then

$$0 = \alpha_2 u'(s_0) - \alpha_3.$$

In the case where $\alpha_2 \neq 0$ we obtain a contradiction with the choice $u'(s_0) \neq \frac{\alpha_3}{\alpha_2}$. In the case where $\alpha_2 = 0$ we obtain a contradiction with $(\alpha_1, \alpha_2, \alpha_3) = \alpha \neq 0$. If $\alpha_1 \neq 0$ we obtain from (3.9)

$$0 = (u''(s_0))^2 - (u''(s))^2$$
$$0 = (u''(s))^2(\alpha_3 - u'(s_0)\alpha_2) - (u''(s_0))^2(\alpha_3 - u'(s)\alpha_2)$$
$$0 = (u''(s_0))^2(3\alpha_1^2((u'(s))^2+1) + (\alpha_3 - \alpha_2 u'(s))^2) - (u''(s))^2(3\alpha_1^2((u'(s_0))^2+1) + (\alpha_3 - \alpha_2 u'(s_0))^2),$$

that is,

$$\begin{aligned}(u''(s))^2 &= (u''(s_0))^2\\ 0 &= (u''(s))^2(\alpha_3 - u'(s_0)\alpha_2) - (u''(s_0))^2(\alpha_3 - u'(s)\alpha_2)\\ 0 &= (u''(s_0))^2(3\alpha_1^2((u'(s))^2+1) + (\alpha_3 - \alpha_2 u'(s))^2)\\ &\quad - (u''(s))^2(3\alpha_1^2((u'(s_0))^2+1) + (\alpha_3 - \alpha_2 u'(s_0))^2),\end{aligned}$$

and substituting the first equality into the others we obtain

$$0 = \alpha_2(u''(s))^2(u'(s) - u'(s_0))$$
$$0 = (u''(s))^2(3\alpha_1^2((u'(s))^2+1) + (\alpha_3 - \alpha_2 u'(s))^2) - (u''(s))^2(3\alpha_1^2((u'(s_0))^2+1) + (\alpha_3 - \alpha_2 u'(s_0))^2),$$

and since $u''(s) \neq 0$ in $U \subset J$ we obtain $(u''(s))^2(u'(s) - u'(s_0)) \neq 0$ in $U \subset J$ and thus $\alpha_2 = 0$. Therefore

$$0 = \alpha_1^2(u''(s))^2((u'(s))^2 - (u'(s_0))^2)$$

a contradiction. Thus, the initial assumption that $u, v$ are not straight lines is false, that is, $u$ or $v$ is a straight line in $J$ or $I$ respectively. □

**Theorem 3.1.1.** *Let be $\Psi(X,Y) = Y$, $\alpha \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J, \mathbb{R})$ and $v \in C^2(I, \mathbb{R})$, Translating Soliton. Then:*

*i) $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Moreover, if $\alpha = \lambda(1,0,\mu)$, for $\lambda \neq 0$, $u$ is arbitrary.*

*ii) $v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Moreover, if $\alpha \neq \lambda(1,0,\mu)$, for $\lambda \neq 0$, we have*

$$u(s) = \frac{\alpha_3 - \alpha_1\mu}{\alpha_2}s + \eta,$$

*where $J \subset \mathbb{R}$.*

*iii) $u(s) = \mu s + \tau$, for constants $\mu, \tau$, and $J \subset \mathbb{R}$. Moreover, if $\alpha = \lambda(1,0,\mu)$, for $\lambda \neq 0$, $v$ is arbitrary.*

*iv) $u(s) = \mu s + \tau$, for constants $\mu, \tau$, and $J \subset \mathbb{R}$. Moreover, if $\alpha \neq \lambda(1,0,\mu)$, for $\lambda \neq 0$, we have*

$$v(t) = \frac{\alpha_3 - \alpha_1\mu}{\alpha_2}t + \eta,$$

*where $I \subset \mathbb{R}$.*

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y) = Y$ and $\alpha \in \mathbb{R}^3$ non-zero. By Proposition 3.1.2 the functions $u, v$ satisfy the equation

$$v''(t)u''(s) = ((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3). \tag{3.10}$$

By Proposition 3.1.3 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for constants $\mu, \tau$.

**-** *i*)**:** If $v(t) = \mu t + \tau$ and $(\alpha_1, \alpha_2, \alpha_3) = \alpha = \lambda(1,0,\mu)$ it is easy to verify, in (3.10), that $u$ is arbitrary.

**-** *ii*)**:** If $v(t) = \mu t + \tau$ and $(\alpha_1, \alpha_2, \alpha_3) = \alpha \neq \lambda(1,0,\mu)$ we have from (3.10)

$$0 = (\mu^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 \mu - \alpha_2 u'(s) + \alpha_3),$$

that is,

$$0 = -\alpha_1 \mu - \alpha_2 u'(s) + \alpha_3,$$

i.e.,

$$u'(s) = \frac{\alpha_3 - \alpha_1 \mu}{\alpha_2}$$

thus

$$u(s) = \frac{\alpha_3 - \alpha_1 \mu}{\alpha_2} s + \eta,$$

where $J \subset \mathbb{R}$.

**-** *iii*) **and** *iv*)**:** Since the differential equation in (3.10) is symmetric in $u$ and $v$ we get the same result as in *i* and *ii*. □

### 3.1.4 *Translating Solitons with* $\Psi(X,Y) = X$

**Proposition 3.1.4.** *Let be $\Psi(X,Y) = X$ and $\alpha \in \mathbb{R}^3$ non-zero. Then $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, is a Translating Soliton if and only if*

$$v''(t)(1 + (u'(s))^2) = 2(1 + (v'(t))^2 + (u'(s))^2)(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3) - u''(s)(1 + (v'(t))^2).$$

**Proof:** By the definition of Translating Soliton, that is,

$$\Psi(H,K)(t,s) = \langle \alpha, N(t,s) \rangle,$$

i.e.,

$$H(t,s) = \langle \alpha, N(t,s) \rangle,$$

which by Proposition 3.1.1 and (3.6) we obtain

$$v''(t)(1 + (u'(s))^2) = 2(1 + (v'(t))^2 + (u'(s))^2)(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3) - u''(s)(1 + (v'(t))^2).$$

□

**Proposition 3.1.5.** *Let be $\Psi(X,Y) = X$, $\alpha \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for $\mu, \tau$ constants.*

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y) = X$ and $\alpha \in \mathbb{R}^3$ non-zero. By Proposition 3.1.4 the functions $u, v$ satisfy the equation

$$v''(t)(1+(u'(s))^2) = 2(1+(v'(t))^2+(u'(s))^2)(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3) - u''(s)(1+(v'(t))^2). \quad (3.11)$$

Suppose that $u, v$ are not straight lines, that is, there exist open set $U \subset J$ and $V \subset I$ such that $u''(s) \neq 0$ in $U$ and $v''(t) \neq 0$ in $V$. Choose $s_0 \in J$. We obtain from (3.11)

$$v''(t)(1+(u'(s_0))^2) = 2(1+(v'(t))^2+(u'(s_0))^2)(-\alpha_1 v'(t) - \alpha_2 u'(s_0) + \alpha_3) - u''(s_0)(1+(v'(t))^2), \quad (3.12)$$

thus from (3.11) and (3.12)

$$\begin{aligned} v''(t)(1+(u'(s))^2)(1+(u'(s_0))^2) &= -(1+(u'(s_0))^2)\big(2(1+(v'(t))^2+(u'(s))^2)(\alpha_1 v'(t)+\alpha_2 u'(s)-\alpha_3)\big) \\ &\quad -(1+(u'(s_0))^2)u''(s)(1+(v'(t))^2) \\ v''(t)(1+(u'(s))^2)(1+(u'(s_0))^2) &= -(1+(u'(s))^2)\big(2(1+(v'(t))^2+(u'(s_0))^2)(\alpha_1 v'(t)+\alpha_2 u'(s_0)-\alpha_3)\big) \\ &\quad -(1+(u'(s))^2)u''(s_0)(1+(v'(t))^2) \end{aligned}$$

and subtracting the second from the first

$$\begin{aligned} 0 &= 2\alpha_1((u'(s))^2-(u'(s_0))^2)(v'(t))^3 \\ &\quad +((1+(u'(s))^2)(u''(s_0)-2\alpha_3+2\alpha_2 u'(s_0)) - (1+(u'(s_0))^2)(u''(s)-2\alpha_3+2\alpha_2 u'(s)))(v'(t))^2 \\ &\quad +(1+(u'(s))^2)(u''(s_0)-2(1+(u'(s_0))^2)(\alpha_3-\alpha_2 u'(s_0))) \\ &\quad -(1+(u'(s_0))^2)(u''(s)-2(1+(u'(s))^2)(\alpha_3-\alpha_2 u'(s))), \end{aligned}$$

that is, $v'(t)$ vanishes the polynomials

$$P_s(Z) = A_s Z^3 + B_s Z^2 + C_s$$

where

$$\begin{aligned} A_s &= 2\alpha_1((u'(s))^2-(u'(s_0))^2) \\ B_s &= (1+(u'(s))^2)(u''(s_0)-2\alpha_3+2\alpha_2 u'(s_0)) - (1+(u'(s_0))^2)(u''(s)-2\alpha_3+2\alpha_2 u'(s)) \\ C_s &= (1+(u'(s))^2)(u''(s_0)-2(1+(u'(s_0))^2)(\alpha_3-\alpha_2 u'(s_0))) \\ &\quad -(1+(u'(s_0))^2)(u''(s)-2(1+(u'(s))^2)(\alpha_3-\alpha_2 u'(s))). \end{aligned}$$

Since $v'(t)$ is non-constant (otherwise $v'' \equiv 0$) and for each fixed $s$ we have $P_s(v'(t)) = 0$ we conclude that the coefficients of $P_s$ are all zero. We obtain

$$\begin{aligned} 0 &= 2\alpha_1((u'(s))^2-(u'(s_0))^2) \\ 0 &= (1+(u'(s))^2)(u''(s_0)-2\alpha_3+2\alpha_2 u'(s_0)) - (1+(u'(s_0))^2)(u''(s)-2\alpha_3+2\alpha_2 u'(s)) \\ 0 &= (1+(u'(s))^2)(u''(s_0)-2(1+(u'(s_0))^2)(\alpha_3-\alpha_2 u'(s_0))) \\ &\quad -(1+(u'(s_0))^2)(u''(s)-2(1+(u'(s))^2)(\alpha_3-\alpha_2 u'(s))). \end{aligned} \quad (3.13)$$

If $\alpha_1 \neq 0$ we obtain from the first equality of (3.13) that $(u'(s))^2 = (u'(s_0))^2$, that is, $u'(s)$ is constant and therefore $u''(s) = 0$ in $U$, a contradiction. If $\alpha_1 = 0$ we obtain from (3.13)

$$\begin{aligned}
0 &= (1+(u'(s))^2)(u''(s_0) - 2\alpha_3 + 2\alpha_2 u'(s_0)) - (1+(u'(s_0))^2)(u''(s) - 2\alpha_3 + 2\alpha_2 u'(s))\\
0 &= (1+(u'(s))^2)(u''(s_0) - 2(1+(u'(s_0))^2)(\alpha_3 - \alpha_2 u'(s_0)))\\
&\quad - (1+(u'(s_0))^2)(u''(s) - 2(1+(u'(s))^2)(\alpha_3 - \alpha_2 u'(s))),
\end{aligned}$$

then

$$\begin{aligned}
u''(s) &= \frac{(1+(u'(s))^2)(u''(s_0) - 2\alpha_3 + 2\alpha_2 u'(s_0)) + (1+(u'(s_0))^2)(2\alpha_3 - 2\alpha_2 u'(s))}{1+(u'(s_0))^2}\\
0 &= (1+(u'(s))^2)(u''(s_0) - 2(1+(u'(s_0))^2)(\alpha_3 - \alpha_2 u'(s_0)))\\
&\quad - (1+(u'(s_0))^2)(u''(s) - 2(1+(u'(s))^2)(\alpha_3 - \alpha_2 u'(s))),
\end{aligned}$$

and substituting the first equality into the second we obtain

$$0 = -2(u'(s) - u'(s_0))(\alpha_2(1+(u'(s_0))^2)(u'(s))^2 + (\alpha_2 u'(s_0) - \alpha_3)u'(s) + u'(s_0)(\alpha_2 u'(s_0) - \alpha_3))$$

and since $u'(s) - u'(s_0) \neq 0$ (otherwise $u'' = 0$ in $U$, a contradiction)

$$0 = \alpha_2(1+(u'(s_0))^2)(u'(s))^2 + (\alpha_2 u'(s_0) - \alpha_3)u'(s) + u'(s_0)(\alpha_2 u'(s_0) - \alpha_3).$$

Observe that $u'(s)$ vanishes the polynomial

$$Q(Z) = \alpha_2(1+(u'(s_0))^2)Z^2 + (\alpha_2 u'(s_0) - \alpha_3)Z + u'(s_0)(\alpha_2 u'(s_0) - \alpha_3)$$

and since $u'(s)$ is non-constant we conclude that the coefficients of such polynomial are all zero, that is,

$$\begin{aligned}
0 &= \alpha_2(1+(u'(s_0))^2)\\
0 &= \alpha_2 u'(s_0) - \alpha_3\\
0 &= u'(s_0)(\alpha_2 u'(s_0) - \alpha_3)
\end{aligned}$$

then

$$\begin{aligned}
0 &= \alpha_2\\
0 &= \alpha_2 u'(s_0) - \alpha_3
\end{aligned}$$

and then $0 = \alpha_2 = \alpha_3$, a contradiction since $0 \neq \alpha = (0, \alpha_2, \alpha_3)$. Thus, the initial assumption that $u, v$ are not straight lines is false, that is, $u$ or $v$ is a straight line in $J$ or $I$ respectively. □

**Theorem 3.1.2.** *Let be $\Psi(X,Y) = X$, $\alpha \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then:*

*i) $v(t) = \mu t + \tau$, for $\mu, \tau$ constants, and $I \subset \mathbb{R}$. Moreover, if $\alpha = \lambda(1,0,\mu)$, for $\lambda \neq 0$, $u(s) = \zeta s + \eta$ for $\zeta, \eta$ constants and $J \subset \mathbb{R}$.*

*ii)* $v(t) = \mu t + \tau$, *for constants* $\mu, \tau$, *and* $I \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, 0, \alpha_3)$, *with* $\alpha_3 \neq \mu\alpha_1$, *we have*

$$u(s) = D + \frac{1+\mu^2}{2(\alpha_3 - \alpha_1\mu)} \ln \left| \sec\left( \frac{2(\alpha_3 - \alpha_1\mu)(s-C)}{\sqrt{1+\mu^2}} + \arctan\left( \frac{F}{\sqrt{1+\mu^2}} \right) \right) \right|,$$

*for constants C, D and F and* $J \subset \mathbb{R} - \left\{ \frac{\sqrt{1+\mu^2}\left((2k+1)\pi - \arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4(\alpha_3-\alpha_1\mu)} + C \in \mathbb{R} \;\middle|\; k \in \mathbb{Z} \right\}$.

*iii)* $v(t) = \mu t + \tau$, *for* $\mu, \tau$ *constants, and* $I \subset \mathbb{R}$. *Moreover, if* $\alpha = (\lambda, \alpha_2, \mu\lambda)$, *with* $\alpha_2 \neq 0$, *we have* $u(s) = D$ *or*

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{(1+\mu^2)F}{(1+F+\mu^2)e^{4\alpha_2(r-C)} - F}} dr.$$

*for* $C, D$ *and* $F > 0$ *constants and* $J \subset \left( \frac{1}{4\alpha_2} \ln\left( \frac{F}{1+F+\mu^2} \right) + C, +\infty \right)$.

*iv)* $v(t) = \mu t + \tau$, *for constants* $\mu, \tau$, *and* $I \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, *with* $\alpha_2 \neq 0$ *and* $\alpha_3 \neq \mu\alpha_1$ *the function u satisfies the differential equation*

$$u''(s) = \frac{2(1+(u'(s))^2+\mu^2)(-\alpha_1\mu - \alpha_2 u'(s) + \alpha_3)}{1+\mu^2}. \tag{3.14}$$

*v)* $u(s) = \mu s + \tau$, *for* $\mu, \tau$ *constants, and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = \lambda(0, 1, \mu)$, *for* $\lambda \neq 0$, $v(t) = \zeta t + \eta$ *for* $\zeta, \eta$ *constants and* $I \subset \mathbb{R}$.

*vi)* $u(s) = \mu s + \tau$, *for* $\mu, \tau$ *constants, and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = (0, \alpha_2, \alpha_3)$, *with* $\alpha_3 \neq \mu\alpha_2$, *we have*

$$v(t) = D + \frac{1+\mu^2}{2(\alpha_3 - \alpha_2\mu)} \ln \left| \sec\left( \frac{2(\alpha_3 - \alpha_2\mu)(t-C)}{\sqrt{1+\mu^2}} + \arctan\left( \frac{F}{\sqrt{1+\mu^2}} \right) \right) \right|,$$

*for constants C, D and F and* $I \subset \mathbb{R} - \left\{ \frac{\sqrt{1+\mu^2}\left((2k+1)\pi - \arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4(\alpha_3-\alpha_2\mu)} + C \in \mathbb{R} \;\middle|\; k \in \mathbb{Z} \right\}$.

*vii)* $u(s) = \mu s + \tau$, *for* $\mu, \tau$ *constants, and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, \lambda, \mu\lambda)$, *with* $\alpha_1 \neq 0$, *we have* $v(t) = D$ *or*

$$v(t) = D \pm \int_{t_0}^{t} \sqrt{\frac{(1+\mu^2)F}{(1+F+\mu^2)e^{4\alpha_1(r-C)} - F}} dr.$$

*for constants* $C, D$ *and* $F > 0$ *and* $I \subset \left( \frac{1}{4\alpha_1} \ln\left( \frac{F}{1+F+\mu^2} \right) + C, +\infty \right)$.

*viii)* $u(s) = \mu s + \tau$, *for constants* $\mu, \tau$, *and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, *with* $\alpha_1 \neq 0$ *and* $\alpha_3 \neq \mu\alpha_2$ *the function v satisfies the differential equation*

$$v''(t) = \frac{2(1+(v'(t))^2+\mu^2)(-\alpha_1 v'(t) - \alpha_2\mu + \alpha_3)}{1+\mu^2}. \tag{3.15}$$

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y)=X$ and $\alpha\in\mathbb{R}^3$ non-zero. By Proposition 3.1.4 the functions $u,v$ satisfy the equation

$$v''(t)(1+(u'(s))^2)=2(1+(v'(t))^2+(u'(s))^2)(-\alpha_1 v'(t)-\alpha_2 u'(s)+\alpha_3)-u''(s)(1+(v'(t))^2). \quad (3.16)$$

By Proposition 3.1.5 we have $v(t)=\mu t+\tau$ in $I\subset\mathbb{R}$ or $u(s)=\mu s+\tau$ in $J\subset\mathbb{R}$, for constant $\mu,\tau$. Suppose the first situation. We obtain from (3.16)

$$u''(s)=\frac{2(1+(u'(s))^2+\mu^2)(-\alpha_1\mu-\alpha_2 u'(s)+\alpha_3)}{1+\mu^2}, \quad (3.17)$$

and setting $w(s)=u'(s)$

$$w'(s)=\frac{2(1+w^2(s)+\mu^2)(-\alpha_1\mu-\alpha_2 w(s)+\alpha_3)}{1+\mu^2}. \quad (3.18)$$

**-** *i)***:** If $(\alpha_1,\alpha_2,\alpha_3)=\alpha=\lambda(1,0,\mu)$ it is easy to verify, in (3.17), that $u''\equiv 0$ and thus there exist constants $\eta,\zeta$ such that $u(s)=\zeta s+\eta$ in $J\subset\mathbb{R}$.

Se $(\alpha_1,\alpha_2,\alpha_3)=\alpha\neq\lambda(1,0,\mu)$ temos as seguintes três situações.

**-** *ii)***:** If $\alpha=(\alpha_1,0,\alpha_3)$, with $\alpha_3\neq\mu\alpha_1$, we obtain from (3.18)

$$\frac{w'(s)}{1+w^2(s)+\mu^2}=\frac{2(-\alpha_1\mu+\alpha_3)}{1+\mu^2}$$

that is,

$$\int_{s_0}^{s}\frac{w'(r)}{1+w^2(r)+\mu^2}dr=\frac{2(-\alpha_1\mu+\alpha_3)}{1+\mu^2}\int_{s_0}^{s}dr,$$

which by the Change of Variables Theorem

$$\int_{w(s_0)}^{w(s)}\frac{dl}{1+l^2+\mu^2}=\frac{2(\alpha_3-\alpha_1\mu)}{1+\mu^2}\int_{s_0}^{s}dr.$$

Consequently,

$$\frac{\arctan\left(\frac{w(s)}{\sqrt{1+\mu^2}}\right)}{\sqrt{1+\mu^2}}-\frac{\arctan\left(\frac{w(s_0)}{\sqrt{1+\mu^2}}\right)}{\sqrt{1+\mu^2}}=\frac{2(\alpha_3-\alpha_1\mu)(s-s_0)}{1+\mu^2}$$

then,

$$\arctan\left(\frac{w(s)}{\sqrt{1+\mu^2}}\right)=\frac{2(\alpha_3-\alpha_1\mu)(s-s_0)}{\sqrt{1+\mu^2}}+\arctan\left(\frac{w(s_0)}{\sqrt{1+\mu^2}}\right)$$

finally,

$$w(s)=\sqrt{1+\mu^2}\tan\left(\frac{2(\alpha_3-\alpha_1\mu)(s-s_0)}{\sqrt{1+\mu^2}}+\arctan\left(\frac{w(s_0)}{\sqrt{1+\mu^2}}\right)\right)$$

that is, there exist constants $C$ and $F$ such that

$$w(s)=\sqrt{1+\mu^2}\tan\left(\frac{2(\alpha_3-\alpha_1\mu)(s-C)}{\sqrt{1+\mu^2}}+\arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right).$$

in $\mathbb{R}-\left\{\frac{\sqrt{1+\mu^2}\left((2k+1)\pi-\arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4(\alpha_3-\alpha_1\mu)}+C\in\mathbb{R}\;\middle|\;k\in\mathbb{Z}\right\}$. Thus, by the definition of $w$

$$u(s)=D+\frac{1+\mu^2}{2(\alpha_3-\alpha_1\mu)}\ln\left|\sec\left(\frac{2(\alpha_3-\alpha_1\mu)(s-C)}{\sqrt{1+\mu^2}}+\arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)\right|.$$

**-** *iii)***:** If $\alpha=(\lambda,\alpha_2,\mu\lambda)$, with $\alpha_2\neq 0$, we obtain from (3.18)

$$w'(s)=-\frac{2\alpha_2 w(s)(1+w^2(s)+\mu^2)}{1+\mu^2}.$$

Observe that $w\equiv 0$ is a solution of this differential equation (and consequently $u$ is constant). By the Existence and Uniqueness Theorem any other solution does not reach zero. Therefore,

$$\frac{w'(s)}{w(s)(1+w^2(s)+\mu^2)}=-\frac{2\alpha_2}{1+\mu^2},$$

then

$$\int_{s_0}^{s}\frac{w'(r)}{w(r)(1+w^2(r)+\mu^2)}dr=-\frac{2\alpha_2}{1+\mu^2}\int_{s_0}^{s}dr,$$

which by the Change of Variables Theorem

$$\int_{w(s_0)}^{w(s)}\frac{dl}{l(1+l^2+\mu^2)}=-\frac{2\alpha_2}{1+\mu^2}\int_{s_0}^{s}dr.$$

Consequently,

$$\frac{\ln\left(\frac{w^2(s)}{1+w^2(s)+\mu^2}\right)}{2(1+\mu^2)}-\frac{\ln\left(\frac{w^2(s_0)}{1+w^2(s_0)+\mu^2}\right)}{2(1+\mu^2)}=-\frac{2\alpha_2(s-s_0)}{1+\mu^2},$$

i.e.,

$$\ln\left(\frac{w^2(s_0)(1+w^2(s)+\mu^2)}{w^2(s)(1+w^2(s_0)+\mu^2)}\right)=4\alpha_2(s-s_0),$$

and applying the exponential

$$0=\left(w^2(s_0)-(1+w^2(s_0)+\mu^2)e^{4\alpha_2(s-s_0)}\right)w^2(s)+w^2(s_0)(1+\mu^2)$$

and since $w^2(s_0)-(1+w^2(s_0)+\mu^2)e^{4\alpha_2(s-s_0)}\neq 0$ (otherwise $0=w^2(s_0)(1+\mu^2)\neq 0$) we have

$$w^2(s)=\frac{w^2(s_0)(1+\mu^2)}{(1+w^2(s_0)+\mu^2)e^{4\alpha_2(s-s_0)}-w^2(s_0)},$$

that is, there exist constants $C$ and $F>0$ such that

$$w^2(s)=\frac{(1+\mu^2)F}{(1+F+\mu^2)e^{4\alpha_2(s-C)}-F},$$

in $\left(\frac{1}{4\alpha_2}\ln\left(\frac{F}{1+F+\mu^2}\right)+C,+\infty\right)$, since $w^2(s)>0$. Hence, we have two distinct solutions

$$w(s)=\pm\sqrt{\frac{(1+\mu^2)F}{(1+F+\mu^2)e^{4\alpha_2(s-C)}-F}}.$$

Then, by the definition of $w$, we have

$$u(s)=D\pm\int_{s_0}^{s}\sqrt{\frac{(1+\mu^2)F}{(1+F+\mu^2)e^{4\alpha_2(r-C)}-F}}dr.$$

**-** *iv*)**:** If $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, with $\alpha_2\neq 0$ and $\alpha_3\neq\mu\alpha_1$ we cannot explicitly show the solution for (3.17). See Corollary 3.1.1

**-** *v*)**,** *vi*)**,** *vii*) **e** *viii*)**:** Since the differential equation in (3.16) is symmetric in $u$ and $v$ we obtain the same result as in *i*), *ii*), *iii*) and *iv*) by inverting $\alpha_2$ with $\alpha_1$. □

**Corollary 3.1.1.** *Assume the conditions of Theorem* 3.1.2. *Then:*

*i) If u is a solution of the differential equation* (3.14)*, we have*

$$u(s)=\frac{\alpha_3-\alpha_1\mu}{\alpha_2}s+\eta,$$

*for a constant $\eta$ and $J\subset\mathbb{R}$, or, $\mathcal{I}(s,u'(s))=0$ where*

$$\mathcal{I}(X,Y)=\frac{\alpha_2\sqrt{1+\mu^2}\ln\left(\frac{(1+Y^2+\mu^2)(-\alpha_1\mu-\alpha_2F+\alpha_3)^2}{(1+F^2+\mu^2)(-\alpha_1\mu-\alpha_2Y+\alpha_3)^2}\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))}$$
$$+\frac{2(\alpha_3-\alpha_1\mu)\left(\arctan\left(\frac{Y}{\sqrt{1+\mu^2}}\right)-\arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))}-\frac{X-C}{1+\mu^2},$$

*for constants C and $F\neq\frac{\alpha_3-\mu\alpha_1}{\alpha_2}$.*

*ii) If v is a solution of the differential equation* (3.15)*, we have*

$$v(t)=\frac{\alpha_3-\alpha_2\mu}{\alpha_1}t+\eta,$$

*for a constant $\eta$ and $I\subset\mathbb{R}$, or, $\tilde{\mathcal{I}}(t,v'(t))=0$ where*

$$\tilde{\mathcal{I}}(X,Y)=\frac{\alpha_1\sqrt{1+\mu^2}\ln\left(\frac{(1+Y^2+\mu^2)(-\alpha_2\mu-\alpha_1F+\alpha_3)^2}{(1+F^2+\mu^2)(-\alpha_2\mu-\alpha_1Y+\alpha_3)^2}\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_2\mu)^2+\alpha_1^2(1+\mu^2))}$$
$$+\frac{2(\alpha_3-\alpha_2\mu)\left(\arctan\left(\frac{Y}{\sqrt{1+\mu^2}}\right)-\arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_2\mu)^2+\alpha_1^2(1+\mu^2))}-\frac{X-C}{1+\mu^2},$$

*for constants C and $F\neq\frac{\alpha_3-\mu\alpha_2}{\alpha_1}$.*

**Proof:** Since the equations (3.14) and (3.15) are similar, we will only prove $i$). Defining $w(s) = u'(s)$ we obtain (3.14)

$$w'(s) = \frac{2(1+w^2(s)+\mu^2)(-\alpha_1\mu - \alpha_2 w(s) + \alpha_3)}{1+\mu^2}. \tag{3.19}$$

since $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_2 \neq 0$ and $\alpha_3 \neq \mu\alpha_1$, notice that $w \equiv \frac{\alpha_3 - \alpha_1\mu}{\alpha_2}$ is a solution of (3.19), therefore $u(s) = \frac{\alpha_3 - \alpha_1\mu}{\alpha_2} s + \eta$. By the Existence and Uniqueness Theorem any other solution does not reach $\frac{\alpha_3 - \alpha_1\mu}{\alpha_2}$ and then $-\alpha_1\mu - \ alpha_2 w(s) + \alpha_3 \neq 0$. We have (3.19)

$$\int_{s_0}^{s} \frac{w'(r)}{2(1+w^2(r)+\mu^2)(-\alpha_1\mu - \alpha_2 w(r) + \alpha_3)} dr = \frac{1}{1+\mu^2} \int_{s_0}^{s} dr,$$

which by the Change of Variables Theorem

$$\int_{w(s_0)}^{w(s)} \frac{dl}{2(1+l^2+\mu^2)(-\alpha_1\mu - \alpha_2 l + \alpha_3)} = \frac{1}{1+\mu^2} \int_{s_0}^{s} dr.$$

Consequently,

$$\begin{aligned}\frac{s-s_0}{1+\mu^2} = & \frac{\alpha_2\sqrt{1+\mu^2}\ln\left(\frac{(1+w^2(s)+\mu^2)(-\alpha_1\mu-\alpha_2 w(s_0)+\alpha_3)^2}{(1+w^2(s_0)+\mu^2)(-\alpha_1\mu-\alpha_2 w(s)+\alpha_3)^2}\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))} \\ & + \frac{2(\alpha_3-\alpha_1\mu)\left(\arctan\left(\frac{w(s)}{\sqrt{1+\mu^2}}\right) - \arctan\left(\frac{w(s_0)}{\sqrt{1+\mu^2}}\right)\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))}\end{aligned}$$

that is, there exist constants $C$ and $F \neq \frac{\alpha_3 - \mu\alpha_1}{\alpha_2}$ such that

$$\begin{aligned}\frac{s-C}{1+\mu^2} = & \frac{\alpha_2\sqrt{1+\mu^2}\ln\left(\frac{(1+w^2(s)+\mu^2)(-\alpha_1\mu-\alpha_2 F+\alpha_3)^2}{(1+F^2+\mu^2)(-\alpha_1\mu-\alpha_2 w(s)+\alpha_3)^2}\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))} \\ & + \frac{2(\alpha_3-\alpha_1\mu)\left(\arctan\left(\frac{w(s)}{\sqrt{1+\mu^2}}\right) - \arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))},\end{aligned}$$

and therefore the graph of $u'$ is given implicitly by the function

$$\begin{aligned}\mathcal{I}(X,Y) = & \frac{\alpha_2\sqrt{1+\mu^2}\ln\left(\frac{(1+Y^2+\mu^2)(-\alpha_1\mu-\alpha_2 F+\alpha_3)^2}{(1+F^2+\mu^2)(-\alpha_1\mu-\alpha_2 Y+\alpha_3)^2}\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))} \\ & + \frac{2(\alpha_3-\alpha_1\mu)\left(\arctan\left(\frac{Y}{\sqrt{1+\mu^2}}\right) - \arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))} - \frac{X-C}{1+\mu^2}.\end{aligned}$$

□

### 3.1.5 *Translating Solitons with* $\Psi(X,Y) = aX + bY$

**Proposition 3.1.6.** *Let be $\Psi(X,Y) = aX + bY$, where $a,b$ are non-zero constants, and $\alpha \in \mathbb{R}^3$ non-zero. Then $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, is a Translating Soliton if and only if*

$$\begin{aligned}0 &= a((v'(t))^2 + (u'(s))^2 + 1)^{\frac{1}{2}}(v''(t)(1+(u'(s))^2) + u''(s)(1+(v'(t))^2)) + 2bv''(t)u''(s)\\ &\quad - 2((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3).\end{aligned}$$

**Proof:** By the definition of Translating Soliton, that is,

$$\Psi(H,K)(t,s) = \langle \alpha, N(t,s)\rangle,$$

i.e.,

$$aH(t,s) + bK(t,s) = \langle \alpha, N(t,s)\rangle,$$

which by Proposition 3.1.1 and (3.6) we obtain

$$\begin{aligned}0 &= a((v'(t))^2 + (u'(s))^2 + 1)^{\frac{1}{2}}(v''(t)(1+(u'(s))^2) + u''(s)(1+(v'(t))^2)) + 2bv''(t)u''(s)\\ &\quad - 2((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3).\end{aligned}$$

□

**Proposition 3.1.7.** *Let be $\Psi(X,Y) = aX + bY$, where $a,b$ are non-zero constants, $\alpha \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for $\mu, \tau$ constants.*

**Proof:** Suppose $\psi$ is a Translating Soliton with $\Psi(X,Y) = aX + bY$ and $a,b$ non-zero constants. By Proposition 3.1.6 the functions $u,v$ satisfy the equation

$$\begin{aligned}0 &= a((v'(t))^2 + (u'(s))^2 + 1)^{\frac{1}{2}}(v''(t)(1+(u'(s))^2) + u''(s)(1+(v'(t))^2)) + 2bv''(t)u''(s)\\ &\quad - 2((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3).\end{aligned} \tag{3.20}$$

We have two possible situations for $\alpha_1$.

-If $\alpha_1 \neq 0$ we have (3.20) (assuming without loss of generality $b = 1$),

$$0 = (2u''(s) + aA_{t,s}((u'(s))^2 + 1))v''(t) + A_{t,s}B_{t,s}, \tag{3.21}$$

where

$$\begin{aligned}A_{t,s} &= \sqrt{(v'(t))^2 + (u'(s))^2 + 1}\\ B_{t,s} &= 2((v'(t))^2 + (u'(s))^2 + 1)(\alpha_2 u'(s) - \alpha_3 + \alpha_1 v'(t)) + au''(s)((v'(t))^2 + 1).\end{aligned}$$

Suppose that $u,v$ are not straight lines, that is, there exist open sets $U \subset J$ and $V \subset I$ such that $u''(s) \neq 0$ in $U$ and $v''(t) \neq 0$ in $V$. We obtain from (3.21)

$$0 = (2u''(s_0) + aA_{t,s_0}((u'(s_0))^2 + 1))v''(t) + A_{t,s_0}B_{t,s_0}, \tag{3.22}$$

thus from (3.21) e (3.22)

$$\begin{aligned}0 &= (2u''(s_0)+aA_{t,s_0}((u'(s_0))^2+1))(2u''(s)+aA_{t,s}((u'(s))^2+1))v''(t)\\ &\quad +A_{t,s}B_{t,s}(2u''(s_0)+aA_{t,s_0}((u'(s_0))^2+1))\\ 0 &= (2u''(s_0)+aA_{t,s_0}((u'(s_0))^2+1))(2u''(s)+aA_{t,s}((u'(s))^2+1))v''(t)\\ &\quad +A_{t,s_0}B_{t,s_0}(2u''(s)+aA_{t,s}((u'(s))^2+1))\end{aligned}$$

and then, subtracting the second equality from the first,

$$A_{t,s}B_{t,s}(2u''(s_0)+aA_{t,s_0}((u'(s_0))^2+1)) = A_{t,s_0}B_{t,s_0}(2u''(s)+aA_{t,s}((u'(s))^2+1))$$

or equivalently,

$$2u''(s)A_{t,s_0}B_{t,s_0} = (B_{t,s}(2u''(s_0)+aA_{t,s_0}(1+(u'(s_0))^2)) - aA_{t,s_0}B_{t,s_0})A_{t,s}$$

which squaring and rewriting

$$\begin{aligned}&(4B^2_{t,s_0}(u''(s))^2 - A^2_{t,s}(aB_{t,s}(1+(u'(s_0))^2) - aB_{t,s_0}(1+(u'(s_0))^2))^2)A^2_{t,s_0} - 4A^2_{t,s}B^2_{t,s}(u''(s_0))^2\\ =&4A^2_{t,s}B_{t,s}u''(s_0)(aB_{t,s}(1+(u'(s_0))^2) - aB_{t,s_0}(1+(u'(s_0))^2))A_{t,s_0}\end{aligned}$$

then squaring and then substituting $B_{t,s}$ and $B_{t,s_0}$ we get

$$\begin{aligned}0 &= Q_1(v'(t))A^4_{t,s}A^4_{t,s_0} + Q_2(v'(t))A^4_{t,s}A^2_{t,s_0} + Q_3(v'(t))A^4_{t,s} + Q_4(v'(t))A^2_{t,s}A^4_{t,s_0}\\ &\quad + Q_5(v'(t))A^2_{t,s}A^2_{t,s_0} + Q_6(v'(t))A^4_{t,s_0}\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}Q_1(Z) &= \rho_1(s)Z^{12} + \tilde{Q}_1(Z)\\ Q_2(Z) &= \rho_2(s)Z^{12} + \tilde{Q}_2(Z)\\ Q_3(Z) &= \rho_3(s)Z^{12} + \tilde{Q}_3(Z)\\ Q_4(Z) &= \rho_4(s)Z^{12} + \tilde{Q}_4(Z)\\ Q_5(Z) &= \rho_5(s)Z^{12} + \tilde{Q}_5(Z)\\ Q_6(Z) &= \rho_6(s)Z^{12} + \tilde{Q}_6(Z),\end{aligned} \tag{3.24}$$

$\deg(\tilde{Q}_i) \leqslant 11$ for $i = 1,..,6$ and

$$\begin{aligned}\rho_1(s) =& -16a^4\alpha_1^4(u'(s)-u'(s_0))^4(u'(s)+u'(s_0))^4\\ \rho_2(s) =& 128a^2\alpha_1^4(u''(s_0))^2(u'(s)-u'(s_0))^2(u'(s)+u'(s_0))^2\\ \rho_3(s) =& -256a^2\alpha_1^4(u''(s_0))^4\\ \rho_4(s) =& 128a^2\alpha_1^4(u''(s))^2(u'(s)-u'(s_0))^2(u'(s)+u'(s_0))^2\\ \rho_5(s) =& 512\alpha_1^4(u''(s_0))^2(u''(s))^2\\ \rho_6(s) =& -256\alpha_1^4(u''(s))^4.\end{aligned} \tag{3.25}$$

One can see

$$\begin{aligned}A^2_{t,s} &= (v'(t))^2 + \tilde{A}\\ A^2_{t,s_0} &= (v'(t))^2 + \tilde{A}_0,\end{aligned} \tag{3.26}$$

where

$$\tilde{A} = 1 + (u'(s))^2$$
$$\tilde{A}_0 = 1 + (u'(s_0))^2.$$

Substituting (3.24) and (3.26) in (3.23) we will get

$$\begin{aligned}
0 &= \rho_1(s)(v'(t))^{20} \\
&+ (\rho_2(s) + \rho_4(s) + 2\rho_1(s)\tilde{A} + 2\rho_1(s)\tilde{A}_0)(v'(t))^{18} \\
&+ (\rho_3(s) + \rho_5(s) + \rho_6(s) + 2\rho_2(s)\tilde{A} + \rho_4(s)\tilde{A} + \rho_2(s)\tilde{A}_0 + 2\rho_4(s)\tilde{A}_0 + \rho_1(s)\tilde{A}^2)(v'(t))^{16} \\
&+ (\rho_1(s)\tilde{A}_0^2 + 4\rho_1(s)\tilde{A}\tilde{A}_0)(v'(t))^{16} \\
&+ ((2\rho_3(s) + \rho_5(s))\tilde{A} + (\rho_5(s) + 2\rho_6(s))\tilde{A}_0 + \rho_2(s)\tilde{A}^2)(v'(t))^{14} \\
&+ (\rho_4(s)\tilde{A}_0^2 + (2\rho_2(s) + 2\rho_4(s))\tilde{A}\tilde{A}_0 + 2\rho_1(s)\tilde{A}\tilde{A}_0^2 + 2\rho_1(s)\tilde{A}^2\tilde{A}_0)(v'(t))^{14} \\
&+ ((\rho_1(s) + \rho_2(s))\tilde{A}^2\tilde{A}_0^2 + \rho_3(s)\tilde{A}^2 + \rho_4(s)\tilde{A}\tilde{A}_0^2 + \rho_5(s)\tilde{A}\tilde{A}_0 + \rho_6(s)\tilde{A}_0^2)(v'(t))^{12} \\
&+ \tilde{Q}_1(v'(t))(v'(t))^8 \\
&+ (\tilde{Q}_2(v'(t)) + \tilde{Q}_4(v'(t)) + 2\tilde{Q}_1(v'(t))\tilde{A} + 2\tilde{Q}_1(v'(t))\tilde{A}_0)(v'(t))^6 \\
&+ (\tilde{Q}_3(v'(t)) + \tilde{Q}_5(v'(t)) + \tilde{Q}_6(v'(t)) + 2\tilde{Q}_2(v'(t))\tilde{A} + \tilde{Q}_4(v'(t))\tilde{A} + \tilde{Q}_2(v'(t))\tilde{A}_0)(v'(t))^4 \\
&+ (2\tilde{Q}_4(v'(t))\tilde{A}_0 + \tilde{Q}_1(v'(t))\tilde{A}^2 + \tilde{Q}_1(v'(t))\tilde{A}_0^2 + 4\tilde{Q}_1(v'(t))\tilde{A}\tilde{A}_0)(v'(t))^4 \\
&+ (2\tilde{Q}_3(v'(t))\tilde{A} + \tilde{Q}_5(v'(t))\tilde{A} + \tilde{Q}_5(v'(t))\tilde{A}_0 + 2\tilde{Q}_6(v'(t))\tilde{A}_0 + \tilde{Q}_2(v'(t))\tilde{A}^2 + \tilde{Q}_4(v'(t))\tilde{A}_0^2)(v'(t))^2 \\
&+ (2\tilde{Q}_1(v'(t))\tilde{A}\tilde{A}_0^2 + 2\tilde{Q}_1(v'(t))\tilde{A}^2\tilde{A}_0 + 2\tilde{Q}_2(v'(t))\tilde{A}\tilde{A}_0 + 2\tilde{Q}_4(v'(t))\tilde{A}\tilde{A}_0)(v'(t))^2 \\
&+ \tilde{Q}_1(v'(t))\tilde{A}^2\tilde{A}_0^2 + \tilde{Q}_2(v'(t))\tilde{A}^2\tilde{A}_0^2 + \tilde{Q}_3(v'(t))\tilde{A}^2 \\
&+ \tilde{Q}_4(v'(t))\tilde{A}\tilde{A}_0^2 + \tilde{Q}_5(v'(t))\tilde{A}\tilde{A}_0 + \tilde{Q}_6(v'(t))\tilde{A}_0^2.
\end{aligned}$$

Let us observe that the right side is a polynomial in $v'$. Since $v'$ is non-constant in $V$ we conclude that such a polynomial is zero, that is, all its coefficients are zero. Analyzing this equality we obtain that the monomial of highest degree is $(v'(t))^{20}$ and its coefficient is $\rho_1(s)$, that is, $-16a^4\alpha_1^4(u'(s) - u'(s_0))^4(u'(s) + u'(s_0))^4$ and being zero we obtain

$$0 = (u'(s) - u'(s_0))(u'(s) + u'(s_0)) = (u'(s))^2 - (u'(s_0))^2$$

that is, $u'(s)$ is constant in $U$ and consequently $u''(s) = 0$ in $U$, a contradiction. Thus, the initial assumption that $u, v$ are not straight lines is false, that is, $u$ or $v$ is a straight line in $J$ or $I$ respectively.
-If $\alpha_1 = 0$ suppose that $u, v$ are not straight lines, that is, there exist open sets $U \subset J$ and $V \subset I$ such that $u''(s) \neq 0$ in $U$ and $v''(t) \neq 0$ in $V$. We have (3.20)

$$\begin{aligned}
2bv''(t)u''(s) = &- a((v'(t))^2 + (u'(s))^2 + 1)^{\frac{1}{2}}(v''(t)(1 + (u'(s))^2) + u''(s)(1 + (v'(t))^2)) \\
&+ 2((v'(t))^2 + (u'(s))^2 + 1)^{\frac{3}{2}}(-\alpha_2 u'(s) + \alpha_3),
\end{aligned} \tag{3.27}$$

which squaring and rewriting

$$
\begin{aligned}
0 = {} & \frac{(au''(s)-2\alpha_3+2u'(s)\alpha_2)^2}{\left(a(u'(s))^2+a\right)^2}(v'(t))^6 - \frac{4b^2(u''(s))^2-\left(a(u'(s))^2+a\right)^2\left((u'(s))^2+1\right)}{\left(a(u'(s))^2+a\right)^2}(v''(t))^2 \\
& + \frac{2\left(a(u'(s))^2+a\right)(au''(s)-2\alpha_3+2u'(s)\alpha_2)}{\left(a(u'(s))^2+a\right)^2}(v'(t))^4 v''(t) + (v'(t))^2(v''(t))^2 \\
& + \frac{\left((u'(s))^2+1\right)(au''(s)-2\alpha_3+2u'(s)\alpha_2)^2}{\left(a(u'(s))^2+a\right)^2}(v'(t))^4 \\
& + \frac{2(au''(s)-2\alpha_3+2u'(s)\alpha_2)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}(v'(t))^4 \\
& + \frac{2\left(a(u'(s))^2+a\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}(v'(t))^2 v''(t) \\
& + \frac{2\left(a(u'(s))^2+a\right)\left((u'(s))^2+1\right)(au''(s)-2\alpha_3+2u'(s)\alpha_2)}{\left(a(u'(s))^2+a\right)^2}(v'(t))^2 v''(t) \\
& + \frac{\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)^2}{\left(a(u'(s))^2+a\right)^2}(v'(t))^2 \\
& + \frac{2(u'(s))^2(au''(s)-2\alpha_3+2u'(s)\alpha_2)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}(v'(t))^2 \\
& + \frac{2(au''(s)-2\alpha_3+2u'(s)\alpha_2)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}(v'(t))^2 \\
& + \frac{2\left(a(u'(s))^2+a\right)\left((u'(s))^2+1\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}v''(t) \\
& + \frac{\left((u'(s))^2+1\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)^2}{\left(a(u'(s))^2+a\right)^2}.
\end{aligned}
$$

Define the family of polynomials

$$P_s(X,Y) = A_sX^3 + B_sX^2Y + C_sX^2 + XY^2 + D_sXY + E_sX + F_sY^2 + G_sY + H_s,$$

where,

$$
\begin{aligned}
A_s &= \frac{(au''(s)-2\alpha_3+2u'(s)\alpha_2)^2}{\left(a(u'(s))^2+a\right)^2} \\
B_s &= \frac{2\left(a(u'(s))^2+a\right)(au''(s)-2\alpha_3+2u'(s)\alpha_2)}{\left(a(u'(s))^2+a\right)^2} \\
C_s &= \frac{\left((u'(s))^2+1\right)(au''(s)-2\alpha_3+2u'(s)\alpha_2)^2}{\left(a(u'(s))^2+a\right)^2} \\
& + \frac{2(au''(s)-2\alpha_3+2u'(s)\alpha_2)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}
\end{aligned}
$$

$$\begin{aligned}
D_s &= \frac{2\left(a(u'(s))^2+a\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}\\
&\quad+\frac{2\left(a(u'(s))^2+a\right)\left((u'(s))^2+1\right)\left(au''(s)-2\alpha_3+2u'(s)\alpha_2\right)}{\left(a(u'(s))^2+a\right)^2}\\
E_s &= \frac{2(u'(s))^2\left(au''(s)-2\alpha_3+2u'(s)\alpha_2\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}\\
&\quad+\frac{2\left(au''(s)-2\alpha_3+2u'(s)\alpha_2\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}\\
F_s &= -\frac{4b^2(u''(s))^2-\left(a(u'(s))^2+a\right)^2\left((u'(s))^2+1\right)}{\left(a(u'(s))^2+a\right)^2}\\
G_s &= \frac{2\left(a(u'(s))^2+a\right)\left((u'(s))^2+1\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)}{\left(a(u'(s))^2+a\right)^2}\\
H_s &= \frac{\left((u'(s))^2+1\right)\left(au''(s)-2\alpha_3\left((u'(s))^2+1\right)+2u'(s)\alpha_2\left((u'(s))^2+1\right)\right)^2}{\left(a(u'(s))^2+a\right)^2},
\end{aligned}$$

and observe that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim 1:** If $A_s \equiv 0$ in $U \subset J$ then both $F_sD_s - G_s \neq 0$ and $E_s + \frac{H_s-E_sF_s}{F_sD_s-G_s}\left(\frac{H_s-E_sF_s}{F_sD_s-G_s}+D_s\right) \neq 0$ in $U \subset J$. Indeed, if $F_sD_s - G_s = 0$ in $U \subset J$ and since $A_s \equiv 0$ in $U \subset J$ then

$$u''(s) = \frac{2\alpha_3 - 2u'(s)\alpha_2}{a}$$

which substituting into the coefficients of $P_s$ we obtain

$$\begin{aligned}
0 = F_sD_s - G_s &= -64b^2(u'(s))^2\frac{(u'(s)\alpha_2-\alpha_3)^3}{a^5(1+(u'(s))^2)^3}\\
&= \frac{-64b^2\alpha_2^3(u'(s))^5+192b^2\alpha_2^2\alpha_3(u'(s))^4-192b^2\alpha_2\alpha_3^2(u'(s))^3+64b^2\alpha_3^3(u'(s))^2}{a^5(1+(u'(s))^2)^3}
\end{aligned}$$

that is, $u'(s)$ is a root of the non-zero polynomial (since $(0,\alpha_2,\alpha_3)=\alpha\neq 0$)

$$Q(Z) = -64b^2\alpha_2^3Z^5+192b^2\alpha_2^2\alpha_3Z^4-192b^2\alpha_2\alpha_3^2Z^3+64b^2\alpha_3^3Z^2$$

and therefore, $u'(s)$ is constant in $U \subset J$ and consequently $u''(s) = 0$ in $U \subset J$, a contradiction. Proceeding analogously we obtain $E_s + \frac{H_s-E_sF_s}{F_sD_s-G_s}\left(\frac{H_s-E_sF_s}{F_sD_s-G_s}+D_s\right) \neq 0$ in $U \subset J$ and thus we conclude the claim.

**Claim 2:** If $A_s \equiv 0$ in $U \subset J$ then $X + F_s$ does not divide $G_s + D_sX$ in $U \subset J$. Indeed, if $X + F_s$ divides $G_s + D_sX$ in $U \subset J$ then there exists a constant $M$ such that

$$D_sX + G_s = M(X + F_s)$$

that is,

$$0 = M(X+F_s) - D_sX + G_s = (M - D_s)X + G_s + MF_s$$

that occurs if, and only if,

$$\begin{aligned} 0 &= M - D_s \\ 0 &= G_s + MF_s. \end{aligned} \tag{3.28}$$

Since $A_s \equiv 0$ in $U \subset J$ then

$$u''(s) = \frac{2\alpha_3 - 2u'(s)\alpha_2}{a}$$

which substituting in the coefficients of $P_s$ we obtain from the first equality of (3.28)

$$0 = M - D_S = M - 4(u'(s))^2 \frac{\alpha_3 + u'(s)\alpha_2}{a(1+(u'(s))^2)} = \frac{-4\alpha_2(u'(s))^3 + (4\alpha_3 + Ma)(u'(s))^2 + Ma}{a(1+(u'(s))^2)}$$

i.e., $u'(s)$ is a root of the non-zero polynomial (since $(0, \alpha_2, \alpha_3) = \alpha \neq 0$)

$$Q(Z) = -4\alpha_2 Z^3 + (4\alpha_3 + Ma)Z^2 + Ma$$

and therefore, $u'(s)$ is constant in $U \subset J$ and consequently $u''(s) = 0$ in $U \subset J$, a contradiction, and thus we conclude the claim.

**Claim 3:** If $A_s \equiv 0$ in $U \subset J$ then the polynomials $\{P_s\}_{s \in U}$ are irreducible. Indeed, if $A_s \equiv 0$ in $U \subset J$ then $B_s \equiv 0$ and $C_s \equiv 0$ in $U \subset J$ and consequently

$$P_s(X,Y) = (X + F_s)Y^2 + (G_s + D_s X)Y + E_s X + H_s$$

then, for $s \in U \subset J$, the polynomial $P_s$ is reducible we have two possible situations. In the first case

$$P_s(X,Y) = (X + F_s)(Y^2 + \lambda Y + \gamma)$$

that occurs if, and only if, $\lambda(X + F_s) = (G_s + D_s X)$, that is, $(X + F_s) | (G_s + D_s X)$,a contradiction with the claim 2. In the second case

$$P_s(X,Y) = (R(X)Y + S(X))\,(T(X)Y + U(X))$$

hence $R(X)T(X) = X + F_s$ and then we can assume without loss of generality $T(X) = 1$ and $R(X) = X + F_s$, that is,

$$P_s(X,Y) = ((F_s + X)Y + S(X))\,(Y + U(X))$$

then $U(X)(F_s + X) + S(X) = D_s X + G_s$ e $S(X)U(X) = E_s X + H_s$. Consequently

$$\deg(S(X)U(X)) = \deg(E_s X + H_s) \leqslant 1$$

and

$$\deg(U(X)(G_s + X) + S(X)) = \deg(D_s X + G_s) \leqslant 1.$$

Then we have the possibilities

$$\begin{aligned}
&\deg(S(X)) = 0 \ \text{ and } \ \deg(U(X)) = 0 \ \text{ then } \ \deg(U(X)(F_s + X) + S(X)) = 1 \\
&\deg(S(X)) = 0 \ \text{ and } \ \deg(U(X)) = 1 \ \text{ then } \ \deg(U(X)(F_s + X) + S(X)) = 2 > 1 \\
&\deg(S(X)) = 1 \ \text{ and } \ \deg(U(X)) = 0 \ \text{ then } \ \deg(U(X)(F_s + X) + S(X)) = 1.
\end{aligned}$$

Consequently,

$$S(X) = nX + o$$
$$U(X) = p$$

hence

$$P_s(X,Y) = ((X+F_s)Y^2 + nX + o)(Y+p)$$

that is,

$$0 = P_s(X,Y) - ((X+F_s)Y^2 + nX + o)(Y+p)$$

which reorganizing

$$0 = (D_s - p - n)XY + (E_s - np)X + (G_s - o - pF_s)Y + H_s - op$$

that occurs if, and only if,

$$\begin{aligned} 0 =& D_s - p - n \\ 0 =& E_s - np \\ 0 =& G_s - o - pF_s \\ 0 =& H_s - op, \end{aligned}$$

i.e.,

$$\begin{aligned} n =& D_s - p \\ 0 =& E_s - np \\ o =& G_s - pF_s \\ 0 =& H_s - op, \end{aligned}$$

and substituting the first and third equalities in the others

$$\begin{aligned} p^2 =& pD_s - E_s \\ 0 =& H_s - pG_s + p^2 F_s \end{aligned} \tag{3.29}$$

which substituting the first equality in the second equality

$$0 = (F_s D_s - G_s)p + H_s - E_s F_s.$$

By claim 1 we have $p = -\frac{H_s - E_s F_s}{F_s D_s - G_s}$ and substituting in the first equality of (3.29)

$$0 = E_s + p(p - D_s) = E_s + \frac{H_s - E_s F_s}{F_s D_s - G_s}\left(\frac{H_s - E_s F_s}{F_s D_s - G_s} + D_s\right)$$

a contradiction with the claim 1 and thus we conclude the claim.

**Claim 4:** If $A_s \neq 0$ in $\tilde{U} \subset U \subset J$ and $G_{s_0} - (D_{s_0} - B_{s_0} F_{s_0})F_{s_0} \neq 0$ or $H_{s_0} - (E_{s_0} - (C_{s_0} - A_{s_0} F_{s_0})F_{s_0})F_{s_0} \neq 0$, for some $s_0 \in \tilde{U}$, then the polynomials $\{P_s\}_{s \in \hat{U}}$ are irreducible in a neighborhood $\hat{U} \subset \tilde{U}$ of $s_0$. Indeed, notice that

$$P_s(X,Y) = (F_s + X)Y^2 + (B_s X^2 + D_s X + G_s)Y + A_s X^3 + C_s X^2 + E_s X + H_s$$

then if given $s \in \hat{U}$ the polynomial $P_s$ is reducible we have two possible situations. In the first case

$$P_s(X,Y) = (R(X)Y + S(X))\,(T(X)Y + U(X))$$

then $R(X)T(X) = X + F_s$ and hence we can assume without loss of generality $T(X) = 1$ and $R(X) = X + F_s$, that is,

$$P_s(X,Y) = ((F_s + X)Y + S(X))\,(Y + U(X))$$

then $U(X)(F_s + X) + S(X) = B_sX^2 + D_sX + G_s$ and $S(X)U(X) = A_sX^3 + C_sX^2 + E_sX + H_s$. Since $A_s \neq 0$ (and consequently $B_s \neq 0$) in $\hat{U}$ we have

$$\deg\left(S(X)U(X)\right) = \deg\left(A_sX^3 + C_sX^2 + E_sX + H_s\right) = 3$$

and

$$\deg\left(U(X)(F_s + X) + S(X)\right) = \deg\left(B_sX^2 + D_sX + G_s\right) = 2.$$

We have the possibilities

$$\begin{aligned}
&\deg(S(X)) = 0 \text{ and } \deg(U(X)) = 3 \text{ then } \deg\left(U(X)(F_s + X) + S(X)\right) = 4 > 2\\
&\deg(S(X)) = 1 \text{ and } \deg(U(X)) = 2 \text{ then } \deg\left(U(X)(F_s + X) + S(X)\right) = 3 > 2\\
&\deg(S(X)) = 2 \text{ and } \deg(U(X)) = 1 \text{ then } \deg\left(U(X)(F_s + X) + S(X)\right) = 2\\
&\deg(S(X)) = 3 \text{ and } \deg(U(X)) = 0 \text{ then } \deg\left(U(X)(F_s + X) + S(X)\right) = 3 > 2,
\end{aligned}$$

that is,

$$\begin{aligned}
S(X) =& mX^2 + nX + o\\
U(X) =& pX + q
\end{aligned}$$

with $m$ and $p$ non-zero. We have

$$0 = P_s(X,Y) - \left((F_s + X)Y + mX^2 + nX + o\right)(Y + pX + q)$$

that is,

$$\begin{aligned}
0 = {}& (F_s + X)Y^2 + (B_sX^2 + D_sX + G_s)Y + A_sX^3 + C_sX^2 + E_sX + H_s\\
& - \left((F_s + X)Y + mX^2 + nX + o\right)(Y + pX + q)
\end{aligned}$$

which reorganizing

$$\begin{aligned}
0 = {}& (A_s - mp)X^3 + (B_s - p - m)X^2Y + (C_s - mq - nq)X^2 + (D_s - q - n - pF_s)XY\\
& + (E_s - nq - op)X + (G_s - o - qF_s)Y + H_s - oq
\end{aligned}$$

that occurs if, only if,

$$\begin{aligned}
0 &= A_s - mp\\
0 &= B_s - p - m\\
0 &= C_s - mq - np\\
0 &= D_s - q - n - pF_s\\
0 &= E_s - nq - op\\
0 &= G_s - o - qF_s\\
0 &= H_s - oq.
\end{aligned}$$

By the second, fourth and sixth equalities we have $m = B_s - p$, $n = D_s - q - pF_s$, $o = G_s - qF_s$ and then

$$\begin{aligned}
0 &= A_s - (B_s - p)p \\
0 &= C_s - (B_s - p)q - (D_s - q - pF_s)p \\
0 &= E_s - (D_s - q - pF_s)q - (G_s - qF_s)p \\
0 &= H_s - (G_s - qF_s)q.
\end{aligned}$$

By the first equality above we have $0 = p^2 - B_s p + A_s$ and therefore $p = \frac{B_s \pm \sqrt{B_s^2 - 4A_s}}{2} = \frac{B_s}{2}$. We have

$$\begin{aligned}
0 &= C_s - \left(B_s - \frac{B_s}{2}\right)q - \left(D_s - q - \frac{B_s}{2}F_s\right)\frac{B_s}{2} \\
0 &= E_s - \left(D_s - q - \frac{B_s}{2}F_s\right)q - (G_s - qF_s)\frac{B_s}{2} \\
0 &= H_s - (G_s - qF_s)q,
\end{aligned}$$

but by the first equality, substituting the coefficients of $P_s$, we obtain in $\hat{U} \subset \tilde{U} \subset U$ that

$$0 = \frac{1}{4}\left(4C_s + B_s^2 F_s - 2B_s D_s\right) = -4b^2(u''(s))^2 \frac{(-2\alpha_3 + au''(s) + 2u'(s)\alpha_2)^2}{a^4((u'(s)) + 1)^4} = -\frac{4b^2(u''(s))^2}{a^2((u'(s)) + 1)^2} A_s \neq 0$$

a contradiction and therefore the first case does not happen. In the second case

$$P_s(X,Y) = U(X)\left(R(X)Y^2 + S(X)Y + T(X)\right)$$

where $U(X)$ is non-constant and $T(X)$ is non-zero. Therefore, $R(X)U(X) = X + F_s$ and then we can assume, without loss of generality, $R(X) = 1$ and $U(X) = X + F_s$, that is,

$$\begin{aligned}
(X + F_s)S(X) &= B_s X^2 + D_s X + G_s \\
(X + F_s)T(X) &= A_s X^3 + C_s X^2 + E_s X + H_s.
\end{aligned}$$

Since $A_s \neq 0$ (and consequently $B_s \neq 0$) in $\hat{U}$ then $\deg(S(X)) = 1$ and $\deg(T(X)) = 2$. Therefore,

$$\begin{aligned}
S(X) &= mX + n \\
T(X) &= oX^2 + pX + q
\end{aligned}$$

with $m$ and $o$ non-zero. We have

$$0 = P_s(X,Y) - (X + F_s)\left(Y^2 + (mX + n)Y + oX^2 + pX + q\right)$$

that is,

$$\begin{aligned}
0 = {} & (F_s + X)Y^2 + (B_s X^2 + D_s X + G_s)Y + A_s X^3 + C_s X^2 + E_s X + H_s \\
& - (X + F_s)\left(Y^2 + (mX + n)Y + oX^2 + pX + q\right)
\end{aligned}$$

which reorganizing

$$\begin{aligned}
0 = {} & (A_s - o)X^3 + (B_s - m)X^2Y + (C_s - p - oF_s)X^2 + (D_s - n - mF_s)XY \\
& + (E_s - q - pF_s)X + (G_s - nF_s)Y + H_s - qF_s
\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}
0 &= A_s - o \\
0 &= B_s - m \\
0 &= C_s - p - oF_s \\
0 &= D_s - n - mF_s \\
0 &= E_s - q - pF_s \\
0 &= G_s - nF_s \\
0 &= H_s - qF_s.
\end{aligned}$$

By the first and second equalities we obtain $o = A_s$ and $m = B_s$. Then

$$\begin{aligned}
0 &= C_s - p - A_sF_s \\
0 &= D_s - n - B_sF_s \\
0 &= E_s - q - pF_s \\
0 &= G_s - nF_s \\
0 &= H_s - qF_s.
\end{aligned}$$

By the first and second equalities we obtain $p = C_s - A_sF_s$ and $n = D_s - B_sF_s$. Thus

$$\begin{aligned}
0 &= E_s - q - (C_s - A_sF_s)F_s \\
0 &= G_s - (D_s - B_sF_s)F_s \\
0 &= H_s - qF_s.
\end{aligned}$$

By the first equality we have $q = E_s - (C_s - A_sF_s)F_s$ and therefore in $\hat{U}$

$$\begin{aligned}
0 &= G_s - (D_s - B_sF_s)F_s \\
0 &= H_s - (E_s - (C_s - A_sF_s)F_s)F_s
\end{aligned}$$

a contradiction with the hypothesis and thus we conclude the claim.

**Claim 5:** If $A_s \neq 0$ in $\tilde{U} \subset U \subset J$ and $G_s - (D_s - B_sF_s)F_s = 0$ and $H_s - (E_s - (C_s - A_sF_s)F_s)F_s = 0$, for all $s \in \tilde{U} \subset U \subset J$, then the polynomials $P_s$ are uniquely written as

$$P_s(X,Y) = (X + F_s)\left(A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + (A_sF_s^2 - C_sF_s + E_s)\right)$$

in which both factors are irreducible. Indeed,

$$\begin{aligned}
&P_s(X,Y) - = (X + F_s)\left(A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + (A_sF_s^2 - C_sF_s + E_s)\right) \\
&= (G_s - (D_s - B_sF_s)F_s)Y + H_s - (E_s - (C_s - A_sF_s)F_s)F_s \\
&= 0.
\end{aligned}$$

Finally, $X + F_s$ is irreducible and therefore it remains to show that

$$A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + (A_sF_s^2 - C_sF_s + E_s)$$

is irreducible. Indeed, suppose that there are polynomials of degree 1 such that

$$A_sX^2+B_sXY+(C_s-A_sF_s)X+Y^2+(D_s-B_sF_s)Y+(A_sF_s^2-C_sF_s+E_s)=(f+gX+hY)(m+nX+oY)$$

that is,

$$\begin{aligned}0=&(A_s-gn)X^2+(B_s-go-hn)XY+(C_s-A_sF_s-fn-gm)X+(1-ho)Y^2\\&+(D_s-B_sF_s-fo-hm)Y+A_sF_s^2-C_sF_s+E_s-fm\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0&=A_s-gn & 0&=1-ho\\0&=B_s-go-hn & 0&=D_s-B_sF_s-fo-hm\\0&=C_s-A_sF_s-fn-gm & 0&=A_sF_s^2-C_sF_s+E_s-fm.\end{aligned}$$

By the symmetry in the factorization of $P_s$ and by the fourth equality we can assume, without loss of generality, $h=o=1$. We have

$$\begin{aligned}0&=A_s-gn & &\\0&=B_s-g-n & 0&=D_s-B_sF_s-f-m\\0&=C_s-A_sF_s-fn-gm & 0&=A_sF_s^2-C_sF_s+E_s-fm,\end{aligned}$$

that by the second and fourth equalities we have $n=B_s-g$ and $m=D_s-B_sF_s-f$ and then

$$\begin{aligned}0&=A_s-g(B_s-g)\\0&=C_s-A_sF_s-f(B_s-g)-g(D_s-B_sF_s-f)\\0&=A_sF_s^2-C_sF_s+E_s-f(D_s-B_sF_s-f)\end{aligned}$$

but, by the first equality above $0=g^2-B_sg+A_s$ and thus $g=\frac{B_s\pm\sqrt{B_s^2-4A_s}}{2}=\frac{B_s}{2}$. We have

$$\begin{aligned}0&=C_s-A_sF_s-f\left(B_s-\frac{B_s}{2}\right)-\frac{B_s}{2}(D_s-B_sF_s-f)\\0&=A_sF_s^2-C_sF_s+E_s-f(D_s-B_sF_s-f)\end{aligned}$$

but by the first equality, substituting the coefficients of $P_s$, we obtain

$$\begin{aligned}0&=-\frac{1}{2}(-2C_s-B_s^2F_s+2A_sF_s+B_sD_s)\\&=-4b^2(u''(s))^2\frac{(-2\alpha_3+au''(s)+2u'(s)\alpha_2)^2}{a^4(1+(u'(s))^2)^4}\\&=-\frac{4b^2(u''(s))^2}{a^2(1+(u'(s))^2)^2}A_s\neq 0\end{aligned}$$

a contradiction and thus we conclude the claim.

**Claim 6:** The polynomial

$$\begin{aligned}S(Z)=&\,4a^3\alpha_2Z^5+(a^4K_2+M_2a^4-4a^3\alpha_3+a^2b^2K_2^3)Z^4\\&+(4a^3\alpha_2-8ab^2K_2^2\alpha_2)Z^3-(8a\alpha_2b^2K_2^2+32\alpha_2\alpha_3b^2K_2)Z\\&+(2a^4K_2+2M_2a^4-4\alpha_3a^3+2a^2b^2K_2^3+8\alpha_3ab^2K_2^2+16b^2K_2\alpha_2^2)Z^2\\&+M_2a^4-a^4K_2+a^2b^2K_2^3+8ab^2K_2^2\alpha_3+16b^2K_2\alpha_3^2\end{aligned}$$

with $K_2 \neq 0$ is non-zero. Indeed, if $\alpha_2 \neq 0$ then $S(Z)$ is clearly non-zero. If $\alpha_2 = 0$ then $\alpha_3 \neq 0$ (since $(0, \alpha_2, \alpha_3) = \alpha \neq 0$). $S(Z)$ is zero if its coefficients are zero. Considering the coefficients of degree 4, degree 2 and degree 0 we have

$$\begin{aligned}
0 &= a^4K_2 + M_2a^4 - 4a^3\alpha_3 + a^2b^2K_2^3 \\
0 &= 2a^4K_2 + 2M_2a^4 - 4\alpha_3a^3 + 2a^2b^2K_2^3 + 8\alpha_3ab^2K_2^2 \\
0 &= M_2a^4 - a^4K_2 + a^2b^2K_2^3 + 8ab^2K_2^2\alpha_3 + 16b^2K_2\alpha_3^2.
\end{aligned}$$

Isolating $\alpha_3$ in the first equality and substituting it in the other two we obtain

$$\begin{aligned}
0 &= \frac{b^6K_2^7 - a^6K_2 + a^6M_2 + 4a^4b^2K_2^3 + 4a^2b^4K_2^5 + a^4b^2K_2M_2^2 + 4a^4b^2K_2^2M_2 + 2a^2b^4K_2^4M_2}{a^2} \\
0 &= (2b^2K_2^2 + a^2)(b^2K_2^3 + a^2K_2 + a^2M_2)
\end{aligned}$$

then

$$\begin{aligned}
0 &= b^6K_2^7 - a^6K_2 + a^6M_2 + 4a^4b^2K_2^3 + 4a^2b^4K_2^5 + a^4b^2K_2M_2^2 + 4a^4b^2K_2^2M_2 + 2a^2b^4K_2^4M_2 \\
0 &= b^2K_2^3 + a^2K_2 + a^2M_2.
\end{aligned}$$

Finally, isolating $M_2$ in the second equality and substituting it in the first we obtain $0 = -2a^6K_2$ a contradiction and thus we conclude the claim.

- If $A_s \equiv 0$ (hence $B_s \equiv 0$ and $C_s \equiv 0$) in $U \subset J$ we have by claim 3 that the polynomials $\{P_s\}_{s \in U}$ are irreducible. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s \in J}$ thus by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Since $v$ is not a straight line we obtain that such a curve is not constant, therefore, $\{P_s\}_{s \in U}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = XY^2 + D_sXY + E_sX + F_sY^2 + G_sY + H_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there exists a constant $K_7$ such that,

$$G_s = K_7$$

and since $A_s \equiv 0$ implies $u''(s) = \frac{2\alpha_3 - 2u'(s)\alpha_2}{a}$ we have,

$$K_7 = 4(u'(s))^2 \frac{-\alpha_3 + u'(s)\alpha_2}{a}$$

then

$$0 = 4\alpha_2(u'(s))^3 - 4\alpha_3(u'(s))^2 - aK_7,$$

i.e., $u'(s)$ is a root of the non-zero polynomial (since $(0, \alpha_2, \alpha_3) = \alpha \neq 0$)

$$R(Z) = 4\alpha_2Z^3 - 4\alpha_3Z^2 - aK_7$$

and therefore, $u'(s)$ is constant in $U \subset J$ and consequently $u''(s) = 0$ in $U \subset J$, a contradiction.

- If $A_s \neq 0$ in $\tilde{U} \subset U \subset J$ and $G_{s_0} - (D_{s_0} - B_{s_0}F_{s_0})F_{s_0} \neq 0$ or $H_{s_0} - (E_{s_0} - (C_{s_0} - A_{s_0}F_{s_0})F_{s_0})F_{s_0} \neq 0$, for some $s_0 \in \tilde{U}$ we have by Claim 4 that the polynomials $\{P_s\}_{s\in\hat{U}}$ are irreducible in a neighborhood $\hat{U} \subset \tilde{U}$ of $s_0$. As previously observed, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ then by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Since $v$ is not a straight line we obtain that such a curve is not constant, therefore, $\{P_s\}_{s\in\hat{U}}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_s(X,Y) = A_sX^3 + B_sX^2Y + C_sX^2 + XY^2 + D_sXY + E_sX + F_sY^2 + G_sY + H_s,$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there are constants $K_2, K_7$ such that,

$$B_s = K_2$$
$$F_s = K_6$$

i.e.,

$$K_2 = \frac{2\left(a(u'(s))^2 + a\right)(au''(s) - 2\alpha_3 + 2u'(s)\alpha_2)}{\left(a(u'(s))^2 + a\right)^2}$$
$$K_7 = -\frac{4b^2(u''(s))^2 - \left(a(u'(s))^2 + a\right)^2\left((u'(s))^2 + 1\right)}{\left(a(u'(s))^2 + a\right)^2}$$

then

$$u''(s) = \frac{a(1 + (u'(s))^2)K_2 - (4u'(s)\alpha_2 - 4\alpha_3)}{2a}$$
$$K_7 = -\frac{4b^2(u''(s))^2 - \left(a(u'(s))^2 + a\right)^2\left((u'(s))^2 + 1\right)}{\left(a(u'(s))^2 + a\right)^2}.$$

Substituting the first equality above into the second we have

$$\begin{aligned} 0 &= a^4(u'(s))^6 + (3a^4 - a^4K_6 - a^2b^2K_2^2)(u'(s))^4 + 8ab^2K_2\alpha_2(u'(s))^3 \\ &+ (3a^4 - 2a^4K_6 - 16b^2\alpha_2^2 - 2a^2b^2K_2^2 - 8ab^2K_2\alpha_3)(u'(s))^2 \\ &+ (32b^2\alpha_2\alpha_3 + 8ab^2K_2\alpha_2)u'(s) + a^4 - a^4K_6 - 16b^2\alpha_3^2 - a^2b^2K_2^2 - 8ab^2K_2\alpha_3, \end{aligned}$$

that is, $u'(s)$ is a root of the non-zero polynomial

$$\begin{aligned} \tilde{R}(Z) &= a^4Z^6 + (3a^4 - a^4K_6 - a^2b^2K_2^2)Z^4 + 8ab^2K_2\alpha_2Z^3 \\ &+ (3a^4 - 2a^4K_6 - 16b^2\alpha_2^2 - 2a^2b^2K_2^2 - 8ab^2K_2\alpha_3)Z^2 \\ &+ (32b^2\alpha_2\alpha_3 + 8ab^2K_2\alpha_2)Z + a^4 - a^4K_6 - 16b^2\alpha_3^2 - a^2b^2K_2^2 - 8ab^2K_2\alpha_3, \end{aligned}$$

and therefore, $u'(s)$ is constant in $\hat{U} \subset \tilde{U} \subset U \subset J$ and consequently $u''(s) = 0$ in $\hat{U} \subset \tilde{U} \subset U \subset J$, a contradiction.

- If $A_s \neq 0$ in $\tilde{U} \subset U \subset J$ and $G_s - (D_s - B_sF_s)F_s = 0$ and $H_s - (E_s - (C_s - A_sF_s)F_s)F_s = 0$ for all $s \in \tilde{U} \subset U \subset J$ we obtain, by claim 5, that holds

$$P_s(X,Y) = (X + F_s)\left(A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + (A_sF_s^2 - C_sF_s + E_s)\right),$$

for all $s \in \tilde{U} \subset U \subset J$. Since $v$ is not a straight line we obtain that the curve $((v'(t))^2, v''(t))$ is non-constant and therefore the factor $(X + F_s)$ does not vanishes on the curve, that is, what vanishes is the factor

$$A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + (A_sF_s^2 - C_sF_s + E_s).$$

By the theory of Plane Algebraic Curves we conclude that these polynomials are multiples by constants. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$A_sX^2 + B_sXY + (C_s - A_sF_s)X + Y^2 + (D_s - B_sF_s)Y + (A_sF_s^2 - C_sF_s + E_s),$$

is constant in the variable $s$, that is, the polynomials are all equal, and therefore, their coefficients are constant, that is, there are constants $K_2, M_6$, with $K_2 \neq 0$ (otherwise $A_s \equiv 0$), such that,

$$B_s = K_2$$
$$D_s - B_sF_S = M_2$$

and isolating $u''(s)$ in the first equality and substituting in the second

$$\begin{aligned} a^4(1+(u'(s))^2)^2M_2 = {} & a^4K_2 + 2a^4(u'(s))^2K_2 + a^4(u'(s))^4K_2 - 4a^3(u'(s))^2\alpha_3 + 4a^3(u'(s))^3\alpha_2 \\ & - 4a^3(u'(s))^4\alpha_3 + 4a^3(u'(s))^5\alpha_2 + 16b^2K_2\alpha_3^2 + a^2b^2K_2^3 + 16b^2(u'(s))^2K_2\alpha_2^2 \\ & + 2a^2b^2(u'(s))^2K_2^3 + a^2b^2(u'(s))^4K_2^3 + 8ab^2K_2^2\alpha_3 + 8ab^2(u'(s))^2K_2^2\alpha_3 \\ & - 8ab^2(u'(s))^3K_2^2\alpha_2 - 32b^2u'(s)K_2\alpha_2\alpha_3 - 8ab^2u'(s)K_2^2\alpha_2 \end{aligned}$$

which rewriting

$$\begin{aligned} 0 = {} & 4a^3\alpha_2(u'(s))^5 + (a^4K_2 + M_2a^4 - 4a^3\alpha_3 + a^2b^2K_2^3)(u'(s))^4 \\ & + (4a^3\alpha_2 - 8ab^2K_2^2\alpha_2)(u'(s))^3 - (8a\alpha_2b^2K_2^2 + 32\alpha_2\alpha_3b^2K_2)u'(s) \\ & + (2a^4K_2 + 2M_2a^4 - 4\alpha_3a^3 + 2a^2b^2K_2^3 + 8\alpha_3ab^2K_2^2 + 16b^2K_2\alpha_2^2)(u'(s))^2 \\ & + M_2a^4 - a^4K_2 + a^2b^2K_2^3 + 8ab^2K_2^2\alpha_3 + 16b^2K_2\alpha_3^2 \end{aligned}$$

i.e., $u'(s)$ is a root of the non-zero polynomial (by claim 6)

$$\begin{aligned} S(Z) = {} & 4a^3\alpha_2Z^5 + (a^4K_2 + M_2a^4 - 4a^3\alpha_3 + a^2b^2K_2^3)Z^4 \\ & + (4a^3\alpha_2 - 8ab^2K_2^2\alpha_2)Z^3 - (8a\alpha_2b^2K_2^2 + 32\alpha_2\alpha_3b^2K_2)Z \\ & + (2a^4K_2 + 2M_2a^4 - 4\alpha_3a^3 + 2a^2b^2K_2^3 + 8\alpha_3ab^2K_2^2 + 16b^2K_2\alpha_2^2)Z^2 \\ & + M_2a^4 - a^4K_2 + a^2b^2K_2^3 + 8ab^2K_2^2\alpha_3 + 16b^2K_2\alpha_3^2 \end{aligned}$$

and therefore, $u'(s)$ is constant in $\tilde{U} \subset U \subset J$ and consequently $u''(s) = 0$ in $\tilde{U} \subset U \subset J$, a contradiction. Thus, with all possibilities verified, the initial assumption that $u, v$ are not straight lines is false, that is, $u$ or $v$ is a straight line in $J$ or $I$ respectively. □

**Theorem 3.1.3.** *Let be $\Psi(X,Y) = aX + bY$, $\alpha \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then:*

*i)* $v(t) = \mu t + \tau$, *for* $\mu, \tau$ *constants, and* $I \subset \mathbb{R}$. *Moreover, if* $\alpha = \lambda(1,0,\mu)$, *with* $\lambda \neq 0$, $u(s) = \zeta s + \eta$ *for* $\zeta, \eta$ *constants and* $J \subset \mathbb{R}$.

*ii)* $v(t) = \mu t + \tau$, *for constants* $\mu, \tau$, *and* $I \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, 0, \alpha_3)$, *with* $\alpha_3 \neq \mu\alpha_1$, *we have*

$$u(s) = D + \frac{a(1+\mu^2)}{2(\alpha_3 - \alpha_1\mu)} \ln \left| \sec\left( \frac{2(\alpha_3 - \alpha_1\mu)(s-C)}{a\sqrt{1+\mu^2}} + \arctan\left( \frac{F}{\sqrt{1+\mu^2}} \right) \right) \right|,$$

*for constants C, D, F and* $J \subset \mathbb{R} - \left\{ \frac{a\sqrt{1+\mu^2}\left((2k+1)\pi - \arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4(\alpha_3-\alpha_1\mu)} + C \in \mathbb{R} \;\middle|\; k \in \mathbb{Z} \right\}$.

*iii)* $v(t) = \mu t + \tau$, *for* $\mu, \tau$ *constants, and* $I \subset \mathbb{R}$. *Moreover, if* $\alpha = (\lambda, \alpha_2, \mu\lambda)$, *with* $\alpha_2 \neq 0$, *we have* $u(s) = D$ *or*

$$u(s) = D \pm \int_{s_0}^{s} \sqrt{\frac{(1+\mu^2)F}{(1+F+\mu^2)e^{4\frac{\alpha_2}{a}(r-C)} - F}}\, dr.$$

*for constants* $C, D$, $F > 0$ *and* $J \subset \left( \frac{a}{4\alpha_2} \ln\left( \frac{F}{1+F+\mu^2} \right) + C, +\infty \right)$.

*iv)* $v(t) = \mu t + \tau$, *for constants* $\mu, \tau$, *and* $I \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, *with* $\alpha_2 \neq 0$ *and* $\alpha_3 \neq \mu\alpha_1$, *the function u satisfies the differential equation*

$$u''(s) = \frac{2(1+(u'(s))^2+\mu^2)(-\frac{\alpha_1}{a}\mu - \frac{\alpha_2}{a}u'(s) + \frac{\alpha_3}{a})}{1+\mu^2}.$$

*v)* $u(s) = \mu s + \tau$, *for* $\mu, \tau$ *constants, and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = \lambda(0,1,\mu)$, *for* $\lambda \neq 0$, $v(t) = \zeta t + \eta$ *for* $\zeta, \eta$ *constants and* $I \subset \mathbb{R}$.

*vi)* $u(s) = \mu s + \tau$, *for* $\mu, \tau$ *constants, and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = (0, \alpha_2, \alpha_3)$, *with* $\alpha_3 \neq \mu\alpha_2$, *we have*

$$v(t) = D + \frac{a(1+\mu^2)}{2(\alpha_3 - \alpha_2\mu)} \ln \left| \sec\left( \frac{2(\alpha_3 - \alpha_2\mu)(t-C)}{a\sqrt{1+\mu^2}} + \arctan\left( \frac{F}{\sqrt{1+\mu^2}} \right) \right) \right|,$$

*for constants C, D, F and* $I \subset \mathbb{R} - \left\{ \frac{a\sqrt{1+\mu^2}\left((2k+1)\pi - \arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4(\alpha_3-\alpha_2\mu)} + C \in \mathbb{R} \;\middle|\; k \in \mathbb{Z} \right\}$.

*vii)* $u(s) = \mu s + \tau$, *for* $\mu, \tau$ *constants, and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, \lambda, \mu\lambda)$, *with* $\alpha_1 \neq 0$, *we have* $v(t) = D$ *or*

$$v(t) = D \pm \int_{t_0}^{t} \sqrt{\frac{(1+\mu^2)F}{(1+F+\mu^2)e^{4\frac{\alpha_1}{a}(r-C)} - F}}\, dr.$$

*for constants* $C, D$, $F > 0$ *and* $I \subset \left( \frac{a}{4\alpha_1} \ln\left( \frac{F}{1+F+\mu^2} \right) + C, +\infty \right)$.

*viii)* $u(s) = \mu s + \tau$, *for constants* $\mu, \tau$, *and* $J \subset \mathbb{R}$. *Moreover, if* $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, *with* $\alpha_1 \neq 0$ *and* $\alpha_3 \neq \mu\alpha_2$, *the function* $v$ *satisfies the differential equation*

$$v''(t) = \frac{2(1+(v'(t))^2+\mu^2)(-\frac{\alpha_1}{a}v'(t) - \frac{\alpha_2}{a}\mu + \frac{\alpha_3}{a})}{1+\mu^2}.$$

**Proof:** Suppose $\psi$ is a Translating Soliton with $\Psi(X,Y) = aX + bY$ and $\alpha \in \mathbb{R}^3$ non-zero. By Proposition 3.1.6 the functions $u, v$ satisfy the equation

$$\begin{aligned} 0 = a((v'(t))^2 + (u'(s))^2+1)^{\frac{1}{2}}(v''(t)(1+(u'(s))^2) + u''(s)(1+(v'(t))^2)) + 2bv''(t)u''(s) \\ -2((v'(t))^2+(u'(s))^2+1)^{\frac{3}{2}}(-\alpha_1 v'(t) - \alpha_2 u'(s) + \alpha_3). \end{aligned} \tag{3.30}$$

By Proposition 3.1.7 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for constants $\mu, \tau$. Suppose the first situation. We obtain from (3.30)

$$u''(s) = \frac{2(1+(u'(s))^2+\mu^2)(-\frac{\alpha_1}{a}\mu - \frac{\alpha_2}{a}u'(s) + \frac{\alpha_3}{a})}{1+\mu^2},$$

and setting $w(s) = u'(s)$

$$w'(s) = \frac{2(1+w^2(s)+\mu^2)(-\frac{\alpha_1}{a}\mu - \frac{\alpha_2}{a}w(s) + \frac{\alpha_3}{a})}{1+\mu^2},$$

and thus, the proof is identical to that of Theorem 3.1.2 just by replacing, in (3.18), $\alpha_i$ by $\dfrac{\alpha_i}{a}$ for $i = 1,2,3$. □

**Corollary 3.1.2.** *Assume the conditions of Theorem* 3.1.3. *Then:*

*i) If* $u$ *is a solution of the differential equation* (3.14), *we have*

$$u(s) = \frac{\alpha_3 - \alpha_1\mu}{\alpha_2}s + \eta,$$

*for a constant* $\eta$ *and* $J \subset \mathbb{R}$, *or,* $\mathcal{I}(s, u'(s)) = 0$ *where*

$$\begin{aligned} \mathcal{I}(X,Y) = {} & \frac{a\alpha_2\sqrt{1+\mu^2}\ln\left(\frac{(1+Y^2+\mu^2)(-\alpha_1\mu-\alpha_2F+\alpha_3)^2}{(1+F^2+\mu^2)(-\alpha_1\mu-\alpha_2Y+\alpha_3)^2}\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))} \\ & + \frac{2a(\alpha_3-\alpha_1\mu)\left(\arctan\left(\frac{Y}{\sqrt{1+\mu^2}}\right) - \arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_1\mu)^2+\alpha_2^2(1+\mu^2))} - \frac{X-C}{1+\mu^2}, \end{aligned}$$

*for constants* $C$ *and* $F \neq \frac{\alpha_3 - \mu\alpha_1}{\alpha_2}$.

*ii) If* $v$ *is a solution of the differential equation* (3.15), *we have*

$$v(t) = \frac{\alpha_3 - \alpha_2\mu}{\alpha_1}t + \eta,$$

*for a constant $\eta$ and $I \subset \mathbb{R}$, or, $\tilde{\mathcal{I}}(t, v'(t)) = 0$ where*

$$\tilde{\mathcal{I}}(X,Y) = \frac{a\alpha_1\sqrt{1+\mu^2}\ln\left(\frac{(1+Y^2+\mu^2)(-\alpha_2\mu-\alpha_1F+\alpha_3)^2}{(1+F^2+\mu^2)(-\alpha_2\mu-\alpha_1Y+\alpha_3)^2}\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_2\mu)^2+\alpha_1^2(1+\mu^2))} + \frac{2a(\alpha_3-\alpha_2\mu)\left(\arctan\left(\frac{Y}{\sqrt{1+\mu^2}}\right)-\arctan\left(\frac{F}{\sqrt{1+\mu^2}}\right)\right)}{4\sqrt{1+\mu^2}((\alpha_3-\alpha_2\mu)^2+\alpha_1^2(1+\mu^2))} - \frac{X-C}{1+\mu^2},$$

*for constants $C$ and $F \neq \frac{\alpha_3-\mu\alpha_2}{\alpha_1}$.*

**Proof:** Identical to Corollary 3.1.1 only changing, in (3.19), $\alpha_i$ by $\dfrac{\alpha_i}{a}$ for $i = 1,2,3$. $\square$

### 3.1.6 *Translating Solitons with* $\Psi(X,Y) = 4X^2 - 2Y$

**Lemma 3.1.6.** *Let be $\delta_1, \delta_2 > 0$, $M$ constant and $\Phi : (M-\delta_1, M+\delta_2) \to \mathbb{R}$ a continuous function such that $\Phi(x) = 0$ if and only if $x = M$. Then $w \in C^1$ is the maximal solution of*

$$\begin{aligned} (w'(s))^2 &= \Phi(w(s)) \\ w(s_0) &= w_0 \end{aligned} \tag{3.31}$$

*if and only if it is the maximal solution of*

$$\begin{aligned} w'(s) &= \sqrt{\Phi(w(s))} \\ w(s_0) &= w_0 \end{aligned} \tag{3.32}$$

*or*

$$\begin{aligned} w'(s) &= -\sqrt{\Phi(w(s))} \\ w(s_0) &= w_0. \end{aligned} \tag{3.33}$$

**Proof:** If $u$ is a maximal (non-constant) solution of (3.31), in the maximal interval $(a,b)$, there exists $s_0$ in which $u'(s_0) \neq 0$ and $u(s_0) \neq M$. Therefore, there exists a maximal interval $(s_0-\varepsilon_1, s_0+\varepsilon_2) \subset (a,b)$, with $\varepsilon_1, \varepsilon_2 > 0$, such that $u' \neq 0$. We have two situations:
If $u' > 0$ in $(s_0-\varepsilon_1, s_0+\varepsilon_2) \subset (a,b)$ take $\overline{u}$ maximal solution of the initial value problem

$$\begin{aligned} w'(s) &= \sqrt{\Phi(w(s))} \\ w(s_0) &= u(s_0) \end{aligned} \tag{3.34}$$

in the maximal interval $(s_0-\overline{\varepsilon}_1, s_0+\overline{\varepsilon}_2)$ with $\overline{\varepsilon}_1, \overline{\varepsilon}_2 > 0$. Observe that $u$ is the maximal solution of

$$\begin{aligned} (w'(s))^2 &= \Phi(w(s)) \\ w(s_0) &= u(s_0). \end{aligned} \tag{3.35}$$

Since $\overline{u}$ a solution of (3.35) we obtain that the interval $(s_0-\overline{\varepsilon}_1, s_0+\overline{\varepsilon}_2) \subset (a,b)$. Since $u$, restricted to the interval $(s_0-\varepsilon_1, s_0+\varepsilon_2)$, is a solution of (3.34), it follows from the Existence and Uniqueness

Theorem that $(s_0-\varepsilon_1,s_0+\varepsilon_2)\subseteq(s_0-\overline{\varepsilon}_1,s_0+\overline{\varepsilon}_2)$ and $u=\overline{u}$ in $(s_0-\varepsilon_1,s_0+\varepsilon_2)$. Suppose $\varepsilon_2<\overline{\varepsilon}_2$ (that is, $s_0+\varepsilon_2<s_0+\overline{\varepsilon}_2$). It follows from the maximality of $\varepsilon_2$ that

$$\overline{u}'(s_0+\varepsilon_2)=u'(s_0+\varepsilon_2)=0$$

therefore $\sqrt{\Phi(\overline{u}(s_0+\varepsilon_2))}=0$. By the hypothesis, $\overline{u}(s_0+\varepsilon_2)=M$. By existence and uniqueness $\overline{u}\equiv M$ consequently, since $u=\overline{u}=M$, we conclude $u'=0$ in $(s_0-\varepsilon_1,s_0+\varepsilon_2)$ , a contradiction. Thus $\varepsilon_2=\overline{\varepsilon}_2$. Proceeding similarly we obtain $\overline{\varepsilon}_1=\varepsilon_1$. Now suppose $s_0+\overline{\varepsilon}_2<b$. It follows from the maximality of $\varepsilon_2$ that

$$\lim_{s\to(s_0+\varepsilon_2)^-}\sqrt{\Phi(\overline{u}(s))}=\lim_{s\to(s_0+\varepsilon_2)^-}\overline{u}'(s)=\lim_{s\to(s_0+\varepsilon_2)^-}u'(s)=0$$

and then

$$\lim_{s\to(s_0+\varepsilon_2)^-}(s,\overline{u}(s))=(s_0+\varepsilon_2,M)$$

a contradiction because, by the Theory of Ordinary Differential Equations, we would have that the graph $(s,\overline{u}(s))$ would tend to the boundary of $\Omega=(s_0-\varepsilon_1,s_0+\varepsilon_2)\times(M-\delta_1,M+\delta_2)$ when $s\to s_0+\varepsilon_2$. Thus $s_0+\overline{\varepsilon}_2=b$. Proceeding in a similar way we obtain $s_0-\overline{\varepsilon}_1=a$.

In the case where $u'<0$ in $(s_0-\varepsilon_1,s_0+\varepsilon_2)\subset(a,b)$ we proceed in an analogous way by replacing (3.34) with

$$\begin{aligned}w'(s)&=-\sqrt{\Phi(w(s))}\\ w(s_0)&=u(s_0)\end{aligned}$$

Conversely, if $\overline{u}$ is a maximal solution of (3.32) or (3.33) on the maximal interval $(\overline{a},\overline{b})$ then $\overline{u}$ is a maximal solution of (3.31) on the maximal interval $(\overline{a},\overline{b})$. Indeed, suppose it is not a maximal solution of (3.31). Then there exists a solution $\tilde{u}$ that extends $\overline{u}$, since $\overline{u}$ verifies (3.31). However, by the first part of the proof, $\tilde{u}$ would be a maximal solution of (3.32) or (3.33) that would extend $\overline{u}$, a contradiction. □

**Proposition 3.1.8.** *Let be $\Psi(X,Y)=4X^2-2Y$ and $\alpha\in\mathbb{R}^3$ non-zero. Then $\psi:\Omega\subset\mathbb{R}^2\to\mathbb{R}^3$ given by $\psi(t,s)=(t,s,v(t)+u(s))$, where $u\in C^2(J,\mathbb{R})$ and $v\in C^2(I,\mathbb{R})$, is a Translating Soliton if and only if*

$$\begin{aligned}0=&(v''(t)(1+(u'(s))^2)+u''(s)(1+(v'(t))^2))^2-2v''(t)u''(s)((v'(t))^2+(u'(s))^2+1)\\ &-((v'(t))^2+(u'(s))^2+1)^{\frac{5}{2}}(-v'(t)\alpha_1-u'(s)\alpha_2+\alpha_3).\end{aligned}$$

**Proof:** By the definition of Translating Soliton,

$$\Psi(H,K)(t,s)=\langle\alpha,N(t,s)\rangle,$$

that is,

$$4H^2(t,s)-2K(t,s)=\langle\alpha,N(t,s)\rangle,$$

which by Proposition 3.1.1 and (3.6) we obtain

$$\begin{aligned}0 &= (v''(t)(1+(u'(s))^2)+u''(s)(1+(v'(t))^2))^2-2v''(t)u''(s)((v'(t))^2+(u'(s))^2+1)\\&-((v'(t))^2+(u'(s))^2+1)^{\frac{5}{2}}(-v'(t)\alpha_1-u'(s)\alpha_2+\alpha_3).\end{aligned}$$

□

**Proposition 3.1.9.** *Let be $\Psi(X,Y)=4X^2-2Y$, $\alpha=(0,\alpha_2,\alpha_3)\in\mathbb{R}^3$ non-zero and $\psi:\Omega\subset\mathbb{R}^2\to\mathbb{R}^3$ given by $\psi(t,s)=(t,s,v(t)+u(s))$, where $u\in C^2(J,\mathbb{R})$ and $v\in C^2(I,\mathbb{R})$, a Translating Soliton. Then $v(t)=\mu t+\tau$ in $I\subset\mathbb{R}$ or $u(s)=\mu s+\tau$ in $J\subset\mathbb{R}$, for $\mu,\tau$ constants.*

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y)=4X^2-2Y$. By Proposition 3.1.8 the functions $u,v$ satisfy the equation

$$\begin{aligned}&(v''(t)(1+(u'(s))^2)+u''(s)(1+(v'(t))^2))^2-2v''(t)u''(s)((v'(t))^2+(u'(s))^2+1)\\=&((v'(t))^2+(u'(s))^2+1)^{\frac{5}{2}}(\alpha_3-u'(s)\alpha_2).\end{aligned}\tag{3.36}$$

Suppose that $u,v$ are not straight lines, that is, there exist open sets $U\subset J$ and $V\subset I$ such that $u''(s)\neq 0$ in $U$ and $v''(t)\neq 0$ in $V$. Since $\alpha_1=0$ we have from (3.36), squaring,

$$\begin{aligned}&\left((v''(t)(1+(u'(s))^2)+u''(s)(1+(v'(t))^2))^2-2v''(t)u''(s)((v'(t))^2+(u'(s))^2+1)\right)^2\\=&((v'(t))^2+(u'(s))^2+1)^5(\alpha_3-u'(s)\alpha_2)^2\end{aligned}$$

or equivalently,

$$\begin{aligned}0=&-\frac{(\alpha_3-u'(s)\alpha_2)^2}{((u'(s))^2+1)^4}(v'(t))^{10}+\frac{(u''(s))^4-5\left((u'(s))^2+1\right)(\alpha_3-u'(s)\alpha_2)^2}{((u'(s))^2+1)^4}(v'(t))^8\\&-\frac{4(u''(s))^3(u'(s))^2}{((u'(s))^2+1)^4}(v'(t))^6v''(t)+\frac{4(u''(s))^4-10(\alpha_3-u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^2}{((u'(s))^2+1)^4}(v'(t))^6\\&+\frac{2(u'(s))^2\left((u'(s))^2+1\right)^2+4(u''(s))^2(u'(s))^4}{((u'(s))^2+1)^4}(v'(t))^4(v''(t))^2\\&-\frac{4(u''(t))^2\left(2u''(s)-2u''(s)\left((u'(s))^2+1\right)\right)}{((u'(s))^2+1)^4}(v'(t))^4v''(t)\\&+\frac{6(u''(s))^4-10(\alpha_3-u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^3}{((u'(s))^2+1)^4}(v'(t))^4-\frac{4u''(s)(u'(s))^2}{((u'(s))^2+1)^2}(v'(t))^2(v''(t))^3\\&+\frac{4(u''(s))^2}{((u'(s))^2+1)^2}(v'(t))^2(v''(t))^2+\frac{4(u''(s))^3(u'(s))^2}{((u'(s))^2+1)^4}(v'(t))^2v''(t)\\&+\frac{4(u''(s))^4-5(\alpha_3-u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^4}{((u'(s))^2+1)^4}(v'(t))^2+(v''(t))^4\\&+\frac{2(u''(s))^2\left((u'(s))^2+1\right)^2}{((u'(s))^2+1)^4}(v''(t))^2+\frac{(u''(s))^4-(\alpha_3-u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^5}{((u'(s))^2+1)^4}.\end{aligned}$$

Define the family of polynomials

$$P_s(X,Y) = A_sX^5 + B_sX^4 + C_sX^3Y + D_sX^3 + E_sX^2Y^2 + F_sX^2Y + G_sX^2 + H_sXY^3 + I_sXY^2 + J_sXY \\ + K_sX + Y^4 + L_sY^2 + M_s,$$

where,

$$\begin{aligned}
A_s &= -\frac{(\alpha_3 - u'(s)\alpha_2)^2}{((u'(s))^2+1)^4}\\
B_s &= \frac{(u''(s))^4 - 5\left((u'(s))^2+1\right)(\alpha_3 - u'(s)\alpha_2)^2}{((u'(s))^2+1)^4}\\
C_s &= -\frac{4(u''(s))^3(u'(s))^2}{((u'(s))^2+1)^4}\\
D_s &= \frac{4(u''(s))^4 - 10(\alpha_3 - u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^2}{((u'(s))^2+1)^4}\\
E_s &= \frac{2(u'(s))^2\left((u'(s))^2+1\right)^2 + 4(u''(s))^2(u'(s))^4}{((u'(s))^2+1)^4}\\
F_s &= -\frac{4(u''(t))^2\left(2u''(s) - 2u''(s)\left((u'(s))^2+1\right)\right)}{((u'(s))^2+1)^4}\\
G_s &= \frac{6(u''(s))^4 - 10(\alpha_3 - u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^3}{((u'(s))^2+1)^4}\\
H_s &= -\frac{4u''(s)(u'(s))^2}{((u'(s))^2+1)^2}\\
I_s &= \frac{4(u''(s))^2}{((u'(s))^2+1)^2}\\
J_s &= \frac{4(u''(s))^3(u'(s))^2}{((u'(s))^2+1)^4}\\
K_s &= \frac{4(u''(s))^4 - 5(\alpha_3 - u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^4}{((u'(s))^2+1)^4}\\
L_s &= \frac{2(u''(s))^2\left((u'(s))^2+1\right)^2}{((u'(s))^2+1)^4}\\
M_s &= \frac{(u''(s))^4 - (\alpha_3 - u'(s)\alpha_2)^2\left((u'(s))^2+1\right)^5}{((u'(s))^2+1)^4},
\end{aligned}$$

and notice that the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$.

**Claim:** The polynomials $\{P_s\}_{s\in J}$ are irreducible. Indeed, if $P_s$ is reducible there are two possibilities. In the first, there are polynomials of degree 1 and degree 4 such that

$$P_s(X,Y) = (a + bX + dY)(f + gX + hY + iXY + jX^2 + kY^2 + lXY^2 + mX^2Y + nX^3 + oY^3) \\ + (a + bX + dY)(pXY^3 + qX^2Y^2 + rX^3Y + tX^4 + wY^4)$$

that is,

$$
\begin{aligned}
0 = P_s(X,Y) &- (a+bX+dY)(f+gX+hY+iXY+jX^2+kY^2+lXY^2+mX^2Y+nX^3+oY^3) \\
&- (a+bX+dY)(pXY^3+qX^2Y^2+rX^3Y+tX^4+wY^4)
\end{aligned}
$$

which reorganizing

$$
\begin{aligned}
0 =&(A_s-bt)X^5-(br+dt)X^4Y+(B_s-bn-at)X^4-(bq+dr)X^3Y^2+(C_s-bm-dn-ar)X^3Y \\
&+(D_s-bj-an)X^3-(bp+dq)X^2Y^3+(E_s-bl-dm-aq)X^2Y^2+(F_s-bi-am-dj)X^2Y \\
&+(G_s-bg-aj)X^2-(dp+bw)XY^4+(H_s-dl-ap-bo)XY^3+(I_s-di-al-bk)XY^2 \\
&+(J_s-ai-bh-dg)XY+(K_s-ag-bf)X-dwY^5+(1-aw-do)Y^4-(dk+ao)Y^3 \\
&+(L_s-ak-dh)Y^2-(ah+df)Y+M_s-af
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{aligned}
0 &=A_s-bt & 0 &=E_s-bl-dm-aq & 0 &=K_s-ag-bf \\
0 &=br+dt & 0 &=F_s-bi-am-dj & 0 &=dw \\
0 &=B_s-bn-at & 0 &=G_s-bg-aj & 0 &=1-aw-do \\
0 &=bq+dr & 0 &=dp+bw & 0 &=dk+ao \\
0 &=C_s-bm-dn-ar & 0 &=H_s-dl-ap-bo & 0 &=L_s-ak-dh \\
0 &=D_s-bj-an & 0 &=I_s-di-al-bk & 0 &=ah+df \\
0 &=bp+dq & 0 &=J_s-ai-bh-dg & 0 &=M_s-af.
\end{aligned} \tag{3.37}
$$

By the second equality of the third column of $(3.37)$ we have either $d=0$ or $w=0$. Suppose that $d=0$. In this case, since $b\neq 0$ (otherwise we would not have factorization), we obtain by the fourth equality of the second column that $w=0$ which substituting in the third equality of the third column we have $1=0$, a contradiction. Then $w=0$ and thus from $(3.37)$ we have

$$
\begin{aligned}
0 &=A_s-bt & 0 &=E_s-bl-dm-aq & & \\
0 &=br+dt & 0 &=F_s-bi-am-dj & 0 &=K_s-ag-bf \\
0 &=B_s-bn-at & 0 &=G_s-bg-aj & 0 &=1-do \\
0 &=bq+dr & 0 &=dp & 0 &=dk+ao \\
0 &=C_s-bm-dn-ar & 0 &=H_s-dl-ap-bo & 0 &=L_s-ak-dh \\
0 &=D_s-bj-an & 0 &=I_s-di-al-bk & 0 &=ah+df \\
0 &=bp+dq & 0 &=J_s-ai-bh-dg & 0 &=M_s-af.
\end{aligned} \tag{3.38}
$$

By line 16 of (3.38) we have $d = \frac{1}{o}$ and consequently, by line 11, $p = 0$. Therefore,

$$\begin{aligned}
0 &= A_s - bt & 0 &= \frac{q}{o} & 0 &= J_s - ai - bh - \frac{g}{o}\\
0 &= br + \frac{t}{o} & 0 &= E_s - bl - \frac{m}{o} - aq & 0 &= K_s - ag - bf\\
0 &= B_s - bn - at & 0 &= F_s - bi - am - \frac{j}{o} & 0 &= \frac{k}{o} + ao\\
0 &= bq + \frac{r}{o} & 0 &= G_s - bg - aj & 0 &= L_s - ak - \frac{h}{o}\\
0 &= C_s - bm - \frac{n}{o} - ar & 0 &= H_s - \frac{l}{o} - bo & 0 &= ah + \frac{f}{o}\\
0 &= D_s - bj - an & 0 &= I_s - \frac{i}{o} - al - bk & 0 &= M_s - af.
\end{aligned} \tag{3.39}$$

By the first equality of the second column of (3.39) we obtain $q = 0$ and then, by the fourth equality, $r = 0$ which consequently, by the second equality, $t = 0$ and we have no factorization.

In the second possibility there are polynomials of degree 2 and degree 3 such that

$$P_s(X,Y) = (a+bX+dY+eXY+fX^2+gY^2)(h+iX+jY+kXY+lX^2+mY^2+nXY^2+oX^2Y+pX^3+qY^3)$$

and proceeding in a similar way to what was done previously we will see that this case also does not occur and thus we conclude the claim.

By the claim the polynomials $\{P_s\}_{s\in J}$ are irreducible. As observed previously, the curve $((v'(t))^2, v''(t))$ is contained in the intersection of the roots of $\{P_s\}_{s\in J}$ thus by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_s$ are multiples by constants. Since $v$ is not a straight line we obtain that such a curve is non-constant, therefore, $\{P_s\}_{s\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$\begin{aligned}
P_s(X,Y) = {} & A_sX^5 + B_sX^4 + C_sX^3Y + D_sX^3 + E_sX^2Y^2 + F_sX^2Y + G_sX^2 + H_sXY^3 + I_sXY^2 + J_sXY\\
& + K_sX + Y^4 + L_sY^2 + M_s,
\end{aligned}$$

is constant in the variable $s$, that is, the polynomials $P_s$ are all equal. Therefore, their coefficients are constant, that is, there exists a constant $K_1$ such that,

$$A_s = K_1$$

that is,

$$K_1 = -\frac{(\alpha_3 - u'(s)\alpha_2)^2}{((u'(s))^2+1)^4}$$

then

$$0 = -K_1(u'(s))^8 - 4K_1(u'(s))^6 - 6K_1(u'(s))^4 - (\alpha_2^2 + 4K_1)(u'(s))^2 + 2\alpha_2\alpha_3 u'(s) - (\alpha_3^2 + K_1),$$

i.e., $u'(s)$ is a root of the non-zero polynomial (since $(0, \alpha_2, \alpha_3) = \alpha \neq 0$)

$$Q(Z) = -K_1Z^8 - 4K_1Z^6 - 6K_1Z^4 - (\alpha_2^2 + 4K_1)Z^2 + 2\alpha_2\alpha_3 Z - (\alpha_3^2 + K_1)$$

Therefore, $u'(s)$ is constant in $J$ and consequently $u''(s) = 0$ in $J$, a contradiction. Thus, the initial assumption that $u, v$ are not straight lines is false, that is, $u$ or $v$ is a straight line in $J$ or $I$ respectively.$\square$

**Proposition 3.1.10.** *Let be $\Psi(X,Y) = 4X^2 - 2Y$, $\alpha = (\alpha_1, 0, \alpha_3) \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for $\mu, \tau$ constants.*

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y) = 4X^2 - 2Y$. By Proposition 3.1.8 the functions $u, v$ satisfy the equation

$$\begin{aligned}&(v''(t)(1+(u'(s))^2) + u''(s)(1+(v'(t))^2))^2 - 2v''(t)u''(s)((v'(t))^2 + (u'(s))^2 + 1)\\ =&((v'(t))^2 + (u'(s))^2 + 1)^{\frac{5}{2}}(\alpha_3 - v'(t)\alpha_1).\end{aligned} \tag{3.40}$$

Observe that this equation is symmetric to the equation (3.36) of Proposition 3.1.9 by inverting $\alpha_2$ with $\alpha_1$ and $u$ with $v$. Thus the proof is analogous. □

**Theorem 3.1.4.** *Let be $\Psi(X,Y) = 4X^2 - 2Y$, $\alpha = (0, \alpha_2, \alpha_3) \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t, s, v(t) + u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then:*

*i)* *$v(t) = \mu t + \tau$, for constants $\mu, \tau$, and $I \subset \mathbb{R}$. Moreover, the function $u$ satisfies one of the* ***two*** *differential equations*

$$u''(s) = \pm \frac{\sqrt{\alpha_3 - \alpha_2 u'(s)}(1 + (u'(s))^2 + \mu^2)^{\frac{5}{4}}}{1+\mu^2}. \tag{3.41}$$

*ii)* *$u(s) = \mu s + \tau$, for constants $\mu, \tau$, and $J \subset \mathbb{R}$. Moreover, $\alpha_3 - \mu\alpha_2 \geqslant 0$ and the function $v$ satisfies one of the* ***two*** *differential equations*

$$v''(t) = \pm \frac{\sqrt{\alpha_3 - \mu\alpha_2}(1 + (v'(t))^2 + \mu^2)^{\frac{5}{4}}}{1+\mu^2}. \tag{3.42}$$

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y) = 4X^2 - 2Y$ and $\alpha = (0, \alpha_2, \alpha_3) \in \mathbb{R}^3$ non-zero. By Proposition 3.1.8 the functions $u, v$ satisfy the equation

$$\begin{aligned}&(v''(t)(1+(u'(s))^2) + u''(s)(1+(v'(t))^2))^2 - 2v''(t)u''(s)((v'(t))^2 + (u'(s))^2 + 1)\\ =&((v'(t))^2 + (u'(s))^2 + 1)^{\frac{5}{2}}(\alpha_3 - u'(s)\alpha_2).\end{aligned} \tag{3.43}$$

By Proposition 3.1.9 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for constants $\mu, \tau$.

**-** *i)***:** Assume the first situation. We obtain from (3.43)

$$(u''(s))^2 = \frac{(1 + (u'(s))^2 + \mu^2)^{\frac{5}{2}}(\alpha_3 - \alpha_2 u'(s))}{(1+\mu^2)^2} \tag{3.44}$$

and setting $w(s) = u'(s)$

$$(w'(s))^2 = \frac{(1 + w^2(s) + \mu^2)^{\frac{5}{2}}(\alpha_3 - \alpha_2 w(s))}{(1+\mu^2)^2}. \tag{3.45}$$

If $\alpha_2 \neq 0$ (the case $\alpha_2 = 0$ is analogous), we obtain from (3.45) that if $\alpha_3 - \alpha_2 w(s) < 0$ we have no solution (since the right-hand side will be negative). If $\alpha_3 - \alpha_2 w(s) \geqslant 0$, by Lemma 3.1.6, with $\Phi(x) = \frac{(\alpha_3-\alpha_2 x)(1+x^2+\mu^2)^{\frac{5}{2}}}{(1+\mu^2)^2}$ and $M = \frac{\alpha_3}{\alpha_2}$, we can take the square root of this differential equation. Thus,

$$w'(s) = \pm\frac{\sqrt{\alpha_3 - \alpha_2 w(s)}(1+w^2(s)+\mu^2)^{\frac{5}{4}}}{1+\mu^2}, \tag{3.46}$$

that is,

$$u''(s) = \pm\frac{\sqrt{\alpha_3 - \alpha_2 u'(s)}(1+(u'(s))^2+\mu^2)^{\frac{5}{4}}}{1+\mu^2}.$$

**-** *ii)***:** Suppose the second situation. We obtain from (3.43)

$$(v''(t))^2 = \frac{(1+(v'(t))^2+\mu^2)^{\frac{5}{2}}(\alpha_3-\mu\alpha_2)}{(1+\mu^2)^2} \tag{3.47}$$

and setting $W(t) = v'(t)$

$$(W'(t))^2 = \frac{(1+W^2(t)+\mu^2)^{\frac{5}{2}}(\alpha_3-\mu\alpha_2)}{(1+\mu^2)^2}. \tag{3.48}$$

We obtain from (3.48) that if $\alpha_3 - \mu\alpha_2 < 0$ we have no solution (since the right-hand side will be negative). If $\alpha_3 - \mu\alpha_2 \geqslant 0$ we can take the square root of this differential equation. Thus,

$$W'(t) = \pm\frac{\sqrt{\alpha_3-\mu\alpha_2}(1+W^2(t)+\mu^2)^{\frac{5}{4}}}{1+\mu^2},$$

that is,

$$v''(t) = \pm\frac{\sqrt{\alpha_3-\mu\alpha_2}(1+(v'(t))^2+\mu^2)^{\frac{5}{4}}}{1+\mu^2}.$$

$\square$

**Corollary 3.1.3.** *Assume the conditions of Theorem* 3.1.4. *If* $\alpha_3 = 0$ *the solution of the differential equation* (3.41) *is either* $u(s) = D$ *or one of the* ***two*** *functions*

$$u(s) = D - \operatorname{sgn}(\alpha_2)\int_{s_0}^{s}\sqrt{\frac{(1+\mu^2)\left(M_{F,\mu,\alpha_2} \mp \frac{\alpha_2(r-C)}{2}\right)^4}{\alpha_2^2 - \left(M_{F,\mu,\alpha_2} \mp \frac{\alpha_2(r-C)}{2}\right)^4}}dr$$

*for constants* $C, D$ *and* $F > 0$ *where*

$$M_{F,\mu,\alpha_2} = \frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}}.$$

*Furthermore, the interval of definition is given by*

$$\mp\operatorname{sgn}(\alpha_2)s \in \left(\operatorname{sgn}(\alpha_2)\left(\mp C - \frac{2M_{F,\mu,\alpha_2}}{\alpha_2}\right), \operatorname{sgn}(\alpha_2)\left(\mp C + \frac{2\left(\sqrt{|\alpha_2|} - M_{F,\mu,\alpha_2}\right)}{\alpha_2}\right)\right)$$

*and the convention is that the sign* $\mp$ *must be taken respecting its position, that is, superior with superior and inferior with inferior.*

**Proof:** Setting $w(s) = u'(s)$ we obtain

$$w'(s) = \pm \frac{\sqrt{-\alpha_2 w(s)}(1 + w^2(s) + \mu^2)^{\frac{5}{4}}}{1 + \mu^2}.$$

Observe that $w \equiv 0$ is a solution of this differential equation (and consequently $u$ is constant). By the Existence and Uniqueness Theorem any other solution does not reach zero. Then,

$$\frac{w'(s)}{\sqrt{-\alpha_2 w(s)}(1 + w^2(s) + \mu^2)^{\frac{5}{4}}} = \pm \frac{1}{1 + \mu^2}$$

consequently

$$\int_{s_0}^{s} \frac{w'(r)}{\sqrt{-\alpha_2 w(r)}(1 + w^2(r) + \mu^2)^{\frac{5}{4}}} dr = \pm \frac{1}{1 + \mu^2} \int_{s_0}^{s} dr,$$

and by the Change of Variables Theorem

$$\int_{w(s_0)}^{w(s)} \frac{dl}{\sqrt{-\alpha_2 l}(1 + l^2 + \mu^2)^{\frac{5}{4}}} = \pm \frac{1}{1 + \mu^2} \int_{s_0}^{s} dr.$$

Thus,

$$-\frac{2}{\alpha_2(1 + \mu^2)} \left( \frac{\sqrt{-\alpha_2 w(s)}}{\sqrt[4]{1 + w^2(s) + \mu^2}} - \frac{\sqrt{-\alpha_2 w(s_0)}}{\sqrt[4]{1 + w^2(s_0) + \mu^2}} \right) = \pm \frac{s - s_0}{1 + \mu^2}$$

i.e.,

$$\frac{\sqrt{-\alpha_2 w(s)}}{\sqrt[4]{1 + w^2(s) + \mu^2}} - \frac{\sqrt{-\alpha_2 w(s_0)}}{\sqrt[4]{1 + w^2(s_0) + \mu^2}} = \mp \frac{\alpha_2(s - s_0)}{2} \tag{3.49}$$

and setting $M = \frac{\sqrt{-\alpha_2 w(s_0)}}{\sqrt[4]{1+w^2(s_0)+\mu^2}}$

$$\frac{\alpha_2^2 w^2(s)}{1 + w^2(s) + \mu^2} = \left( M \mp \frac{\alpha_2(s - s_0)}{2} \right)^4$$

that is,

$$0 = \left( \alpha_2^2 - \left( M \mp \frac{\alpha_2(s - s_0)}{2} \right)^4 \right) w^2(s) - (1 + \mu^2) \left( M \mp \frac{\alpha_2(s - s_0)}{2} \right)^4. \tag{3.50}$$

Notice that $\alpha_2^2 - \left( M \mp \frac{\alpha_2(s-s_0)}{2} \right)^4 > 0$. Indeed, if there exists $\tilde{s}$ such that $\alpha_2^2 - \left( M \mp \frac{\alpha_2(\tilde{s}-s_0)}{2} \right)^4 \leqslant 0$ we will have

$$(1 + \mu^2) \left( M \mp \frac{\alpha_2(\tilde{s} - s_0)}{2} \right)^4 = \left( \alpha_2^2 - \left( M \mp \frac{\alpha_2(\tilde{s} - s_0)}{2} \right)^4 \right) w^2(\tilde{s}) \leqslant 0$$

thus,

$$0 = \left( M \mp \frac{\alpha_2(\tilde{s} - s_0)}{2} \right)^4,$$

consequently

$$\alpha_2^2 = \alpha_2^2 - \left( M \mp \frac{\alpha_2(\tilde{s} - s_0)}{2} \right)^4 \leqslant 0$$

i.e., $\alpha_2 = 0$, a contradiction. Therefore, from (3.50), we have four distinct solutions

$$w(s) = \pm\sqrt{\frac{(1+\mu^2)\left(M \mp \frac{\alpha_2(s-s_0)}{2}\right)^4}{\alpha_2^2 - \left(M \mp \frac{\alpha_2(s-s_0)}{2}\right)^4}}$$

and since $\alpha_2 w(s) < 0$, that is, the sign of $w$ is the inverse of the sign of $\alpha_2$, we obtain only two distinct solutions

$$w(s) = -\operatorname{sgn}(\alpha_2)\sqrt{\frac{(1+\mu^2)\left(M \mp \frac{\alpha_2(s-s_0)}{2}\right)^4}{\alpha_2^2 - \left(M \mp \frac{\alpha_2(s-s_0)}{2}\right)^4}}$$

i.e., for constants $C, D$ and $F > 0$,

$$u(s) = D - \operatorname{sgn}(\alpha_2)\int_{s_0}^{s}\sqrt{\frac{(1+\mu^2)\left(\frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(r-C)}{2}\right)^4}{\alpha_2^2 - \left(\frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(r-C)}{2}\right)^4}}\,dr.$$

The domain of $u$ is determined by analyzing the denominator of the above fraction, i.e.,

$$\alpha_2^2 - \left(\frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(s-C)}{2}\right)^4 > 0$$

that is,

$$\left(\alpha_2 - \left(\frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(s-C)}{2}\right)^2\right)\left(\alpha_2 + \left(\frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(s-C)}{2}\right)^2\right) > 0, \quad (3.51)$$

and therefore both factors have the same sign. We have two situations: If $\alpha_2 > 0$ we obtain from (3.51)

$$\alpha_2 - \left(\frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(s-C)}{2}\right)^2 > 0$$

then

$$-\sqrt{\alpha_2} < \frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(s-C)}{2} < \sqrt{\alpha_2}$$

and since, from (3.49), we have $\frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(s-C)}{2} > 0$ we conclude

$$0 < \frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}} \mp \frac{\alpha_2(s-C)}{2} < \sqrt{\alpha_2}$$

hence

$$\mp C - \frac{2\sqrt{|\alpha_2|F}}{\alpha_2\sqrt[4]{1+F^2+\mu^2}} < \mp s < \mp C + \frac{2}{\alpha_2}\left(\sqrt{\alpha_2} - \frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}}\right). \quad (3.52)$$

The case where $\alpha_2 < 0$ we proceed similarly to obtain

$$\mp C + \frac{2}{\alpha_2}\left(\sqrt{-\alpha_2} - \frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}}\right) < \mp s < \mp C - \frac{2\sqrt{|\alpha_2|F}}{\alpha_2\sqrt[4]{1+F^2+\mu^2}},$$

that is,

$$\pm C + \frac{2\sqrt{|\alpha_2|F}}{\alpha_2\sqrt[4]{1+F^2+\mu^2}} < \pm s < \pm C - \frac{2}{\alpha_2}\left(\sqrt{-\alpha_2} - \frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}}\right). \tag{3.53}$$

We will gather the intervals (3.52) and (3.53) depending on the sign of $\alpha_2$ as follows:

$$\mp \operatorname{sgn}(\alpha_2)s \in \left(\operatorname{sgn}(\alpha_2)\left(\mp C - \frac{2\sqrt{|\alpha_2|F}}{\alpha_2\sqrt[4]{1+F^2+\mu^2}}\right), \operatorname{sgn}(\alpha_2)\left(\mp C + \frac{2}{\alpha_2}\left(\sqrt{|\alpha_2|} - \frac{\sqrt{|\alpha_2|F}}{\sqrt[4]{1+F^2+\mu^2}}\right)\right)\right).$$

□

It is important to emphasize that the explicit solution found in this corollary uses the integral

$$\int \frac{dl}{\sqrt{\alpha_3 - \alpha_2 l}(1+l^2+\mu^2)^{\frac{5}{4}}}$$

whose antiderivative uses the hypergeometric function ${}_2F_1(a,b;c;l)$. This makes it difficult to isolate the solution of the differential equation. When $\alpha_3 = 0$ the factor $\sqrt{l}$ simplifies the integration reduzing us to

$$\int \frac{dl}{\sqrt{-\alpha_2 l}(1+l^2+\mu^2)^{\frac{5}{4}}} = -\frac{2}{\alpha_2(1+\mu^2)}\frac{\sqrt{-\alpha_2 l}}{\sqrt[4]{1+l^2+\mu^2}}.$$

The same occurs for the differential equation in (3.42). The integral in this case is

$$\int \frac{dl}{(1+l^2+\mu^2)^{\frac{5}{4}}}$$

whose primitive also uses the hypergeometric function. Replacing the vector $(0,\alpha_2,\alpha_3)$ with the vector $(\alpha_1,0,\alpha_3)$ in the previous Theorem and Corollary we obtain the following similar results:

**Theorem 3.1.5.** *Let be $\Psi(X,Y) = 4X^2 - 2Y$, $\alpha = (\alpha_1,0,\alpha_3) \in \mathbb{R}^3$ non-zero and $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(t,s) = (t,s,v(t)+u(s))$, where $u \in C^2(J,\mathbb{R})$ and $v \in C^2(I,\mathbb{R})$, a Translating Soliton. Then:*

*i) $u(s) = \mu s + \tau$, for constants $\mu,\tau$, and $J \subset \mathbb{R}$. Moreover, the function $v$ satisfies one of the* ***two*** *differential equations*

$$v''(t) = \pm\frac{\sqrt{\alpha_3 - \alpha_1 v'(t)}(1+(v'(s))^2+\mu^2)^{\frac{5}{4}}}{1+\mu^2}. \tag{3.54}$$

*ii) $v(t) = \mu t + \tau$, for constants $\mu,\tau$, and $I \subset \mathbb{R}$. Moreover, $\alpha_3 - \mu\alpha_2 \geqslant 0$ and the function $u$ satisfies one of the* ***two*** *differential equations*

$$u''(s) = \pm\frac{\sqrt{\alpha_3 - \mu\alpha_1}(1+(u'(s))^2+\mu^2)^{\frac{5}{4}}}{1+\mu^2}. \tag{3.55}$$

**Proof:** Suppose that $\psi$ is a Translating Soliton with $\Psi(X,Y) = 4X^2 - 2Y$ and $\alpha = (\alpha_1, 0, \alpha_3) \in \mathbb{R}^3$ non-zero. By Proposition 3.1.8 the functions $u, v$ satisfy the equation

$$\begin{aligned} &(v''(t)(1+(u'(s))^2)+u''(s)(1+(v'(t))^2))^2 - 2v''(t)u''(s)((v'(t))^2+(u'(s))^2+1) \\ =&((v'(t))^2+(u'(s))^2+1)^{\frac{5}{2}}(\alpha_3 - v'(t)\alpha_1). \end{aligned} \tag{3.56}$$

By Proposition 3.1.10 we have $v(t) = \mu t + \tau$ in $I \subset \mathbb{R}$ or $u(s) = \mu s + \tau$ in $J \subset \mathbb{R}$, for constants $\mu, \tau$. Therefore, the proof follows in a similar way to that of Theorem 3.1.4. $\square$

**Corollary 3.1.4.** *Assume the conditions of Theorem* 3.1.5. *If* $\alpha_3 = 0$ *the solution of the differential equation* (3.54) *is either* $u(s) = D$ *or one of the* ***two*** *functions*

$$v(t) = D - \operatorname{sgn}(\alpha_1)\int_{t_0}^{t}\sqrt{\frac{(1+\mu^2)\left(M_{F,\mu,\alpha_1} \mp \frac{\alpha_1(r-C)}{2}\right)^4}{\alpha_1^2 - \left(M_{F,\mu,\alpha_1} \mp \frac{\alpha_1(r-C)}{2}\right)^4}}dr$$

*for constants* $C, D$ *and* $F > 0$ *where*

$$M_{F,\mu,\alpha_1} = \frac{\sqrt{|\alpha_1|F}}{\sqrt[4]{1+F^2+\mu^2}}.$$

*Furthermore, the interval of definition is given by*

$$\mp\operatorname{sgn}(\alpha_1)t \in \left(\operatorname{sgn}(\alpha_1)\left(\mp C - \frac{2M_{F,\mu,\alpha_1}}{\alpha_1}\right), \operatorname{sgn}(\alpha_1)\left(\mp C + \frac{2\left(\sqrt{|\alpha_1|} - M_{F,\mu,\alpha_1}\right)}{\alpha_1}\right)\right)$$

*and the convention is that the sign* $\mp$ *must be taken respecting its position, that is, superior with superior and inferior with inferior.*

**Proof:** Setting $w(s) = u'(s)$ we obtain

$$w'(s) = \pm\frac{\sqrt{-\alpha_1 w(s)}(1+w^2(s)+\mu^2)^{\frac{5}{4}}}{1+\mu^2}.$$

Therefore, the proof follows in a similar way to that of the Corollary 3.1.3. $\square$

# CHAPTER 4

# Separation of Variables for nonlinear Partial Differential Equations

In the previous two chapters, many of the problems were reduced to taking a surface parametrization, for example, $\psi(t,x) = (t,x,f(t)+g(x))$, that is, a Translation Surface, calculating its mean and Gaussian curvatures, and then trying to determine which properties $f$ and $g$ should have so that their curvatures are constant. If we analyze carefully, we will see that, in reality, we could consider such a surface parametrized by a graph, that is, $\phi(t,x) = (t,x,u(t,x))$ and, thus, in the curvatures differential equations, try to obtain separable solutions. As an example ([1]), let us consider this graph in the Euclidean space $\mathbb{E}^3$. The mean curvature of this surface is described as

$$H_\phi = \operatorname{div}\left(\frac{\nabla u}{2\sqrt{1+|\nabla u|^2}}\right) = \frac{(1+u_x^2)u_{tt} - 2u_t u_x u_{tx} + (1+u_t^2)u_{xx}}{2(1+u_t^2+u_x^2)^{\frac{3}{2}}}, \tag{4.1}$$

in other words, finding a Translation Surface of zero constant Mean Curvature is equivalent to finding solutions of the form $u(t,x) = f(t)+g(x)$ for the equation

$$0 = (1+u_x^2)u_{tt} - 2u_t u_x u_{tx} + (1+u_t^2)u_{xx},$$

i.e.,

$$0 = (1+(g')^2)f'' + (1+(f')^2)g''. \tag{4.2}$$

If we define the family of polynomials,

$$P_x(W,Y) = g''(x)W + (1+(g'(x))^2)Y + g''(x)$$

we conclude that the curve $((f'(t))^2, f''(t))$ vanishes this family. In the previous sections, the approach was essentially to study this polynomial in order to obtain information about the function $f$ that satisfies (4.2). Consider now the heat equation

$$u_t = u_{xx}. \tag{4.3}$$

In an introductory course of Fourier Analysis, we apply the method of separation of variables in order to find an explicit solution to this equation. In this case, this is equivalent to imposing on the heat equation the existence of solutions of the form $u(t,x) = f(t)g(x)$. Therefore, the equation becomes

$$0 = f'g - fg''.$$

Assuming $fg \neq 0$ we obtain

$$\frac{f'}{f} = \frac{g''}{g}.$$

Thus, since the variables $t$ and $x$ are independent, there exists a constant $K$ such that $\frac{f'}{f} = K = \frac{g''}{g}$ and, therefore,

$$f' = Kf$$
$$g'' = Kg.$$

We can interpret this system as having separated the equation $(4.3)$ into two independent equations. In a sense, analyzing the equation $(4.2)$ by means of the polynomial $P_x$ yields something similar. In view of this discussion, and since $(4.1)$ is nonlinear, this chapter is devoted to trying to find separable solutions to some nonlinear partial differential equations. Being precise, we investigate particular solutions for the Camassa-Holm, modified KdV and Novikov equations, all equations involved in the field of integrable PDEs.

# 4.1 Camassa-Holm Equation

In 1993 Robert Camassa and Darryl D. Holm introduced a new integrable dispersive nonlinear partial differential equation as a model for waves in shallow water (see [12]) using Hamiltonian methods.

**Definition 4.1.1** (Camassa-Holm)**.** Let $k \in \mathbb{R}$ be constant. We say that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution of the Camassa-Holm Equation if

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{4.4}$$

in the open set $\Omega \subset \mathbb{R}^2$.

They showed the existence of solitary wave solutions and, in the special case $k = 0$, "peakon" solutions (a type of solitary wave introduced by the authors). In the beginning of section **Modified KdV Equation** we give a sketch of how to find traveling wave solutions for the KdV equation. Recently, Guoping Zhang found new periodic exact traveling wave solutions for the Camassa-Holm equation (see [13]). In this section, we investigate the existence of explicit solutions called Homothetic and Translation solutions.

## 4.1.1 *Homothetic solutions*

**Definition 4.1.2.** We say that the **Camassa-Holm equation** admits **Homothetic** solutions if there exist $f \in C^1(I, \mathbb{R})$ and $g \in C^3(J, \mathbb{R})$ such that $u(t, x) = f(t)g(x)$ is a solution of (4.4).

**Proposition 4.1.1.** *Let $f \in C^1(I, \mathbb{R})$ and $g \in C^3(J, \mathbb{R})$ non-constant. If u is a **Homothetic** solution of the **Camassa-Holm equation**, with $k \neq 0$, then f is constant.*

**Proof:** Suppose that $u(t, x) = f(t)g(x)$ is a solution of the equation (4.4). Thus, the functions $f, g$ satisfy the equation

$$f'(t)g(x) + 2kf(t)g'(x) - f'(t)g''(x) + 3f^2(t)g(x)g'(x) = 2f^2(t)g'(x)g''(x) + f^2(t)g(x)g'''(x). \tag{4.5}$$

Since $g$ is non-constant there exists an open set $V \subset J$ such that $g'(x) \neq 0$ in $V$. Rewriting we obtain

$$0 = -\frac{g''(x) - g(x)}{2kg'(x)} f'(t) - \frac{g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x)}{2kg'(x)} f^2(t) + f(t)$$

in $I \times V$. Define the family of polynomials

$$P_x(W, Y) = A_x W + B_x Y^2 + Y,$$

where,

$$A_x = -\frac{g''(x) - g(x)}{2kg'(x)}$$
$$B_x = -\frac{g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x)}{2kg'(x)}$$

and notice that the curve $(f'(t), f(t))$ is contained in the intersection of the roots of $\{P_x\}_{x\in V}$.

**Claim 1:** If there exists $x_0 \in V \subset J$ such that $A_{x_0} \neq 0$ then the polynomials $\{P_x\}_{x\in\tilde{V}}$ are irreducible in a neighborhood $\tilde{V}$, contained in $V$, of $x_0$. Indeed, take $\tilde{V} \subset V \subset J$ a neighborhood of $x_0$ such that $A_x \neq 0$ in $\tilde{V}$ and then observe that the polynomials $P_x$ are irreducible in this open set.

**Claim 2:** If $A_x \equiv 0$ in $V \subset J$ then $B_x \equiv 0$ in $V \subset J$. Indeed, if $A_x \equiv 0$ in $V$ we obtain $g''(x) = g(x)$ in $V$ and then $g'''(x) = g'(x)$ in $V$. Therefore,

$$B_x = -\frac{g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x)}{2kg'(x)} = -\frac{g(x)g'(x) - 3g(x)g'(x) + 2g'(x)g(x)}{2kg'(x)} = 0$$

thus we prove the claim.

**Claim 3:** If $A_x, B_x$ are mutually constant in an open set $\tilde{V} \subset V \subset J$, that is, there exist constants $K_1, K_2$ such that the pair $(A_x, B_x) = (K_1, K_2)$, then $(K_1, K_2) \neq \left(\pm\frac{\sqrt{2}}{k}, 0\right)$. Indeed, if $(K_1, K_2) = \left(\pm\frac{\sqrt{2}}{k}, 0\right)$ then

$$\begin{aligned} \pm\frac{\sqrt{2}}{k} = A_x &= -\frac{g''(x) - g(x)}{2kg'(x)} \\ 0 = B_x &= -\frac{g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x)}{2kg'(x)} \end{aligned}$$

i.e.

$$\begin{aligned} g''(x) &= \mp 2\sqrt{2}g'(x) + g(x) \\ 0 &= g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x) \end{aligned} \tag{4.6}$$

and differentiating the first equality and listing

$$\begin{aligned} g''(x) &= \mp 2\sqrt{2}g'(x) + g(x) \\ g'''(x) &= \mp 2\sqrt{2}g''(x) + g'(x) = \mp 2\sqrt{2}(\mp 2\sqrt{2}g'(x) + g(x)) + g'(x) = 9g'(x) \mp 2\sqrt{2}g(x) \\ 0 &= g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x). \end{aligned}$$

Finally, substituting the first and second equalities above into the last one we obtain

$$0 = \mp\sqrt{2}(\mp 2g'(x) + \sqrt{2}g(x))^2$$

that is,

$$g'(x) = \mp\frac{\sqrt{2}}{2}g(x)$$

which substituting in the first equality of (4.6) we obtain $g''(x) = 3g(x)$ and consequently $g'''(x) = 3g'(x)$. Therefore, by the second equality of (4.6), we have

$$0 = g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x) = 6g'(x)g(x) = 3\frac{d}{dx}[g^2(x)]$$

and then $g(x)$ is constant in $\tilde{V}$, a contradiction, and thus we have provedthe claim.

-If $A_{x_0} \neq 0$ for some $x_0 \in V \subset J$, by claim 1, the polynomials $\{P_x\}_{x\in\tilde{V}}$ are irreducible in a neighborhood $\tilde{V}$ of $x_0$, that is, for $x \in \tilde{V} \subset V \subset J$. As notice previously, the curve $(f'(t), f(t))$ is contained in the roots of $\{P_x\}_{x\in V}$ so by the theory of Plane Algebraic Curves we conclude that either such a curve is constant or the polynomials $P_x$ are multiples by constants. Suppose that $\{P_x\}_{x\in\tilde{V}}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_x(W,Y) = A_xW + B_xY^2 + Y,$$

is constant in the variable $x$, that is, the polynomials $P_x$ are all equal. Then, their coefficients are constant, that is, there are $K_1, K_2$ constants such that,

$$A_x = K_1$$
$$B_x = K_2$$

that is,

$$\begin{aligned} K_1 &= -\frac{g''(x) - g(x)}{2kg'(x)} \\ K_2 &= -\frac{g(x)g'''(x) - 3g(x)g'(x) + 2g'(x)g''(x)}{2kg'(x)}, \end{aligned} \tag{4.7}$$

By the first equality of (4.7) we obtain

$$g''(x) = g(x) - 2K_1kg'(x) \tag{4.8}$$

that differentiating and substituting the same equality from (4.8)

$$\begin{aligned} g'''(x) &= g'(x) - 2K_1kg''(x) \\ &= g'(x) - 2K_1k(g(x) - 2K_1kg'(x)) \\ &= (4K_1^2k^2 + 1)g'(x) - 2K_1kg(x) \end{aligned}$$

and again substituting $g''(x)$ and $g'''(x)$ in the second equality of (4.7)

$$0 = -2K_1(g'(x))^2 + 2K_1^2kg'(x)g(x) + K_2g'(x) - K_1g^2(x). \tag{4.9}$$

Define $Q : \mathbb{R}^2 \to \mathbb{R}$ by

$$Q(w,y) = -2K_1y^2 + 2K_1^2kyw + K_2y - K_1w^2.$$

Observe that $Q(g(x), g'(x)) = 0$. We have

$$\frac{\partial Q}{\partial w}(w,y) = 2K_1^2ky - 2K_1w$$

and

$$\frac{\partial Q}{\partial y}(w,y) = -4K_1y + 2K_1^2kw + K_2.$$

**Claim 4:** There exists $\tilde{x} \in \tilde{V}$ such that $\frac{\partial Q}{\partial y}(g(\tilde{x}), g'(\tilde{x})) \neq 0$. Indeed, if $\frac{\partial Q}{\partial y}(g(x), g'(x)) = 0$ for all $x \in \tilde{V}$ then

$$0 = \frac{\partial Q}{\partial y}(g(x), g'(x)) = -4K_1g'(x) + 2K_1^2kg(x) + K_2$$

that is, as $K_1 \neq 0$ (otherwise $A_x \equiv 0$ in $\tilde{V}$),

$$g'(x) = \frac{2K_1^2 kg(x) + K_2}{4K_1}$$

which substituting in (4.9)

$$0 = K_1(K_1^2k^2 - 2)g^2(x) + kK_1K_2g(x) + \frac{K_2^2}{4K_1}$$

and therefore $g(x)$ vanishes the polynomial

$$S(Z) = K_1(K_1^2k^2 - 2)Z + kK_1K_2Z + \frac{K_2^2}{4K_1}.$$

Since $g$ is non-constant in $\tilde{V}$ the only way to vanish $S$ is in the case where $S$ is the null polynomial, that is, if and only if its coefficients are null. We have,

$$\begin{aligned} 0 =& K_1(K_1^2k^2 - 2) \\ 0 =& kK_1K_2 \\ 0 =& \frac{K_2^2}{4K_1}. \end{aligned}$$

or equivalently,

$$\begin{aligned} K_1 =& \pm\frac{\sqrt{2}}{k} \\ K_2 =& 0 \end{aligned}$$

a contradiction with the claim 3 and thus we prove the claim.

By claim 4 we can apply the Implicit Function Theorem obtaining

$$g''(x) = g'(x)\frac{\frac{\partial Q}{\partial w}(g(x), g'(x))}{\frac{\partial Q}{\partial y}(g(x), g'(x))} = g'(x)\frac{2K_1^2kg'(x) - 2K_1g(x)}{-4K_1g'(x) + 2K_1^2kg(x) + K_2}$$

for some neighborhood of $x_0 \in \tilde{V}$. As from (4.8)

$$g''(x) = g(x) - 2K_1kg'(x)$$

we have

$$g(x) - 2K_1kg'(x) = g'(x)\frac{2K_1^2kg'(x) - 2K_1g(x)}{-4K_1g'(x) + 2K_1^2kg(x) + K_2}$$

that rewriting

$$0 = 6kK_1^2(g'(x))^2 - K_1(4k^2K_1^2 - 2)g'(x)g(x) - 2kK_1K_2g'(x) + 2kK_1^2g^2(x) + K_2g(x). \tag{4.10}$$

Define $\tilde{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{Q}(w,y) = 6kK_1^2y^2 - K_1(4k^2K_1^2 - 2)yw - 2kK_1K_2y + 2kK_1^2w^2 + K_2w.$$

Observe that, from (4.9) and (4.10), $Q(g(x),g'(x)) = 0 = \tilde{Q}(g(x),g'(x))$. Since the curve $(g(x),g'(x))$ is non-constant we conclude from the theory of Plane Algebraic Curves that the polynomials $Q$ and $\tilde{Q}$ must have at least one factor in common, a contradiction to Lemma $A$.0.3. Then, the curve $(f'(t),f(t))$ is constant and therefore $f$ is constant in $I \subset \mathbb{R}$.

-If $A_x \equiv 0$ in $V \subset J$ we have by claim 2 that $B_x \equiv 0$ in $V \subset J$ therefore,

$$0 = P_x(f'(t),f(t)) = A_x f'(t) + B_x f^2(t) + f(t) = f(t)$$

i.e., $f$ is zero and therefore constant in $I \subset \mathbb{R}$. □

**Theorem 4.1.1.** *Let be $f \in C^1(I,\mathbb{R})$ and $g \in C^3(J,\mathbb{R})$. If $u$ is a* ***Homothetic*** *solution of the* ***Camassa-Holm equation****, with $k \neq 0$, then the possibilities for $f$ and $g$ are:*

*i) $f$ is arbitrary in $I \subset \mathbb{R}$ and $g \equiv 0$ in $J$.*

*ii) $f(t) = \mu$ constant in $I$ and $g$ satisfies the differential equation*

$$\mu^2 g(x)g'''(x) = \mu g'(x)(3\mu g(x) - 2\mu g''(x) + 2k).$$

**Proof:** Suppose that $u(t,x) = f(t)g(x)$ is a solution of the equation (4.4). Thus, the functions $f,g$ satisfy the equation

$$f'(t)g(x) + 2kf(t)g'(x) - f'(t)g''(x) + 3f^2(t)g(x)g'(x) = 2f^2(t)g'(x)g''(x) + f^2(t)g(x)g'''(x). \quad (4.11)$$

Notice that if $g \equiv 0$ in $J \subset \mathbb{R}$ so $f$ is arbitrary, that is, we obtain $i)$. The next two claims ensure that disregarding this specific case for $g$ we obtain that $f$ is constant.

**Claim 1:** If $g \equiv \lambda \neq 0$ in $J \subset \mathbb{R}$ the equation in $(4.11)$ reduces to

$$0 = \lambda f'(t)$$

thus we conclude the claim.

**Claim 2:** If $g$ is non-constant in $J \subset \mathbb{R}$ then $f$ is constant in $I \subset \mathbb{R}$. Indeed, it follows from Proposition 4.1.1.

From claim 1 and 2 we conclude that $f$ is constant. Then, if $f(t) = \mu$ we obtain from $(4.11)$ that $g$ satisfies the differential equation

$$\mu^2 g(x)g'''(x) = \mu g'(x)(3\mu g(x) - 2\mu g''(x) + 2k).$$

□

### 4.1.2 *Translation Solutions*

**Definition 4.1.3.** We say that the **Camassa-Holm equation** admits **Translation** solutions if there exist $f \in C^1(I,\mathbb{R})$ and $g \in C^3(J,\mathbb{R})$ such that $u(t,x) = f(t) + g(x)$ is a solution of (4.4).

**Proposition 4.1.2.** *Let be $f \in C^1(I,\mathbb{R})$ and $g \in C^4(J,\mathbb{R})$ non-constant. If $u$ is a **Translation** solution of the **Camassa-Holm equation**, with $k \neq 0$, then $f$ is constant.*

**Proof:** Suppose that $u(t,x) = f(t) + g(x)$ is a solution of the equation (4.4). Thus, the functions $f, g$ satisfy the equation

$$f'(t) + 2kg'(x) + 3(f(t) + g(x))g'(x) = 2g'(x)g''(x) + (f(t) + g(x))g'''(x). \tag{4.12}$$

Rewriting we get

$$0 = f'(t) + (3g'(x) - g'''(x))f(t) + 3g(x)g'(x) - g(x)g'''(x) + 2kg'(x) - 2g'(x)g''(x).$$

Since $g$ is non-constant there exist an open set $V \subset J$ such that $g'(x) \neq 0$ in $V$. Define the family of irreducible polynomials

$$P_x(W,Y) = W + A_xY + B_x,$$

where,

$$A_x = 3g'(x) - g'''(x)$$
$$B_x = 3g(x)g'(x) - g(x)g'''(x) + 2kg'(x) - 2g'(x)g''(x)$$

and observe that the curve $(f'(t), f(t))$ is contained in the intersection of the roots of $\{P_x\}_{x\in J}$.

**Claim 1:** $A_x$ and $B_x$ are not mutually constant zero in $V \subset J$. Indeed, if they are mutually constant zero we have

$$0 = 3g'(x) - g'''(x)$$
$$0 = 3g(x)g'(x) - g(x)g'''(x) + 2kg'(x) - 2g'(x)g''(x)$$

and then isolating $g'''$ in the first equality and substituting it in the second we have $0 = 2g'(x)(k - g''(x))$. Since $g'$ is non-zero in $V$ we conclude $g''(x) = k$. Differentiating,

$$0 = g'''(x) = 3g'(x)$$

in $V \subset J$, a contradiction and thus we conclude the claim.

By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_x$ are multiples by constants. Suppose that $\{P_x\}_{x\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_x(W,Y) = W + A_xY + B_x,$$

is constant in the variable $x$, that is, the polynomials $P_x$ are all equal. Then, their coefficients are constant, that is, there are constant $K_1, K_2$ such that,

$$A_x = K_1$$
$$B_x = K_2$$

that is,

$$\begin{aligned} K_1 &= 3g'(x) - g'''(x) \\ K_2 &= 3g(x)g'(x) - g(x)g'''(x) + 2kg'(x) - 2g'(x)g''(x). \end{aligned} \tag{4.13}$$

by the first equality of (4.13) we obtain by isolating $g'''(x)$ and differentiating

$$\begin{aligned} g'''(x) &= 3g'(x) - K_1 \\ g''''(x) &= 3g''(x) \end{aligned} \tag{4.14}$$

and differentiating the second equality of (4.13)

$$\begin{aligned} 0 &= 3g(x)g'(x) - g(x)g'''(x) + 2kg'(x) - 2g'(x)g''(x) - K_2 \\ 0 &= 3((g'(x))^2 + g(x)g''(x)) - (g'(x)g'''(x) + g(x)g''''(x)) + 2kg'(x) - 2((g''(x))^2 + g'(x)g'''(x)) \end{aligned} \tag{4.15}$$

and finally, replacing (4.14) in (4.15)

$$\begin{aligned} 2g'(x)g''(x) =& 3g(x)K_1 + 2kg'(x) - K_2 \\ 0 =& 2kg''(x) - 6(g'(x))^2 - 2(g''(x))^2 + 3g'(x)K_1 \end{aligned}$$

and multiplying the second equality above by $-2(g'(x))^2$

$$\begin{aligned} 2g'(x)g''(x) =& 3g(x)K_1 + 2kg'(x) - K_2 \\ 0 =& -4k(g'(x))^2g''(x) + 12(g'(x))^4 + (2g'(x)g''(x))^2 - 6(g'(x))^3K_1 \end{aligned}$$

thus, substituting the first equality into the second,

$$0 = 12(g'(x))^4 - 6K_1(g'(x))^3 + 2kK_1g'(x)g(x) - 2kK_2g'(x) + K_1^2g^2(x) - 2K_1K_2g(x) + K_2^2. \tag{4.16}$$

**Claim 2:** $K_1 \neq 0$. Indeed, if $K_1 = 0$ we obtain from (4.16) and the first equality of (4.13)

$$\begin{aligned} 0 =& g'(x) - g'''(x) \\ 0 =& 12(g'(x))^4 - 2kK_2g'(x) + K_2^2. \end{aligned}$$

By the second equality $g'$ is root of a non-zero polynomial, that is, it is constant, a contradiction with the first equality since we would have $0 = g'(x) - g'''(x) = g'(x)$ and thus we conclude the claim.

Define $Q : \mathbb{R}^2 \to \mathbb{R}$ by

$$Q(w,y) = 12y^4 - 6K_1y^3 + 2kK_1yw - 2kK_2y + K_1^2w^2 - 2K_1K_2w + K_2^2.$$

Observe that $Q(g(x), g'(x)) = 0$. It follows that

$$\frac{\partial Q}{\partial w}(w, y) = 2kK_1 y + 2K_1^2 w - 2K_1 K_2$$

and

$$\frac{\partial Q}{\partial y}(w, y) = 48y^3 - 18K_1 y^2 + 2kK_1 w - 2kK_2.$$

**Claim 3:** There exists $x_0 \in V$ such that $\frac{\partial Q}{\partial y}(g(x_0), g'(x_0)) \neq 0$. Indeed, if $\frac{\partial Q}{\partial y}(g(x), g'(x)) = 0$ for all $x \in V$, and since the curve $(g(x), g'(x))$ is non-constant, we have $Q(g(x), g'(x)) = 0 = \frac{\partial Q}{\partial y}(g(x), g'(x))$, that is, we conclude by the theory of Plane Algebraic Curves that these polynomials should have at least one factor in common, a contradiction with Lemma $A.0.4$ and thus we conclude the claim.

By claim 3 we can apply the Implicit Function Theorem obtaining

$$g''(x) = g'(x) \frac{\frac{\partial Q}{\partial w}(g(x), g'(x))}{\frac{\partial Q}{\partial y}(g(x), g'(x))} = g'(x) \frac{2kK_1 g'(x) + 2K_1^2 g(x) - 2K_1 K_2}{48(g'(x))^3 - 18K_1 (g'(x))^2 + 2kK_1 g(x) - 2kK_2}$$

for a neighborhood of $x_0 \in V$. Thus, from (4.13),

$$\begin{aligned} g'''(x) &= 3g'(x) - K_1 \\ K_2 &= 3g(x)g'(x) - g(x)g'''(x) + 2kg'(x) - \frac{2(g'(x))^2 (2kK_1 g'(x) + 2K_1^2 g(x) - 2K_1 K_2)}{48(g'(x))^3 - 18K_1 (g'(x))^2 + 2kK_1 g(x) - 2kK_2}. \end{aligned}$$

Substituting the first equality into the second equality we obtain

$$\begin{aligned} 0 = &-2kK_1^2 g^2(x) - 48g(x)(g'(x))^3 + 22K_1^2 g(x)(g'(x))^2 - 4k^2 K_1 g(x)g'(x) + 4kK_1 K_2 g(x) - 96k(g'(x))^4 \\ &+ (48K_2 + 40kK_1)(g'(x))^3 - 22K_1 K_2 (g'(x))^2 + 4k^2 K_2 g'(x) - 2kK_2^2. \end{aligned} \tag{4.17}$$

Define $\tilde{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} \tilde{Q}(w, y) = &-2kK_1^2 w^2 - 48wy^3 + 22K_1^2 wy^2 - 4k^2 K_1 wy + 4kK_1 K_2 w - 96ky^4 \\ &+ (48K_2 + 40kK_1)y^3 - 22K_1 K_2 y^2 + 4k^2 K_2 y - 2kK_2^2. \end{aligned}$$

Notice that, from (4.16) and (4.17), $Q(g(x), g'(x)) = 0 = \tilde{Q}(g(x), g'(x))$. Since the curve $(g(x), g'(x))$ is non-constant we conclude from the theory of Plane Algebraic Curves that the polynomials $Q$ and $\tilde{Q}$ must have at least one factor in common, a contradiction to Lemma $A.0.5$. Therefore, the curve $(f'(t), f(t))$ is constant and then $f$ is constant in $I \subset \mathbb{R}$. □

**Theorem 4.1.2.** *Let be $f \in C^1(I, \mathbb{R})$ and $g \in C^4(J, \mathbb{R})$. If $u$ is a* ***Translation*** *solution of the* ***Camassa-Holm equation****, with $k \neq 0$, then $f(t) = \mu$ is constant in $I$ and $g$ satisfies the differential equation*

$$(\mu + g(x))g'''(x) = 2kg'(x) + 3(\mu + g(x))g'(x) - 2g'(x)g''(x).$$

**Proof:** Suppose that $u(t,x) = f(t) + g(x)$ is a solution of the equation $(4.4)$. Thus, the functions $f,g$ satisfy the equation

$$f'(t) + 2kg'(x) + 3(f(t) + g(x))g'(x) = 2g'(x)g''(x) + (f(t) + g(x))g'''(x). \tag{4.18}$$

**Claim:** $f$ is constant in $I \subset \mathbb{R}$. Indeed, if $g$ is non-constant in $J$ it follows from Proposition 4.1.2 that $f$ is constant. If $g$ is constant in $J$ we obtain from $(4.18)$ that $f'(t) = 0$ and therefore $f$ is constant.

From the claim we obtain that $f$ is constant. Therefore, if $f(t) = \mu$ we obtain from $(4.18)$ that $g$ satisfies the differential equation

$$(\mu + g(x))g'''(x) = 2kg'(x) + 3(\mu + g(x))g'(x) - 2g'(x)g''(x).$$

$\square$

## 4.2 Modified KdV Equation

According to the literature, the first person to study the phenomenon of solitary waves was John Scott Russell, a British naval engineer. Russell observed this phenomenon for the first time in 1834, in the Edinburgh-Glasgow canal. By laboratory experiments, Russel reproduced numerous solitary waves. The experiment consisted of throwing weights over the water in order to produce elevations. Using this data, Russel was able to deduce the following equation:

$$c^2 = g(h+a)$$

where $g$ is gravity, $h$ is the height of water at rest, $a$ is the height of the wave and $c$ is the wave propagation speed. In short, we can say that a solitary wave is a localized wave that travels without deformation. Russel died without being able to find a purely theoretical deduction, which was only possible with the work of Korteweg and de Vries in 1895. Korteweg and his student Gustav de Vries, when studying the propagation of waves in shallow waters, deduced what is now known as the KdV equation. Although KdV is a non-linear partial differential equation, it admits solutions whose shape represents the phenomenon observed by Russel, a "well-behaved" profile. The KdV equation

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0$$

admits solitary waves as solutions. We say that a function is a solitary wave if it is a solution of a non-linear partial differential equation and satisfies the following properties:

*i*) It represents a wave of constant shape, that is, it is of the form $\phi(x-ct)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function and $c$ is a real constant that corresponds to the wave propagation speed;

*ii*) It's localized, that is, $\phi^n(\xi) \to 0$ when $|\xi| \to \infty$ for some large enough $n \in \mathbb{N}$.

With this definition, we can plug $u(x,t) = \phi(x-ct)$ into the KdV equation

$$0 = \partial_t u - 6u\partial_x u + \partial_x^3 u = -c\phi' - 6\phi\phi' + \phi'''$$

which we rewrite as

$$0 = -c\phi' - 3(\phi^2)' + \phi'''.$$

Integrating this equation, we get for a constant $k$

$$k = -c\phi - 3\phi^2 + \phi''.$$

By the property of being localized, we conclude, applying the limits, that $k = 0$. Multiplying this equation by $\phi'$ we have

$$0 = -c\left(\frac{\phi^2}{2}\right)' - (\phi^3)' + [(\phi')^2]'.$$

Repeating what we did before, we will end with the separable ODE

$$0 = -c\frac{\phi^2}{2} - \phi^3 + (\phi')^2,$$

i.e.

$$\frac{d\phi}{d\xi} = \pm\sqrt{c\frac{\phi}{2}2 + \phi^3}$$

whose solution is the hyperbolic secant. Thus,

$$u(x,t) = \phi(x-ct) = -4a^2 sech^2(a(x-ct)),$$

where $a = \frac{1}{2}\sqrt{\frac{c}{2}}$. Graphically, one can see that this solution is consistent with the phenomenon observed by Russel. Note also that the hypothesis that $\phi$ does not vanish at any point was implicitly assumed. To know more about solitary waves and how the equation was deduced, see references [10] and [11]. In what follows, we introduce the modified KdV equation and investigate the existence of separable solutions for this equation.

**Lemma 4.2.1.** *Let $m \geqslant 3$, $J \subset \mathbb{R}$ open intervals, $g \in C^m(J,\mathbb{R})$ and $a,c,k,K_1$ constants be such that $c,k,K_1$ are nonzero. If*

$$g'(x) = \frac{K_1}{kc(a+cg(x))}$$

*then*

$$g^{(m)}(x) = \frac{M_m}{(a+cg(x))^{2m-1}}$$

*for $M_m \neq 0$ being a constant that depends on m.*

**Proof:** Set

$$M_m = (-1)^{m+1}\left(\frac{K_1}{kc}\right)^m \prod_{i=1}^{m-2}(2i+1)$$

and the proof will follow by induction. □

**Definition 4.2.1** (Modified KdV)**.** Let be $k \in \mathbb{R}$ a non-zero constant and $m \geqslant 3$ an integer number. We say that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution of the modified KdV equation if

$$\partial_t u + \partial_x^m u + ku^2 \partial_x u = 0, \tag{4.19}$$

in the open set $\Omega \subset \mathbb{R}^2$.

**Definition 4.2.2.** We say that the **modified KdV equation** admits **separable** solutions if there exist $f \in C^1(I,\mathbb{R})$, $g \in C^m(J,\mathbb{R})$ and constants $a,b,c$ such that $u(t,x) = af(t) + bg(x) + cf(t)g(x)$ is a solution of (4.19).

**Proposition 4.2.1.** *Let be $f \in C^1(I,\mathbb{R})$ and $g \in C^m(J,\mathbb{R})$ non-constant. Moreover, assume $a,b,c$ to be constants such that if $c = 0$ then $ab \neq 0$. If u is a* ***separable*** *solution of the* ***modified KdV equation*** *then f is constant.*

**Proof:** Suppose that $u(x,t) = af(t) + bg(x) + cf(t)g(x)$ is a solution of (4.19). Thus, the functions $f,g$ satisfy

$$\begin{aligned}0 =&(a+cg(x))f'(t) + ck(a+cg(x))^2g'(x)f^3(t) + bk(a+3cg(x))(a+cg(x))g'(x)f^2(t)\\ &+ (cg^{(m)}(x) + 3cb^2kg^2(x)g'(x) + 2ab^2kg(x)g'(x))f(t) + b(b^2kg^2(x)g'(x) + g^{(m)}(x)).\end{aligned} \tag{4.20}$$

Let's split it into two cases depending on $c$:

- If $c = 0$ our equation reduces to

$$0 = f'(t) + abkg'(x)f^2(t) + 2b^2kg(x)g'(x)f(t) + \frac{b(b^2kg^2(x)g'(x) + g^{(m)}(x))}{a}.$$

Define the family of irreducible polynomials

$$P_x(W,Y) = W + A_xY^2 + B_xY + C_x,$$

where,

$$\begin{aligned}A_x &= abkg'(x)\\ B_x &= 2b^2kg(x)g'(x)\\ C_x &= \frac{b(b^2kg^2(x)g'(x) + g^{(m)}(x))}{a}\end{aligned}$$

and notice that the curve $(f'(t), f(t))$ is contained in the intersection of the roots of $\{P_x\}_{x\in J}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_x$ are multiples by constants. Suppose that $\{P_x\}_{x\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_x(W,Y) = W + A_xY^2 + B_xY + C_x,$$

is constant in the variable $x$, that is, the polynomials $P_x$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2, K_3$ constants such that,

$$\begin{aligned}A_x &= K_1\\ B_x &= K_2\\ C_x &= K_3\end{aligned}$$

that is,

$$\begin{aligned}K_1 &= abkg'(x)\\ K_2 &= 2b^2kg(x)g'(x)\\ K_3 &= \frac{b(b^2kg^2(x)g'(x) + g^{(m)}(x))}{a}.\end{aligned} \tag{4.21}$$

Since $g$ is non-constant, there exists an open set $V \subset J$ such that $g' \neq 0$. Thus, isolating $g'$ in the first equality and substituting it into the second we obtain

$$g(x) = \frac{aK_2}{2bK_1}$$

a contradiction since $g'(x) \neq 0$ in $V \subset J$. Therefore, the curve $(f'(t), f(t))$ is constant and then $f$ is constant in $I \subset \mathbb{R}$.
- If $c \neq 0$ take an open set $\hat{V} \subset J$ such that $g'(x) \neq 0$ (which exists, otherwise $g$ would be constant). Moreover, there exists an open set $\tilde{V} \subset \hat{V}$ such that $a + cg(x) \neq 0$ (otherwise $g'(x) = 0$ in $\tilde{V}$). Then, from (4.20),

$$\begin{aligned} 0 =& f'(t) + ck(a+cg(x))g'(x)f^3(t) + bk(a+3cg(x))g'(x)f^2(t) \\ &+ \frac{cg^{(m)}(x) + 3cb^2kg^2(x)g'(x) + 2ab^2kg(x)g'(x)}{a+cg(x)} f(t) + b\frac{g^{(m)}(x) + b^2kg^2(x)g'(x)}{a+cg(x)}. \end{aligned}$$

Define the family of polynomials

$$\tilde{P}_x(W,Y) = W + \tilde{A}_xY^3 + \tilde{B}_xY^2 + \tilde{C}_xY + \tilde{D}_x$$

where

$$\begin{aligned} \tilde{A}_x &= ck(a+cg(x))g'(x) \\ \tilde{B}_x &= bk(a+3cg(x))g'(x) \\ \tilde{C}_x &= \frac{cg^{(m)}(x) + 3cb^2kg^2(x)g'(x) + 2ab^2kg(x)g'(x)}{a+cg(x)} \\ \tilde{D}_x &= b\frac{g^{(m)}(x) + b^2kg^2(x)g'(x)}{a+cg(x)} \end{aligned}$$

and observe that the curve $(f'(t), f(t))$ is contained in the intersection of the roots of $\{\tilde{P}_x\}_{x\in\tilde{V}}$.

**Claim 1:** The coefficients $\tilde{A}_x, \tilde{B}_x, \tilde{C}_x$ and $\tilde{D}_x$ are not mutually constant in $\tilde{V}$. Indeed, if there exist constants $\tilde{K}_1, \tilde{K}_2, \tilde{K}_3$ and $\tilde{K}_4$ such that

$$\begin{aligned} \tilde{K}_1 &= ck(a+cg(x))g'(x) \\ \tilde{K}_2 &= bk(a+3cg(x))g'(x) \\ \tilde{K}_3 &= \frac{cg^{(m)}(x) + 3cb^2kg^2(x)g'(x) + 2ab^2kg(x)g'(x)}{a+cg(x)} \\ \tilde{K}_4 &= b\frac{g^{(m)}(x) + b^2kg^2(x)g'(x)}{a+cg(x)}. \end{aligned} \tag{4.22}$$

Isolating $g'(x)$ in the first equality and substituting it in the second and isolating $g^{(m)}(x)$ in the third and substituting it in the fourth we will have

$$\begin{aligned} 0 &= c(cK_2 - 3bK_1)g(x) + a(cK_2 - 1) \\ 0 &= (2b^3K_1 + c^3K_4 - bc^2K_3)g(x) - ac(bK_3 - cK_4). \end{aligned}$$

Since $g(x)$ is non-constant in $\tilde{V}$ we conclude that the coefficients are all zero. We have,

$$\begin{aligned} 0 &= cK_2 - 3bK_1) \\ 0 &= a(cK_2 - 1) \\ 0 &= 2b^3K_1 + c^3K_4 - bc^2K_3 \\ 0 &= a(bK_3 - cK_4). \end{aligned}$$

Isolating $bK_1$ in the first equality and substituting it in the third and then isolating $bK_3$ in the third and substituting it in the last we will obtain

$$0 = a(cK_2 - 1)$$
$$0 = ab^2 K_2,$$

and then from the first equality $aK_2 = \frac{a}{c}$ which substituting into the second we conclude $ab^2 = 0$, that is, $a = 0$ or $b = 0$. Returning to (4.22)

$$g'(x) = \frac{\tilde{K}_1}{ck(a+cg(x))}$$
$$\tilde{K}_3 = \frac{cg^{(m)}(x)}{a+cg(x)} + \frac{3cb^2kg^2(x)g'(x)}{a+cg(x)} + \frac{2ab^2kg(x)g'(x)}{a+cg(x)}$$

that substituting the first equality into the second and applying the Lemma 4.2.1 we conclude

$$\tilde{K}_3 = \frac{M_m}{(a+cg(x))^{2m}} + \frac{3b^2g^2(x)\tilde{K}_1}{(a+cg(x))^2} + \frac{2ab^2g(x)}{c(a+cg(x))^2}. \tag{4.23}$$

Thus, if $b = 0$,

$$\tilde{K}_3 = \frac{M_m}{(a+cg(x))^{2m}}$$

and since $M_m \neq 0$ we conclude $\tilde{K}_3 \neq 0$. Thus, $g$ is constant in $\tilde{V}$, a contradiction. Therefore $a = 0$. Then, from (4.23),

$$\tilde{K}_3 = \frac{M_m}{(cg(x))^{2m}} + \frac{3b^2\tilde{K}_1}{c^2}$$

and then, since $\overline{K}_1 \neq 0$, $g$ is constant in $\tilde{V}$, a contradiction. Thus we conclude the claim.

**Claim 2:** The family of polynomials $\tilde{P}_x$ is irreducible in $\tilde{V}$. Indeed, if $P_x$ is reducible there exist polynomials of degree 1 and degree 2 such that

$$\tilde{P}_x(W,Y) = (F + GW + HY)(I + JW + KY + LWY + MW^2 + NY^2)$$

that is,

$$0 = W + \tilde{A}_xY^3 + \tilde{B}_xY^2 + \tilde{C}_xY + \tilde{D}_x - (F + GW + HY)(I + JW + KY + LWY + MW^2 + NY^2)$$

that rewriting

$$\begin{aligned} 0 =& -HMW^3 - (GM + HL)W^2Y - (HJ + FM)W^2 - (GL + HN)WY^2 - (GJ + FL + HK)WY \\ &+ (1 - FJ - HI)W + (\tilde{A}_x - GN)Y^3 + (\tilde{B}_x - GK - FN)Y^2 + (\tilde{C}_x - GI - FK)Y + \tilde{D}_x - FI, \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} 0 =&HM \\ 0 =&GM + HL \\ 0 =&HJ + FM \\ 0 =&GL + HN \end{aligned} \qquad \begin{aligned} 0 =&GJ + FL + HK \\ 0 =&1 - FJ - HI \\ 0 =&\tilde{A}_x - GN \end{aligned} \qquad \begin{aligned} 0 =&\tilde{B}_x - GK - FN \\ 0 =&\tilde{C}_x - GI - FK \\ 0 =&\tilde{D}_x - FI. \end{aligned} \tag{4.24}$$

By the first equality $H=0$ or $M=0$. Suppose $H=0$. We have,

$$\begin{aligned} 0 &= GM & 0 &= GJ+FL & 0 &= \tilde{B}_x - GK - FN \\ 0 &= FM & 0 &= 1-FJ & 0 &= \tilde{C}_x - GI - FK \\ 0 &= GL & 0 &= \tilde{A}_x - GN & 0 &= \tilde{D}_x - FI. \end{aligned}$$

Since $G \neq 0$ (otherwise we would not have factorization) we obtain, from the third equality, $L=0$. Therefore, by the fourth equality, $J=0$ which substituting into the fifth equality we obtain $1=0$, a contradiction. Thus, $H \neq 0$ and $M=0$. From (4.24) we have

$$\begin{aligned} 0 &= HL & 0 &= GJ+FL+HK & 0 &= \tilde{B}_x - GK - FN \\ 0 &= HJ & 0 &= 1-FJ-HI & 0 &= \tilde{C}_x - GI - FK \\ 0 &= GL+HN & 0 &= \tilde{A}_x - GN & 0 &= \tilde{D}_x - FI. \end{aligned}$$

By the first equality $L=0$. Substituting in the third equality we obtain $N=0$ and therefore we have no factorization. Thus, we conclude the claim.

By claim 2, the polynomials $\{\tilde{P}_x\}_{x\in\tilde{V}}$ are irreducible. As observed previously, the curve $(f'(t), f(t))$ is contained in the roots of $\{\tilde{P}_x\}_{x\in\tilde{V}}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $\tilde{P}_x$ are multiples by constants. Suppose that $\{\tilde{P}_x\}_{x\in\tilde{V}}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$\tilde{P}_x(W,Y) = W + \tilde{A}_x Y^3 + \tilde{B}_x Y^2 + \tilde{C}_x Y + \tilde{D}_x,$$

is constant in the variable $x$, that is, the polynomials $\tilde{P}_x$ are all equal. Therefore, their coefficients are constant, that is, there are $\tilde{K}_1, \tilde{K}_2, \tilde{K}_3, \tilde{K}_4$ constants such that,

$$\begin{aligned} \tilde{A}_x &= \tilde{K}_1 \\ \tilde{B}_x &= \tilde{K}_2 \\ \tilde{C}_x &= \tilde{K}_3 \\ \tilde{D}_x &= \tilde{K}_4, \end{aligned}$$

a contradiction with claim 1. Therefore, the curve $(f'(t), f(t))$ is constant and then $f$ is constant in $I \subset \mathbb{R}$. $\square$

The definition of separable solution, given at the beginning of this section, depends on the non-zero $(a,b,c) \in \mathbb{R}^3$. In the previous proposition, we required that, if $c=0$, then $ab \neq 0$. Therefore, to obtain all possible situations of separable solutions, it would remain to consider the case in which $c=0$ and $ab=0$ or, more precisely, the non-zero $(a,0,0)$ and $(0,b,0)$. However, these situations are equivalent to searching for solutions of the form $u(x,t) = af(t)$ or $u(x,t) = bg(x)$, solutions that do not bring the idea of separation.

**Theorem 4.2.1.** *Let $f \in C^1(I,\mathbb{R})$ and $g \in C^m(J,\mathbb{R})$. Moreover, assume $(a,b,c) \in \mathbb{R}^3$ nonzero such that if $c=0$ then $ab \neq 0$. If $u$ is a* ***separable*** *solution of the* ***modified KdV equation*** *then the possibilities for $f$ and $g$ are:*

*i)* $f$ *is arbitrary in* $I \subset \mathbb{R}$ *and* $g \equiv \lambda$ *inJ, with* $a + c\lambda = 0$.

*ii)* $f \equiv \mu$ *in* $I \subset \mathbb{R}$, *with* $b + c\mu = 0$, *and* $g$ *é is arbitrary in* $J \subset \mathbb{R}$.

*iii)* $f \equiv \mu$ *in* $I \subset \mathbb{R}$, *with* $b + c\mu \neq 0$, *and* $g$ *satisfies the differential equation*

$$g^{(m)}(x) = -k(a\mu + (b + c\mu)g(x))^2 g'(x).$$

**Proof:** Suppose that $u(x,t) = af(t) + bg(x) + cf(t)g(x)$ is a solution of (4.19). Thus, the functions $f, g$ satisfy

$$\begin{aligned} 0 =& (a + cg(x))f'(t) + ck(a + cg(x))^2 g'(x) f^3(t) + bk(a + 3cg(x))(a + cg(x))g'(x) f^2(t) \\ &+ (cg^{(m)}(x) + 3cb^2 kg^2(x)g'(x) + 2ab^2 kg(x)g'(x))f(t) + b(b^2 kg^2(x)g'(x) + g^{(m)}(x)). \end{aligned} \tag{4.25}$$

Note that if $g \equiv \lambda$ is constant in $J \subset \mathbb{R}$, with $a + \lambda c = 0$, then $f$ is arbitrary, that is, we obtain *i*). The following two claims ensure that disregarding this specific case for $g$ we obtain that $f$ is constant.

**Claim 1:** If $g \equiv \lambda$ in $J \subset \mathbb{R}$, with $a + c\lambda \neq 0$, the equation in (4.25) reduces to

$$0 = (a + c\lambda)f'(t)$$

and we conclude the claim.

**Claim 2:** If $g$ is non-constant in $J \subset \mathbb{R}$ then $f$ is constant in $I \subset \mathbb{R}$. Indeed, it follows from Proposition 4.2.1.

By claim 1 and 2 we obtain that $f$ is constant. Then,
**-***ii*) **and** *iii*) If $f \equiv \mu$ is constant in $I \subset \mathbb{R}$ the equation in (4.25) reduces to

$$0 = (b + c\mu)\left(g^{(m)}(x) + k(a\mu + (b + c\mu)g(x))^2 g'(x)\right).$$

If $b + c\mu = 0$ we obtain *ii*). If $b + c\mu \neq 0$ we obtain

$$g^{(m)}(x) = -k(a\mu + (b + c\mu)g(x))^2 g'(x).$$

□

## 4.3 Novikov Equation

Interested in a generalization of the Camassa-Holm equation, Vladimir Novikov introduced a new integrable equation which now receives its name (see [15]).

**Definition 4.3.1** (Novikov)**.** We say that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution of the Novikov Equation if

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{4.26}$$

in the open set $\Omega \subset \mathbb{R}^2$.

It is well known that the Camassa-Holm equation is bi-Hamiltonian, possesses an infinite hierarchy of local conservation laws, and admits peakons as solutions. Motivated by the work of Novikov, Hone and Wang investigate the existence of peakons for this equation (see [14]). As we did in the two previous sections, here we investigate the existence of explicit separable solutions for the Novikov equation.

**Definition 4.3.2.** We say that the **Novikov equation** admits **separable** solutions if there exist $f \in C^1(I,\mathbb{R})$, $g \in C^3(J,\mathbb{R})$ and constants $a,b,c$ such that $u(t,x) = af(t) + bg(x) + cf(t)g(x)$ is a solution of (4.26).

**Proposition 4.3.1.** *Let be $f \in C^1(I,\mathbb{R})$ and $g \in C^3(J,\mathbb{R})$ non-constant. Moreover, assume $a,b,c$ to be constants such that $ab \neq 0$. If $u$ is a **separable** solution of the **Novikov equation** then:*

*i) $f$ is constant and $I \subset \mathbb{R}$ or;*

*ii) There exist constants $C$, $D \neq \frac{c}{b}$, $\mu \neq 0$ such that*

$$f(t) = \frac{1}{\frac{c}{b} + \left(D - \frac{c}{b}\right) e^{\mu(t-C)}}$$

*and*

$$I \subset \begin{cases} \mathbb{R}, & \text{if } 1 - \frac{bD}{c} < 0 \\ \mathbb{R} - \{C - \frac{1}{\mu}\ln\left(1 - \frac{bD}{c}\right)\}, & \text{if } 1 - \frac{bD}{c} > 0. \end{cases}$$

**Proof:** Suppose that $u(x,t) = af(t) + bg(x) + cf(t)g(x)$ is a solution of (4.26). Thus, the functions $f,g$ satisfy

$$\begin{aligned}
0 =& (a - cg''(x) + cg(x))f'(t) \\
&+ c(a + cg(x))(4ag'(x) - ag'''(x) + 4cg(x)g'(x) - cg(x)g'''(x) - 3cg'(x)g''(x))f^3(t) \\
&+ b(4a^2 g'(x) - a^2 g'''(x) + 12c^2 g^2(x)g'(x) - 3c^2 g^2(x)g'''(x) + 16acg(x)g'(x))f^2(t) \\
&- b(4acg(x)g'''(x) + 6acg'(x)g''(x) + 9c^2 g(x)g'(x)g''(x))f^2(t) \\
&+ b^3 g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x)) + b^2(8ag(x)g'(x) - 2ag(x)g'''(x))f(t) \\
&+ b^2(-3ag'(x)g''(x) + 12cg^2(x)g'(x) - 3cg^2(x)g'''(x) - 9cg(x)g'(x)g''(x))f(t).
\end{aligned} \tag{4.27}$$

Let's split it into two cases depending on $c$:

- If $c = 0$ our equation reduces to

$$\begin{aligned}0 =& f'(t) + ba(4g'(x) - g'''(x))f^2(t) + b^2(8g(x)g'(x) - 2g(x)g'''(x) - 3g'(x)g''(x))f(t) \\ &+ \frac{b^3 g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x))}{a}.\end{aligned}$$

Define the family of irreducible polynomials

$$P_x(W,Y) = W + A_x Y^2 + B_x Y + C_x,$$

where,

$$\begin{aligned}A_x &= ba(4g'(x) - g'''(x)) \\ B_x &= b^2(8g(x)g'(x) - 2g(x)g'''(x) - 3g'(x)g''(x)) \\ C_x &= \frac{b^3 g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x))}{a}\end{aligned}$$

and notice that the curve $(f'(t), f(t))$ is contained in the intersection of the roots of $\{P_x\}_{x\in J}$. By the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $P_x$ are multiples by constants. Suppose that $\{P_x\}_{x\in J}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$P_x(W,Y) = W + A_x Y^2 + B_x Y + C_x,$$

is constant in the variable $x$, that is, the polynomials $P_x$ are all equal. Therefore, their coefficients are constant, that is, there are $K_1, K_2, K_3$ constants such that,

$$A_x = K_1 \qquad B_x = K_2 \qquad C_x = K_3$$

that is,

$$\begin{aligned}K_1 &= ba(4g'(x) - g'''(x)) \\ K_2 &= b^2(8g(x)g'(x) - 2g(x)g'''(x) - 3g'(x)g''(x)) \\ K_3 &= \frac{b^3 g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x))}{a}.\end{aligned} \tag{4.28}$$

Isolating $g'''(x)$ in the first equality and substituting it in the others we obtain

$$K_2 = \frac{b}{a}(2K_1 g(x) - 3abg'(x)g''(x)) \qquad K_3 = \frac{b^2}{a^2} g(x)(K_1 g(x) - 3abg'(x)g''(x)).$$

Isolating $3abg'(x)g''(x)$ in the first and substituting it in the second we conclude

$$0 = -b^2 K_1 g^2(x) + abK_2 g(x) - a^2 K_3.$$

Since $g$ is non-constant we conclude that the coefficients of this polynomial are all zero. Thus,

$$0 = P_x(f'(t), f(t)) = f'(t) + K_1 f^2(t) + K_2 f(t) + K_3 = f'(t)$$

that is, $f$ is constant in $I$. In the case where the curve $(f'(t), f(t))$ is constant we also conclude that $f$ is constant in $I$.
- If $c \neq 0$ takes open $\hat{V} \subset J$ such that $g'(x) \neq 0$ (which exists, otherwise $g$ would be constant). Also, there exists open set $\tilde{V} \subset \hat{V}$ such that $a + cg(x) \neq 0$ (otherwise $g'(x) = 0$ in $\tilde{V}$). If $a - cg''(x) + cg(x) \equiv 0$ in $\tilde{V}$ then $g''(x) = \frac{a+cg(x)}{c}$ and consequently $g'(x) = g'''(x)$. From (4.27) we conclude

$$0 = abg'(x)(b + cf(t))(af(t) + bg(x) + cf(t)g(x))$$

in $\tilde{V}$. Then, for $x_0 \in \tilde{V}$

$$0 = (b + cf(t))(af(t) + bg(x_0) + cf(t)g(x_0))$$

in $I$. Thus, $f(t) \in \{-\frac{b}{c}, -\frac{bg(x_0)}{a+cg(x_0)}\}$ for all $t \in I$. By continuity, $f$ is constant in $I$. If $a - cg''(x) + cg(x) \neq 0$ in an open set $\mathcal{V} \subset \tilde{V}$ then from (4.27) we conclude

$$\begin{aligned}
0 = f'(t) &+ \frac{c(a+cg(x))(4ag'(x) - ag'''(x) + 4cg(x)g'(x) - cg(x)g'''(x) - 3cg'(x)g''(x))}{a - cg''(x) + cg(x)} f^3(t) \\
&+ \frac{b(4a^2g'(x) - a^2g'''(x) + 12c^2g^2(x)g'(x) - 3c^2g^2(x)g'''(x) + 16acg(x)g'(x)}{a - cg''(x) + cg(x)} f^2(t) \\
&+ \frac{b(-4acg(x)g'''(x) - 6acg'(x)g''(x) - 9c^2g(x)g'(x)g''(x))}{a - cg''(x) + cg(x)} f^2(t) \\
&+ \frac{b^2(8ag(x)g'(x) - 2ag(x)g'''(x) - 3ag'(x)g''(x) + 12cg^2(x)g'(x) - 3cg^2(x)g'''(x))}{a - cg''(x) + cg(x)} f(t) \\
&- \frac{9b^2cg(x)g'(x)g''(x)}{a - cg''(x) + cg(x)} f(t) + \frac{b^3g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x))}{a - cg''(x) + cg(x)}.
\end{aligned}$$

Define the family of polynomials

$$\tilde{P}_x(W,Y) = W + \tilde{A}_xY^3 + \tilde{B}_xY^2 + \tilde{C}_xY + \tilde{D}_x$$

where

$$\begin{aligned}
\tilde{A}_x &= \frac{c(a+cg(x))(4ag'(x) - ag'''(x) + 4cg(x)g'(x) - cg(x)g'''(x) - 3cg'(x)g''(x))}{a - cg''(x) + cg(x)} \\
\tilde{B}_x &= \frac{b(4a^2g'(x) - a^2g'''(x) + 12c^2g^2(x)g'(x) - 3c^2g^2(x)g'''(x) + 16acg(x)g'(x))}{a - cg''(x) + cg(x)} \\
&+ \frac{b(-4acg(x)g'''(x) - 6acg'(x)g''(x) - 9c^2g(x)g'(x)g''(x))}{a - cg''(x) + cg(x)} \\
\tilde{C}_x &= \frac{b^2(8ag(x)g'(x) - 2ag(x)g'''(x) - 3ag'(x)g''(x) + 12cg^2(x)g'(x) - 3cg^2(x)g'''(x))}{a - cg''(x) + cg(x)} \\
&- \frac{9b^2cg(x)g'(x)g''(x)}{a - cg''(x) + cg(x)} \\
\tilde{D}_x &= \frac{b^3g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x))}{a - cg''(x) + cg(x)}
\end{aligned}$$

and observe that the curve $(f'(t), f(t))$ is contained in the intersection of the roots of $\{\tilde{P}_x\}_{x\in\mathcal{V}}$.
**Claim 1:** If there exist constants $\tilde{K}_1, \tilde{K}_2, \tilde{K}_3, \tilde{K}_4$ such that

$$\begin{aligned}
\tilde{A}_x &= \tilde{K}_1\\
\tilde{B}_x &= \tilde{K}_2\\
\tilde{C}_x &= \tilde{K}_3\\
\tilde{D}_x &= \tilde{K}_4
\end{aligned}$$

in $\mathcal{V}$, then $\tilde{K}_1 = \tilde{K}_4 = 0$ and $\tilde{K}_2 = -\frac{c}{b}\tilde{K}_3$. Indeed, if there are such constants then

$$\begin{aligned}
\tilde{K}_1 &= \frac{c(a+cg(x))(4ag'(x) - ag'''(x) + 4cg(x)g'(x) - cg(x)g'''(x) - 3cg'(x)g''(x))}{a - cg''(x) + cg(x)}\\
\tilde{K}_2 &= \frac{b(4a^2g'(x) - a^2g'''(x) + 12c^2g^2(x)g'(x) - 3c^2g^2(x)g'''(x) + 16acg(x)g'(x))}{a - cg''(x) + cg(x)}\\
&\quad + \frac{b(-4acg(x)g'''(x) - 6acg'(x)g''(x) - 9c^2g(x)g'(x)g''(x))}{a - cg''(x) + cg(x)}\\
\tilde{K}_3 &= \frac{b^2(8ag(x)g'(x) - 2ag(x)g'''(x) - 3ag'(x)g''(x) + 12cg^2(x)g'(x) - 3cg^2(x)g'''(x))}{a - cg''(x) + cg(x)}\\
&\quad - \frac{9b^2cg(x)g'(x)g''(x)}{a - cg''(x) + cg(x)}\\
\tilde{K}_4 &= \frac{b^3g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x))}{a - cg''(x) + cg(x)}.
\end{aligned}$$

Rewriting we get

$$\begin{aligned}
0 &= -c\,(a+cg(x))^2\,g'''(x) + c\,(a+cg(x))\left(4ag'(x) + c\left(4g(x)g'(x) - 3g'(x)g''(x)\right)\right)\\
&\quad - \tilde{K}_1\left(a - cg''(x) + cg(x)\right)\\
0 &= -b\,(a+cg(x))\,(a+3cg(x))\,g'''(x) - \tilde{K}_2\left(a - cg''(x) + cg(x)\right)\\
&\quad + b\left(4a^2g'(x) + 3c^2g(x)\left(4g(x)g'(x) - 3g'(x)g''(x)\right)\right)\\
&\quad + b\left(4ac\left(4g(x)g'(x) - 3g'(x)g''(x)\right) + 6acg'(x)g''(x)\right)\\
0 &= -b^2g(x)\,(2a+3cg(x))\,g'''(x) + b^2\left(2a\left(4g(x)g'(x) - 3g'(x)g''(x)\right) + 3ag'(x)g''(x)\right)\\
&\quad + 3b^2cg(x)\left(4g(x)g'(x) - 3g'(x)g''(x)\right) - \tilde{K}_3\left(a - cg''(x) + cg(x)\right)\\
0 &= -b^3g^2(x)g'''(x) + b^3g(x)\left(4g(x)g'(x) - 3g'(x)g''(x)\right) - \tilde{K}_4\left(a - cg''(x) + cg(x)\right).
\end{aligned}$$

Multiplying the first equality by $-b(a+cg(x))(a+3cg(x))$, the second equality by $-c(a+cg(x))^2$, the third equality by $-b^3g^2(x)$ and the last equality by $-b^2g(x)(2a+3cg(x))$

$$
\begin{aligned}
0 &= bc(a+cg(x))^3(a+3cg(x))g'''(x)\\
&\quad -b(a+cg(x))(a+3cg(x))\left(c(a+cg(x))\left(4ag'(x)+c\left(4g(x)g'(x)-3g'(x)g''(x)\right)\right)\right)\\
&\quad +b\tilde{K}_1(a+cg(x))(a+3cg(x))\left(a-cg''(x)+cg(x)\right)\\
0 &= bc(a+cg(x))^3(a+3cg(x))g'''(x)+\tilde{K}_2c(a+cg(x))^2\left(a-cg''(x)+cg(x)\right)\\
&\quad -bc(a+cg(x))^2\left(4a^2g'(x)+3c^2g(x)\left(4g(x)g'(x)-3g'(x)g''(x)\right)\right)\\
&\quad -bc(a+cg(x))^2\left(4ac\left(4g(x)g'(x)-3g'(x)g''(x)\right)+6acg'(x)g''(x)\right)\\
0 &= b^5g^3(x)(2a+3cg(x))g'''(x)+\tilde{K}_3b^3g^2(x)\left(a-cg''(x)+cg(x)\right)\\
&\quad -b^3g^2(x)\left(b^2\left(2a\left(4g(x)g'(x)-3g'(x)g''(x)\right)+3ag'(x)g''(x)+3cg(x)\left(4g(x)g'(x)-3g'(x)g''(x)\right)\right)\right)\\
0 &= b^5g^3(x)(2a+3cg(x))g'''(x)\\
&\quad -b^2g(x)(2a+3cg(x))\left(b^3g(x)\left(4g(x)g'(x)-3g'(x)g''(x)\right)-\tilde{K}_4\left(a-cg''(x)+cg(x)\right)\right)
\end{aligned}
$$

and subtracting the second equality from the first equality and the last equality from the third equality

$$
\begin{aligned}
0 &= -c(a+cg(x))\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g(x)\tilde{K}_2+3bcg(x)\tilde{K}_1+3a^2bcg'(x)+3abc^2g(x)g'(x)\right)g''(x)\\
&\quad +(a+cg(x))^2\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g(x)\tilde{K}_2+3bcg(x)\tilde{K}_1\right)\\
0 &= b^2g(x)\left(2ac\tilde{K}_4+3c^2g(x)\tilde{K}_4-bcg(x)\tilde{K}_3-3ab^3g(x)g'(x)\right)g''(x)\\
&\quad -b^2g(x)(a+cg(x))\left(2a\tilde{K}_4-bg(x)\tilde{K}_3+3cg(x)\tilde{K}_4\right)
\end{aligned}
$$

Multiplying the first equality by

$$b^2g(x)\left(2ac\tilde{K}_4+3c^2g(x)\tilde{K}_4-bcg(x)\tilde{K}_3-3ab^3g(x)g'(x)\right)$$

and the second equality by

$$-c(a+cg(x))\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g(x)\tilde{K}_2+3bcg(x)\tilde{K}_1+3a^2bcg'(x)+3abc^2g(x)g'(x)\right)$$

we get

$$
\begin{aligned}
0 &= -b^2cg(a+cg)\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g\tilde{K}_2+3bcg\tilde{K}_1+3a^2bcg'+3abc^2gg'\right)\left(2ac\tilde{K}_4+3c^2g\tilde{K}_4\right)g''\\
&\quad +b^2cg(a+cg)\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g\tilde{K}_2+3bcg\tilde{K}_1+3a^2bcg'+3abc^2gg'\right)\left(bcg\tilde{K}_3+3ab^3gg'\right)g''\\
&\quad +b^2g(a+cg)^2\left(2ac\tilde{K}_4+3c^2g\tilde{K}_4-bcg\tilde{K}_3-3ab^3gg'\right)\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g\tilde{K}_2+3bcg\tilde{K}_1\right)\\
0 &= -b^2cg(a+cg)\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g\tilde{K}_2+3bcg\tilde{K}_1+3a^2bcg'+3abc^2gg'\right)\left(2ac\tilde{K}_4+3c^2g\tilde{K}_4\right)g''\\
&\quad +b^2cg(a+cg)\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g\tilde{K}_2+3bcg\tilde{K}_1+3a^2bcg'+3abc^2gg'\right)\left(bcg\tilde{K}_3+3ab^3gg'\right)g''\\
&\quad +b^2cg(a+cg)^2\left(ab\tilde{K}_1-ac\tilde{K}_2-c^2g\tilde{K}_2+3bcg\tilde{K}_1+3a^2bcg'+3abc^2gg'\right)\left(2a\tilde{K}_4-bg\tilde{K}_3+3cg\tilde{K}_4\right)
\end{aligned}
$$

and subtracting the second equality from the first equality

$$
\begin{aligned}
0 &= -3ab^3g(x)g'(x)(a+cg(x))^2\left(c\left(3b^3\tilde{K}_1+3c^3\tilde{K}_4-b^2c\tilde{K}_2-bc^2\tilde{K}_3\right)g^2(x)\right)\\
&\quad -3ab^3g(x)g'(x)(a+cg(x))^2\left(a\left(b^3\tilde{K}_1+5c^3\tilde{K}_4-b^2c\tilde{K}_2-bc^2\tilde{K}_3\right)g(x)+2a^2c^2\tilde{K}_4\right)
\end{aligned}
$$

and since $g(x)g'(x)\left(a+cg(x)\right)^2 \neq 0$ in $\mathcal{V}$

$$0 = c\left(3b^3\tilde{K}_1 + 3c^3\tilde{K}_4 - b^2c\tilde{K}_2 - bc^2\tilde{K}_3\right)g^2(x) + a\left(b^3\tilde{K}_1 + 5c^3\tilde{K}_4 - b^2c\tilde{K}_2 - bc^2\tilde{K}_3\right)g(x) + 2a^2c^2\tilde{K}_4.$$

Since $g(x)$ is non-constant in $\mathcal{V}$ we conclude that the coefficients of this polynomial are zero. We have,

$$\begin{aligned} 0 &= c\left(3b^3\tilde{K}_1 + 3c^3\tilde{K}_4 - b^2c\tilde{K}_2 - bc^2\tilde{K}_3\right) \\ 0 &= a\left(b^3\tilde{K}_1 + 5c^3\tilde{K}_4 - b^2c\tilde{K}_2 - bc^2\tilde{K}_3\right) \\ 0 &= 2a^2c^2\tilde{K}_4 \end{aligned}$$

that is, $\tilde{K}_4 = 0$, and consequently

$$\begin{aligned} 0 &= 3b^3\tilde{K}_1 - b^2c\tilde{K}_2 - bc^2\tilde{K}_3 \\ 0 &= b^3\tilde{K}_1 - b^2c\tilde{K}_2 - bc^2\tilde{K}_3 \end{aligned}$$

and subtracting the second equality from the first equality we conclude $\tilde{K}_1 = 0$. Then,

$$0 = -b^2c\tilde{K}_2 - bc^2\tilde{K}_3$$

and thus we proof the claim.

**Claim 2:** The family of polynomials $\tilde{P}_x$ is irreducible in $\mathcal{V}$. Indeed, just check claim 2 of Proposition 4.2.1.

By claim 2, the polynomials $\{\tilde{P}_x\}_{x\in\mathcal{V}}$ are irreducible. As noticed previously, the curve $(f'(t), f(t))$ is contained in the roots of $\{\tilde{P}_x\}_{x\in\mathcal{V}}$ so by the theory of Plane Algebraic Curves we conclude that such a curve is constant or the polynomials $\tilde{P}_x$ are multiples by constants. In the case where the curve $(f'(t), f(t))$ is constant we conclude that $f$ is constant in $I$. Suppose that $\{\tilde{P}_x\}_{x\in\mathcal{V}}$ are all multiples. Since in this case the only possible multiple is $\lambda = 1$ we conclude that

$$\tilde{P}_x(W,Y) = W + \tilde{A}_xY^3 + \tilde{B}_xY^2 + \tilde{C}_xY + \tilde{D}_x,$$

is constant in the variable $x$, that is, the polynomials $\tilde{P}_x$ are all equal. Therefore, their coefficients are constant, that is, there are $\tilde{K}_1, \tilde{K}_2, \tilde{K}_3, \tilde{K}_4$ constants such that,

$$\tilde{A}_x = \tilde{K}_1 \qquad \tilde{B}_x = \tilde{K}_2 \qquad \tilde{C}_x = \tilde{K}_3 \qquad \tilde{D}_x = \tilde{K}_4,$$

and by claim 1 we conclude $\tilde{K}_1 = \tilde{K}_4 = 0$ and $\tilde{K}_2 = -\frac{c}{b}\tilde{K}_3$. Then,

$$0 = \tilde{P}_x(f'(t), f(t)) = f'(t) - \frac{c\tilde{K}_3}{b}f^2(t) + \tilde{K}_3f(t),$$

that is, setting $\tilde{K}_3 = \mu$,

$$f'(t) = \mu\left(\frac{c}{b}f^2(t) - f(t)\right) \tag{4.29}$$

in $I$. If $\mu = 0$ then $f$ is constant in $I$. Let us consider $\mu \neq 0$. The equation in (4.29) is a Bernoulli differential equation. To solve it we will make the change $h(t) = \frac{1}{f(t)}$ obtaining a linear differential equation. Observe that $f \equiv 0$ and $f \equiv \frac{b}{c}$ are solutions of (4.29) and therefore, by the Existence and Uniqueness Theorem, any other solution does not reach them. If $f \neq 0$ and $f \neq \frac{b}{c}$ we obtain

$$h'(t) = \mu\left(h(t) - \frac{c}{b}\right)$$

which multiplying by the integration factor $e^{-\mu t}$,

$$\frac{d}{dt}[e^{-\mu t}h(t)] = -\frac{c\mu}{b}e^{-\mu t},$$

and integrating,

$$e^{-\mu t}h(t) - e^{-\mu t_0}h(t_0) = \int_{t_0}^{t}\frac{d}{dr}[e^{-\mu r}h(r)]dr = -\frac{c\mu}{b}\int_{t_0}^{t}e^{-\mu r}dr = \frac{c}{b}\left(e^{-\mu t} - e^{-\mu t_0}\right),$$

i.e.,

$$h(t) = \frac{c}{b} + \left(h(t_0) - \frac{c}{b}\right)e^{\mu(t-t_0)},$$

thus there exist constants $C$ and $D \neq \frac{c}{b}$ such that

$$f(t) = \frac{1}{h(t)} = \frac{1}{\frac{c}{b} + \left(D - \frac{c}{b}\right)e^{\mu(t-C)}}$$

in the domain

$$I \subset \begin{cases} \mathbb{R}, & \text{if } 1 - \frac{bD}{c} < 0 \\ \mathbb{R} - \{C - \frac{1}{\mu}\ln\left(1 - \frac{bD}{c}\right)\}, & \text{if } 1 - \frac{bD}{c} > 0. \end{cases}$$

□

**Theorem 4.3.1.** *Let $f \in C^1(I,\mathbb{R})$ and $g \in C^3(J,\mathbb{R})$. Moreover, assume $(a,b,c) \in \mathbb{R}^3$ nonzero such that $ab \neq 0$. If $u$ is a* ***separable*** *solution of the* ***Novikov equation*** *then the possibilities for $f$ and $g$ are:*

*i) $f$ is arbitrary in $I \subset \mathbb{R}$ and $g \equiv \gamma$ constant in $J \subset \mathbb{R}$, with $a + c\gamma = 0$.*

*ii) $f \equiv \lambda$ constant in $I \subset \mathbb{R}$, with $b + c\lambda = 0$, and $g$ is arbitrary in $J \subset \mathbb{R}$.*

*iii) $f \equiv \lambda$ constant in $I \subset \mathbb{R}$, with $b + c\lambda \neq 0$, and the function $g$ satisfies the differential equation*

$$(g(x) + \Lambda)^2 g'''(x) = (g(x) + \Lambda)\left(4(g(x) + \Lambda) - 3g''(x)\right)g'(x) \tag{4.30}$$

*where $\Lambda = \frac{a\lambda}{b+c\lambda}$.*

**Proof:** Suppose that $u(x,t) = af(t) + bg(x) + cf(t)g(x)$ is a solution of (4.26). Thus, the functions $f, g$ satisfy

$$\begin{aligned}
0 =& (a - cg''(x) + cg(x))f'(t) \\
&+ c(a + cg(x))(4ag'(x) - ag'''(x) + 4cg(x)g'(x) - cg(x)g'''(x) - 3cg'(x)g''(x))f^3(t) \\
&+ b(4a^2g'(x) - a^2g'''(x) + 12c^2g^2(x)g'(x) - 3c^2g^2(x)g'''(x) + 16acg(x)g'(x))f^2(t) \\
&- b(4acg(x)g'''(x) + 6acg'(x)g''(x) + 9c^2g(x)g'(x)g''(x))f^2(t) \\
&+ b^3g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x)) + b^2(8ag(x)g'(x) - 2ag(x)g'''(x))f(t) \\
&+ b^2(-3ag'(x)g''(x) + 12cg^2(x)g'(x) - 3cg^2(x)g'''(x) - 9cg(x)g'(x)g''(x))f(t).
\end{aligned} \tag{4.31}$$

Notice that if $g \equiv \gamma$ is constant in $J \subset \mathbb{R}$, with $a+\gamma c = 0$, then $f$ is arbitrary, that is, we obtain $i$). The following two claim ensure that disregarding this specific case for $g$ we obtain that $f$ is constant.

**Claim 1:** If $g \equiv \gamma$ is constant in $J \subset \mathbb{R}$, with $a+\gamma c \neq 0$, then $f$ is constant in $I \subset \mathbb{R}$. Indeed, if $g \equiv \gamma$ the equation in $(4.31)$ reduces to

$$0 = (a + c\gamma) f'(t)$$

and we conclude the claim.

**Claim 2:** If $g$ is non-constant in $J \subset \mathbb{R}$ then $f$ is constant in $I \subset \mathbb{R}$. Indeed, if $g$ is non-constant we conclude by Proposition 4.3.1 that either $f$ is constant or there exist constants $C$, $D \neq \frac{c}{b}$, $\mu \neq 0$ such that

$$f(t) = \frac{1}{\frac{c}{b} + \left(D - \frac{c}{b}\right) e^{\mu(t-C)}}.$$

We will show that assuming the second case leads us to a contradiction. Substituting this $f$ into the equation $(4.31)$ we obtain

$$\begin{aligned}
0 =& b^3(bD-c)^3 g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x)) \left(e^{\mu(t-C)}\right)^3 \\
&+ b(bD-c)^2 \left(-a\mu - c\mu g(x) + c\mu g''(x) + 24b^2cg^2(x)g'(x) - 6b^2cg^2(x)g'''(x)\right) \left(e^{\mu(t-C)}\right)^2 \\
&+ b(bD-c)^2 \left(8ab^2g(x)g'(x) - 2ab^2g(x)g'''(x) - 3ab^2g'(x)g''(x) - 18b^2cg(x)g'(x)g''(x)\right) \left(e^{\mu(t-C)}\right)^2 \\
&+ b(bD-c) \left(4a^2b^2g'(x) - a^2b^2g'''(x) - ac\mu - c^2\mu g(x) + c^2\mu g''(x) + 48b^2c^2g^2(x)g'(x)\right) e^{\mu(t-C)} \\
&+ b(bD-c) \left(-12b^2c^2g^2(x)g'''(x) - 36b^2c^2g(x)g'(x)g''(x) + 32ab^2cg(x)g'(x)\right) e^{\mu(t-C)} \\
&- b(bD-c) \left(8ab^2cg(x)g'''(x) + 12ab^2cg'(x)g''(x)\right) e^{\mu(t-C)} \\
&+ 2b^3c(a+2cg(x))(4ag'(x) - ag'''(x) + 8cg(x)g'(x) - 2cg(x)g'''(x) - 6cg'(x)g''(x)),
\end{aligned}$$

and since $e^{\mu(t-C)}$ is not constant, this polynomial equality holds if, and only if, all coefficients are zero

$$\begin{aligned}
0 =& b^3(bD-c)^3 g(x)(4g(x)g'(x) - g(x)g'''(x) - 3g'(x)g''(x)) \\
0 =& b(bD-c)^2 \left(-a\mu - c\mu g(x) + c\mu g''(x) + 24b^2cg^2(x)g'(x) - 6b^2cg^2(x)g'''(x)\right) \\
&+ b(bD-c)^2 \left(8ab^2g(x)g'(x) - 2ab^2g(x)g'''(x) - 3ab^2g'(x)g''(x) - 18b^2cg(x)g'(x)g''(x)\right) \\
0 =& b(bD-c) \left(4a^2b^2g'(x) - a^2b^2g'''(x) - ac\mu - c^2\mu g(x) + c^2\mu g''(x) + 48b^2c^2g^2(x)g'(x)\right) \\
&+ b(bD-c) \left(-12b^2c^2g^2(x)g'''(x) - 36b^2c^2g(x)g'(x)g''(x) + 32ab^2cg(x)g'(x)\right) \\
&- b(bD-c) \left(8ab^2cg(x)g'''(x) + 12ab^2cg'(x)g''(x)\right) \\
0 =& 2b^3c(a+2cg(x))(4ag'(x) - ag'''(x) + 8cg(x)g'(x) - 2cg(x)g'''(x) - 6cg'(x)g''(x)),
\end{aligned}$$

and since $b(bD-c)\neq 0$

$$\begin{aligned}
0 =&g(x)(4g(x)g'(x)-g(x)g'''(x)-3g'(x)g''(x))\\
0 =&-a\mu-c\mu g(x)+c\mu g''(x)+24b^2cg^2(x)g'(x)-6b^2cg^2(x)g'''(x)+8ab^2g(x)g'(x)\\
&-2ab^2g(x)g'''(x)-3ab^2g'(x)g''(x)-18b^2cg(x)g'(x)g''(x)\\
0 =&4a^2b^2g'(x)-a^2b^2g'''(x)-ac\mu-c^2\mu g(x)+c^2\mu g''(x)+48b^2c^2g^2(x)g'(x)-12b^2c^2g^2(x)g'''(x)\\
&-36b^2c^2g(x)g'(x)g''(x)+32ab^2cg(x)g'(x)-8ab^2cg(x)g'''(x)-12ab^2cg'(x)g''(x)\\
0 =&c(a+2cg(x))(4ag'(x)-ag'''(x)+8cg(x)g'(x)-2cg(x)g'''(x)-6cg'(x)g''(x)).
\end{aligned} \tag{4.32}$$

We have two cases depending on $c$. If $c=0$ we obtain from (4.32)

$$\begin{aligned}
0 =&g(x)(4g(x)g'(x)-g(x)g'''(x)-3g'(x)g''(x))\\
0 =&-\mu+8b^2g(x)g'(x)-2b^2g(x)g'''(x)-3b^2g'(x)g''(x)\\
0 =&4g'(x)-g'''(x)
\end{aligned}$$

that isolating $g'''(x)$ in the third equality and substituting it in the first two we have

$$\begin{aligned}
0 =&3gg'(x)g''(x)\\
g'(x)g''(x) =&-\frac{\mu}{3b^2},
\end{aligned}$$

that is, substituting the second equality into the first, we conclude $g\equiv 0$ in $J$, a contradiction since $g$ is non-constant. If $c\neq 0$ we have from(4.32)

$$\begin{aligned}
0 =&g(x)(4g(x)g'(x)-g(x)g'''(x)-3g'(x)g''(x))\\
0 =&-a\mu-c\mu g(x)+c\mu g''(x)+24b^2cg^2(x)g'(x)-6b^2cg^2(x)g'''(x)+8ab^2g(x)g'(x)\\
&-2ab^2g(x)g'''(x)-3ab^2g'(x)g''(x)-18b^2cg(x)g'(x)g''(x)\\
0 =&4a^2b^2g'(x)-a^2b^2g'''(x)-ac\mu-c^2\mu g(x)+c^2\mu g''(x)+48b^2c^2g^2(x)g'(x)-12b^2c^2g^2(x)g'''(x)\\
&-36b^2c^2g(x)g'(x)g''(x)+32ab^2cg(x)g'(x)-8ab^2cg(x)g'''(x)-12ab^2cg'(x)g''(x)\\
0 =&(a+2cg(x))(4ag'(x)-ag'''(x)+8cg(x)g'(x)-2cg(x)g'''(x)-6cg'(x)g''(x)).
\end{aligned} \tag{4.33}$$

Suppose that exists an open set $V\subset J$ such that $4g(x)g'(x)-g(x)g'''(x)-3g'(x)g''(x)\neq 0$. We obtain by the first equality that $g\equiv 0$ in $V$ and therefore $4g(x)g'(x)-g(x)g'''(x)-3g'(x)g''(x)\equiv 0$ in $V$, a contradiction. Hence $4g(x)g'(x)-g(x)g'''(x)-3g'(x)g''(x)\equiv 0$ in $J$. Then $g(x)g'''(x)=4g(x)g'(x)-3g'(x)g''(x)$ which substituting into (4.33) we obtain

$$\begin{aligned}
0 =&-a\mu-c\mu g(x)+c\mu g''(x)+3ab^2g'(x)g''(x)\\
0 =&4a^2b^2g'(x)-a^2b^2g'''(x)-ac\mu-c^2\mu g(x)+c^2\mu g''(x)+12ab^2cg'(x)g''(x)\\
0 =&(a+2cg(x))(4g'(x)-g'''(x))
\end{aligned}$$

which multiplying the last two equalities by $g$ and substituting again $g(x)g'''(x)=4g(x)g'(x)-3g'(x)g''(x)$

$$\begin{aligned}
0 =&\ 3ab^2g'(x)g''(x)+c\mu g''(x)-(a\mu+c\mu g(x))\\
0 =&\ (3a^2b^2+12cab^2g(x))g'(x)g''(x)+c^2\mu g(x)g''(x)-(\mu c^2g^2(x)+ac\mu g(x))\\
0 =&\ (a+2cg(x))g'(x)g''(x).
\end{aligned}$$

Multiplying the first two equalities by $a+2cg(x)$ and substituting the last one into these two we have

$$\begin{aligned}0 =& c\mu(a+2cg(x))g''(x)-(a\mu+c\mu g(x))(a+2cg(x))\\ 0 =& c^2\mu(a+2cg(x))g(x)g''(x)-c^2\mu g^2(x)+ac\mu(a+2cg(x))g(x).\end{aligned}$$

Isolating $c\mu(a+2cg(x))g''(x)$ in the first equality and substituting it in the second we obtain the non-zero polynomial

$$0=2c^2g^3(x)+c(5a-1)g^2(x)+2a^2g(x),$$

i.e., $g(x)$ is constant, a contradiction and thus we conclude the claim.

From claim 1 and 2 we obtain that $f$ is constant. Then:

**-***ii*) **and** *iii*) If $f\equiv\lambda$ is constant in $I\subset\mathbb{R}$ the equation in (4.31) reduces to

$$\begin{aligned}0=&-(b+c\lambda)(a\lambda+bg(x)+c\lambda g(x))^2g'''(x)\\ &+(b+c\lambda)(a\lambda+bg(x)+c\lambda g(x))g'(x)(-3bg''(x)+4a\lambda+4bg(x)+4c\lambda g(x)-3c\lambda g''(x)).\end{aligned}\tag{4.34}$$

If $b+c\lambda=0$ we get *ii*). If $b+c\lambda\neq0$ the equation (4.34) reduces to

$$\left(g(x)+\frac{a\lambda}{b+c\lambda}\right)^2g'''(x)=\left(g(x)+\frac{a\lambda}{b+c\lambda}\right)\left(4\left(g(x)+\frac{a\lambda}{b+c\lambda}\right)-3g''(x)\right)g'(x).$$

□

**Corollary 4.3.1.** *Assume the setting of Theorem* 4.3.1. *If* $g(x)\neq-\Lambda$ *the change of variable* $h(x)=(g(x)+\Lambda)^2$ *reduces the equation* (4.30) *to*

$$h'''(x)=4h'(x)\tag{4.35}$$

*whose general solution is* $h(x)=k_1e^{2x}+k_2e^{-2x}-\frac{C}{4}$.

**Proof:** Observe that

$$\begin{aligned}h'(x)&=2\,(g(x)+\Lambda)\,g'(x)\\ h'''(x)&=6g'(x)g''(x)+2\,(g(x)+\Lambda)\,g'''(x)\end{aligned}$$

which substituting in (4.35)

$$(g(x)+\Lambda)\,g'''(x)=\left(4\,(g(x)+\Lambda)-3g''(x)\right)g'(x).$$

Integrating (4.35) we obtain the second order linear differential equation

$$h''(x)-4h(x)=C.$$

We can find the general solution of this equation by adding a particular solution $\left(h_p(x)=-\frac{C}{4}\right)$ with the general solution of the homogeneous differential equation

$$H''(x)-4H(x)=0,$$

whose solution $H(x) = k_1 e^{2x} + k_2 e^{-2x}$ is obtained by substituting $e^{\alpha x}$ into the differential equation and then solving the characteristic polynomial

$$\alpha^2 - 4 = 0.$$

$\square$

# APPENDIX A

# Factorization of Polynomials

This appendix will be dedicated to list and proof lemmas used throughout the text.

**Lemma A.0.1.** *Let $K_1, K_2, K_3, c$ be constants such that $K_2 \neq$ and $c \neq 0$. Define $G : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{G} : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$G(x,y) = 4y^6 + 9xy^4 - K_2(1+4c)xy^3 + 6x^2y^2 - K_2(1+4c)x^2y + (1-K_3)x^3 + cK_2^2x^2$$

*and*

$$\overline{G}(x,y) = y^4 + 2xy^2 - K_2xy + (1-K_3)x^2.$$

*Then the polynomials $G$ and $\overline{G}$ have no factors in common.*

**Proof:** We will split the proof into three claims.

**Claim 1:** $\overline{G}$ does not divide $G$. Indeed, if $\overline{G}$ divides $G$ then there exists a polynomial of degree 2 such that

$$G(x,y) = \overline{G}(x,y)(f + gx + hy + ixy + jx^2 + ky^2),$$

which reorganizing,

$$\begin{aligned}
0 =\, & j(K_3-1)x^4 - 2jx^3y^2 + (jK_2 + i(K_3-1))x^3y + (g(K_3-1) + 1 - K_3)x^3 \\
& - jx^2y^4 - 2ix^2y^3 + (iK_2 - 2g + k(K_3-1) + 6)x^2y^2 \\
& + (gK_2 - K_2(1+4c) + h(K_3-1))x^2y + (cK_2^2 + f(K_3-1))x^2 \\
& - ixy^5 + (9 - 2k - g)xy^4 + (kK_2 - K_2(1+4c) - 2h)xy^3 + (hK_2 - 2f)xy^2 \\
& + fK_2xy + (4-k)y^6 - hy^5 - fy^4,
\end{aligned}$$

that occurs if, and only if,

$$
\begin{aligned}
0 &= j(K_3-1) & 0 &= iK_2-2g+k(K_3-1)+6 & 0 &= hK_2-2f\\
0 &= 2j & 0 &= gK_2-K_2(1+4c)+h(K_3-1) & 0 &= fK_2\\
0 &= jK_2+i(K_3-1) & 0 &= cK_2^2+f(K_3-1) & 0 &= 4-k\\
0 &= g(K_3-1)+1-K_3 & 0 &= i & 0 &= h\\
0 &= j & 0 &= 9-2k-g & 0 &= f.\\
0 &= 2i & 0 &= kK_2-K_2(1+4c)-2h
\end{aligned}
$$

Then $f=0$, which substituting into the third equality of the second column we obtain $0 = cK_2^2 \neq 0$, a contradiction and thus we conclude the claim.

**Claim 2:** If $K_3 \neq 1$ then $\overline{G}$ is irreducible. Indeed, if $\overline{G}$ is reducible then there are two possibilities. In the first, there exist polynomials of degree 1 and degree 3 such that

$$\overline{G}(x,y) = (f+gx+hy)(i+jx+ky+lxy+mx^2+ny^2+oxy^2+px^2y+qx^3+ry^3)$$

which reorganizing,

$$
\begin{aligned}
0 = &-gqx^4-(gp+hq)x^3y-(gm+fq)x^3-(go+hp)x^2y^2-(gl+hm+fp)x^2y\\
&+(1-gj-fm-K_3)x^2-(ho+gr)xy^3+(2-fo-gn-hl)xy^2-(K_2+fl+gk+hj)xy\\
&-(gi+fj)x+(1-hr)y^4-(hn+fr)y^3-(hk+fn)y^2+(hi+fk)y-fi,
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{aligned}
0 &= gq & 0 &= 1-gj-fm-K_3 & 0 &= 1-hr\\
0 &= gp+hq & 0 &= ho+gr & 0 &= hn+fr\\
0 &= gm+fq & 0 &= 2-fo-gn-hl & 0 &= hk+fn\\
0 &= go+hp & 0 &= K_2+fl+gk+hj & 0 &= hi+fk\\
0 &= gl+hm+fp & 0 &= gi+fj & 0 &= fi.
\end{aligned}
\tag{A.1}
$$

By the first equality of the first column $g=0$ or $q=0$. Suppose $g=0$. We have,

$$
\begin{aligned}
0 &= hq & 0 &= ho & 0 &= hn+fr\\
0 &= fq & 0 &= 2-fo-hl & 0 &= hk+fn\\
0 &= hp & 0 &= K_2+fl+hj & 0 &= hi+fk\\
0 &= hm+fp & 0 &= fj & 0 &= fi.\\
0 &= 1-fm-K_3 & 0 &= 1-hr
\end{aligned}
$$

By the fifth equality of the second column we have $h \neq 0$. Thus, by the third equality of the first column $p=0$ which substituting into the fourth equality of the first column we conclude $m=0$.

Therefore, by the fifth equality of the first column, $K_3 = 1$, a contradiction. Therefore, $g \neq 0$ and $q = 0$. From $(A.1)$,

$$
\begin{aligned}
0 &= gp & 0 &= ho + gr & 0 &= hn + fr \\
0 &= gm & 0 &= 2 - fo - gn - hl & 0 &= hk + fn \\
0 &= go + hp & 0 &= K_2 + fl + gk + hj & 0 &= hi + fk \\
0 &= gl + hm + fp & 0 &= gi + fj & 0 &= fi. \\
0 &= 1 - gj - fm - K_3 & 0 &= 1 - hr & &
\end{aligned}
$$

By the first equality of the first column we have $p = 0$. By the third equality of the first column $o = 0$ which substituting in the first equality of the second column $r = 0$ we obtain a contradiction with the fifth equality of the second column. Therefore we do not have this factorization. In the second possibility there are polynomials of degree 2 such that

$$\overline{G}(x,y) = (f + gx + hy + ixy + jx^2 + ky^2)(l + mx + ny + oxy + px^2 + qy^2)$$

which reorganizing,

$$
\begin{aligned}
0 = &-jpx^4 - (pi + jo)x^3y - (gp + jm)x^3 - (oi + jq + kp)x^2y^2 - (mi + go + hp + jn)x^2y \\
&+ (1 - gm - fp - jl - K_3)x^2 - (qi + ko)xy^3 + (2 - ho - gq - km - ni)xy^2 \\
&- (K_2 + li + fo + gn + hm)xy - (fm + gl)x + (1 - kq)y^4 - (hq + kn)y^3 - (hn + fq + kl)y^2 \\
&- (fn + hl)y - fl,
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{aligned}
0 &= jp & 0 &= 1 - gm - fp - jl - K_3 & 0 &= 1 - kq \\
0 &= pi + jo & 0 &= qi + ko & 0 &= hq + kn \\
0 &= gp + jm & 0 &= 2 - ho - gq - km - ni & 0 &= hn + fq + kl \\
0 &= oi + jq + kp & 0 &= K_2 + li + fo + gn + hm & 0 &= fn + hl \\
0 &= mi + go + hp + jn & 0 &= fm + gl & 0 &= fl.
\end{aligned}
$$

By the first equality of the third column $kq = 1$. Suppose, without loss of generality, $k = q = 1$. By the first equality of the first column we have that $j = 0$ or $p = 0$. Suppose, without loss of generality, $j = 0$. We have,

$$
\begin{aligned}
0 &= pi & 0 &= i + o & 0 &= h + n \\
0 &= gp & 0 &= 2 - ho - g - m - ni & 0 &= hn + f + l \\
0 &= oi + p & 0 &= K_2 + li + fo + gn + hm & 0 &= fn + hl \\
0 &= mi + go + hp & 0 &= fm + gl & 0 &= fl. \\
0 &= 1 - gm - fp - K_3 & & & &
\end{aligned}
\tag{A.2}
$$

By the first equality of the first column $p = 0$ or $i = 0$. Suppose $p = 0$. We have,

$$\begin{array}{lll} 0 = oi & 0 = 2 - ho - g - m - ni & 0 = hn + f + l \\ 0 = mi + go & 0 = K_2 + li + fo + gn + hm & 0 = fn + hl \\ 0 = 1 - gm - K_3 & 0 = fm + gl & 0 = fl. \\ 0 = i + o & 0 = h + n & \end{array}$$

From the first and fourth equalities of the first column we conclude $i = o = 0$. We have,

$$\begin{array}{lll} 0 = 1 - gm - K_3 & 0 = fm + gl & 0 = fn + hl \\ 0 = 2 - g - m & 0 = h + n & 0 = fl. \\ 0 = K_2 + gn + hm & 0 = hn + f + l & \end{array}$$

By the second equalities of the first and second columns, $m = 2 - g$ and $n = -h$. We have,

$$\begin{array}{lll} 0 = 1 - g(2-g) - K_3 & 0 = f(2-g) + gl & 0 = h(l-f) \\ 0 = K_2 - gh + h(2-g) & 0 = -h^2 + f + l & 0 = fl. \end{array}$$

By the second equality of the second column $l = h^2 - f$. We have,

$$\begin{array}{lll} 0 = 1 - g(2-g) - K_3 & 0 = f(2-g) + g(h^2 - f) & 0 = f(h^2 - f). \\ 0 = K_2 - gh + h(2-g) & 0 = h(2f - h^2) & \end{array}$$

By the second equality of the first column we have $h \neq 0$ (otherwise $K_2 = 0$, a contradiction). Thus, by the second equality of the second column, $f = \frac{h^2}{2}$. Substituting this into the last equality we conclude $h^4 = 0$, a contradiction. Thus $p \neq 0$ and $i = 0$. We have, from$(A.2)$,

$$\begin{array}{lll} 0 = gp & 0 = o & 0 = h + n \\ 0 = p & 0 = 2 - ho - g - m & 0 = hn + f + l \\ 0 = go + hp & 0 = K_2 + fo + gn + hm & 0 = fn + hl \\ 0 = 1 - gm - fp - K_3 & 0 = fm + gl & 0 = fl. \end{array}$$

By the second equality of the first column $p = 0$, a contradiction and therefore the second possibility also does not occur. Thus we proof the claim.

**Claim 3:** If $K_3 = 1$ then $\overline{G}$ is reducible but none of its factors divides $G$. Indeed, if $K_3 = 1$, then

$$\begin{aligned} G(x,y) &= 4y^6 + 9xy^4 - K_2(1+4c)xy^3 + 6x^2y^2 - K_2(1+4c)x^2y + cK_2^2x^2 \\ \overline{G}(x,y) &= y(y^3 + 2xy - K_2x). \end{aligned}$$

By claim 1 the polynomial $\overline{G}$ does not divide $G$. $y$ also does not divide $G$. The only possibility is if $(y^3 + 2xy - K_2x)|G$. If this occurs there exists a polynomial of 3 such that

$$G(x,y) = (f + gx + hy + ixy + jx^2 + ky^2 + lx^2y + mxy^2 + nx^3 + oy^3)(y^3 + 2xy - K_2x)$$

which reorganizing,

$$\begin{aligned}0 = &-2nx^4y + nK_2x^4 - nx^3y^3 - 2mx^3y^2 + (mK_2 - 2j)x^3y + jK_2x^3 - mx^2y^4 - (j+2l)x^2y^3 \\ &+ (lK_2 - 2i + 6)x^2y^2 + (iK_2 - K_2(1+4c) - 2g)x^2y + (cK_2^2 + gK_2)x^2 - lxy^5 + (9 - i - 2o)xy^4 \\ &+ (oK_2 - 2k - K_2(1+4c) - g)xy^3 + (kK_2 - 2h)xy^2 + (hK_2 - 2f)xy + fK_2x + (4-o)y^6 \\ &- ky^5 - hy^4 - fy^3\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = n & 0 = j + 2l & 0 = kK_2 - 2h \\ 0 = nK_2 & 0 = lK_2 - 2i + 6 & 0 = hK_2 - 2f \\ 0 = n & 0 = iK_2 - K_2(1+4c) - 2g & 0 = fK_2 \\ 0 = 2m & 0 = cK_2^2 + gK_2 & 0 = 4 - o \\ 0 = mK_2 - 2j & 0 = l & 0 = k \\ 0 = jK_2 & 0 = 9 - i - 2o & 0 = h \\ 0 = m & 0 = oK_2 - 2k - K_2(1+4c) - g & 0 = f. \end{array}$$

By the fifth equality of the second column and the fourth equality of the third column we have $l = 0$ and $o = 4$. Substituting in the second and sixth equalities of the second column we conclude $3 = i = 1$, a contradiction, and therefore such factorization does not exist. Let us verify that the factor $(y^3 + 2xy - K_2x)$ is irreducible. Indeed, if it were reducible there would exist polynomials of degree 1 and degree 2 such that

$$y^3 + 2xy - K_2x = (f + gx + hy)(i + jx + ky + lxy + mx^2 + ny^2)$$

which reorganizing,

$$\begin{aligned}0 = &-gmx^3 - (gl + hm)x^2y - (gj + fm)x^2 - (hl + gn)xy^2 \\ &+ (2 - gk - hj - fl)xy - (K_2 + gi + fj)x + (1 - hn)y^3 \\ &- (hk + fn)y^2 - (hi + fk)y - fi,\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = gm & & \\ 0 = gl + hm & 0 = 2 - gk - hj - fl & 0 = hk + fn \\ 0 = gj + fm & 0 = K_2 + gi + fj & 0 = hi + fk \\ 0 = hl + gn & 0 = 1 - hn & 0 = fi. \end{array} \tag{A.3}$$

By the first equality of the first column $g = 0$ or $m = 0$. Suppose $g = 0$. We have,

$$\begin{array}{lll} 0 = hm & 0 = 2 - hj - fl & 0 = hk + fn \\ 0 = fm & 0 = K_2 + fj & 0 = hi + fk \\ 0 = hl & 0 = 1 - hn & 0 = fi. \end{array}$$

By the third equality of the second column $n = \frac{1}{h}$. Therefore, by the first and third equalities of the first column, $m = l = 0$. We have,

$$\begin{array}{lll} 0 = 2 - hj & 0 = hk + \dfrac{f}{h} & \\ & & 0 = fi. \\ 0 = K_2 + fj & 0 = hi + fk & \end{array}$$

By the first equalities of the first and second columns $j = \frac{2}{h}$ and $k = \frac{f}{h^2}$. We have,

$$0 = K_2 + \frac{2f}{h} \qquad 0 = hi + \frac{f^2}{h^2} \qquad 0 = fi.$$

Isolating $i$ in the second equality and substituting it in the third will imply $f = 0$, a contradiction with the first equality. Thus, $g \neq 0$ and $m = 0$. We have$(A.3)$

$$\begin{array}{lll} 0 = gl & 0 = 2 - gk - hj - fl & 0 = hk + fn \\ 0 = gj & 0 = K_2 + gi + fj & 0 = hi + fk \\ 0 = hl + gn & 0 = 1 - hn & 0 = fi. \end{array}$$

By the first equality $l = 0$. Substituting in the third we obtain $n = 0$ and therefore we have no factorization. Thus, we conclude the claim. $\square$

**Lemma A.0.2.** *Define $G : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{G} : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$G(x, y) = y^5 - y - 2c^2 K_3 x^2 + 2c^2 K_2 x$$

*and*

$$\overline{G}(x, y) = y^4 + 2y^2 + 4c^2 K_1 x + 1,$$

*where $K_1, K_2, K_3, c$ are constants being $c \neq 0$ and $K_1 \neq 0$. Then the polynomials $G$ and $\overline{G}$ have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $\overline{G}$ does not divide $G$. Indeed, if $\overline{G}$ divides $G$ then there exists a polynomial of degree 1 such that

$$G(x, y) = \overline{G}(x, y)(Ax + By + C),$$

that is,

$$0 = y^5 - y - 2c^2 K_3 x^2 + 2c^2 K_2 x - \left(y^4 + 2y^2 + 4c^2 K_1 x + 1\right)(Ax + By + C)$$

which reorganizing

$$\begin{aligned} 0 =& (-2c^2 K_3 - 4c^2 K_1 A)x^2 + (-A)xy^4 + (-2A)xy^2 + (-4c^2 K_1 B)xy + (2c^2 K_2 - A - 4Cc^2 K_1)x \\ &+ (1 - B)y^5 + (-C)y^4 + (-2B)y^3 + (-2C)y^2 + (-1 - B)y + C \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} -2c^2K_3-4c^2K_1A&=0 & 2c^2K_2-A-4Cc^2K_1&=0 & -2C&=0\\ -A&=0 & 1-B&=0 & -1-B&=0\\ -2A&=0 & -C&=0 & C&=0.\\ -4c^2K_1B&=0 & -2B&=0 \end{aligned}$$

Therefore $B=0$ and $B=1$, a contradiction and thus we proof the claim 1.

**Claim 2:** $\overline{G}$ is irreducible. Indeed, if $\overline{G}$ is reducible then there exist polynomials of degree 3 and degree 1 such that

$$\overline{G}(x,y)=(Ax^3+By^3+Cx^2y+Dxy^2+Ex^2+Fxy+Gy^2+Hx+Iy+J)(Kx+Ly+M)$$

or there are polynomials of degree 2 such that

$$\overline{G}(x,y)=(Ax^2+Bxy+Cy^2+Dx+Ey+F)(Gx^2+Hxy+Iy^2+Jx+Ky+L).$$

In the first case, that is,

$$0=y^4+2y^2+4c^2K_1x+1-(Ax^3+By^3+Cx^2y+Dxy^2+Ex^2+Fxy+Gy^2+Hx+Iy+J)(Kx+Ly+M)$$

which reorganizing

$$\begin{aligned} 0=&(-AK)x^4+(-CK-AL)x^3y+(-EK-AM)x^3+(-DK-CL)x^2y^2+(-FK-EL-CM)x^2y\\ &+(-HK-EM)x^2+(-BK-DL)xy^3+(-GK-FL-DM)xy^2+(-IK-HL-FM)xy\\ &+(4c^2K_1-JK-HM)x+(1-BL)y^4+(-GL-BM)y^3+(2-IL-GM)y^2\\ &+(-JL-IM)y+1-JM \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} -AK&=0 & -HK-EM&=0 & 1-BL&=0\\ -CK-AL&=0 & -BK-DL&=0 & -GL-BM&=0\\ -EK-AM&=0 & -GK-FL-DM&=0 & 2-IL-GM&=0\\ -DK-CL&=0 & -IK-HL-FM&=0 & -JL-IM&=0\\ -FK-EL-CM&=0 & 4c^2K_1-JK-HM&=0 & 1-JM&=0. \end{aligned}$$

If $A\neq 0$ by the first equality we conclude $K=0$. By the second equality we conclude $L=0$, a contradiction with the first equality of the third column. Therefore $A=0$. We have

$$\begin{aligned} -CK&=0 & -BK-DL&=0 & -GL-BM&=0\\ -EK&=0 & -GK-FL-DM&=0 & 2-IL-GM&=0\\ -DK-CL&=0 & -IK-HL-FM&=0 & -JL-IM&=0\\ -FK-EL-CM&=0 & 4c^2K_1-JK-HM&=0 & 1-JM&=0.\\ -HK-EM&=0 & 1-BL&=0 \end{aligned}$$

Repeating the argument successively we obtain $C = D = E = F = H = 0$ and therefore,

$$\begin{aligned} -BK &= 0 & 4c^2K_1 - JK &= 0 & 2 - IL - GM &= 0 \\ -GK &= 0 & 1 - BL &= 0 & -JL - IM &= 0 \\ -IK &= 0 & -GL - BM &= 0 & 1 - JM &= 0. \end{aligned}$$

If $B \neq 0$ by the first equality we conclude $K = 0$. By the first equality of the second column we conclude $4c^2K_1 = 0$, a contradiction. Then $B = 0$ and by the second equality of the second column $1 = 0$, a contradiction and therefore the first case does not occur. In the second case, that is,

$$0 = y^4 + 2y^2 + 4c^2K_1x + 1 - (Ax^2 + Bxy + Cy^2 + Dx + Ey + F)(Gx^2 + Hxy + Iy^2 + Jx + Ky + L)$$

which reorganizing

$$\begin{aligned} 0 =&(-AG)x^4 + (-AH - BG)x^3y + (-GD - AJ)x^3 + (-AI - BH - CG)x^2y^2 \\ &+ (-GE - HD - AK - BJ)x^2y + (-JD - AL - FG)x^2 + (-BI - CH)xy^3 \\ &+ (-HE - DI - BK - CJ)xy^2 + (-JE - KD - BL - FH)xy + (4c^2K_1 - LD - FJ)x \\ &+ (1 - CI)y^4 + (-EI - CK)y^3 + (2 - FI - CL - KE)y^2 + (-LE - FK)y + 1 - FL \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} -AG &= 0 & -JD - AL - FG &= 0 & 1 - CI &= 0 \\ -AH - BG &= 0 & -BI - CH &= 0 & -EI - CK &= 0 \\ -GD - AJ &= 0 & -HE - DI - BK - CJ &= 0 & 2 - FI - CL - KE &= 0 \\ -AI - BH - CG &= 0 & -JE - KD - BL - FH &= 0 & -LE - FK &= 0 \\ -GE - HD - AK - BJ &= 0 & 4c^2K_1 - LD - FJ &= 0 & 1 - FL &= 0. \end{aligned}$$

If $A \neq 0$ by the first equality we conclude $G = 0$. By the second equality we conclude $H = 0$ and consequently by the fourth equality $I = 0$, a contradiction with the first equality of the third column. Therefore $A = 0$. We have

$$\begin{aligned} -BG &= 0 & -BI - CH &= 0 & & \\ -GD &= 0 & -HE - DI - BK - CJ &= 0 & -EI - CK &= 0 \\ -BH - CG &= 0 & -JE - KD - BL - FH &= 0 & 2 - FI - CL - KE &= 0 \\ -GE - HD - BJ &= 0 & 4c^2K_1 - LD - FJ &= 0 & -LE - FK &= 0 \\ -JD - FG &= 0 & 1 - CI &= 0 & 1 - FL &= 0. \end{aligned}$$

Repeating the argument successively we obtain $B = D = 0$ and then

$$\begin{aligned} -CG &= 0 & -HE - CJ &= 0 & -EI - CK &= 0 \\ -GE &= 0 & -JE - FH &= 0 & 2 - FI - CL - KE &= 0 \\ -FG &= 0 & 4c^2K_1 - FJ &= 0 & -LE - FK &= 0 \\ -CH &= 0 & 1 - CI &= 0 & 1 - FL &= 0. \end{aligned}$$

If $C \neq 0$ by the fourth equality we conclude $H = 0$. By the second equality of the second column we conclude $J = 0$ and consequently by the third equality of the second column $4c^2K_1 = 0$, a contradiction. Then $C = 0$ and by the fourth equality of the second column $1 = 0$, a contradiction and therefore the second case does not occur either, i.e., we proof the claim 2. □

**Lemma A.0.3.** *Let $K_1, K_2$ be constants such that $K_1 \neq 0$ and $(K_1, K_2) \neq \left(\pm\frac{\sqrt{2}}{k}, 0\right)$. Define $Q : \mathbb{R}^2 \to \mathbb{R}$, $\tilde{Q} : \mathbb{R}^2 \to \mathbb{R}$ by*

$$Q(w,y) = -2K_1y^2 + 2K_1^2kyw + K_2y - K_1w^2$$

*and*

$$\tilde{Q}(w,y) = 6kK_1^2y^2 - K_1(4k^2K_1^2 - 2)yw - 2kK_1K_2y + 2kK_1^2w^2 + K_2w.$$

*Then the polynomials $Q$ and $\tilde{Q}$ have no factors in common.*

**Proof:** First, let us observe that since $Q$ and $\tilde{Q}$ have the same degree, the only way for one to divide the other is if they are multiples, a situation that does not occur. Thus, we will split the proof depending on $K_2$.

- If $K_2 \neq 0$: Notice that $Q$ is irreducible. Indeed, if $Q$ is reducible there exist polynomials of degree 1 such that

$$Q(w,y) = (a + by + cw)(d + ey + fw),$$

that is,

$$0 = -2K_1y^2 + 2K_1^2kyw + K_2y - K_1w^2 - (a + by + cw)(d + ey + fw)$$

which reorganizing

$$0 = -(2K_1 + be)y^2 + (2kK_1^2 - ce - bf)yw + (K_2 - ae - bd)y - (K_1 + cf)w^2 - (af + cd)w - ad$$

that occurs if, and only if,

$$0 = 2K_1 + be \qquad 0 = 2kK_1^2 - ce - bf \qquad 0 = K_2 - ae - bd \qquad 0 = K_1 + cf \qquad 0 = af + cd \qquad 0 = ad.$$

By the last equality we have that $a = 0$ or $d = 0$. Assume without loss of generality that $a = 0$. Then,

$$0 = 2K_1 + be \qquad 0 = 2kK_1^2 - ce - bf \qquad 0 = K_2 - bd \qquad 0 = K_1 + cf \qquad 0 = cd.$$

Since $K_1 \neq 0$ we obtain by the fourth equality that $c \neq 0$ and consequently by the last equality $d = 0$. Thus, by the third equality we conclude $K_2 = 0$, a contradiction and therefore $Q$ is irreducible. Since $Q$ is irreducible we conclude that $Q$ and $\tilde{Q}$ have no factors in common.

- If $K_2 = 0$: If $Q$ is irreducible we conclude as before that $Q$ and $\tilde{Q}$ have no factors in common. If $Q$ is reducible it is easy to verify that

$$Q(w,y) = -K_1(by + cw)\left(\frac{2}{b}y + \frac{1}{c}w\right)$$

with $0 = 2kK_1 + \frac{b}{c} + \frac{2c}{b}$ or, equivalently,

$$0 = b^2 + 2c^2 + 2bckK_1. \tag{A.4}$$

**Claim 1:** The factor $by + cw$ does not divide $\tilde{Q}$. Indeed, if it does divide, there exists a polynomial of degree 1 such that

$$\tilde{Q}(w,y) = (by + cw)(F + Gy + Hw)$$

that is,

$$0 = 6kK_1^2y^2 - K_1(4k^2K_1^2 - 2)yw + 2kK_1^2w^2 - (by + cw)(F + Gy + Hw)$$

which reorganizing

$$0 = (6kK_1^2 - Gb)y^2 - (K_1(4k^2K_1^2 - 2) + Gc + Hb)yw - Fby + (2kK_1^2 - Hc)w^2 - Fcw$$

that occurs if, and only if,

$$\begin{aligned} 0 &= 6kK_1^2 - Gb & 0 &= Fb & & \\ 0 &= K_1(4k^2K_1^2 - 2) + Gc + Hb & 0 &= 2kK_1^2 - Hc & 0 &= Fc. \end{aligned}$$

Since $c \neq 0$ we obtain, by the last equality, that $F = 0$. We have,

$$0 = 6kK_1^2 - Gb \qquad 0 = K_1(4k^2K_1^2 - 2) + Gc + Hb \qquad 0 = 2kK_1^2 - Hc.$$

Isolating $G$ in the first equality and $H$ in the last equality and substituting both in the second equality we conclude

$$0 = -bc + b^2kK_1 + 3c^2kK_1 + 2bck^2K_1^2$$

but from $(A.4)$ we have $b^2 = -(2c^2 + 2bckK_1)$, hence

$$0 = -bc + b^2kK_1 + 3c^2kK_1 + 2bck^2K_1^2 = -bc - (2c^2 + 2bckK_1)kK_1 + 3c^2kK_1 + 2bck^2K_1^2 = -c(b - ckK_1)$$

thus, $b = ckK_1$ $b = ckK_1$ which substituting in $(A.4)$ we have $0 = c^2(3k^2K_1^2 + 2) \neq 0$ a contradiction and therefore we conclude the claim.

**Claim 2:** The factor $\frac{2}{b}y + \frac{1}{c}w$ does not divide $\tilde{Q}$. Indeed, if it does divide, there exists a polynomial of degree 1 such that

$$\tilde{Q}(w,y) = \left(\frac{2}{b}y + \frac{1}{c}w\right)(F + Gy + Hw)$$

that is,

$$0 = 6kK_1^2y^2 - K_1(4k^2K_1^2 - 2)yw + 2kK_1^2w^2 - \left(\frac{2}{b}y + \frac{1}{c}w\right)(F + Gy + Hw)$$

which reorganizing

$$0 = \left(6kK_1^2 - 2\frac{G}{b}\right)y^2 - \left(K_1(4k^2K_1^2 - 2) + \frac{G}{c} + \frac{2H}{b}\right)yw - \frac{2F}{b}y + \left(2kK_1^2 - \frac{H}{c}\right)w^2 - \frac{F}{c}w$$

that occurs if, and only if,

$$0 = 6kK_1^2 - 2\frac{G}{b} \qquad 0 = \frac{2F}{b}y \qquad 0 = \frac{F}{c}.$$
$$0 = K_1(4k^2K_1^2 - 2) + \frac{G}{c} + \frac{2H}{b} \qquad 0 = 2kK_1^2 - \frac{H}{c}$$

By the last equality $F = 0$. We have

$$0 = 6kK_1^2 - 2\frac{G}{b} \qquad 0 = K_1(4k^2K_1^2 - 2) + \frac{G}{c} + \frac{2H}{b} \qquad 0 = 2kK_1^2 - \frac{H}{c}.$$

Isolating $G$ in the first equality and $H$ in the last equality and substituting both in the second equality we conclude

$$0 = 2bc - 3b^2kK_1 - 4c^2kK_1 - 4bck^2K_1^2$$

but from $(A.4)$ we have $b^2 = -(2c^2 + 2bckK_1)$, hence

$$\begin{aligned} 0 &= 2bc - 3b^2kK_1 - 4c^2kK_1 - 4bck^2K_1^2 \\ &= 2bc + 3(2c^2 + 2bckK_1)kK_1 - 4c^2kK_1 - 4bck^2K_1^2 \\ &= 2c(b + bk^2K_1^2 + ckK_1) \end{aligned}$$

thus, $b = -\frac{ckK_1}{k^2K_1^2+1}$ which substituting in $(A.4)$ we have $0 = c^2\frac{3k^2K_1^2+2}{k^2K_1^2+1} \neq 0$ a contradiction and therefore we conclude the claim. Finally, with statements 1 and 2 we conclude the case $K_2 = 0$. □

**Lemma A.0.4.** *Let $K_1, K_2, k$ be constants such that $K_1, k$ are non-zero. Define $Q : \mathbb{R}^2 \to \mathbb{R}$ by*

$$Q(w,y) = 12y^4 - 6K_1y^3 + 2kK_1yw - 2kK_2y + K_1^2w^2 - 2K_1K_2w + K_2^2.$$

*Then the polynomials $Q$ and $\frac{\partial Q}{\partial y}$ have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $\frac{\partial Q}{\partial y}$ does not divide $Q$. Indeed, if $\frac{\partial Q}{\partial y}$ divides $Q$ then there exists a polynomial of degree 1 such that

$$Q(w,y) = (a + by + cw)(48y^3 - 18K_1y^2 + 2kK_1w - 2kK_2)$$

which reorganizing

$$\begin{aligned} 0 = &(K_1^2 - 2ckK_1)w^2 - 48cwy^3 + 18cK_1wy^2 + (2kK_1 - 2bkK_1)wy + (2ckK_2 - 2akK_1 - 2K_1K_2)w \\ &+ (12 - 48b)y^4 + (18bK_1 - 6K_1 - 48a)y^3 + 18aK_1y^2 + (2bkK_2 - 2kK_2)y + K_2^2 + 2akK_2 \end{aligned}$$

that occurs if, and only if,

$$0 = K_1^2 - 2ckK_1 \qquad 0 = 2ckK_2 - 2akK_1 - 2K_1K_2 \qquad 0 = 18aK_1$$
$$0 = 48c \qquad 0 = 12 - 48b \qquad 0 = 2bkK_2 - 2kK_2$$
$$0 = 8cK_1 \qquad 0 = 18bK_1 - 6K_1 - 48a \qquad 0 = K_2^2 + 2akK_2.$$
$$0 = 2kK_1 - 2bkK_1$$

By the second equality we obtain $c=0$ which, substituting into the first equality, we have $K_1=0$, a contradiction, and thus we conclude the claim.

**Claim 2:** $\frac{\partial Q}{\partial y}$ is irreducible. Indeed, if $\frac{\partial Q}{\partial y}$ is reducible there exist polynomials of degree 1 and degree 2 such that

$$\frac{\partial Q}{\partial y}(w,y) = (a+by+cw)(d+ey+fw+gyw+hy^2+iw^2)$$

that is,

$$0 = 48y^3 - 18K_1y^2 + 2kK_1w - 2kK_2 - (a+by+cw)(d+ey+fw+gyw+hy^2+iw^2)$$

which reorganizing

$$\begin{aligned}0 =&(48-bh)y^3 - (bg+ch)y^2w - (18K_1+be+ah)y^2 - (bi+cg)yw^2 - (ce+ag+bf)yw \\ &- (ae+bd)y - ciw^3 - (ai+cf)w^2 + (2kK_1 - af - cd)w - (2kK_2+ad)\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0 &=48-bh \\ 0 &=bg+ch \\ 0 &=18K_1+be+ah \\ 0 &=bi+cg\end{aligned} \qquad \begin{aligned}0 &=ce+ag+bf \\ 0 &=ae+bd \\ 0 &=ci\end{aligned} \qquad \begin{aligned}0 &=ai+cf \\ 0 &=2kK_1-af-cd \\ 0 &=2kK_2+ad.\end{aligned} \tag{A.5}$$

By the third equality of the second column $c=0$ or $i=0$. Suppose $c=0$. We have,

$$\begin{aligned}0 &=48-bh \\ 0 &=bg \\ 0 &=18K_1+be+ah\end{aligned} \qquad \begin{aligned}0 &=bi \\ 0 &=ag+bf \\ 0 &=ae+bd\end{aligned} \qquad \begin{aligned}0 &=ai \\ 0 &=2kK_1-af \\ 0 &=2kK_2+ad.\end{aligned}$$

By the first equality $b\neq 0$. Consequently by the second equality of the first column and first equality of the second column, respectively, $g=0=i$. We have,

$$\begin{aligned}0 &=48-bh \\ 0 &=18K_1+be+ah\end{aligned} \qquad \begin{aligned}0 &=bf \\ 0 &=ae+bd\end{aligned} \qquad \begin{aligned}0 &=2kK_1-af \\ 0 &=2kK_2+ad.\end{aligned}$$

By the second equality of the second column $f=0$ which substituting into the first equality of the third column we conclude $K_1=0$ a contradiction. Thus, $c\neq 0$ and $i=0$. We have from $(A.5)$

$$\begin{aligned}0 &=48-bh \\ 0 &=bg+ch \\ 0 &=18K_1+be+ah\end{aligned} \qquad \begin{aligned}0 &=cg \\ 0 &=ce+ag+bf \\ 0 &=ae+bd\end{aligned} \qquad \begin{aligned}0 &=cf \\ 0 &=2kK_1-af-cd \\ 0 &=2kK_2+ad.\end{aligned}$$

By the first equality of the second column we have $g=0$ which, substituting into the second equality of the first column, we conclude $h=0$ and therefore we have no factorization and, thus, we conclude the claim. $\square$

**Lemma A.0.5.** *Let $K_1, K_2, k$ be constants such that $K_1, k$ are non-zero. Define $Q : \mathbb{R}^2 \to \mathbb{R}$, $\tilde{Q} : \mathbb{R}^2 \to \mathbb{R}$ by*

$$Q(w,y) = 12y^4 - 6K_1y^3 + 2kK_1yw - 2kK_2y + K_1^2w^2 - 2K_1K_2w + K_2^2.$$

*and*

$$\begin{aligned}\tilde{Q}(w,y) =& -2kK_1^2w^2 - 48wy^3 + 22K_1^2wy^2 - 4k^2K_1wy + 4kK_1K_2w - 96ky^4 \\ &+ (48K_2 + 40kK_1)y^3 - 22K_1K_2y^2 + 4k^2K_2y - 2kK_2^2.\end{aligned}$$

*Then the polynomials $Q$ and $\tilde{Q}$ have no factors in common.*

**Proof:** First, let us observe that since $Q$ and $\tilde{Q}$ have the same degree, the only way for one to divide the other is if they are multiples, a situation that does not occur. Let us show that $\tilde{Q}$ is irreducible. If $\tilde{Q}$ were reducible, we have two possibilities. In the first possibility, there are polynomials of degree 1 and degree 3 such that

$$\tilde{Q}(w,y) = (a + by + cw)(d + ey + fw + gyw + hy^2 + iw^2 + jyw^2 + lwy^2 + my^3 + nw^3)$$

which reorganizing

$$\begin{aligned}0 =& -(96k + bm)y^4 - (48K_1 + bl + cm)y^3w + (48K_2 + 40kK_1 - bh - am)y^3 - (bj + cl)y^2w^2 \\ &+ (22K_1^2 - bg - ch - al)y^2w - (be + 22K_1K_2 + ah)y^2 - (cj + bn)yw^3 - (bi + cg + aj)yw^2 \\ &- (4K_1k^2 + ce + ag + bf)yw + (4k^2K_2 - ae - bd)y - cnw^4 - (ci + an)w^3 - (2kK_1^2 + ai + cf)w^2 \\ &+ (4kK_1K_2 - cd - af)w - (2kK_2^2 + ad)\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}
0 &= 96k + bm & 0 &= be + 22K_1K_2 + ah & 0 &= cn \\
0 &= 48K_1 + bl + cm & 0 &= cj + bn & 0 &= ci + an \\
0 &= 48K_2 + 40kK_1 - bh - am & 0 &= bi + cg + aj & 0 &= 2kK_1^2 + ai + cf \\
0 &= bj + cl & 0 &= 4K_1k^2 + ce + ag + bf & 0 &= 4kK_1K_2 - cd - af \\
0 &= 22K_1^2 - bg - ch - al & 0 &= 4k^2K_2 - ae - bd & 0 &= 2kK_2^2 + ad.
\end{aligned}$$

By the first equality in the first column $m = -\frac{96k}{b}$. Thus,

$$\begin{aligned}
&0 = 48K_1 + bl - \frac{96kc}{b} && 0 = cj + bn && \\
&0 = 48K_2 + 40kK_1 - bh + \frac{96ka}{b} && 0 = bi + cg + aj && 0 = ci + an \\
&0 = bj + cl && 0 = 4K_1k^2 + ce + ag + bf && 0 = 2kK_1^2 + ai + cf \\
&0 = 22K_1^2 - bg - ch - al && 0 = 4k^2K_2 - ae - bd && 0 = 4kK_1K_2 - cd - af \\
&0 = be + 22K_1K_2 + ah && 0 = cn && 0 = 2kK_2^2 + ad.
\end{aligned} \tag{A.6}$$

By the last equality in the second column we have $c = 0$ or $n = 0$. Suppose $c = 0$. We have,

$$\begin{aligned}
0 &= 48K_1 + bl & 0 &= bn & & \\
0 &= 48K_2 + 40kK_1 - bh + \frac{96ka}{b} & 0 &= bi + aj & 0 &= 2kK_1^2 + ai \\
0 &= bj & 0 &= 4K_1k^2 + ag + bf & 0 &= 4kK_1K_2 - af \\
0 &= 22K_1^2 - bg - al & 0 &= 4k^2K_2 - ae - bd & 0 &= 2kK_2^2 + ad. \\
0 &= be + 22K_1K_2 + ah & 0 &= an & &
\end{aligned}$$

By the third equality of the first column we have $j = 0$. Therefore, by the second equality of the second column $i = 0$. Hence, by the first equality of the third column, $0 = 2kK_1^2 + ai = 2kK_1^2 \neq 0$ a contradiction and therefore the assumption does not hold. Thus $c \neq 0$ and $n = 0$. We have from $(A.6)$

$$\begin{aligned}
0 &= 48K_1 + bl - \frac{96kc}{b} & 0 &= cj & & \\
0 &= 48K_2 + 40kK_1 - bh + \frac{96ka}{b} & 0 &= bi + cg + aj & 0 &= 2kK_1^2 + ai + cf \\
0 &= bj + cl & 0 &= 4K_1k^2 + ce + ag + bf & 0 &= 4kK_1K_2 - cd - af \\
0 &= 22K_1^2 - bg - ch - al & 0 &= 4k^2K_2 - ae - bd & 0 &= 2kK_2^2 + ad. \\
0 &= be + 22K_1K_2 + ah & 0 &= ci & &
\end{aligned}$$

By the first and last equalities of the second column we have $j = i = 0$. Thus,

$$\begin{aligned}
0 &= 48K_1 + bl - \frac{96kc}{b} & 0 &= be + 22K_1K_2 + ah & 0 &= 2kK_1^2 + cf \\
0 &= 48K_2 + 40kK_1 - bh + \frac{96ka}{b} & 0 &= cg & 0 &= 4kK_1K_2 - cd - af \\
0 &= cl & 0 &= 4K_1k^2 + ce + ag + bf & 0 &= 2kK_2^2 + ad. \\
0 &= 22K_1^2 - bg - ch - al & 0 &= 4k^2K_2 - ae - bd & &
\end{aligned}$$

By the third equality of the first column and by the second equality of the second column we obtain $l = g = 0$. We have,

$$\begin{aligned}
0 &= 48K_1 - \frac{96kc}{b} & 0 &= be + 22K_1K_2 + ah & 0 &= 2kK_1^2 + cf \\
0 &= 48K_2 + 40kK_1 - bh + \frac{96ka}{b} & 0 &= 4K_1k^2 + ce + bf & 0 &= 4kK_1K_2 - cd - af \\
0 &= 22K_1^2 - ch & 0 &= 4k^2K_2 - ae - bd & 0 &= 2kK_2^2 + ad.
\end{aligned}$$

By the third equality of the first column and by the first equality of the third column we have, respectively, $h = \frac{22K_1^2}{c}$ and $f = -\frac{2kK_1^2}{c}$. Therefore,

$$\begin{aligned}
0 &= 48K_1 - \frac{96kc}{b} & & & & \\
0 &= 48K_2 + 40kK_1 - \frac{22K_1^2b}{c} + \frac{96ka}{b} & 0 &= 4K_1k^2 + ce - \frac{2kK_1^2b}{c} & 0 &= 4kK_1K_2 - cd + \frac{2kK_1^2a}{c} \\
0 &= be + 22K_1K_2 + \frac{22K_1^2a}{c} & 0 &= 4k^2K_2 - ae - bd & 0 &= 2kK_2^2 + ad.
\end{aligned}$$

By the first equality in the third column we have $d = \frac{1}{c}\left(4kK_1K_2 + \frac{2kK_1^2 a}{c}\right)$ and substituting in the last equality in the third column we obtain $0 = \frac{2k(aK_1+cK_2)^2}{c^2}$, that is, $a = -\frac{cK_2}{K_1}$. Thus,

$$
\begin{aligned}
0 =& 48K_1 - \frac{96kc}{b} & 0 =& be & & \\
0 =& 48K_2 + 40kK_1 - \frac{22K_1^2 b}{c} - \frac{96ckK_2}{bK_1} & 0 =& 4K_1k^2 + ce - \frac{2kK_1^2 b}{c} & 0 =& 4k^2K_2 + \frac{ceK_2}{K_1} - \frac{2kbK_1K_2}{c}.
\end{aligned}
$$

By the first equality of the second column $e = 0$ which, substituting into the second equality of the second column, we obtain $b = \frac{2kc}{K_1}$. Therefore, by the second equality, we conclude

$$0 = 48K_2 + 40kK_1 - \frac{22K_1^2 b}{c} - \frac{96ckK_2}{bK_1} = -4kK_1 \neq 0$$

a contradiction and then we do not have the factorization by polynomials of degree 1 and degree 3. In the second possibility there are polynomials of degree 2 such that

$$\tilde{Q}(w,y) = (a + by + cw + dyw + ey^2 + fw^2)(g + hy + iw + jyw + ly^2 + mw^2)$$

which reorganizing

$$
\begin{aligned}
0 =& -(96k + le)y^4 - (48K_1 + je + dl)y^3w + (48K_2 - he + 40kK_1 - bl)y^3 - (me + dj + fl)y^2w^2 \\
&+ (22K_1^2 - ei - bj - dh - cl)y^2w - (ge + 22K_1K_2 + bh + al)y^2 - (fj + dm)yw^3 \\
&- (di + cj + fh + bm)yw^2 - (4K_1k^2 + bl + aj + ch + dg)yw + (4k^2K_2 - ah - bg)y \\
&- fmw^4 - (fi + cm)w^3 - (2kK_1^2 + ci + fg + am)w^2 + (4kK_1K_2 - cg - ai)w - (2kK_2^2 + ag),
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{aligned}
0 =& 96k + le & 0 =& ge + 22K_1K_2 + bh + al & 0 =& fm \\
0 =& 48K_1 + je + dl & 0 =& fj + dm & 0 =& fi + cm \\
0 =& 48K_2 - he + 40kK_1 - bl & 0 =& di + cj + fh + bm & 0 =& 2kK_1^2 + ci + fg + am \\
0 =& me + dj + fl & 0 =& 4K_1k^2 + bl + aj + ch + dg & 0 =& 4kK_1K_2 - cg - ai \\
0 =& 22K_1^2 - ei - bj - dh - cl & 0 =& 4k^2K_2 - ah - bg & 0 =& 2kK_2^2 + ag.
\end{aligned}
$$

By the first equality of the third column we have $f = 0$ or $m = 0$. Let us assume, without loss of generality, that $f = 0$. Also, by the first equality of the first column, $l = -\frac{96k}{e}$. We have,

$$
\begin{aligned}
0 =& 48K_1 + je - \frac{96kd}{e} & 0 =& dm & & \\
0 =& 48K_2 - he + 40kK_1 + \frac{96kb}{e} & 0 =& di + cj + bm & 0 =& 2kK_1^2 + ci + am \\
0 =& me + dj & 0 =& 4K_1k^2 - \frac{96kb}{e} + aj + ch + dg & 0 =& 4kK_1K_2 - cg - ai \\
0 =& 22K_1^2 - ei - bj - dh + \frac{96kc}{e} & 0 =& 4k^2K_2 - ah - bg & 0 =& 2kK_2^2 + ag. \\
0 =& ge + 22K_1K_2 + bh - \frac{96ka}{e} & 0 =& cm & &
\end{aligned}
\tag{A.7}
$$

By the first equality in the second column we have $d = 0$ or $m = 0$. Suppose $d = 0$. Thus,

$$\begin{aligned} 0 &= 48K_1 + je \\ 0 &= 48K_2 - he + 40kK_1 + \frac{96kb}{e} \\ 0 &= me \\ 0 &= 22K_1^2 - ei - bj + \frac{96kc}{e} \end{aligned} \qquad \begin{aligned} 0 &= ge + 22K_1K_2 + bh - \frac{96ka}{e} \\ 0 &= cj + bm \\ 0 &= 4K_1k^2 - \frac{96kb}{e} + aj + ch \\ 0 &= 4k^2K_2 - ah - bg \end{aligned} \qquad \begin{aligned} 0 &= cm \\ 0 &= 2kK_1^2 + ci + am \\ 0 &= 4kK_1K_2 - cg - ai \\ 0 &= 2kK_2^2 + ag. \end{aligned}$$

By the first equality $j = -\frac{48K_1}{e}$. Then, by the third equality, $m = 0$. Thus,

$$\begin{aligned} 0 &= 48K_2 - he + 40kK_1 + \frac{96kb}{e} \\ 0 &= 22K_1^2 - ei + \frac{48K_1b}{e} + \frac{96kc}{e} \\ 0 &= ge + 22K_1K_2 + bh - \frac{96ka}{e} \end{aligned} \qquad \begin{aligned} 0 &= \frac{48K_1c}{e} \\ 0 &= 4K_1k^2 - \frac{96kb}{e} - \frac{48K_1a}{e} + ch \\ 0 &= 4k^2K_2 - ah - bg \end{aligned} \qquad \begin{aligned} 0 &= 2kK_1^2 + ci \\ 0 &= 4kK_1K_2 - cg - ai \\ 0 &= 2kK_2^2 + ag. \end{aligned}$$

By the first equality of the second column $c = 0$ which substituting in the first equality of the third column we obtain $0 = 2kK_1^2 + ci = 2kK_1^2 \neq 0$ a contradiction and therefore the assumption does not hold. Thus $d \neq 0$ and $m = 0$. We have from $(A.7)$

$$\begin{aligned} 0 &= 48K_1 + je - \frac{96kd}{e} \\ 0 &= 48K_2 - he + 40kK_1 + \frac{96kb}{e} \\ 0 &= dj \\ 0 &= 22K_1^2 - ei - bj - dh + \frac{96kc}{e} \end{aligned} \qquad \begin{aligned} 0 &= ge + 22K_1K_2 + bh - \frac{96ka}{e} \\ 0 &= di + cj \\ 0 &= 4K_1k^2 - \frac{96kb}{e} + aj + ch + dg \\ 0 &= 4k^2K_2 - ah - bg \end{aligned} \qquad \begin{aligned} 0 &= 2kK_1^2 + ci \\ 0 &= 4kK_1K_2 - cg - ai \\ 0 &= 2kK_2^2 + ag. \end{aligned}$$

By the third equality of the first column $j = 0$ which, substituting into the second equality of the second column, we obtain $i = 0$. Thus, by the first equality of the third column, $0 = 2kK_1^2 + ci = 2kK_1^2 \neq 0$ a contradiction and therefore this factorization also does not occur. $\square$

**Lemma A.0.6.** *Let $K_1, K_2, K_3, c$ be constants such that $K_2 \neq 0$, $c \neq 0$, $c \neq -\frac{1}{2}$ and $c \neq \frac{1}{K_1 - 2}$. Define $G : \mathbb{R}^2 \to \mathbb{R}$, $\overline{G} : \mathbb{R}^2 \to \mathbb{R}$ by*

$$G(x,y) = 2cy^2x^2 + 4cy^2x + 2cy^2 - cK_1x^2 + (8c - 2cK_1 + 4)x - cK_1$$

*and*

$$\begin{aligned} \overline{G}(x,y) = {} & cx^4y^4 + 4cx^3y^4 - 2cK_2x^4y + 6cx^2y^4 + 8cx^3y^2 - cK_3x^4 - 4cK_2x^3y + 4cxy^4 + 16cx^2y^2 \\ & - 4cK_3x^3 + cy^4 + 8cxy^2 + (16c - 6cK_3)x^2 + 4cK_2xy - 4cK_3x + 2cK_2y - cK_3. \end{aligned}$$

*Then the polynomials $G$ and $\overline{G}$ have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $G$ does not divide $\overline{G}$. Indeed, if $G$ divides $\overline{G}$ then there exists a polynomial of degree 4 such that

$$\overline{G}(x,y) = G(x,y)(f+gx+hy+ixy+jx^2+ky^2+lxy^2+mx^2y+nx^3+oy^3+pxy^3+qx^2y^2+rx^3y+sx^4+ty^4)$$

which reorganizing

$$\begin{aligned}
0 = & -2csx^6y^2 + csK_1x^6 - 2crx^5y^3 - (2cn+4cs)x^5y^2 + crK_1x^5y + (cnK_1 - s(8c-2cK_1+4))x^5 \\
& + (c-2cq)x^4y^4 - (2cm+4cr)x^4y^3 + (cqK_1 - 4cn - 2cs - 2cj)x^4y^2 \\
& + (cmK_1 - r(8c-2cK_1+4) - 2cK_2)x^4y + (cjK_1 - n(8c-2cK_1+4) - cK_3 + csK_1)x^4 \\
& - 2cpx^3y^5 + (4c - 2cl - 4cq)x^3y^4 + (cpK_1 - 4cm - 2cr - 2ci)x^3y^3 \\
& + (8c - q(8c-2cK_1+4) - 2cg - 4cj - 2cn + clK_1)x^3y^2 \\
& + (crK_1 - m(8c-2cK_1+4) - 4cK_2 + ciK_1)x^3y + (cgK_1 - j(8c-2cK_1+4) - 4cK_3 + cnK_1)x^3 \\
& - 2ctx^2y^6 - (2co+4cp)x^2y^5 + (6c - 2ck - 4cl - 2cq + ctK_1)x^2y^4 \\
& + (coK_1 - p(8c-2cK_1+4) - 2ch - 2cm - 4ci)x^2y^3 \\
& + (16c - l(8c-2cK_1+4) - 2cf - 4cg - 2cj + ckK_1 + cqK_1)x^2y^2 \\
& + (chK_1 - i(8c-2cK_1+4) + cmK_1)x^2y \\
& + (16c - 6cK_3 - g(8c-2cK_1+4) + cfK_1 + cjK_1)x^2 - 4ctxy^6 - (4co+2cp)xy^5 \\
& + (4c - t(8c-2cK_1+4) - 4ck - 2cl)xy^4 + (cpK_1 - o(8c-2cK_1+4) - 4ch - 2ci)xy^3 \\
& + (8c - k(8c-2cK_1+4) - 4cf - 2cg + clK_1)xy^2 + (4cK_2 - h(8c-2cK_1+4) + ciK_1)xy \\
& + (cgK_1 - f(8c-2cK_1+4) - 4cK_3)x - 2cty^6 - 2coy^5 + (c - 2ck + ctK_1)y^4 + (coK_1 - 2ch)y^3 \\
& + (ckK_1 - 2cf)y^2 + (2cK_2 + chK_1)y + cfK_1 - cK_3
\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}
0 &= 2cs & 0 &= cjK_1 - n(8c-2cK_1+4) - cK_3 + csK_1 \\
0 &= csK_1 & 0 &= 2cp \\
0 &= 2cr & 0 &= 4c - 2cl - 4cq \\
0 &= 2cn + 4cs & 0 &= cpK_1 - 4cm - 2cr - 2ci \\
0 &= crK_1 & 0 &= 8c - q(8c-2cK_1+4) - 2cg - 4cj - 2cn + clK_1 \\
0 &= cnK_1 - s(8c-2cK_1+4) & 0 &= crK_1 - m(8c-2cK_1+4) - 4cK_2 + ciK_1 \\
0 &= c - 2cq & 0 &= cgK_1 - j(8c-2cK_1+4) - 4cK_3 + cnK_1 \\
0 &= 2cm + 4cr & 0 &= 2ct \\
0 &= cqK_1 - 4cn - 2cs - 2cj & 0 &= 2co + 4cp \\
0 &= cmK_1 - r(8c-2cK_1+4) - 2cK_2 & 0 &= 6c - 2ck - 4cl - 2cq + ctK_1
\end{aligned}$$

$$0 = coK_1 - p(8c - 2cK_1 + 4) - 2ch - 2cm - 4ci$$
$$0 = 16c - l(8c - 2cK_1 + 4) - 2cf - 4cg - 2cj + ckK_1 + cqK_1$$
$$0 = chK_1 - i(8c - 2cK_1 + 4) + cmK_1$$
$$0 = 16c - 6cK_3 - g(8c - 2cK_1 + 4) + cfK_1 + cjK_1$$
$$0 = 4ct$$
$$0 = 4co + 2cp$$
$$0 = 4c - t(8c - 2cK_1 + 4) - 4ck - 2cl$$
$$0 = cpK_1 - o(8c - 2cK_1 + 4) - 4ch - 2ci$$
$$0 = 8c - k(8c - 2cK_1 + 4) - 4cf - 2cg + clK_1$$
$$0 = 4cK_2 - h(8c - 2cK_1 + 4) + ciK_1$$
$$0 = cgK_1 - f(8c - 2cK_1 + 4) - 4cK_3$$
$$0 = 2ct$$
$$0 = 2co$$
$$0 = c - 2ck + ctK_1$$
$$0 = coK_1 - 2ch$$
$$0 = ckK_1 - 2cf$$
$$0 = 2cK_2 + chK_1$$
$$0 = cfK_1 - cK_3.$$

By the first and third equalities of the first column, second and eighth equalities of the second column and the fifith equality of the last column we obtain $s = r = p = t = o = 0$. We have,

$$0 = 2cn$$
$$0 = cnK_1$$
$$0 = c - 2cq$$
$$0 = 2cm$$
$$0 = cqK_1 - 4cn - 2cj$$
$$0 = cmK_1 - 2cK_2$$
$$0 = cjK_1 - n(8c - 2cK_1 + 4) - cK_3$$
$$0 = 4c - 2cl - 4cq$$
$$0 = -4cm - 2ci$$
$$0 = 8c - q(8c - 2cK_1 + 4) - 2cg - 4cj - 2cn + clK_1$$
$$0 = -m(8c - 2cK_1 + 4) - 4cK_2 + ciK_1$$
$$0 = cgK_1 - j(8c - 2cK_1 + 4) - 4cK_3 + cnK_1$$
$$0 = 6c - 2ck - 4cl - 2cq$$
$$0 = -2ch - 2cm - 4ci$$
$$0 = 16c - l(8c - 2cK_1 + 4) - 2cf - 4cg - 2cj + ckK_1 + cqK_1$$
$$0 = chK_1 - i(8c - 2cK_1 + 4) + cmK_1$$
$$0 = 16c - 6cK_3 - g(8c - 2cK_1 + 4) + cfK_1 + cjK_1$$
$$0 = 4c - 4ck - 2cl$$
$$0 = -4ch - 2ci$$
$$0 = 8c - k(8c - 2cK_1 + 4) - 4cf - 2cg + clK_1$$
$$0 = 4cK_2 - h(8c - 2cK_1 + 4) + ciK_1$$
$$0 = cgK_1 - f(8c - 2cK_1 + 4) - 4cK_3$$
$$0 = c - 2ck$$
$$0 = -2ch$$
$$0 = ckK_1 - 2cf$$
$$0 = 2cK_2 + chK_1$$
$$0 = cfK_1 - cK_3.$$

By the tenth equality fo the second column we obtain $h = 0$. Thus, by the twelfth equality we have $K_2 = 0$, a contradiction, and thus we proof the claim.

**Claim 2:** $G$ is irreducible. Indeed, if $G$ is reducible we have two possible situations. In the first, there are polynomials of degree 1 and degree 3 such that

$$G(x,y) = (f+gx+hy)(i+jx+ky+lxy+mx^2+ny^2+oxy^2+px^2y+qx^3+ry^3)$$

that is,

$$0 = G(x,y) - (f+gx+hy)(i+jx+ky+lxy+mx^2+ny^2+oxy^2+px^2y+qx^3+ry^3)$$

which reorganizing

$$\begin{aligned}0 = &-gqx^4 - (gp+hq)x^3y - (gm+fq)x^3 + (2c-go-hp)x^2y^2 - (gl+hm+fp)x^2y\\ &-(cK_1+gj+fm)x^2 - (ho+gr)xy^3 + (4c-hl-fo-gn)xy^2 - (fl+gk+hj)xy\\ &+(8c-gi-2cK_1-fj+4)x - hry^4 - (hn+fr)y^3 + (2c-hk-fn)y^2 - (hi+fk)y - (fi+cK_1)\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = gq & 0 = cK_1+gj+fm & 0 = hr \\ 0 = gp+hq & 0 = ho+gr & 0 = hn+fr \\ 0 = gm+fq & 0 = 4c-hl-fo-gn & 0 = 2c-hk-fn \\ 0 = 2c-go-hp & 0 = fl+gk+hj & 0 = hi+fk \\ 0 = gl+hm+fp & 0 = 8c-gi-2cK_1-fj+4 & 0 = fi+cK_1. \end{array} \tag{A.8}$$

By the first equality $g$ or $q$ is zero. Suppose $g = 0$. We have,

$$\begin{array}{lll} 0 = hq & 0 = ho & \\ 0 = fq & 0 = 4c-hl-fo & 0 = hn+fr \\ 0 = 2c-hp & 0 = fl+hj & 0 = 2c-hk-fn \\ 0 = hm+fp & 0 = 8c-2cK_1-fj+4 & 0 = hi+fk \\ 0 = cK_1+fm & 0 = hr & 0 = fi+cK_1. \end{array}$$

Since $h \neq 0$ (otherwise we would not have factorization) we obtain by the first equalities od the first and second columns and the last equality of the second column $q = o = r = 0$. We have,

$$\begin{array}{lll} 0 = 2c-hp & 0 = fl+hj & 0 = 2c-hk-fn \\ 0 = hm+fp & 0 = 8c-2cK_1-fj+4 & 0 = hi+fk \\ 0 = cK_1+fm & 0 = hn & 0 = fi+cK_1. \\ 0 = 4c-hl & & \end{array}$$

Since $h \neq 0$ (otherwise we would not have factorization) we obtain by the first equality of the first column, the second equality of the second column and last equality of the third column $q = o = r = 0$. We have,

$$\begin{array}{lll} 0 = hm + \dfrac{2cf}{h} & 0 = \dfrac{4cf}{h} + hj & 0 = hi + \dfrac{2cf}{h} \\ 0 = cK_1+fm & 0 = 8c-2cK_1-fj+4 & 0 = fi+cK_1. \end{array}$$

By the first equality of the first, second and third columns we have $m = i = -\frac{2cf}{h^2}$ and $j = -\frac{4cf}{h^2}$. Thus,

$$0 = cK_1 - \frac{2cf^2}{h^2} \qquad 0 = 8c - 2cK_1 + \frac{4cf^2}{h^2} + 4.$$

By the first equality we obtain $\frac{f^2}{h^2} = \frac{K_1}{2}$ which substituting into the second we conclude $0 = 4(2c+1)$, that is, $c = -\frac{1}{2}$ a contradiction. Thus, $g \neq 0$ and $q = 0$. From $(A.8)$ we have

$$\begin{array}{lll}
0 = gp & 0 = ho + gr & 0 = hn + fr \\
0 = gm & 0 = 4c - hl - fo - gn & 0 = 2c - hk - fn \\
0 = 2c - go - hp & 0 = fl + gk + hj & 0 = hi + fk \\
0 = gl + hm + fp & 0 = 8c - gi - 2cK_1 - fj + 4 & 0 = fi + cK_1. \\
0 = cK_1 + gj + fm & 0 = hr &
\end{array}$$

Proceeding in a similar way to the previous situation we will obtain $p = m = l = 0$, $o = \frac{2c}{g}$, $j = -\frac{cK_1}{g}$, $r = -\frac{2ch}{g^2}$, $h = k = 0$, $n = \frac{2c}{g}\left(2 - \frac{f}{g}\right)$ and $f = g$. Finally, $i = -\frac{cK_1}{g}$ and consequently we will conclude $c = -\frac{1}{2}$, a contradiction, and so we do not have this situation. In the second possible situation, there are polynomials of degree 2 such that

$$0 = G(x,y) - (f + gx + hy + ixy + jx^2 + ky^2)(l + mx + ny + oxy + px^2 + qy^2)$$

which reorganizing

$$\begin{aligned}
0 = &-jpx^4 - (pi + jo)x^3y - (gp + jm)x^3 + (2c - oi - jq - kp)x^2y^2 - (mi + go + hp + jn)x^2y \\
&- (cK_1 + gm + fp + jl)x^2 - (qi + ko)xy^3 + (4c - ni - ho - gq - km)xy^2 - (li + fo + gn + hm)xy \\
&+ (8c - 2cK_1 - fm - gl + 4)x - kqy^4 - (hq + kn)y^3 + (2c - hn - fq - kl)y^2 \\
&- (fn + hl)y - (cK_1 + fl)
\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll}
0 = jp & 0 = cK_1 + gm + fp + jl & 0 = kq \\
0 = pi + jo & 0 = qi + ko & 0 = hq + kn \\
0 = gp + jm & 0 = 4c - ni - ho - gq - km & 0 = 2c - hn - fq - kl \\
0 = 2c - oi - jq - kp & 0 = li + fo + gn + hm & 0 = fn + hl \\
0 = mi + go + hp + jn & 0 = 8c - 2cK_1 - fm - gl + 4 & 0 = cK_1 + fl.
\end{array}$$

By the first equality $j$ is zero or $p$ is zero. Suppose, without loss of generality, $j = 0$. We have,

$$\begin{array}{lll}
0 = pi & 0 = qi + ko & 0 = hq + kn \\
0 = gp & 0 = 4c - ni - ho - gq - km & 0 = 2c - hn - fq - kl \\
0 = 2c - oi - kp & 0 = li + fo + gn + hm & 0 = fn + hl \\
0 = mi + go + hp & 0 = 8c - 2cK_1 - fm - gl + 4 & 0 = cK_1 + fl. \\
0 = cK_1 + gm + fp & 0 = kq &
\end{array} \tag{A.9}$$

By the first equality $p$ or $i$ is zero. Suppose $p=0$. We have,

$$\begin{array}{lll} 0=2c-oi & 0=4c-ni-ho-gq-km & 0=hq+kn \\ 0=mi+go & 0=li+fo+gn+hm & 0=2c-hn-fq-kl \\ 0=cK_1+gm & 0=8c-2cK_1-fm-gl+4 & 0=fn+hl \\ 0=qi+ko & 0=kq & 0=cK_1+fl. \end{array}$$

By the first equality, since $c\neq 0$, we conclude that $o\neq 0$ and consequently $o=\frac{2c}{i}$. Thus,

$$\begin{array}{lll} 0=mi+\dfrac{2cg}{i} & 0=li+\dfrac{2cf}{i}+gn+hm & 0=2c-hn-fq-kl \\ 0=cK_1+gm & 0=8c-2cK_1-fm-gl+4 & 0=fn+hl \\ 0=qi+\dfrac{2ck}{i} & 0=kq & 0=cK_1+fl. \\ 0=4c-ni-\dfrac{2ch}{i}-gq-km & 0=hq+kn & \end{array}$$

By the first and third equalities we obtain $m=-\frac{2cg}{i^2}$ and $q=-\frac{2ck}{i^2}$. Therefore,

$$\begin{array}{lll} 0=cK_1-\dfrac{2cg^2}{i^2} & 0=8c-2cK_1+\dfrac{2cgf}{i^2}-gl+4 & 0=2c-hn+\dfrac{2ckf}{i^2}-kl \\ 0=4c-ni-\dfrac{2ch}{i}+\dfrac{2cgk}{i^2}+\dfrac{2cgk}{i^2} & 0=\dfrac{2ck^2}{i^2} & 0=fn+hl \\ 0=li+\dfrac{2cf}{i}+gn-\dfrac{2cgh}{i^2} & 0=-\dfrac{2ckh}{i^2}+kn & 0=cK_1+fl. \end{array}$$

By the second equality of the second column $k=0$. Thus,

$$\begin{array}{lll} 0=cK_1-\dfrac{2cg^2}{i^2} & 0=8c-2cK_1+\dfrac{2cgf}{i^2}-gl+4 & 0=fn+hl \\ 0=4c-ni-\dfrac{2ch}{i} & 0=2c-hn & 0=cK_1+fl. \\ 0=li+\dfrac{2cf}{i}+gn-\dfrac{2cgh}{i^2} & & \end{array}$$

By the second equality of the second column $n=\frac{2c}{h}$, which substituting into the first equality of the third column we obtain $l=-\frac{2cf}{h^2}$. Therefore,

$$\begin{array}{lll} 0=cK_1-\dfrac{2cg^2}{i^2} & 0=-\dfrac{2cfi}{h^2}+\dfrac{2cf}{i}+\dfrac{2cg}{h}-\dfrac{2cgh}{i^2} & 0=cK_1-\dfrac{2cf^2}{h^2}. \\ 0=4c-\dfrac{2ci}{h}-\dfrac{2ch}{i} & 0=8c-2cK_1+\dfrac{2cgf}{i^2}+\dfrac{2cfg}{h^2}+4 & \end{array}$$

By the second equality we obtain $0=-2c(h-i)^2$, that is, $h=i$. Then,

$$0=cK_1-\frac{2cg^2}{h^2} \qquad 0=8c-2cK_1+\frac{4cgf}{h^2}+4 \qquad 0=cK_1-\frac{2cf^2}{h^2}.$$

Subtracting the third equality from the first we conclude that $f=\pm g$. Thus,

$$\frac{g^2}{h^2}=\frac{K_1}{2} \qquad 0=8c-2cK_1\pm\frac{4cg^2}{h^2}+4,$$

i.e.,

$$0 = 8c - 2cK_1 \pm 2cK_1 + 4$$

and then $c = -\frac{1}{2}$ or $c = \frac{1}{K_1-2}$, a contradiction. Therefore, $p \neq 0$ and $i = 0$. We have from $(A.9)$, proceeding in a similar way, $g = h = 0$, $k = \frac{2c}{p}$, $o = q = n = 0$, $f = -\frac{cK_1}{p}$, $m = 2p$ and $l = p$ and thus we will conclude $c = -\frac{1}{2}$, a contradiction, that is, this factorization does not occur. Hence we proof the claim. □

**Lemma A.0.7.** *Let $K_1, K_2$ be constants such that $K_2 \neq K_1$. Define $F : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $G : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$F(x,y) = 2y^3 - (K_2 - K_1)xy^2 - x(K_2 - K_1)$$

*and*

$$\begin{aligned} G(x,y) = {} & (8K_2 - 14K_1)xy^4 - 24y^5 - 2K_1(K_1 - K_2)x^2y^3 - 12y^3 \\ & + (3K_2 - 9K_1)xy^2 - 2K_1(K_1 - K_2)x^2y + (K_1 - K_2)x. \end{aligned}$$

*Then the polynomials $F$ and $G$ have no factors in common.*

**Proof:** We will divide the proof into two claims.

**Claim 1:** $F$ does not divide $G$. Indeed, if $F$ divides $G$ then there exists a polynomial of degree 2 such that

$$G(x,y) = F(x,y)\left(a + bc + cy + dxy + ex^2 + fy^2\right),$$

which reorganizing

$$\begin{aligned} 0 = & -e(K_1 - K_2)x^3y^2 - e(K_1 - K_2)x^3 - (2e + d(K_1 - K_2) + 2K_1(K_1 - K_2))x^2y^3 - b(K_1 - K_2)x^2y^2 \\ & -(K_1 - K_2)(d + 2K_1)x^2y - b(K_1 - K_2)x^2 + (8K_2 - 14K_1 - 2d - f(K_1 - K_2))xy^4 \\ & -(2b + c(K_1 - K_2))xy^3 + (3K_2 - 9K_1 - a(K_1 - K_2) - f(K_1 - K_2))xy^2 - c(K_1 - K_2)xy \\ & +(K_1 - K_2 - a(K_1 - K_2))x - (2f + 24)y^5 - 2cy^4 - (2a + 12)y^3 \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} 0 &= e(K_1 - K_2) & 0 &= 2b + c(K_1 - K_2) \\ 0 &= e(K_1 - K_2) & 0 &= 3K_2 - 9K_1 - a(K_1 - K_2) - f(K_1 - K_2) \\ 0 &= 2e + d(K_1 - K_2) + 2K_1(K_1 - K_2) & 0 &= c(K_1 - K_2) \\ 0 &= b(K_1 - K_2) & 0 &= K_1 - K_2 - a(K_1 - K_2) \\ 0 &= (K_1 - K_2)(d + 2K_1) & 0 &= 2f + 24 \\ 0 &= b(K_1 - K_2) & 0 &= 2c \\ 0 &= 8K_2 - 14K_1 - 2d - f(K_1 - K_2) & 0 &= 2a + 12. \end{aligned}$$

By the last equality we have $a = -6$ which substituting into the fourth equality of the second column we conclude $K_1 = K_2$, a contradiction, and thus we proof the claim.

**Claim 2:** $F$ is irreducible. Indeed, if $F$ is reducible then there exist polynomials of degree 1 and degree 2 such that,

$$F(x,y) = (g+hx+iy)\left(a+bx+cy+dxy+ex^2+fy^2\right)$$

that is,

$$0 = 2y^3-(K_2-K_1)xy^2-x(K_2-K_1)-(g+hx+iy)\left(a+bx+cy+dxy+ex^2+fy^2\right)$$

which reorganizing

$$\begin{aligned}0 &= -hex^3-(ie+dh)x^2y-(ge+bh)x^2+(2ih-id+K_1-K_2)xy^2-(ib+ch+dg)xy\\ &\quad +(K_1-K_2-ah-bg)x+(2ig-ic)y^2-(ia+cg)y-ag\end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0=he & 0=2ih-id+K_1-K_2 & 0=2ig-ic\\ 0=ie+dh & 0=ib+ch+dg & 0=ia+cg\\ 0=ge+bh & 0=K_1-K_2-ah-bg & 0=ag. \end{array} \tag{A.10}$$

If $i \neq 0$ we have by the first equality of the third column that $c = 2g$, then

$$\begin{array}{lll} 0=he & 0=2ih-id+K_1-K_2 & \\ 0=ie+dh & 0=ib+2gh+dg & 0=ia+2g^2\\ 0=ge+bh & 0=K_1-K_2-ah-bg & 0=ag, \end{array}$$

that is, by the penultimate equality, $a=-\frac{2g^2}{i}$ which substituting into the last equality we obtain $g=0$. Therefore,

$$\begin{array}{lll} 0=he & 0=bh & 0=ib\\ 0=ie+dh & 0=2ih-id+K_1-K_2 & 0=K_1-K_2, \end{array}$$

and by the last equality $K_1=K_2$, a contradiction, hence $i=0$. From $(A.10)$ we have

$$\begin{array}{lll} 0=he & 0=K_1-K_2 & \\ 0=dh & 0=ch+dg & 0=cg\\ 0=ge+bh & 0=K_1-K_2-ah-bg & 0=ag, \end{array}$$

and by the first equality of the second column we obtain $K_1=K_2$, a contradiction, and thus we proof the claim. □

**Lemma A.0.8.** *Define* $G:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ *by*

$$G(x,y) = K_1x^2y^4+2K_1x^2y^2+K_1x^2+4c^2$$

*and*

$$\overline{G}(x,y) = (K_5-K_3)xy^2-4y^3+(K_5-K_3)x,$$

*where* $K_1, K_3, K_5, c$ *are constants with* $K_5 \neq K_3$ *and* $c \neq 0$. *We conclude that the polynomials* $G$ *and* $\overline{G}$ *have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $\overline{G}$ does not divide $G$. Indeed, if $\overline{G}$ divides $G$ then there exists a polynomial of degree 3 such that

$$G(x,y) = \overline{G}(x,y)(f+gx+hy+ixy+jx^2+ky^2+lx^2y+mxy^2+nx^3+oy^3),$$

that is,

$$\begin{aligned}0 = & K_1x^2y^4+2K_1x^2y^2+K_1x^2+4c^2\\ & -((K_5-K_3)xy^2-4y^3+(K_5-K_3)x)(f+gx+hy+ixy+jx^2+ky^2+lx^2y+mxy^2+nx^3+oy^3)\end{aligned}$$

which reorganizing

$$\begin{aligned}0 = & n(K_3-K_5)x^4y^2+n(K_3-K_5)x^4+(4n+l(K_3-K_5))x^3y^3+j(K_3-K_5)x^3y^2+l(K_3-K_5)x^3y\\ & +j(K_3-K_5)x^3+(4l+K_1+m(K_3-K_5))x^2y^4+(4j+iK_3-iK_5)x^2y^3\\ & +(2K_1+g(K_3-K_5)+m(K_3-K_5))x^2y^2+i(K_3-K_5)x^2y+(K_1+g(K_3-K_5))x^2\\ & +(4m+o(K_3-K_5))xy^5+(k(K_3-K_5)+4i)xy^4+(4g+h(K_3-K_5)+o(K_3-K_5))xy^3\\ & +(f(K_3-K_5)+k(K_3-K_5))xy^2+h(K_3-K_5)xy+f(K_3-K_5)x+4oy^6\\ & +4ky5+4hy^4+4fy^3+4c^2\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0 =& n(K_3-K_5)\\ 0 =& n(K_3-K_5)\\ 0 =& 4n+l(K_3-K_5)\\ 0 =& j(K_3-K_5)\\ 0 =& l(K_3-K_5)\\ 0 =& j(K_3-K_5)\\ 0 =& 4l+K_1+m(K_3-K_5)\\ 0 =& 4j+iK_3-iK_5\end{aligned}
\qquad
\begin{aligned}0 =& 2K_1+g(K_3-K_5)+m(K_3-K_5)\\ 0 =& i(K_3-K_5)\\ 0 =& K_1+g(K_3-K_5)\\ 0 =& 4m+o(K_3-K_5)\\ 0 =& k(K_3-K_5)+4i\\ 0 =& 4g+h(K_3-K_5)+o(K_3-K_5)\\ 0 =& f(K_3-K_5)+k(K_3-K_5)\end{aligned}
\qquad
\begin{aligned}0 =& h(K_3-K_5)\\ 0 =& f(K_3-K_5)\\ 0 =& 4o\\ 0 =& 4k\\ 0 =& 4h\\ 0 =& 4f\\ 0 =& 4c^2.\end{aligned}$$

Hence, by the last equality, $c=0$, a contradiction and thus we have proof the claim.

**Claim 2:** $\overline{G}$ is irreducible. Indeed, if $\overline{G}$ is reducible then there exist polynomials of degree 1 and degree 2 such that

$$\overline{G}(x,y) = (f+gx+hy)(i+jx+ky+lxy+mx^2+ny^2)$$

that is,

$$0 = (K_5-K_3)xy^2-4y^3+(K_5-K_3)x-(f+gx+hy)(i+jx+ky+lxy+mx^2+ny^2)$$

which reorganizing

$$0 = -gmx^3 - (gl+hm)x^2y - (gj+fm)x^2 + (K_5 - K_3 - hl - gn)xy^2 - (fl+gk+hj)xy \\ + (K_5 - K_3 - ig - fj)x - (hn+4)y^3 - (hk+fn)y^2 - (ih+fk)y - if$$

that occurs if, and only if,

$$\begin{aligned} 0 &= gm \\ 0 &= gl+hm \\ 0 &= gj+fm \\ 0 &= K_5 - K_3 - hl - gn \end{aligned} \qquad \begin{aligned} 0 &= fl+gk+hj \\ 0 &= K_5 - K_3 - ig - fj \\ 0 &= hn+4 \end{aligned} \qquad \begin{aligned} 0 &= hk+fn \\ 0 &= ih+fk \\ 0 &= if. \end{aligned} \tag{A.11}$$

By the first equality $g$ or $m$ is zero. Suppose $g = 0$. We have

$$\begin{aligned} 0 &= hm \\ 0 &= fm \\ 0 &= K_5 - K_3 - hl \end{aligned} \qquad \begin{aligned} 0 &= fl+hj \\ 0 &= K_5 - K_3 - fj \\ 0 &= hn+4 \end{aligned} \qquad \begin{aligned} 0 &= hk+fn \\ 0 &= ih+fk \\ 0 &= if, \end{aligned}$$

Since $h \neq 0$ (otherwise we would not have factorization) we obtain $m = 0$. We have

$$\begin{aligned} 0 &= K_5 - K_3 - hl \\ 0 &= fl+hj \\ 0 &= K_5 - K_3 - fj \end{aligned} \qquad \begin{aligned} 0 &= hn+4 \\ 0 &= hk+fn \end{aligned} \qquad \begin{aligned} 0 &= ih+fk \\ 0 &= if, \end{aligned}$$

that is,

$$\begin{aligned} 0 &= K_5 - K_3 - hl \\ 0 &= fl+hj \\ 0 &= K_5 - K_3 - fj \end{aligned} \qquad \begin{aligned} n &= -\frac{4}{h} \\ 0 &= hk+fn \end{aligned} \qquad \begin{aligned} 0 &= ih+fk \\ 0 &= if, \end{aligned}$$

then,

$$\begin{aligned} 0 &= K_5 - K_3 - hl \\ 0 &= fl+hj \end{aligned} \qquad \begin{aligned} 0 &= K_5 - K_3 - fj \\ 0 &= hk - \frac{4f}{h} \end{aligned} \qquad \begin{aligned} 0 &= ih+fk \\ 0 &= if, \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &= K_5 - K_3 - hl \\ 0 &= fl+hj \end{aligned} \qquad \begin{aligned} 0 &= K_5 - K_3 - fj \\ k &= \frac{4f}{h^2} \end{aligned} \qquad \begin{aligned} 0 &= ih+fk \\ 0 &= if, \end{aligned}$$

hence

$$\begin{aligned} 0 &= K_5 - K_3 - hl \\ 0 &= fl+hj \end{aligned} \qquad \begin{aligned} 0 &= K_5 - K_3 - fj \\ 0 &= ih + \frac{4f^2}{h^2} \end{aligned} \quad [c] \qquad 0 = if,$$

thus

$$\begin{aligned} 0 &= K_5 - K_3 - hl \\ 0 &= fl+hj \end{aligned} \qquad \begin{aligned} 0 &= K_5 - K_3 - fj \\ i &= -\frac{4f^2}{h^3} \end{aligned} \qquad 0 = if,$$

that is,

$$\begin{aligned} 0 &= K_5 - K_3 - hl & \quad 0 &= K_5 - K_3 - fj & \quad & \\ 0 &= fl + hj & \quad k &= \frac{4f}{h^2} & \quad 0 &= -\frac{4f^3}{h^3}, \end{aligned}$$

then, by the last equality, $f = 0$ concluding by the first equality of the second column that $K_5 = K_3$ a contradiction. Therefore, $m = 0$. From $(A.11)$ we have,

$$\begin{aligned} 0 &= gl & \quad 0 &= fl + gk + hj & \quad 0 &= hk + fn \\ 0 &= gj & \quad 0 &= K_5 - K_3 - ig - fj & \quad 0 &= ih + fk \\ 0 &= K_5 - K_3 - hl - gn & \quad 0 &= hn + 4 & \quad 0 &= if, \end{aligned}$$

i.e., as $h \neq 0$ (otherwise we would have 4=0),

$$\begin{aligned} 0 &= gl & \quad 0 &= fl + gk + hj & \quad 0 &= hk + fn \\ 0 &= gj & \quad 0 &= K_5 - K_3 - ig - fj & \quad 0 &= ih + fk \\ 0 &= K_5 - K_3 - hl - gn & \quad n &= -\frac{4}{n} & \quad 0 &= if, \end{aligned}$$

and repeating the steps in an similar way we conclude $K_5 = K_3$, a contradiction, and thus we proof the claim. □

**Lemma A.0.9.** *Let be $K_1, K_2$ non-zero constants. Defining $G : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{G} : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$G(x,y) = -2xK_2 - 2yK_1^2 + 2xyK_1K_2 - K_1 + 3y^2K_1$$

*and*

$$\overline{G}(x,y) = K_1 + y^2K_1 - xK_2 - xy^2K_2 + y - y^3.$$

*we conclude that the polynomials $G$ and $\overline{G}$ have no factors in common.*

**Proof:** We will split the proof into four claims.

**Claim 1:** $G$ does not divide $\overline{G}$. Indeed, if $G$ divides $\overline{G}$ then there exists a polynomial of degree 1 such that

$$\overline{G}(x,y) = G(x,y)\,(Ax + By + C)\,,$$

that is,

$$0 = K_1 + y^2K_1 - xK_2 - xy^2K_2 + y - y^3 - \left(-2xK_2 - 2yK_1^2 + 2xyK_1K_2 - K_1 + 3y^2K_1\right)(Ax + By + C)$$

which reorganizing

$$\begin{aligned} 0 = & -2AK_1K_2x^2y + 2AK_2x^2 + (-K_2 - 3AK_1 - 2BK_1K_2)xy^2 + (2AK_1^2 + 2BK_2 - 2K_1K_2C)xy \\ & + (-K_2 + AK_1 + 2CK_2)x + (-1 - 3BK_1)y^3 + (K_1 + 2BK_1^2 - 3CK_1)y^2 \\ & + (1 + BK_1 + 2CK_1^2)y + K_1 + CK_1 \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{rrr} -2AK_1K_2=0 & (2AK_1^2+2BK_2-2K_1K_2C)=0 & (K_1+2BK_1^2-3CK_1)=0 \\ 2AK_2=0 & (-K_2+AK_1+2CK_2)=0 & (1+BK_1+2CK_1^2)=0 \\ (-K_2-3AK_1-2BK_1K_2)=0 & (-1-3BK_1)=0 & K_1+CK_1=0. \end{array}$$

Since $K_1$ and $K_2$ are non-zero we obtain from the first equality above that $A=0$. Consequently, by the second equality of the second column and last equality of the third column, we conclude $C=-1$ and $C=\frac{1}{2}$, a contradiction, and thus we proof the claim.

**Claim 2:** If $K_1 \neq \pm 1$ then $G$ is irreducible. Indeed, if $G$ is reducible then there exist polynomials of degree 1 such that,

$$G(x,y)=(Ax+By+C)(Dx+Ey+F)$$

that is,

$$\begin{aligned} 0 = &-2xK_2-2yK_1^2+2xyK_1K_2-K_1+3y^2K_1-ADx^2-BDxy-CDx-AExy-BEy^2 \\ &-CEy-AFx-BFy-CF \end{aligned}$$

which reorganizing

$$\begin{aligned} 0 = &-ADx^2+(2K_1K_2-BD-AE)xy+(-2K_2-AF-CD)x+(3K_1-BE)y^2 \\ &+\left(-2K_1^2-CE-BF\right)y-K_1-CF \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{rrr} -AD=0 & -2K_2-AF-CD=0 & -2K_1^2-CE-BF=0 \\ 2K_1K_2-BD-AE=0 & 3K_1-BE=0 & -K_1-CF=0, \end{array}$$

that is,

$$\begin{array}{lll} AD=0 & K_2=-\left(\dfrac{AF+CD}{2}\right) & K_1^2=-\left(\dfrac{CE+BF}{2}\right) \\ K_1K_2=\dfrac{BD+AE}{2} & K_1=\dfrac{BE}{3} & K_1=-CF. \end{array}$$

Since $K_1$ and $K_2$ are non-zero we obtain from the second equality that $A$ and $D$ are not mutually zero and, from the fourth and sixth equalities, that $B$ and $C$ are non-zero. Therefore,

$$\begin{array}{lll} AD=0 & E=\dfrac{3K_1}{B} & \\ K_1=-\dfrac{BD+AE}{AF+CD} & K_1^2=-\left(\dfrac{CE+BF}{2}\right) & F=-\dfrac{K_1}{C}, \end{array}$$

finally,

$$AD=0 \qquad K_1=\frac{BD+\frac{3AK_1}{B}}{\frac{AK_1}{C}-CD} \qquad -2K_1^2=K_1\left(\frac{3C}{B}-\frac{B}{C}\right)$$

concluding that in both cases $A=0$ or $D=0$ we have, substituting the second equality in the third, $K_1^2=1$, that is, a contradiction and thus we proof the claim.

**Claim 3:** If $K_1 = 1$ then $G(x,y) = (2xK_2 + 3y + 1)(y-1)$ and none of these factors divides $\overline{G}(x,y)$. Indeed, if $y-1$ divides $\overline{G}(x,y)$ there exists a polynomial of degree 2 such that,

$$\overline{G}(x,y) = \left(Ax^2 + Bxy + Cy^2 + Dx + Ey + F\right)(y-1),$$

that is,

$$\begin{aligned} 0 = 1 &+ y^2 - xK_2 - xy^2K_2 + y - y^3 - Ax^2y - Bxy^2 - Cy^3 - Dxy - Ey^2 - Fy + Ax^2 \\ &+ Bxy + Cy^2 + Dx + Ey + F \end{aligned}$$

which reorganizing

$$\begin{aligned} 0 = &-Ax^2y + Ax^2 + (-B-K_2)xy^2 + (B-D)xy + (D-K_2)x \\ &+ (-1-C)y^3 + (1+C-E)y^2 + (1+E-F)y + F + 1 \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} -A &= 0 & B-D &= 0 & 1+C-E &= 0 \\ A &= 0 & D-K_2 &= 0 & 1+E-F &= 0 \\ -B-K_2 &= 0 & -1-C &= 0 & F+1 &= 0. \end{aligned}$$

From the last four equalities we obtain $C = -1$ and $C = -3$, a contradiction. Also, if $2xK_2 + 3y + 1$ divides $\overline{G}(x,y)$ there exists a polynomial of degree 2 such that,

$$\overline{G}(x,y) = \left(Ax^2 + Bxy + Cy^2 + Dx + Ey + F\right)(2xK_2 + 3y + 1),$$

that is,

$$\begin{aligned} 0 = &1 + y^2 - xK_2 - xy^2K_2 + y - y^3 - 2AK_2x^3 - 2BK_2x^2y - 2CK_2xy^2 - 2DK_2x^2 - 2EK_2xy \\ &- 2FK_2x - 3Ax^2y - 3Bxy^2 - 3Cy^3 - 3Dxy - 3Ey^2 - 3Fy - Ax^2 - Bxy - Cy^2 - Dx - Ey - F \end{aligned}$$

which reorganizing

$$\begin{aligned} 0 = &-2AK_2x^3 + (-2BK_2 - 3A)x^2y + (-K_2 - 2CK_2 - 3B)xy^2 + (-2DK_2 - A)x^2 \\ &+ (-2EK_2 - 3D - B)xy + (-K_2 - 2FK_2 - D)x + (-1-3C)y^3 + (1-3E-C)y^2 \\ &+ (1-3F-E)y + 1 - F \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} -2A &= 0 & & & & \\ -2BK_2 - 3A &= 0 & -2EK_2 - 3D - B &= 0 & 1-3E-C &= 0 \\ -K_2 - 2CK_2 - 3B &= 0 & -K_2 - 2FK_2 - D &= 0 & 1-3F-E &= 0 \\ -2DK_2 - A &= 0 & -1-3C &= 0 & 1-F &= 0. \end{aligned}$$

From the last four equalities we obtain $C=\frac{1}{3}$ and $C=7$, a contradiction and thus we proof the claim.

**Claim 4:** If $K_1=-1$ then $G(x,y)=(-2xK_2-3y+1)(y+1)$ and none of these factors divides $\overline{G}$. Indeed, if $y+1$ divides $\overline{G}(x,y)$ there exists a polynomial of degree 2 such that,

$$\overline{G}(x,y)=\left(Ax^2+Bxy+Cy^2+Dx+Ey+F\right)(y+1),$$

that is,

$$\begin{aligned}0=&-1-y^2-xK_2-xy^2K_2+y-y^3-Ax^2y-Bxy^2-Cy^3-Dxy-Ey^2-Fy-Ax^2-Bxy\\&-Cy^2-Dx-Ey-F\end{aligned}$$

which reorganizing,

$$\begin{aligned}0=&-Ax^2y-Ax^2+(-K_2-B)xy^2+(-D-B)xy+(-K_2-D)x+(-1-C)y^3+(-1-E-C)y^2\\&+(1-F-E)y-1-F\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}-A&=0 & -D-B&=0 & -1-E-C&=0\\ -A&=0 & -K_2-D&=0 & 1-F-E&=0\\ -K_2-B&=0 & -1-C&=0 & -1-F&=0.\end{aligned}$$

From the last four equalities we obtain $C=1$ and $C=-1$, a contradiction. Also, if $-2xK_2-3y+1$ divides $\overline{G}(x,y)$ there exists a polynomial of degree 2 such that,

$$\overline{G}(x,y)=\left(Ax^2+Bxy+Cy^2+Dx+Ey+F\right)(-2xK_2-3y+1),$$

that is,

$$\begin{aligned}0=&-1-y^2-xK_2-xy^2K_2+y-y^3+2AK_2x^3+2BK_2x^2y+2CK_2xy^2+2DK_2x^2+2EK_2xy\\&+2FK_2x+3Ax^2y+3Bxy^2+3Cy^3+3Dxy+3Ey^2+3Fy-Ax^2-Bxy-Cy^2-Dx-Ey-F\end{aligned}$$

which reorganizing

$$\begin{aligned}0=&2AK_2x^3+(2BK_2+3A)x^2y+(-K_2+2CK_2+3B)xy^2+(2DK_2-A)x^2+(2EK_2+3D-B)xy\\&+(-K_2+2FK_2-D)x+(-1+3C)y^3+(-1+3E-C)y^2+(1+3F-E)y-1-F\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}2A&=0\\ 2BK_2+3A&=0\\ -K_2+2CK_2+3B&=0\\ 2DK_2-A&=0\end{aligned}\qquad \begin{aligned}2EK_2+3D-B&=0 & 1+3E-C&=0\\ -K_2+2FK_2-D&=0 & 1+3F-E&=0\\ -1+3C&=0 & -1-F&=0.\end{aligned}$$

From the last four equalities we obtain $C=-\frac{1}{3}$ and $C=-5$, a contradiction and thus we proof the claim. □

**Lemma A.0.10.** *Define* $G:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G}:(0,+\infty)\times\mathbb{R}\to\mathbb{R}$ *by*

$$\begin{aligned}G(x,y) &= y^6-K_1xy^4+\left(\frac{3a^2-K_2^2}{a^2}\right)y^4-\frac{8K_2}{a^2}y^3-2K_1xy^2-\left(\frac{2K_2^2-3a^2+16}{a^2}\right)y^2\\ &\quad -\frac{8K_2}{a^2}y-K_1x-\left(\frac{K_2^2-a^2}{a^2}\right)\end{aligned}$$

*and*

$$\begin{aligned}\overline{G}(x,y) &= \left(2a^2K_2+4K_2\right)x^2-2a^2x^2y^5+\left(5a^2K_2+4K_2\right)x^2y^2-a^2K_3x^3+\left(2a^2+16\right)x^2y\\ &\quad +4a^2K_2x^2y^4-2a^2K_3x^3y^2+a^2K_2x^2y^6-a^2K_3x^3y^4,\end{aligned}$$

*where* $K_1,K_2,K_3,a$ *are constants and* $a,K_1,K_2$ *are non-zero. We conclude that the polynomials* $G$ *and* $\overline{G}$ *have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $G$ does not divide $\overline{G}$. Indeed, if $G$ divides $\overline{G}$ then there exists a polynomial of degree 2 such that

$$\overline{G}(x,y)=G(x,y)\left(Ax^2+Bxy+Cy^2+Dx+Ey+F\right),$$

that is,

$$0=\overline{G}(x,y)-G(x,y)\left(Ax^2+Bxy+Cy^2+Dx+Ey+F\right)$$

which reorganizing

$$\begin{aligned}0 &= (AK_1-a^2K_3)x^3y^4+(2AK_1-2a^2K_3)x^3y^2+(AK_1-a^2K_3)x^3+(a^2-A)x^2y^6+(BK_1-2a^2)x^2y^5\\
&+\left(DK_1-3A+4a^2K_2+\frac{A}{a^2}K_2^2\right)x^2y^4+\left(2BK_1+8\frac{A}{a^2}K_2\right)x^2y^3\\
&+\left(4K_2-3A+16\frac{A}{a^2}+2DK_1+5a^2K_2+2\frac{A}{a^2}K_2^2\right)x^2y^2\\
&+\left(BK_1+2a^2+8\frac{A}{a^2}K_2+16\right)x^2y+\left(4K_2-A+DK_1+2a^2K_1+\frac{A}{a^2}K_2^2\right)x^2\\
&+(-B)xy^7+(CK_1-D)xy^6+\left(EK_1-3B+\frac{B}{a^2}K_2^2\right)xy^5\\
&+\left(2CK_1-3D+FK_1+\frac{1}{a^2}DK_2^2+8\frac{B}{a^2}\right)xy^4+\left(16\frac{B}{a^2}-3B+2EK_1+2\frac{B}{a^2}K_2^2+\frac{8}{a^2}DK_2\right)xy^3\\
&+\left(CK_1-3D+2FK_1+\frac{16}{a^2}D+\frac{2}{a^2}DK_2^2+8\frac{B}{a^2}K_2\right)xy^2+\left(EK_1-B+\frac{B}{a^2}K_2^2+\frac{8}{a^2}DK_2\right)xy\\
&+\left(FK_1-D+\frac{1}{a^2}DK_2^2\right)x+(-C)y^8+(-E)y^7+\left(\frac{C}{a^2}K_2^2-F-3C\right)y^6\\
&+\left(\frac{1}{a^2}EK_2^2-3E+8\frac{C}{a^2}\right)y^5+\left(16\frac{C}{a^2}-3F-3C+2\frac{C}{a^2}K_2^2+\frac{F}{a^2}K_2^2+\frac{8}{a^2}EK_2\right)y^4\\
&+\left(\frac{16}{a^2}E-3E+\frac{2}{a^2}EK_2^2+8\frac{C}{a^2}K_2+8\frac{F}{a^2}K_2\right)y^3+\left(16\frac{F}{a^2}-3F-C+\frac{C}{a^2}K_2^2+2\frac{F}{a^2}K_2^2+\frac{8}{a^2}EK_2\right)y^2\\
&+\left(\frac{1}{a^2}EK_2^2-E+8\frac{F}{a^2}K_2\right)y+\left(\frac{F}{a^2}K_2^2-F\right)\end{aligned}$$

that occurs if, and only if,

$$
\begin{aligned}
0 &= AK_1 - a^2K_3\\
0 &= 2AK_1 - 2a^2K_3\\
0 &= AK_1 - a^2K_3\\
0 &= a^2 - A\\
0 &= BK_1 - 2a^2\\
0 &= DK_1 - 3A + 4a^2K_2 + \frac{A}{a^2}K_2^2\\
0 &= 2BK_1 + 8\frac{A}{a^2}K_2\\
0 &= 4K_2 - 3A + 16\frac{A}{a^2} + 2DK_1 + 5a^2K_2 + 2\frac{A}{a^2}K_2^2\\
0 &= BK_1 + 2a^2 + 8\frac{A}{a^2}K_2 + 16\\
0 &= 4K_2 - A + DK_1 + 2a^2K_1 + \frac{A}{a^2}K_2^2\\
0 &= -B\\
0 &= CK_1 - D\\
0 &= EK_1 - 3B + \frac{B}{a^2}K_2^2\\
0 &= 2CK_1 - 3D + FK_1 + \frac{1}{a^2}DK_2^2 + 8\frac{B}{a^2}
\end{aligned}
$$

$$
\begin{aligned}
0 &= 16\frac{B}{a^2} - 3B + 2EK_1 + 2\frac{B}{a^2}K_2^2 + \frac{8}{a^2}DK_2\\
0 &= CK_1 - 3D + 2FK_1 + \frac{16}{a^2}D + \frac{2}{a^2}DK_2^2 + 8\frac{B}{a^2}K_2\\
0 &= EK_1 - B + \frac{B}{a^2}K_2^2 + \frac{8}{a^2}DK_2\\
0 &= FK_1 - D + \frac{1}{a^2}DK_2^2\\
0 &= -C\\
0 &= -E\\
0 &= \frac{C}{a^2}K_2^2 - F - 3C\\
0 &= \frac{1}{a^2}EK_2^2 - 3E + 8\frac{C}{a^2}\\
0 &= 16\frac{C}{a^2} - 3F - 3C + 2\frac{C}{a^2}K_2^2 + \frac{F}{a^2}K_2^2 + \frac{8}{a^2}EK_2\\
0 &= \frac{16}{a^2}E - 3E + \frac{2}{a^2}EK_2^2 + 8\frac{C}{a^2}K_2 + 8\frac{F}{a^2}K_2\\
0 &= 16\frac{F}{a^2} - 3F - C + \frac{C}{a^2}K_2^2 + 2\frac{F}{a^2}K_2^2 + \frac{8}{a^2}EK_2\\
0 &= \frac{1}{a^2}EK_2^2 - E + 8\frac{F}{a^2}K_2\\
0 &= \frac{F}{a^2}K_2^2 - F.
\end{aligned}
$$

By the fourth and eleventh equalities of the first column we obtain $a^2 = A$ and $B = 0$. Therefore, by the seventh equality of the first column, we conclude that $K_2 = 0$, that is, a contradiction and thus we proof the claim.

**Claim 2:** $G$ is irreducible. Indeed, notice that

$$G(x,y) = -K_1(y^2+1)^2x + y^6 - \frac{K_2^2 - 3a^2}{a^2}y^4 - \frac{8}{a^2}K_2y^3 - \frac{2K_2^2 - 3a^2 + 16}{a^2}y^2 - \frac{8}{a^2}K_2y - \frac{K_2^2 - a^2}{a^2}$$

i.e., $G(x,y) = R(y)x + S(y)$ where

$$
\begin{aligned}
R(y) =& -K_1(y^2+1)^2\\
S(y) =& y^6 - \frac{K_2^2 - 3a^2}{a^2}y^4 - \frac{8}{a^2}K_2y^3 - \frac{2K_2^2 - 3a^2 + 16}{a^2}y^2 - \frac{8}{a^2}K_2y - \frac{K_2^2 - a^2}{a^2}.
\end{aligned}
$$

If $G$ is reducible there are polynomials $J, Q, P$ such that $G(x,y) = J(y)(Q(y)x + P(y))$, that is, $J|R$ and $J|S$. This is possible only if $(y^2+1)|S$, which occurs if and only if there exists a polynomial of degree 4 such that

$$S(y) = (y^2+1)(f + gy + hy^2 + iy^3 + jy^4)$$

i.e.,

$$0 = y^6 - \frac{K_2^2 - 3a^2}{a^2}y^4 - \frac{8}{a^2}K_2 y^3 - \frac{2K_2^2 - 3a^2 + 16}{a^2}y^2 - \frac{8}{a^2}K_2 y - \frac{K_2^2 - a^2}{a^2}$$
$$- (y^2+1)(f + gy + hy^2 + iy^3 + jy^4)$$

which reorganizing,

$$\begin{aligned} 0 =& (1-j)y^6 - iy^5 - \left(h + j + \frac{K_2^2 - 3a^2}{a^2}\right)y^4 - \left(g + i + \frac{8K_2}{a^2}\right)y^3 - \left(f + h + \frac{2K_2^2 - 3a^2 + 16}{a^2}\right)y^2 \\ & - \left(g + \frac{8K_2}{a^2}\right)y - \left(f + \frac{K_2^2 - a^2}{a^2}\right) \end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} 0 &= 1 - j & 0 &= g + i + \frac{8K_2}{a^2} & 0 &= g + \frac{8K_2}{a^2} \\ 0 &= i & & & & \\ 0 &= h + j + \frac{K_2^2 - 3a^2}{a^2} & 0 &= f + h + \frac{2K_2^2 - 3a^2 + 16}{a^2} & 0 &= f + \frac{K_2^2 - a^2}{a^2}. \end{aligned}$$

By the first and second equalities we obtain $j = 1$ and $i = 0$. We have,

$$\begin{aligned} 0 &= h + 1 + \frac{K_2^2 - 3a^2}{a^2} & 0 &= f + h + \frac{2K_2^2 - 3a^2 + 16}{a^2} & & \\ & & & & 0 &= f + \frac{K_2^2 - a^2}{a^2}. \\ 0 &= g + \frac{8K_2}{a^2} & 0 &= g + \frac{8K_2}{a^2} & & \end{aligned}$$

By the first equality $h = -\left(1 + \frac{K_2^2 - 3a^2}{a^2}\right)$. We have,

$$0 = g + \frac{8K_2}{a^2} \qquad 0 = f - \left(1 + \frac{K_2^2 - 3a^2}{a^2}\right) + \frac{2K_2^2 - 3a^2 + 16}{a^2} \qquad 0 = f + \frac{K_2^2 - a^2}{a^2},$$

and subtracting the third equality from the second equality we obtain $0 = \frac{16}{a^2}$, a contradiction, and thus we conclude the claim. □

**Lemma A.0.11.** *Define* $T : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{T} : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ *by*

$$\begin{aligned} T(x,y) =& 4a^2M_2y^5 + 4a^2M_4xy^4 - (2a^2M_2^2 + M_2^4)y^4 - 8M_2^3y^3 + 8a^2M_4xy^2 - (4a^2M_2^2 + 2M_2^4 + 16M_2^2)y^2 \\ & - (4a^2M_2 + 8M_2^3)y + 4a^2M_4x - 2a^2M_2^2 - M_2^4 + 16 \end{aligned}$$

*and*

$$\begin{aligned} \overline{T}(x,y) =& 2a^2y^7 + a^2M_1xy^6 - a^2M_2y^6 + (2a^2 - 2M_2^2)y^5 - M_1(M_2^2 - 3a^2)xy^4 - (3M_2a^2 + 16M_2)y^4 \\ & - 8M_1M_2xy^3 - (2a^2 + 4M_2^2 + 32)y^3 - M_1(2M_2^2 - 3a^2 + 16)xy^2 - (3M_2a^2 + 16M_2)y^2 \\ & - 8M_1M_2xy - (2a^2 + 2M_2^2)y - M_1(M_2^2 - a^2)x - a^2M_2. \end{aligned}$$

*where* $M_1, M_2, M_4, a$ *are constants and* $a, M_4$ *are non-zero. We conclude that the polynomials* $T$ *and* $\overline{T}$ *have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $T$ does not divide $\overline{T}$. Indeed, if $T$ divides $\overline{T}$ then there exists a polynomial of degree 2 such that

$$\overline{T}(x,y) = T(x,y)(f + gx + hy + ixy + jx^2 + ky^2)$$

that is,

$$0 = \overline{T}(x,y) - T(x,y)(f + gx + hy + ixy + jx^2 + ky^2)$$

which reorganizing

$$\begin{aligned}
0 &= -4aj^2M_4x^3y^4 - 8a^2jM_4x^3y^2 - 4a^2jM_4x^3 - (4a^2jM_2 + 4a^2iM_4)x^2y^5 \\
&+ (j(2a^2M_2^2 + M_2^4) - 4a^2gM_4)x^2y^4 + (8jM_2^3 - 8a^2iM_4)x^2y^3 \\
&+ (j(4a^2M_2^2 + 2M_2^4 + 16M_2^2) - 8a^2gM_4)x^2y^2 + (j(4a^2M_2 8M_2^3) - 4a^2iM_4)x^2y \\
&+ (j(2a^2M_2^2 + M_2^4 - 16) - 4a^2gM_4)x^2 + (a^2M_1 - 4a^2kM_4 - 4a^2iM_2)xy^6 \\
&+ (i(2a^2M_2^2 + M_2^4) - 4a^2gM_2 - 4a^2hM_4)xy^5 \\
&+ (3a^2M_1 - M_1M_2^2 + g(2a^2M_2^2 + M_2^4) + 8iM_2^3 - 4a^2fM_4 - 8a^2kM_4)xy^4 \\
&+ (i(4a^2M_2^2 + 16M_2^2) - 8M_1M_2 + 8gM_2^3 - 8a^2hM_4)xy^3 \\
&+ (i(4a^2M_2 + 8M_2^3) - 16M_1 - 2M_1M_2^2 + g(4a^2M_2^2 + 2M_2^4 + 16M_2^2) + 3a^2M_1 - 8a^2fM_4 - 4a^2kM_4)xy^2 \\
&+ (i(2a^2M_2^2 + M_2^4 - 16) - 8M_1M_2 + g(4a^2M_2 + 8M_2^3) - 4a^2hM_4)xy \\
&+ (a^2M_1 - M_1M_2^2 + g(2a^2M_2^2 + M_2^4 - 16) - 4a^2fM_4)x + (2a^2 - 4a^2kM_2)y^7 \\
&+ (k(2a^2M_2^2 + M_2^4) - a^2M_2 - 4a^2hM_2)y^6 \\
&+ (8kM_2^3 - 2M_2^2 + h(2a^2M_2^2 + M_2^4) + 2a^2 - 4a^2fM_2)y^5 \\
&+ (k(4a^2M_2^2 + 2M_2^4 + 16M_2^2) - 16M_2 - 3a^2M_2 + 8hM_2^3 + f(2a^2M_2^2 + M_2^4))y^4 \\
&+ (h(4a^2M_2^2 + 2M_2^4 + 16M_2^2) - 4M_2^2 + 8fM_2^3 + k(4a^2M_2 + 8M_2^3) - 2a^2 - 32)y^3 \\
&+ (f(4a^2M_2^2 + 2M_2^4 + 16M_2^2) - 16M_2 - 3a^2M_2 + h(4a^2M_2 + 8M_2^3) + k(2a^2M_2^2 + M_2^4 - 16))y^2 \\
&+ (f(4a^2M_2 + 8M_2^3) - 2M_2^2 + h(2a^2M_2^2 + M_2^4 - 16) - 2a^2)y + f(2a^2M_2^2 + M_2^4 - 16) - a^2M_2
\end{aligned}$$

that occurs if, and only if,

$$
\begin{aligned}
0 &= 4aj^2M_4\\
0 &= 8a^2jM_4\\
0 &= 4a^2jM_4\\
0 &= 4a^2jM_2+4a^2iM_4\\
0 &= j(2a^2M_2^2+M_2^4)-4a^2gM_4\\
0 &= 8jM_2^3-8a^2iM_4\\
0 &= j(4a^2M_2^2+2M_2^4+16M_2^2)-8a^2gM_4\\
0 &= j(4a^2M_28M_2^3)-4a^2iM_4\\
0 &= j(2a^2M_2^2+M_2^4-16)-4a^2gM_4\\
0 &= a^2M_1-4a^2kM_4-4a^2iM_2\\
0 &= i(2a^2M_2^2+M_2^4)-4a^2gM_2-4a^2hM_4\\
0 &= 3a^2M_1-M_1M_2^2+g(2a^2M_2^2+M_2^4)\\
&\quad +8iM_2^3-4a^2fM_4-8a^2kM_4
\end{aligned}
$$

$$
\begin{aligned}
0 &= i(4a^2M_2^2+16M_2^2)-8M_1M_2+8gM_2^3-8a^2hM_4\\
0 &= i(4a^2M_2+8M_2^3)-16M_1-2M_1M_2^2\\
&\quad +g(4a^2M_2^2+2M_2^4+16M_2^2)+3a^2M_1-8a^2fM_4\\
&\quad -4a^2kM_4\\
0 &= i(2a^2M_2^2+M_2^4-16)-8M_1M_2+g(4a^2M_2+8M_2^3)\\
&\quad -4a^2hM_4\\
0 &= a^2M_1-M_1M_2^2+g(2a^2M_2^2+M_2^4-16)-4a^2fM_4\\
0 &= 2a^2-4a^2kM_2\\
0 &= k(2a^2M_2^2+M_2^4)-a^2M_2-4a^2hM_2\\
0 &= 8kM_2^3-2M_2^2+h(2a^2M_2^2+M_2^4)+2a^2-4a^2fM_2\\
0 &= k(4a^2M_2^2+2M_2^4+16M_2^2)-16M_2-3a^2M_2\\
&\quad +8hM_2^3+f(2a^2M_2^2+M_2^4)\\
0 &= h(4a^2M_2^2+2M_2^4+16M_2^2)-4M_2^2+8fM_2^3\\
&\quad +k(4a^2M_2+8M_2^3)-2a^2-32\\
0 &= f(4a^2M_2^2+2M_2^4+16M_2^2)-16M_2-3a^2M_2\\
&\quad +h(4a^2M_2+8M_2^3)+k(2a^2M_2^2+M_2^4-16)\\
0 &= f(4a^2M_2+8M_2^3)-2M_2^2+h(2a^2M_2^2+M_2^4-16)-2a^2\\
0 &= f(2a^2M_2^2+M_2^4-16)-a^2M_2.
\end{aligned}
$$

By the first equality we obtain $j=0$. Substituting in the fourth and fifth equalities we obtain $i=g=0$. Therefore, by the eleventh equality we have $h=0$. Hence,

$$
\begin{aligned}
0 &= a^2M_1-4a^2kM_4\\
0 &= 3a^2M_1-M_1M_2^2-4a^2fM_4\\
&\quad -8a^2kM_4\\
0 &= -8M_1M_2\\
0 &= -16M_1-2M_1M_2^2+3a^2M_1\\
&\quad -8a^2fM_4-4a^2kM_4\\
0 &= -8M_1M_2\\
0 &= a^2M_1-M_1M_2^2-4a^2fM_4\\
0 &= 2a^2-4a^2kM_2
\end{aligned}
$$

$$
\begin{aligned}
0 &= k(2a^2M_2^2+M_2^4)-a^2M_2\\
0 &= 8kM_2^3-2M_2^2+2a^2-4a^2fM_2\\
0 &= k(4a^2M_2^2+2M_2^4+16M_2^2)-16M_2-3a^2M_2+f(2a^2M_2^2+M_2^4)\\
0 &= -4M_2^2+8fM_2^3+k(4a^2M_2+8M_2^3)-2a^2-32\\
0 &= f(4a^2M_2^2+2M_2^4+16M_2^2)-16M_2-3a^2M_2\\
&\quad +k(2a^2M_2^2+M_2^4-16)\\
0 &= f(4a^2M_2+8M_2^3)-2M_2^2-2a^2\\
0 &= f(2a^2M_2^2+M_2^4-16)-a^2M_2,
\end{aligned}
$$

that by the third equality $M_1M_2 = 0$, that is,

$$
\begin{aligned}
0 &= a^2M_1 - 4a^2kM_4\\
0 &= 3a^2M_1 - 4a^2fM_4\\
&- 8a^2kM_4\\
0 &= -16M_1 + 3a^2M_1\\
&- 8a^2fM_4 - 4a^2kM_4\\
0 &= a^2M_1 - 4a^2fM_4\\
0 &= 2a^2 - 4a^2kM_2\\
0 &= k(2a^2M_2^2 + M_2^4) - a^2M_2
\end{aligned}
\qquad
\begin{aligned}
0 &= 8kM_2^3 - 2M_2^2 + 2a^2 - 4a^2fM_2\\
0 &= k(4a^2M_2^2 + 2M_2^4 + 16M_2^2) - 16M_2 - 3a^2M_2 + f(2a^2M_2^2 + M_2^4)\\
0 &= -4M_2^2 + 8fM_2^3 + k(4a^2M_2 + 8M_2^3) - 2a^2 - 32\\
0 &= f(4a^2M_2^2 + 2M_2^4 + 16M_2^2) - 16M_2 - 3a^2M_2\\
&+ k(2a^2M_2^2 + M_2^4 - 16)\\
0 &= f(4a^2M_2 + 8M_2^3) - 2M_2^2 - 2a^2\\
0 &= f(2a^2M_2^2 + M_2^4 - 16) - a^2M_2.
\end{aligned}
$$

By the first and fourth equalities we obtain $k = f = \frac{M_1}{4M_4}$. Substituting in the third equalities we obtain $M_1 = 0$, that is, $k = f = 0$ and then we have no factorization. Therefore we conclude the claim.

**Claim 2:** $\overline{T}$ is irreducible. Indeed, notice that

$$
\begin{aligned}
\overline{T}(x,y) = 4a^2M_4(y^2+1)^2x + 4a^2M_2y^5 - (2a^2M_2^2 + M_2^4)y^4 - 8M_2^3y^3\\
- (4a^2M_2^2 + 2M_2^4 + 16M_2^2)y^2 - (4a^2M_2 + 8M_2^3)y - 2a^2M_2^2 - M_2^4 + 16
\end{aligned}
$$

i.e., $\overline{T}(x,y) = R(y)x + S(y)$ where

$$
\begin{aligned}
R(y) =& 4a^2M_4(y^2+1)^2\\
S(y) =& 4a^2M_2y^5 - (2a^2M_2^2 + M_2^4)y^4 - 8M_2^3y^3 - (4a^2M_2^2 + 2M_2^4 + 16M_2^2)y^2\\
&- (4a^2M_2 + 8M_2^3)y - 2a^2M_2^2 - M_2^4 + 16.
\end{aligned}
$$

If $\overline{T}$ is reducible there are polynomials $J,Q,P$ such that $\overline{T}(x,y) = J(y)(Q(y)x + P(y))$, that is, $J|R$ and $J|S$. This is only possible if $(y^2+1)|S$, which occurs if and only if there exists a polynomial of degree 3 such that

$$S(y) = (y^2+1)(f + gy + hy^2 + iy^3)$$

which reoganizing,

$$
\begin{aligned}
0 =& (4a^2M_2 - i)y^5 - (2a^2M_2^2 + M_2^4 + h)y^4 - (8M_2^3 + g + i)y^3 - (4a^2M_2^2 + 2M_2^4 + 16M_2^2 + f + h)y^2\\
&-(4a^2M_2 + 8M_2^3 + g)y + 16 - M_2^4 - f - 2a^2M_2^4
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{aligned}
0 =& 4a^2M_2 - i & 0 =& 8M_2^3 + g + i & 0 =& 4a^2M_2 + 8M_2^3 + g\\
0 =& 2a^2M_2^2 + M_2^4 + h & 0 =& 4a^2M_2^2 + 2M_2^4 + 16M_2^2 + f + h & 0 =& 16 - M_2^4 - f - 2a^2M_2^4.
\end{aligned}
$$

By the first and second equalities we have $i = 4a^2M_2^2$ and $h = -(2a^2M_2^2 + M_2^4)$. Substituting we have

$$
\begin{aligned}
0 =& 8M_2^3 + g + 4a^2M_2^2 & 0 =& 4a^2M_2 + 8M_2^3 + g\\
0 =& 2a^2M_2^2 + M_2^4 + 16M_2^2 + f & 0 =& 16 - M_2^4 - f - 2a^2M_2^4,
\end{aligned}
$$

that is,

$$0 = 8M_2^3 + g + 4a^2M_2^2 \qquad 0 = 2a^2M_2^2 + M_2^4 + 16M_2^2 + f \qquad 0 = 16 - M_2^4 - f - 2a^2M_2^4.$$

Adding the last two equalities we obtain $0 = 16(M_2^2 + 1)$, a contradiction, and thus we conclude the claim. □

**Lemma A.0.12.** *Define* $G : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{G} : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ *by*

$$G(x,y) = 6y^3 + K_2y^2x + 6yx + K_2x^2 - K_7$$

*and*

$$\begin{aligned}\overline{G}(x,y) = {} & 9y^6 + 3K_2y^5x + \frac{K_2^2}{4}y^4x^2 + 18y^4x + 6K_2y^3x^2 + \frac{K_2^2}{2}y^2x^3 + 9y^2x^2 \\ & - K_8y^2 + 3K_2yx^3 + \frac{K_2^2}{4}x^4 - K_8x.\end{aligned}$$

*where* $K_2, K_7, K_8$ *are constants and* $K_7, K_8$ *are non-zero. We conclude that the polynomials* $G$ *and* $\overline{G}$ *have no factors in common.*

**Proof:** We will split the proof into two statements.

**Claim 1:** $G$ does not divide $\overline{G}$. Indeed, if $G$ divides $\overline{G}$ then there exists a polynomial of degree 3 such that

$$\overline{G}(x,y) = G(x,y)(f + gx + hy + ixy + jx^2 + ky^2 + lxy^2 + mx^2y + nx^3 + oy^3)$$

that is,

$$\begin{aligned}0 = {} & -nK_2x^5 - nK_2x^4y^2 - (6n + mK_2)x^4y + \left(\frac{K_2^2}{4} - jK_2\right)x^4 - (6n + mK_2)x^3y^3 \\ & + \left(\frac{K_2^2}{2} - jK_2 - lK_2 - 6m\right)x^3y^2 + (3K_2 - 6j - iK_2)x^3y + (nK_7 - gK_2)x^3 + \left(\frac{K_2^2}{4} - lK_2 - 6m\right)x^2y^4 \\ & + (6K_2 - 6l - 6j - oK_2 - iK_2)x^2y^3 + (9 - gK_2 - kK_2 - 6i)x^2y^2 + (mK_7 - hK_2 - 6g)x^2y \\ & + (jK_7 - fK_2)x^2 + (3K_2 - 6l - oK_2)xy^5 + (18 - 6i - kK_2 - 6o)xy^4 - (6g + 6k + hK_2)xy^3 \\ & + (lK_7 - fK_2 - 6h)xy^2 + (iK_7 - 6f)xy + (gK_7 - K_8)x + (9 - 6o)y^6 - 6ky^5 - 6hy^4 \\ & + (oK_7 - 6f)y^3 + (kK_7 - K_8)y^2 + hK_7y + fK_7\end{aligned}$$

that occurs if, and only if,

$$0 = nK_2 \qquad 0 = 6K_2 - 6l - 6j - oK_2 - iK_2 \qquad 0 = gK_7 - K_8$$
$$0 = nK_2 \qquad 0 = 9 - gK_2 - kK_2 - 6i \qquad 0 = 9 - 6o$$
$$0 = 6n + mK_2 \qquad 0 = mK_7 - hK_2 - 6g \qquad 0 = 6k$$
$$0 = \frac{K_2^2}{4} - jK_2 \qquad 0 = jK_7 - fK_2 \qquad 0 = 6h$$
$$0 = 6n + mK_2 \qquad 0 = 3K_2 - 6l - oK_2 \qquad 0 = oK_7 - 6f$$
$$0 = \frac{K_2^2}{2} - jK_2 - lK_2 - 6m \qquad 0 = 18 - 6i - kK_2 - 6o \qquad 0 = kK_7 - K_8$$
$$0 = 3K_2 - 6j - iK_2 \qquad 0 = 6g + 6k + hK_2 \qquad 0 = hK_7$$
$$0 = nK_7 - gK_2 \qquad 0 = lK_7 - fK_2 - 6h \qquad 0 = fK_7.$$
$$0 = \frac{K_2^2}{4} - lK_2 - 6m \qquad 0 = iK_7 - 6f$$

By the third equality of the third column $k = 0$, which substituting into sixth equality fo the third column we obtain $K_8 = 0$, a contradiction and thus we conclude the claim.

**Claim 2:** $G$ is irreducible. Indeed, if $G$ is reducible there exist polynomials of degree 1 and degree 2 such that

$$G(x,y) = (f + gx + hy)(i + jx + ky + lxy + mx^2 + ny^2)$$

that is,

$$\begin{aligned} 0 =& -gmx^3 - (gl + hm)x^2y + (K_2 - gj - fm)x^2 + (K_2 - hl - gn)xy^2 + (6 - gk - hj - fl)xy \\ & - (gi + fj)x + (6 - hn)y^3 - (hk + fn)y^2 - (hi + fk)y - (K_7 + fi) \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = gm & 0 = 6 - gk - hj - fl & 0 = hk + fn \\ 0 = gl + hm & 0 = gi + fj & 0 = hi + fk \\ 0 = K_2 - gj - fm & 0 = 6 - hn & 0 = K_7 + fi. \\ 0 = K_2 - hl - gn & & \end{array} \tag{A.12}$$

By the first equality $g$ or $m$ is zero. Suppose $g = 0$. We have

$$\begin{array}{lll} 0 = hm & 0 = 6 - hj - fl & 0 = hk + fn \\ 0 = K_2 - fm & 0 = fj & 0 = hi + fk \\ 0 = K_2 - hl & 0 = 6 - hn & 0 = K_7 + fi. \end{array}$$

Since $h \neq 0$ (otherwise we would have a contradiction with the third equality of the second column) we conclude, by the first equality, $m = 0$. We have

$$\begin{array}{lll} 0 = K_2 & 0 = fj & 0 = hi + fk \\ 0 = K_2 - hl & 0 = 6 - hn & 0 = K_7 + fi. \\ 0 = 6 - hj - fl & 0 = hk + fn & \end{array}$$

If $K_2 \neq 0$ we have a contradiction. If $K_2 = 0$ then by the second equality we obtain $l = 0$. We have,

$$\begin{aligned} 0 &= 6 - hj & 0 &= 6 - hn & 0 &= hi + fk \\ 0 &= fj & 0 &= hk + fn & 0 &= K_7 + fi. \end{aligned}$$

By the first equality $j \neq 0$, therefore, by the second equality, $f = 0$ and then, by the last equality, $K_7 = 0$ a contradiction. Thus, assuming that $g = 0$ leads us to a contradiction thus $m = 0$. From $(A.12)$

$$\begin{aligned} 0 &= gl & 0 &= 6 - gk - hj - fl & 0 &= hk + fn \\ 0 &= K_2 - gj & 0 &= gi + fj & 0 &= hi + fk \\ 0 &= K_2 - hl - gn & 0 &= 6 - hn & 0 &= K_7 + fi. \end{aligned}$$

Since assuming $g = 0$ leads us to a contradiction we conclude by the first equality that $l = 0$. We have

$$\begin{aligned} 0 &= K_2 - gj & 0 &= gi + fj & & \\ 0 &= K_2 - gn & 0 &= 6 - hn & 0 &= hi + fk \\ 0 &= 6 - gk - hj & 0 &= hk + fn & 0 &= K_7 + fi. \end{aligned}$$

By the first and second equalities we obtain $j = n = \frac{K_2}{g}$. Then

$$\begin{aligned} 0 &= 6 - gk - h\frac{K_2}{g} & 0 &= 6 - h\frac{K_2}{g} & 0 &= hi + fk \\ 0 &= gi + f\frac{K_2}{g} & 0 &= hk + f\frac{K_2}{g} & 0 &= K_7 + fi. \end{aligned}$$

By the first and second equalities we obtain $k = \frac{1}{g}\left(6 - h\frac{K_2}{g}\right)$ and $i = -\frac{fK_2}{g^2}$. Thus

$$\begin{aligned} 0 &= 6 - h\frac{K_2}{g} & 0 &= -\frac{fhK_2}{g^2} + \frac{f}{g}\left(6 - h\frac{K_2}{g}\right) \\ 0 &= \frac{h}{g}\left(6 - h\frac{K_2}{g}\right) + f\frac{K_2}{g} & 0 &= K_7 - \frac{f^2K_2}{g^2}. \end{aligned}$$

By the first equality $h \neq 0$ and $K_2 \neq 0$. Substituting the first equality into the third equality we obtain that $f = 0$. Substituting into the last equality we obtain $K_7 = 0$, a contradiction, and thus we conclude the claim. □

**Lemma A.0.13.** *Define $H : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{H} : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$\begin{aligned} H(x,y) = {} & 6a^2y^3 + a^2M_2y^2x + M_3(36b^2 - a^2)y^2 + (6a^2 + 12M_2M_3b^2)yx \\ & + (a^2M_2 + M_3M_2^2b^2)x^2 - a^2M_3x \end{aligned}$$

*and*

$$\begin{aligned} \overline{H}(x,y) = {} & 36a^2y^4 + 12a^2M_2y^3x + a^2M_2^2y^2x^2 + 36a^2y^2x + M_4(144b^2 - 4a^2)y^2 \\ & + 12a^2M_2yx^2 + 48b^2M_2M_4yx + a^2M_2^2x^3 + 4b^2M_2^2M_4x^2 - 4a^2M_4x. \end{aligned}$$

*where $M_2, M_3, M_4$ are constants and $M_3$ is non-zero. We conclude that the polynomials $H$ and $\overline{H}$ have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $H$ does not divide $\overline{H}$. Indeed, if $H$ divides $\overline{H}$ then there exists a polynomial of degree 1 such that

$$\overline{H}(x,y) = H(x,y)(f+gx+hy)$$

that is,

$$\begin{aligned}
0 = &(a^2M_2^2 - g(a^2M_2+M_3b^2M_2^2))x^3 + (a^2M_2^2 - a^2gM_2)x^2y^2\\
&+(12a^2M_2 - h(a^2M_2+M_3b^2M_2^2) - g(6a^2+12M_2M_3b^2))x^2y\\
&+(4b^2M_2^2M_4 - f(a^2M_2+M_3b^2M_2^2) + a^2gM_3)x^2 + (12a^2M_2 - 6a^2g - a^2hM_2)xy^3\\
&+(g(a^2M_3-36b^2M_3) - h(6a^2+12M_2M_3b^2) + 36a^2 - a^2fM_2)xy^2\\
&+(48b^2M_2M_4 - f(6a^2+12M_2M_3b^2) + a^2hM_3)xy + (a^2fM_3 - 4a^2M_4)x + (36a^2 - 6a^2h)y^4\\
&+(h(a^2M_3-36b^2M_3) - 6a^2f)y^3 + (144b^2M_4 - 4a^2M_4 + f(a^2M_3 - 36b^2M_3))y^2
\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}
0 &= a^2M_2^2 - g(a^2M_2+M_3b^2M_2^2)\\
0 &= a^2M_2^2 - a^2gM_2\\
0 &= 12a^2M_2 - h(a^2M_2+M_3b^2M_2^2)\\
&\quad - g(6a^2+12M_2M_3b^2)\\
0 &= 4b^2M_2^2M_4 - f(a^2M_2+M_3b^2M_2^2) + a^2gM_3\\
0 &= 12a^2M_2 - 6a^2g - a^2hM_2
\end{aligned}$$

$$\begin{aligned}
0 &= g(a^2M_3-36b^2M_3) - h(6a^2+12M_2M_3b^2)\\
&\quad + 36a^2 - a^2fM_2\\
0 &= 48b^2M_2M_4 - f(6a^2+12M_2M_3b^2) + a^2hM_3\\
0 &= a^2fM_3 - 4a^2M_4\\
0 &= 36a^2 - 6a^2h\\
0 &= h(a^2M_3-36b^2M_3) - 6a^2f\\
0 &= 144b^2M_4 - 4a^2M_4 + f(a^2M_3-36b^2M_3).
\end{aligned}$$

By the ninth equality we have $h=6$. Therefore,

$$\begin{aligned}
0 &= a^2M_2^2 - g(a^2M_2+M_3b^2M_2^2)\\
0 &= a^2M_2^2 - a^2gM_2\\
0 &= 12a^2M_2 - 6(a^2M_2+M_3b^2M_2^2)\\
&\quad - g(6a^2+12M_2M_3b^2)\\
0 &= 4b^2M_2^2M_4 - f(a^2M_2+M_3b^2M_2^2) + a^2gM_3\\
0 &= 6a^2M_2 - 6a^2g
\end{aligned}$$

$$\begin{aligned}
0 &= g(a^2M_3-36b^2M_3) - 6(6a^2+12M_2M_3b^2)\\
&\quad + 36a^2 - a^2fM_2\\
0 &= 48b^2M_2M_4 - f(6a^2+12M_2M_3b^2) + 6a^2M_3\\
0 &= a^2fM_3 - 4a^2M_4\\
0 &= 6(a^2M_3-36b^2M_3) - 6a^2f\\
0 &= 144b^2M_4 - 4a^2M_4 + f(a^2M_3-36b^2M_3).
\end{aligned}$$

By the fifth equality we obtain $g=M_2$. Thus

$$\begin{aligned}
0 &= M_3b^2M_2^3\\
0 &= 12a^2M_2 - 6(a^2M_2+M_3b^2M_2^2)\\
&\quad - M_2(6a^2+12M_2M_3b^2)\\
0 &= 4b^2M_2^2M_4 - f(a^2M_2+M_3b^2M_2^2) + a^2M_2M_3\\
0 &= M_2(a^2M_3-36b^2M_3) - 6(6a^2+12M_2M_3b^2)\\
&\quad + 36a^2 - a^2fM_2
\end{aligned}$$

$$\begin{aligned}
0 &= 48b^2M_2M_4 - f(6a^2+12M_2M_3b^2) + 6a^2M_3\\
0 &= a^2fM_3 - 4a^2M_4\\
0 &= 6(a^2M_3-36b^2M_3) - 6a^2f\\
0 &= 144b^2M_4 - 4a^2M_4 + f(a^2M_3-36b^2M_3).
\end{aligned}$$

By the first equality $M_2 = 0$. If such equality does not occur we have a contradiction. If such equality occurs we have

$$\begin{aligned} c0 &= -6a^2 f + 6a^2 M_3 & 0 &= 6(a^2 M_3 - 36b^2 M_3) - 6a^2 f \\ 0 &= a^2 f M_3 - 4a^2 M_4 & 0 &= 144b^2 M_4 - 4a^2 M_4 + f(a^2 M_3 - 36b^2 M_3). \end{aligned}$$

By the first equality $f = M_3$ which, substituting into the third equality, we conclude $M_3 = 0$, a contradiction and thus we proof the claim.

**Claim 2:** $H$ is irreducible. Indeed, if $H$ is reducible there are polynomials of degree 1 and degree 2 such that

$$H(x,y) = (f + gx + hy)(i + jx + ky + lxy + mx^2 + ny^2)$$

that is,

$$\begin{aligned} 0 = &-gmx^3 - (gl + hm)x^2 y + (a^2 M_2 + M_3 b^2 M_2^2 - gj - fm)x^2 + (M_2 a^2 - hl - gn)xy^2 \\ &+ (6a^2 + 12M_2 M_3 b^2 - fl - gk - hj)xy - (M_3 a^2 + gi + fj)x + (6a^2 - hn)y^3 \\ &+ (36M_3 b^2 - M_3 a^2 - hk - fn)y^2 - (hi + fk)y - fi \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = gm & & 0 = 36M_3 b^2 - M_3 a^2 \\ & 0 = 6a^2 + 12M_2 M_3 b^2 - fl - gk - hj & \quad - hk - fn \\ 0 = gl + hm & 0 = M_3 a^2 + gi + fj & 0 = hi + fk \\ 0 = a^2 M_2 + M_3 b^2 M_2^2 - gj - fm & 0 = 6a^2 - hn & 0 = fi. \\ 0 = M_2 a^2 - hl - gn & & \end{array} \tag{A.13}$$

By the first equality $g$ or $m$ is zero. Suppose $g = 0$. Then

$$\begin{array}{lll} 0 = hm & 0 = 6a^2 + 12M_2 M_3 b^2 - fl - hj & 0 = 36M_3 b^2 - M_3 a^2 - hk - fn \\ 0 = a^2 M_2 + M_3 b^2 M_2^2 - fm & 0 = M_3 a^2 + fj & 0 = hi + fk \\ 0 = M_2 a^2 - hl & 0 = 6a^2 - hn & 0 = fi. \end{array}$$

Since $h \neq 0$ (otherwise we would not have factorization) we obtain by the first equality $m = 0$. We have

$$\begin{array}{lll} 0 = a^2 M_2 + M_3 b^2 M_2^2 & 0 = M_3 a^2 + fj & \\ 0 = M_2 a^2 - hl & 0 = 6a^2 - hn & 0 = hi + fk \\ 0 = 6a^2 + 12M_2 M_3 b^2 - fl - hj & 0 = 36M_3 b^2 - M_3 a^2 - hk - fn & 0 = fi. \end{array}$$

By the second and fifth equalities we obtain $l = \frac{M_2 a^2}{h}$ and $n = \frac{6a^2}{h}$. We have

$$\begin{array}{ll} 0 = a^2 M_2 + M_3 b^2 M_2^2 & 0 = 36M_3 b^2 - M_3 a^2 - hk - f\left(\dfrac{6a^2}{h}\right) \\ 0 = 6a^2 + 12M_2 M_3 b^2 - f\left(\dfrac{M_2 a^2}{h}\right) - hj & 0 = hi + fk \\ 0 = M_3 a^2 + fj & 0 = fi. \end{array}$$

By the second and fourth equalities we obtain

$$j = \frac{1}{h}\left(6a^2 + 12M_2M_3b^2 - f\left(\frac{M_2a^2}{h}\right)\right)$$

and

$$k = \frac{1}{h}\left(36M_3b^2 - M_3a^2 - f\left(\frac{6a^2}{h}\right)\right).$$

We have

$$\begin{aligned} 0 &= a^2M_2 + M_3b^2M_2^2 \\ 0 &= M_3a^2 + \frac{f}{h}\left(6a^2 + 12M_2M_3b^2 - f\left(\frac{M_2a^2}{h}\right)\right) \end{aligned} \qquad \begin{aligned} 0 &= hi + \frac{f}{h}\left(36M_3b^2 - M_3a^2 - f\left(\frac{6a^2}{h}\right)\right) \\ 0 &= fi. \end{aligned}$$

by the third equality we obtain $i = -\frac{f}{h^2}\left(36M_3b^2 - M_3a^2 - f\left(\frac{6a^2}{h}\right)\right)$ and therefore

$$\begin{aligned} 0 &= a^2M_2 + M_3b^2M_2^2 \\ 0 &= M_3a^2 + \frac{f}{h}\left(6a^2 + 12M_2M_3b^2 - f\left(\frac{M_2a^2}{h}\right)\right) \\ 0 &= \frac{f^2}{h^2}\left(36M_3b^2 - M_3a^2 - f\left(\frac{6a^2}{h}\right)\right). \end{aligned}$$

Since $f \neq 0$ (otherwise we would have $M_3a^2 = 0$ in the third equality, a contradiction) we obtain, by the last equality, $\frac{f}{h} = \frac{(36M_3b^2 - M_3a^2)}{6a^2}$ and then

$$\begin{aligned} 0 &= a^2M_2 + M_3b^2M_2^2 \\ 0 &= M_3a^2 + \frac{(36M_3b^2 - M_3a^2)}{6a^2}\left(6a^2 + 12M_2M_3b^2 - M_2\frac{(36M_3b^2 - M_3a^2)}{6}\right), \end{aligned}$$

which simplifying,

$$\begin{aligned} 0 &= M_2(a^2 + b^2M_2M_3) \\ 0 &= -1296a^2b^2 + a^4M_2M_3 - 1296b^4M_2M_3. \end{aligned}$$

By the second equality we conclude that $M_2 \neq 0$ (otherwise $a$ or $b$ would be zero). Therefore, by the first equality, we have $M_3 = -\frac{a^2}{b^2M_2}$ which substituting into the second equality we conclude

$$0 = a^2\frac{a^4 + 2592b^4}{b^2} \neq 0$$

a contradiction, that is, the case $g = 0$ does not occur, and therefore $m = 0$. We have $(A.13)$

$$\begin{aligned} 0 &= gl \\ 0 &= a^2M_2 + M_3b^2M_2^2 - gj \\ 0 &= M_2a^2 - hl - gn \end{aligned} \qquad \begin{aligned} 0 &= 6a^2 + 12M_2M_3b^2 - fl - gk - hj \\ 0 &= M_3a^2 + gi + fj \\ 0 &= 6a^2 - hn \end{aligned} \qquad \begin{aligned} 0 &= 36M_3b^2 - M_3a^2 \\ &\quad - hk - fn \\ 0 &= hi + fk \\ 0 &= fi. \end{aligned}$$

As seen before, assuming $g = 0$ leads us to a contradiction. Therefore, by the first equality $l = 0$. We have,

$$0 = a^2M_2 + M_3b^2M_2^2 - gj \qquad 0 = M_3a^2 + gi + fj \qquad 0 = hi + fk$$
$$0 = M_2a^2 - gn \qquad 0 = 6a^2 - hn \qquad 0 = fi.$$
$$0 = 6a^2 + 12M_2M_3b^2 - gk - hj \qquad 0 = 36M_3b^2 - M_3a^2 - hk - fn$$

By the first and second equalities we obtain $j = \frac{a^2M_2+M_3b^2M_2^2}{g}$ and $n = \frac{a^2M_2}{g}$. We have,

$$0 = 6a^2 + 12M_2M_3b^2 - gk - h\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right) \qquad 0 = 36M_3b^2 - M_3a^2 - hk - f\left(\frac{a^2M_2}{g}\right)$$
$$0 = M_3a^2 + gi + f\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right) \qquad 0 = hi + fk$$
$$0 = 6a^2 - h\left(\frac{a^2M_2}{g}\right) \qquad 0 = fi.$$

By the first and second equalities we have

$$i = -\frac{1}{g}\left(M_3a^2 + f\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right)$$

and

$$k = \frac{1}{g}\left(6a^2 + 12M_2M_3b^2 - h\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right).$$

Then

$$0 = 6a^2 - h\left(\frac{a^2M_2}{g}\right)$$
$$0 = 36M_3b^2 - M_3a^2 - \frac{h}{g}\left(6a^2 + 12M_2M_3b^2 - h\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right) - f\left(\frac{a^2M_2}{g}\right)$$
$$0 = -\frac{h}{g}\left(M_3a^2 + f\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right) + \frac{f}{g}\left(6a^2 + 12M_2M_3b^2 - h\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right)$$
$$0 = \frac{f}{g}\left(M_3a^2 + f\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right).$$

By the first equality $h$ and $M_2$ are non-zero. Therefore, $h = \frac{6g}{M_2}$. Thus,

$$0 = 36M_3b^2 - M_3a^2 - \frac{6}{M_2}\left(6a^2 + 12M_2M_3b^2 - \frac{6g}{M_2}\left(a^2M_2 + M_3b^2M_2^2\right)\right) - f\left(\frac{a^2M_2}{g}\right)$$
$$0 = -\frac{6}{M_2}\left(M_3a^2 + f\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right) + \frac{f}{g}\left(6a^2 + 12M_2M_3b^2 - \frac{6}{M_2}\left(a^2M_2 + M_3b^2M_2^2\right)\right)$$
$$0 = \frac{f}{g}\left(M_3a^2 + f\left(\frac{a^2M_2 + M_3b^2M_2^2}{g}\right)\right).$$

Simplifying the first equality we obtain $0 = \frac{fM_2+gM_3}{g}$, that is, $f = -\frac{gM_3}{M_2}$. Substituting in the last equality we obtain $0 = -M_3^3b^2 \neq 0$, a contradiction and thus we conclude the claim. $\square$

**Lemma A.0.14.** *Let be $K_1, K_2, K_3, K_4$ constants such that $K_1, K_2$ are non-zero. Define $G : (0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $\overline{G} : (0,+\infty)\times\mathbb{R}\to\mathbb{R}$ by*

$$\begin{aligned} G(x,y) =& 12K_1xy^6 + 12K_1x^2y^4 + (K_2^2 - 4K_1K_3)xy^4 + 32K_1^2y^4 + 16K_1K_2xy^3 + 2K_2^2x^2y^2 + 36K_1^2xy^2 \\ &+ 12K_1K_2x^2y + K_2^2x^3 \end{aligned}$$

*and*

$$\begin{aligned} \overline{G}(x,y) =& 6K_1xy^8 + 12K_1x^2y^6 + 16K_1^2y^6 + 8K_1K_2xy^5 + 6K_1x^3y^4 + K_2^2x^2y^4 + (52K_1^2 - 2K_1K_4)xy^4 \\ &+ 20K_1K_2x^2y^3 + 2K_2^2x^3y^2 + 36K_1^2x^2y^2 + 12K_1K_2x^3y + K_2^2x^4. \end{aligned}$$

*Then the polynomials $G$ and $\overline{G}$ have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $G$ does not divide $\overline{G}$. Indeed, if $G$ divides $\overline{G}$ then there exists a polynomial of degree 2 such that

$$\overline{G}(x,y) = G(x,y)(f + gx + hy + ixy + jx^2 + ky^2)$$

which reorganizing

$$\begin{aligned} 0 =& -jK_2^2x^5 - 12jK_1x^4y^4 - 2jK_2^2x^4y^2 \\ &- \left(iK_2^2 + 12jK_1K_2\right)x^4y \\ &+ \left(K_2^2 - gK_2^2\right)x^4 - 12jK_1x^3y^6 - 12iK_1x^3y^5 \\ &+ \left(6K_1 - j\left(K_2^2 - 4K_1K_3\right) - 12gK_1\right)x^3y^4 \\ &- \left(2iK_2^2 + 16jK_1K_2\right)x^3y^3 \\ &+ \left(2K_2^2 - 2gK_2^2 - 36jK_1^2 - kK_2^2 - 12iK_1K_2\right)x^3y^2 \\ &+ \left(12K_1K_2 - hK_2^2 - 12gK_1K_2\right)x^3y - fK_2^2x^3 \\ &- 12iK_1x^2y^7 + (12K_1 - 12gK_1 - 12kK_1)x^2y^6 \\ &- \left(12hK_1 + i\left(K_2^2 - 4K_1K_3\right)\right)x^2y^5 \\ &+ \left(K_2^2 - 12fK_1 - g\left(K_2^2 - 4K_1K_3\right) - 32jK_1^2 - 2kK_2^2 - 16iK_1K_2\right)x^2y^4 \\ &+ \left(20K_1K_2 - 2hK_2^2 - 36iK_1^2 - 16gK_1K_2 - 12kK_1K_2\right)x^2y^3 \\ &+ \left(36K_1^2 - 2fK_2^2 - 36gK_1^2 - 12hK_1K_2\right)x^2y^2 - 12fK_1K_2x^2y \\ &+ (6K_1 - 12kK_1)xy^8 - 12hK_1xy^7 - \left(k\left(K_2^2 - 4K_1K_3\right) + 12fK_1\right)xy^6 \\ &+ \left(8K_1K_2 - h\left(K_2^2 - 4K_1K_3\right) - 32iK_1^2 - 16kK_1K_2\right)xy^5 \\ &+ \left(52K_1^2 - f\left(K_2^2 - 4K_1K_3\right) - 2K_1K_4 - 32gK_1^2 - 36kK_1^2 - 16hK_1K_2\right)xy^4 \\ &- \left(36hK_1^2 + 16fK_2K_1\right)xy^3 \\ &- 36fK_1^2xy^2 + \left(16K_1^2 - 32kK_1^2\right)y^6 \\ &- 32hK_1^2y^5 - 32fK_1^2y^4 \end{aligned}$$

that occurs if, and only if,

$$
\begin{aligned}
0 &= jK_2^2\\
0 &= 12jK_1\\
0 &= 2jK_2^2\\
0 &= iK_2^2+12jK_1K_2\\
0 &= K_2^2-gK_2^2\\
0 &= 12jK_1\\
0 &= 12iK_1\\
0 &= 6K_1-j\left(K_2^2-4K_1K_3\right)-12gK_1\\
0 &= 2iK_2^2+16jK_1K_2\\
0 &= 2K_2^2-2gK_2^2-36jK_1^2-kK_2^2-12iK_1K_2\\
0 &= 12K_1K_2-hK_2^2-12gK_1K_2\\
0 &= fK_2^2\\
0 &= 12iK_1\\
0 &= 12K_1-12gK_1-12kK_1\\
0 &= 12hK_1+i\left(K_2^2-4K_1K_3\right)
\end{aligned}
$$

$$
\begin{aligned}
0 &= K_2^2-12fK_1-g\left(K_2^2-4K_1K_3\right)-32jK_1^2\\
&\quad -2kK_2^2-16iK_1K_2\\
0 &= 20K_1K_2-2hK_2^2-36iK_1^2-16gK_1K_2-12kK_1K_2\\
0 &= 36K_1^2-2fK_2^2-36gK_1^2-12hK_1K_2\\
0 &= 12fK_1K_2\\
0 &= 6K_1-12kK_1\\
0 &= 12hK_1\\
0 &= k\left(K_2^2-4K_1K_3\right)+12fK_1\\
0 &= 8K_1K_2-h\left(K_2^2-4K_1K_3\right)-32iK_1^2-16kK_1K_2\\
0 &= 52K_1^2-f\left(K_2^2-4K_1K_3\right)-2K_1K_4-32gK_1^2\\
&\quad -36kK_1^2-16hK_1K_2\\
0 &= 36hK_1^2+16fK_2K_1\\
0 &= 36fK_1^2\\
0 &= 16K_1^2-32kK_1^2\\
0 &= 32hK_1^2\\
0 &= 32fK_1^2.
\end{aligned}
$$

By the fifth and sixth equalities we obtain $g=1$ and $j=0$ which, substituting into the eighth equalities, we conclude $K_1=0$, a contradiction, and thus proof the claim.

**Claim 2:** $G$ is irreducible. Indeed, notice that

$$
\begin{aligned}
G(x,y) &= K_2^2x^3+(12K_1y^4+2K_2^2y^2+12K_1K_2y)x^2\\
&\quad +(12K_1y^6+(K_2^2-4K_1K_3)y^4+16K_1K_2y^3+36K_1^2y^2)x\\
&\quad +32K_1^2y^4
\end{aligned}
$$

therefore if such a polynomial is reducible it will be of the form $G(x,y)=(K_2^2x+S(y))(x^2+T(y)x+U(y))$. Therefore,

$$
\begin{aligned}
S(y)U(y) &= 32K_1^2y^4\\
K_2^2U(y)+S(y)T(y) &= 12K_1y^6+(K_2^2-4K_1K_3)y^4+16K_1K_2y^3+36K_1^2y^2\\
K_2^2T(y)+S(y) &= 12K_1y^4+2K_2^2y^2+12K_1K_2y.
\end{aligned}
$$

Then

$$\deg(SU)=4 \qquad \deg(K_2^2U+ST)=6 \qquad \deg(K_2^2T+S)=4$$

and consequently the possibilities

$$
\begin{aligned}
&\deg(S)=0 \text{ and } \deg(U)=4 \text{ then } \deg(T)=4 \text{ thus } \deg(K_2^2U+ST)\leqslant 4<6\\
&\deg(S)=1 \text{ and } \deg(U)=3 \text{ then } \deg(T)=4 \text{ thus } \deg(K_2^2U+ST)=5<6\\
&\deg(S)=3 \text{ and } \deg(U)=1 \text{ then } \deg(T)=4 \text{ thus } \deg(K_2^2U+ST)7>6,
\end{aligned}
$$

do not occur. There are two situations left. In the first situation

$$S(y) = \alpha y^2 \qquad U(y) = \beta y^2 \qquad Y(y) = A + By + Cy^2 + Dy^3 + Ey^4$$

where $\alpha, \beta, E$ are non-zero. Then

$$G(x,y) = (K_2^2 x + \alpha y^2)(x^2 + (A + By + Cy^2 Dy^3 + Ey^4)x + \beta y^2)$$

which reorganizing

$$\begin{aligned}0 = &\left(12K_1 - EK_2^2\right)x^2y^4 - DK_2^2x^2y^3 + \left(2K_2^2 - \alpha - CK_2^2\right)x^2y^2 + \left(12K_1K_2 - BK_2^2\right)x^2y - AK_2^2x^2 \\ &+ (12K_1 - \alpha E)xy^6 - \alpha Dxy^5 + \left(K_2^2 - C\alpha - 4K_1K_3\right)xy^4 + (16K_1K_2 - B\alpha)xy^3 \\ &+ \left(36K_1^2 - \beta K_2^2 - A\alpha\right)xy^2 + \left(32K_1^2 - \alpha\beta\right)y^4,\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned} 0 &= 12K_1 - EK_2^2 & 0 &= AK_2^2 & 0 &= 16K_1K_2 - B\alpha \\ 0 &= DK_2^2 & 0 &= 12K_1 - \alpha E & 0 &= 36K_1^2 - \beta K_2^2 - A\alpha \\ 0 &= 2K_2^2 - \alpha - CK_2^2 & 0 &= \alpha D & 0 &= 32K_1^2 - \alpha\beta. \\ 0 &= 12K_1K_2 - BK_2^2 & 0 &= K_2^2 - C\alpha - 4K_1K_3 & & \end{aligned}$$

By the second and fifth equalities we have $D = A = 0$. Substituting in the others,

$$\begin{aligned} 0 &= 12K_1 - EK_2^2 & 0 &= 12K_1 - \alpha E & 0 &= 36K_1^2 - \beta K_2^2 \\ 0 &= 2K_2^2 - \alpha - CK_2^2 & 0 &= K_2^2 - C\alpha - 4K_1K_3 & 0 &= 32K_1^2 - \alpha\beta. \\ 0 &= 12K_1K_2 - BK_2^2 & 0 &= 16K_1K_2 - B\alpha & & \end{aligned}$$

By the first, third and fourth equalities we obtain $E = \frac{12K_1}{K_2^2}$, $B = \frac{12K_1}{K_2}$ and $\alpha = K_2^2$. Thus, by the sixth equality, $4K_1K_2 = 0$, a contradiction. In the second situation

$$\begin{aligned} S(y) &= \alpha y^4 \\ U(y) &= \beta \\ Y(y) &= A + By + Cy^2 \end{aligned}$$

where $\alpha, \beta, C$ are non-zero. We have

$$G(x,y) = (K_2^2 x + \alpha y^4)(x^2 + (A + By + Cy^2)x + \beta)$$

which reorganizing

$$\begin{aligned}0 = (12K_1 - \alpha)x^2y^4 + (2K_2^2 - CK_2^2)x^2y^2 + (12K_1K_2 - BK_2^2)x^2y - AK_2^2x^2 + (12K_1 - C\alpha)xy^6 \\ - B\alpha xy^5 + (K_2^2 - A\alpha - 4K_1K_3)xy^4 + 16K_1K_2xy^3 + 36K_1^2xy^2 - \beta K_2^2 x + (32K_1^2 - \alpha\beta)y^4\end{aligned}$$

therefore $K_1 = 0$, a contradiction and thus we conclude the claim. □

**Lemma A.0.15.** *Let $K_1, K_2, K_3, c$ be constants where $K_3, c$ are non-zero. Define $G : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{G} : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$G(x,y) = 2cy^2x - cK_2y + 2cx^2 - cK_1x + K_3$$

*and*

$$\overline{G}(x,y) = y^4 + 2y^2x + x^2 - K_3.$$

*Then the polynomials G and $\overline{G}$ have no factors in common.*

**Proof:** We will split the proof into two claims.
**Claim 1:** $G$ does not divide $\overline{G}$. Indeed, if $G$ divides $\overline{G}$ then there exists a polynomial of degree 1 such that

$$\overline{G}(x,y) = G(x,y)(f + gx + hy)$$

that is,

$$0 = y^4 + 2y^2x + x^2 - K_3 - (2cy^2x - cK_2y + 2cx^2 - cK_1x + K_3)(f + gx + hy)$$

Which reorganizing

$$\begin{aligned} 0 = &-2cgx^3 - 2cgx^2y^2 - 2chx^2y^+(cgK_1 - 2cf + 1)x^2 - 2chxy^3 + (2 - 2cf)xy^2 \\ &+ (cgK_2 + chK_1)xy + (cfK_1 - gK_3)x + y^4 + chK_2y^2 + (cfK_2 - hK_3)y - (K_3 + fK_3) \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = 2cg & 0 = 2ch & 0 = 1 \\ 0 = 2cg & 0 = 2 - 2cf & 0 = chK_2 \\ 0 = 2ch & 0 = cgK_2 + chK_1 & 0 = cfK_2 - hK_3 \\ 0 = cgK_1 - 2cf + 1 & 0 = cfK_1 - gK_3 & 0 = K_3 + fK_3, \end{array}$$

and therefore by the first equality of the third column we obtain $1 = 0$, a contradiction.
**Claim 2:** $G$ is irreducible. Indeed, if $G$ is reducible there exist polynomials of degree 1 and degree 2 such that

$$G(x,y) = (f + gx + hy)(i + jx + ky + lxy + mx^2 + ny^2)$$

that is,

$$0 = 2cy^2x - cK_2y + 2cx^2 - cK_1x + K_3 - (f + gx + hy)(i + jx + ky + lxy + mx^2 + ny^2)$$

which reorganizing

$$\begin{aligned} 0 = &-gmx^3 - (gl + hm)x^2y + (2c - gj - fm)x^2 + (2c - hl - gn)xy^2 - (fl + gk + hj)xy \\ &- (gi + cK_1 + fj)x - hny^3 - (hk + fn)y^2 - (hi + cK_2 + fk)y + K_3 - fi \end{aligned}$$

that occurs if, and only if,

$$\begin{array}{lll} 0 = gm & & \\ & 0 = fl + gk + hj & 0 = hk + fn \\ 0 = gl + hm & & \\ & 0 = gi + cK_1 + fj & 0 = hi + cK_2 + fk \\ 0 = 2c - gj - fm & & \\ & 0 = hn & 0 = K_3 - fi. \\ 0 = 2c - hl - gn & & \end{array} \tag{A.14}$$

By the first equality, $g$ or $m$ is zero. Suppose $g = 0$. We have,

$$\begin{array}{lll} 0 = hm & 0 = fl + hj & 0 = hk + fn \\ 0 = 2c - fm & 0 = cK_1 + fj & 0 = hi + cK_2 + fk \\ 0 = 2c - hl & 0 = hn & 0 = K_3 - fi. \end{array}$$

Since $h \neq 0$ (otherwise we would not have factorization) we obtain by the first equality $m = 0$ which substituting in the second equality we have $c = 0$, a contradiction. Then $m = 0$. From $(A.14)$

$$\begin{array}{lll} 0 = gl & 0 = fl + gk + hj & 0 = hk + fn \\ 0 = 2c - gj & 0 = gi + cK_1 + fj & 0 = hi + cK_2 + fk \\ 0 = 2c - hl - gn & 0 = hn & 0 = K_3 - fi. \end{array}$$

By the first equality $g$ or $l$ is zero. If $g = 0$, by the second equality, we obtain $c = 0$, a contradiction. Then $l = 0$. Hence

$$\begin{array}{lll} 0 = 2c - gj & 0 = gi + cK_1 + fj & \\ & & 0 = hi + cK_2 + fk \\ 0 = 2c - gn & 0 = hn & \\ & & 0 = K_3 - fi, \\ 0 = gk + hj & 0 = hk + fn & \end{array}$$

and since $n \neq 0$ (otherwise $c = 0$ by the second equality) we obtain by the fifth equality $h = 0$. We have

$$\begin{array}{lll} 0 = 2c - gj & & \\ & 0 = gi + cK_1 + fj & 0 = cK_2 + fk \\ 0 = 2c - gn & & \\ & 0 = fn & 0 = K_3 - fi. \\ 0 = gk & & \end{array}$$

By the fifth equality $f = 0$, which substituting into the last equality we obtain $K_3 = 0$, a contradiction and thus we proof the claim. $\square$

**Lemma A.0.16.** *Define* $G : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{G} : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *by*

$$G(x, y) = a^2 K_1 y^4 + 2a^2 K_1 y^2 x + a^2 K_1 x^2 + 4c^2$$

*and*

$$\overline{G}(x, y) = (8K_1 a^4 + 32bcK_1 a^2) y^2 x + a^4 K_1 K_7 y + (4K_1 a^4 + 16bcK_1 a^2) x^2 + 8a^2 c^2 + 32bc^3.$$

*where $K_1, K_7, a, b, c$ are constants and $K_1, a, b, c$ are non-zero. We conclude that the polynomials $G$ and $\overline{G}$ have no factors in common.*

**Proof:** We will split the proof into two statements.

**Claim 1:** $\overline{G}$ does not divide $G$. Indeed, if $\overline{G}$ divides $G$ then there exists a polynomial of degree 1 such that

$$G(x,y) = \overline{G}(x,y)(f+gx+hy)$$

that is,

$$\begin{aligned}
0 = &-g(4K_1a^4+16bcK_1a^2)x^3 - g(8K_1a^4+32bcK_1a^2)x^2y^2 - h(4K_1a^4+16bcK_1a^2)x^2y \\
&+(a^2K_1 - f(4K_1a^4+16bcK_1a^2))x^2 - h(8K_1a^4+32bcK_1a^2)xy^3 \\
&+(2a^2K_1 - f(8K_1a^4+32bcK_1a^2))xy^2 - a^4gK_1K_7xy - g(8a^2c^2+32bc^3)x + a^2K_1y^4 - a^4hK_1K_7y^2 \\
&+(-h(8a^2c^2+32bc^3) - a^4fK_1K_7)y + 4c^2 - f(8a^2c^2+32bc^3),
\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}
0 &= g(4K_1a^4+16bcK_1a^2) & 0 &= a^4gK_1K_7 \\
0 &= g(8K_1a^4+32bcK_1a^2) & 0 &= g(8a^2c^2+32bc^3) \\
0 &= h(4K_1a^4+16bcK_1a^2) & 0 &= a^2K_1 \\
0 &= a^2K_1 - f(4K_1a^4+16bcK_1a^2) & 0 &= a^4hK_1K_7 \\
0 &= h(8K_1a^4+32bcK_1a^2) & 0 &= -h(8a^2c^2+32bc^3) - a^4fK_1K_7 \\
0 &= 2a^2K_1 - f(8K_1a^4+32bcK_1a^2) & 0 &= 4c^2 - f(8a^2c^2+32bc^3).
\end{aligned}$$

By the third equality of the second column we obtain $a^2K_1 = 0$, a contradiction and thus we proof the claim.

**Claim 2:** $\overline{G}$ is irreducible. Indeed, if $8K_1a^4+32bcK_1 = 0$ then $\overline{G}$ is irreducible. If $8K_1a^4+32bcK_1 \neq 0$ and $\overline{G}$ is reducible, then there exist polynomials of degree 1 and of degree 2 such that

$$\overline{G}(x,y) = (f+gx+hy)(i+jx+ky+lxy+mx^2+ny^2)$$

that is,

$$\begin{aligned}
0 = &-gmx^3 - (gl+hm)x^2y + (4K_1a^4+16bcK_1a^2 - gj - fm)x^2 \\
&+(8K_1a^4+32bcK_1a^2 - hl - gn)xy^2 - (fl+gk+hj)xy - (gi+fj)x - hny^3 \\
&-(hk+fn)y^2 + (K_1K_7a^4 - hi - fk)y + 8a^2c^2 + 32bc^3 - fi
\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}
0 &= gm & & & & \\
0 &= gl+hm & 0 &= fl+gk+hj & 0 &= hk+fn \\
0 &= 4K_1a^4+16bcK_1a^2 - gj - fm & 0 &= gi+fj & 0 &= K_1K_7a^4 - hi - fk \\
0 &= 8K_1a^4+32bcK_1a^2 - hl - gn & 0 &= hn & 0 &= 8a^2c^2+32bc^3 - fi.
\end{aligned}$$

By the first equality we obtain that $g = 0$ or $m = 0$. Let us suppose $g = 0$. We have

$$\begin{aligned}
0 &= hm & 0 &= fl+hj & 0 &= hk+fn \\
0 &= 4K_1a^4+16bcK_1a^2 - fm & 0 &= fj & 0 &= K_1K_7a^4 - hi - fk \\
0 &= 8K_1a^4+32bcK_1a^2 - hl & 0 &= hn & 0 &= 8a^2c^2+32bc^3 - fi,
\end{aligned}$$

and since $h \neq 0$ (otherwise we would not have factorization) we obtain $m = 0$ and therefore, by the second equality, $4K_1a^4 + 16bcK_1a^2 = 0$, a contradiction. Therefore $m = 0$. Hence

$$\begin{aligned}
&0 = gl && 0 = fl + gk + hj && 0 = hk + fn\\
&0 = 4K_1a^4 + 16bcK_1a^2 - gj && 0 = gi + fj && 0 = K_1K_7a^4 - hi - fk\\
&0 = 8K_1a^4 + 32bcK_1a^2 - hl - gn && 0 = hn && 0 = 8a^2c^2 + 32bc^3 - fi.
\end{aligned}$$

As seen before, assuming $g = 0$ leads us to a contradiction, so $g \neq 0$ and therefore by the first equality $l = 0$. We have,

$$\begin{aligned}
&0 = 4K_1a^4 + 16bcK_1a^2 - gj && 0 = gi + fj && \\
&0 = 8K_1a^4 + 32bcK_1a^2 - gn && 0 = hn && 0 = K_1K_7a^4 - hi - fk\\
&0 = gk + hj && 0 = hk + fn && 0 = 8a^2c^2 + 32bc^3 - fi.
\end{aligned}$$

Since $n \neq 0$ (otherwise we would not have factorization) we obtain by the fifth equality $h = 0$ Then,

$$\begin{aligned}
&0 = 4K_1a^4 + 16bcK_1a^2 - gj && 0 = gi + fj && 0 = K_1K_7a^4 - fk\\
&0 = 8K_1a^4 + 32bcK_1a^2 - gn && 0 = fn && 0 = 8a^2c^2 + 32bc^3 - fi.\\
&0 = gk && &&
\end{aligned}$$

Finally, by the fifth equality we have $f = 0$, which substituting in the last equality we obtain $8a^2c^2 + 32bc^3 = 0$, a contradiction and thus we conclude the claim. $\square$

**Lemma A.0.17.** *Define $H : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $\overline{H} : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$H(x,y) = a^2M_1y^2 + 2a^2M_1xy + a^2M_1x^2 + 4c^2$$

*and*

$$\begin{aligned}
\overline{H}(x,y) =&(16c^2 - 4a^2)y^5 + (64c^2 - 20a^2)xy^4 + M_9(4a^2 - 16b^2)y^4 + (96c^2 - 40a^2)x^2y^3\\
&+ M_9(12a^2 - 64b^2)xy^3 + (64c^2 - 40a^2)x^3y^2 + M_9(12a^2 - 96b^2)x^2y^2 + (16c^2 - 20a^2)x^4y\\
&+ M_9(4a^2 - 64b^2)x^3y - 4a^2x^5 - 16M_9b^2x^4,
\end{aligned}$$

*where $M_1, M_9, a, b, c$ are non-zero constants. We conclude that the polynomials $H$ and $\overline{H}$ have no factors in common.*

**Proof:** We will split the proof into two claims.

**Claim 1:** $H$ does not divide $\overline{H}$. Indeed, if $H$ divides $\overline{H}$ there exists a polynomial of degree 3 such that

$$\overline{H}(x,y) = (a^2M_1y^2 + 2a^2M_1xy + a^2M_1x^2 + 4c^2)(f + gx + hy + ixy + jx^2 + ky^2 + lxy^2 + mx^2y + nx^3 + oy^3)$$

that is,

$$\begin{aligned}0 =& -\left(4a^2+a^2nM_1\right)x^5+\left(16c^2-20a^2-a^2mM_1-2a^2nM_1\right)x^4y-\left(jK_1a^2+16M_9b^2\right)x^4\\ &+\left(64c^2-40a^2-a^2lM_1-2a^2mM_1-a^2nM_1\right)x^3y^2+\left(4a^2M_9-64b^2M_9-2a^2jM_1-a^2iM_1\right)x^3y\\ &-\left(gM_1a^2+4nc^2\right)x^3+\left(96c^2-40a^2-2a^2lM_1-a^2mM_1-a^2oM_1\right)x^2y^3\\ &+\left(12a^2M_9-96b^2M_9-a^2jM_1-a^2kM_1-2a^2iM_1\right)x^2y^2-\left(4c^2m+2a^2gM_1+a^2hM_1\right)x^2y\\ &-\left(fM_1a^2+4jc^2\right)x^2+\left(64c^2-20a^2-a^2lM_1-2a^2oM_1\right)xy^4\\ &+\left(12a^2M_9-64b^2M_9-2a^2kM_1-a^2iM_1\right)xy^3-\left(4c^2l+a^2gM_1+2a^2hM_1\right)xy^2\\ &-\left(2fM_1a^2+4ic^2\right)xy-\left(4c^2g\right)x+\left(16c^2-4a^2-a^2oM_1\right)y^5\\ &+\left(4a^2M_9-16b^2M_9-a^2kM_1\right)y^4-\left(hM_1a^2+4oc^2\right)y^3-\left(fM_1a^2+4kc^2\right)y^2-\left(4c^2h\right)y-4c^2f\end{aligned}$$

that occurs if, and only if,

$$\begin{aligned}0 &= 4a^2+a^2nM_1\\ 0 &= 16c^2-20a^2-a^2mM_1-2a^2nM_1\\ 0 &= jK_1a^2+16M_9b^2\\ 0 &= 64c^2-40a^2-a^2lM_1-2a^2mM_1-a^2nM_1\\ 0 &= 4a^2M_9-64b^2M_9-2a^2jM_1-a^2iM_1\\ 0 &= gM_1a^2+4nc^2\\ 0 &= 96c^2-40a^2-2a^2lM_1-a^2mM_1-a^2oM_1\\ 0 &= 12a^2M_9-96b^2M_9-a^2jM_1-a^2kM_1-2a^2iM_1\\ 0 &= 4c^2m+2a^2gM_1+a^2hM_1\\ 0 &= fM_1a^2+4jc^2\\ 0 &= 64c^2-20a^2-a^2lM_1-2a^2oM_1\end{aligned}$$

$$\begin{aligned}0 &= 12a^2M_9-64b^2M_9-2a^2kM_1-a^2iM_1\\ 0 &= 4c^2l+a^2gM_1+2a^2hM_1\\ 0 &= 2fM_1a^2+4ic^2\\ 0 &= 4c^2g\\ 0 &= 16c^2-4a^2-a^2oM_1\\ 0 &= 4a^2M_9-16b^2M_9-a^2kM_1\\ 0 &= hM_1a^2+4oc^2\\ 0 &= fM_1a^2+4kc^2\\ 0 &= 4c^2h\\ 0 &= 4c^2f.\end{aligned}$$

By the last equality $f=0$ and consequently, by the tenth equality, $j=0$. Finally, by the third equality, $M_9=0$ a contradiction and thus we conclude the claim.

**Claim 2:** The factors of $H$ do not divide $\overline{H}$. Indeed, notice that

$$H(x,y) = -M_1\left(2c\sqrt{-\frac{1}{M_1}}+ax+ay\right)\left(2c\sqrt{-\frac{1}{M_1}}-ax-ay\right).$$

If $b^2=\frac{a^4}{16c^2}$ then

$$\overline{H}(x,y) = -\frac{1}{c^2}(x+y)^3(a^2x+(a^2-4c^2)y)(4c^2x+4c^2y+a^2M_9)$$

then the factors of $H$ do not divide $\overline{H}$. If $b^2\neq\frac{a^4}{16c^2}$, that is, $0\neq(4bc+a^2)(-4bc+a^2)$ suppose that $2c\sqrt{-\frac{1}{M_1}}+ax+ay$ divides $\overline{H}$, then there exists a polynomial of degree 4 such that

$$\begin{aligned}\overline{H}(x,y) =& \left(2c\sqrt{-\frac{1}{M_1}}+ax+ay\right)(i+jx+ky+lxy+mx^2+ny^2+oxy^2+px^2y+qx^3+ry^3)\\ &+\left(2c\sqrt{-\frac{1}{M_1}}+ax+ay\right)(sxy^3+tx^2y^2+ux^3y+vx^4+wy^4)\end{aligned}$$

that is,

$$
\begin{aligned}
0 = & -\left(4a^2+va\right)x^5+\left(16c^2-av-20a^2-au\right)x^4y-\left(16M_9b^2+aq+2\sqrt{-\frac{1}{M_1}}cv\right)x^4\\
& +\left(64c^2-au-40a^2-at\right)x^3y^2+\left(4a^2M_9-64b^2M_9-ap-aq-2\sqrt{-\frac{1}{M_1}}cu\right)x^3y\\
& +\left(-am-2\sqrt{-\frac{1}{M_1}}cq\right)x^3+\left(96c^2-at-40a^2-as\right)x^2y^3\\
& +\left(12a^2M_9-96b^2M_9-ao-ap-2\sqrt{-\frac{1}{M_1}}ct\right)x^2y^2-\left(al+am+2\sqrt{-\frac{1}{M_1}}cp\right)x^2y\\
& -\left(aj+2\sqrt{-\frac{1}{M_1}}cm\right)x^2+\left(64c^2-aw-20a^2-as\right)xy^4\\
& +\left(12a^2M_2-64b^2M_9-ao-ar-2\sqrt{-\frac{1}{M_1}}cs\right)xy^3+\left(-al-an-2\sqrt{-\frac{1}{M_1}}co\right)xy^2\\
& +\left(-aj-ak-2\sqrt{-\frac{1}{M_1}}cl\right)xy+\left(-ai-2\sqrt{-\frac{1}{M_1}}cj\right)x+\left(16c^2-wa-4a^2\right)y^5\\
& +\left(4M_9a^2-ra-16M_2b^2-2\sqrt{-\frac{1}{M_1}}cw\right)y^4-\left(an+2\sqrt{-\frac{1}{M_1}}cr\right)y^3\\
& -\left(ak+2\sqrt{-\frac{1}{M_1}}cn\right)y^2-\left(ai+2\sqrt{-\frac{1}{M_1}}ck\right)y-2\sqrt{-\frac{1}{M_1}}ci
\end{aligned}
$$

that occurs if, and only if,

$$
\begin{aligned}
0 =& 4a^2+va\\
0 =& 16c^2-av-20a^2-au\\
0 =& 16M_9b^2+aq+2\sqrt{-\frac{1}{M_1}}cv\\
0 =& 64c^2-au-40a^2-at\\
0 =& 4a^2M_9-64b^2M_9-ap-aq-2\sqrt{-\frac{1}{M_1}}cu\\
0 =& -am-2\sqrt{-\frac{1}{M_1}}cq\\
0 =& 96c^2-at-40a^2-as\\
0 =& 12a^2M_9-96b^2M_9-ao-ap-2\sqrt{-\frac{1}{M_1}}ct\\
0 =& al+am+2\sqrt{-\frac{1}{M_1}}cp\\
0 =& aj+2\sqrt{-\frac{1}{M_1}}cm\\
0 =& 64c^2-aw-20a^2-as
\end{aligned}
$$

$$
\begin{aligned}
0 =& 12a^2M_2-64b^2M_9-ao-ar-2\sqrt{-\frac{1}{M_1}}cs\\
0 =& -al-an-2\sqrt{-\frac{1}{M_1}}co\\
0 =& -aj-ak-2\sqrt{-\frac{1}{M_1}}cl\\
0 =& -ai-2\sqrt{-\frac{1}{M_1}}cj\\
0 =& 16c^2-wa-4a^2\\
0 =& 4M_9a^2-ra-16M_2b^2-2\sqrt{-\frac{1}{M_1}}cw\\
0 =& an+2\sqrt{-\frac{1}{M_1}}cr\\
0 =& ak+2\sqrt{-\frac{1}{M_1}}cn\\
0 =& ai+2\sqrt{-\frac{1}{M_1}}ck\\
0 =& 2\sqrt{-\frac{1}{M_1}}ci.
\end{aligned}
$$

By the first equality $v=-4a$. By the last equality $i=0$ and consequently, by the other equalities,

$k = n = r = n = 0$. Then

$$
\begin{aligned}
0 =& 16c^2 + 4a^2 - 20a^2 - au\\
0 =& 16M_9b^2 + aq - 8a\sqrt{-\frac{1}{M_1}}c\\
0 =& 64c^2 - au - 40a^2 - at\\
0 =& 4a^2M_9 - 64b^2M_9 - ap - aq - 2\sqrt{-\frac{1}{M_1}}cu\\
0 =& -am - 2\sqrt{-\frac{1}{M_1}}cq\\
0 =& 96c^2 - at - 40a^2 - as\\
0 =& 12a^2M_9 - 96b^2M_9 - ao - ap - 2\sqrt{-\frac{1}{M_1}}ct\\
0 =& al + am + 2\sqrt{-\frac{1}{M_1}}cp
\end{aligned}
\qquad
\begin{aligned}
0 =& aj + 2\sqrt{-\frac{1}{M_1}}cm\\
0 =& 64c^2 - aw - 20a^2 - as\\
0 =& 12a^2M_2 - 64b^2M_9 - ao - 2\sqrt{-\frac{1}{M_1}}cs\\
0 =& -al - 2\sqrt{-\frac{1}{M_1}}co\\
0 =& -aj - 2\sqrt{-\frac{1}{M_1}}cl\\
0 =& -2\sqrt{-\frac{1}{M_1}}cj\\
0 =& 16c^2 - wa - 4a^2\\
0 =& 4M_9a^2 - ra - 16M_2b^2 - 2\sqrt{-\frac{1}{M_1}}cw.
\end{aligned}
$$

By the sixth equality of the second column $j = 0$ and consequently, by the other equalities, $l = o = m = p = q = 0$. Thus

$$
\begin{aligned}
0 =& 16c^2 + 4a^2 - 20a^2 - au\\
0 =& 16M_9b^2 - 8a\sqrt{-\frac{1}{M_1}}c\\
0 =& 64c^2 - au - 40a^2 - at\\
0 =& 4a^2M_9 - 64b^2M_9 - 2\sqrt{-\frac{1}{M_1}}cu\\
0 =& 96c^2 - at - 40a^2 - as
\end{aligned}
\qquad
\begin{aligned}
0 =& 12a^2M_9 - 96b^2M_9 - 2\sqrt{-\frac{1}{M_1}}ct\\
0 =& 64c^2 - aw - 20a^2 - as\\
0 =& 12a^2M_2 - 64b^2M_9 - ao - 2\sqrt{-\frac{1}{M_1}}cs\\
0 =& 16c^2 - wa - 4a^2\\
0 =& 4M_9a^2 - ra - 16M_2b^2 - 2\sqrt{-\frac{1}{M_1}}cw.
\end{aligned}
$$

By the first equality $u = \frac{16c^2+4a^2-20a^2}{a}$ and then

$$
\begin{aligned}
0 =& 16M_9b^2 - 8a\sqrt{-\frac{1}{M_1}}c\\
0 =& 64c^2 - 16c^2 + 4a^2 - 20a^2 - 40a^2 - at\\
0 =& 4a^2M_9 - 64b^2M_9 - 2\sqrt{-\frac{1}{M_1}}c\left(\frac{16c^2 + 4a^2 - 20a^2}{a}\right)\\
0 =& 96c^2 - at - 40a^2 - as\\
0 =& 12a^2M_9 - 96b^2M_9 - 2\sqrt{-\frac{1}{M_1}}ct
\end{aligned}
\qquad
\begin{aligned}
0 =& 64c^2 - aw - 20a^2 - as\\
0 =& 12a^2M_2 - 64b^2M_9 - ao - 2\sqrt{-\frac{1}{M_1}}cs\\
0 =& 16c^2 - wa - 4a^2\\
0 =& 4M_9a^2 - ra - 16M_2b^2 - 2\sqrt{-\frac{1}{M_1}}cw.
\end{aligned}
$$

By the second and third equalities

$$\sqrt{-\frac{1}{M_1}} = \frac{2b^2M_9}{ac} \qquad 0 = 4a^2M_9 - 64b^2M_9 - 2\frac{c}{a}\sqrt{-\frac{1}{M_1}}\left(16c^2 - 16a^2\right).$$

If $M_1, M_9$ do not verify the first equality we have a contradiction. If $M_1, M_9$ verify the first equality we obtain, substituting in the second equality, $0 = (4bc + a^2)(-4bc + a^2)$ a contradiction. The case for the factor $2c\sqrt{-\frac{1}{M_1}} - ax - ay$ is similar and thus we conclude the claim. □

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